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Distance magic-type and distance antimagic-type labelings of graphs

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DISTANCE MAGIC-TYPE AND DISTANCE ANTMAGIC-TYPE LABELINGS
OF GRAPHS

By

Bryan J. Freyberg

A DISSERTATION

Submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

In Mathematical Sciences

MICHIGAN TECHNOLOGICAL UNIVERSITY

2017

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This dissertation has been approved in partial fulfillment of the requirements for the Degree of DOCTOR OF PHILOSOPHY in Mathematical Sciences.

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Dedication

For Meryl and the pack,

who sacrificed more for this than I.

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Preface

Chapter 2, “ d -Handicap graphs,” was authored by B. Freyberg and M. Keranen and submitted for publication in Discrete Mathematics and is currently under review. All results were obtained by B. Freyberg and improved through the mutual cooperation of the two authors. The paper was written by B. Freyberg and reviewed by M. Keranen.

Chapter 3, “Orientable \mathbb{Z}_n -distance magic graphs,” was authored by S. Cichacz, B. Freyberg, and D. Froncek and submitted for publication in Discussiones Mathematicae Graph Theory and is currently under review. The paper was written by S. Cichacz and reviewed by B. Freyberg and D. Froncek. In addition to reviewing the paper, my main contributions to this work were Theorems 3.2.1, 3.2.9, 3.2.10, 3.3.5. The remaining results were obtained by S. Cichacz and D. Froncek.

Chapter 4, “Orientable \mathbb{Z}_n -distance magic graphs via products,” was authored by B. Freyberg and M. Keranen and submitted for publication in Australasian Journal of Combinatorics and is currently under review. The results of this work were obtained by B. Freyberg and improved through the mutual cooperation of the two authors. The paper was written by B. Freyberg and reviewed by M. Keranen.

Chapter 5, “Orientable \mathbb{Z}_n -distance magic labeling of Cartesian product of two cycles,” was authored by B. Freyberg and M. Keranen and submitted for publication in

Australasian Journal of Combinatorics and is currently under review. The results of this work are a product of the mutual cooperation of both authors. The paper was written by B. Freyberg and reviewed by M. Keranen.

Chapter 6, “Orientable \mathbb{Z}_n -distance magic labeling of Cartesian product of many cycles,” was authored by B. Freyberg and M. Keranen and submitted for publication in Electronic Journal of Graph Theory and Applications and is currently under review. The results of this work were obtained by B. Freyberg and improved through conversations between the two authors. The paper was written by B. Freyberg and reviewed by M. Keranen.

The Introduction to this dissertation is a compilation of ideas and excerpts taken from the aforementioned papers.

Figure 1.5 is licensed under a Creative Commons Attribution-Noncommercial-Share Alike License.

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Abstract

Generally speaking, a distance magic-type labeling of a graph G of order n is a bijection ℓ from the vertex set of the graph to the first n natural numbers or to the elements of a group of order n , with the property that the weight of each vertex is the same. The weight of a vertex x is defined as the sum (or appropriate group operation) of all the labels of vertices adjacent to x . If instead we require that all weights differ, then we refer to the labeling as a distance antimagic-type labeling. This idea can be generalized for directed graphs; the weight will take into consideration the direction of the arcs.

In this manuscript, we provide new results for d -handicap labeling, a distance antimagic-type labeling, and introduce a new distance magic-type labeling called orientable Γ -distance magic labeling.

A *d -handicap distance antimagic labeling* (or just *d -handicap labeling* for short) of a graph $G = (V, E)$ of order n is a bijection $\ell : V \rightarrow \{1, 2, \dots, n\}$ with induced *weight function*

$$w(x_i) = \sum_{x_j \in N(x_i)} \ell(x_j)$$

such that $\ell(x_i) = i$ and the sequence of weights $w(x_1), w(x_2), \dots, w(x_n)$ forms an arithmetic sequence with constant difference $d \geq 1$. If a graph G admits a d -handicap

labeling, we say G is a d -handicap graph.

A d -handicap incomplete tournament, $H(n, k, d)$ is an incomplete tournament of n teams ranked with the first n natural numbers such that each team plays exactly k games and the strength of schedule of the i^{th} ranked team is d more than the $i + 1^{st}$ ranked team. That is, strength of schedule increases arithmetically with strength of team. Constructing an $H(n, k, d)$ is equivalent to finding a d -handicap labeling of a k -regular graph of order n .

In Chapter 2 we provide general constructions for every $d \geq 1$ for large classes of both n and k , providing breadth and depth to the catalog of known $H(n, k, d)$'s.

In Chapters 3 - 6, we introduce a new type of labeling called orientable Γ -distance magic labeling. Let Γ be an abelian group of order n . If for a graph $G = (V, E)$ of order n there exists an orientation $\vec{G}(V, A)$ and a companion bijection $\vec{\ell} : V \rightarrow \Gamma$ with the property that there is a $\mu \in \Gamma$ (called the *magic constant*) such that

$$w(x) = \sum_{y \in N_G^+(x)} \vec{\ell}(y) - \sum_{y \in N_G^-(x)} \vec{\ell}(y) = \mu \text{ for every } x \in V(G),$$

where $w(x)$ is the weight of vertex x , we say that G is *orientable Γ -distance magic*.

In addition to introducing the concept, we provide numerous results on orientable \mathbb{Z}_n -distance magic graphs, where \mathbb{Z}_n is the cyclic group of order n .

In Chapter 7, we summarize the results of this dissertation and provide suggestions for future work.

Chapter 1

Introduction

For many members of society, “recreational mathematics” may sound like an oxymoron. Nevertheless, people have found entertainment in mathematical puzzles for time immemorial. In fact, all one has to do is page through a newspaper to find evidence that modern society still embraces recreational mathematics (e.g. Sudoku).

One of the most popular and oldest sources of recreational mathematics provides the motivation for this study. A *magic square* is an $n \times n$ array containing the first n^2 natural numbers without repeats such that all entries in each row, each column, main diagonal, or back diagonal have the same sum. As all things so ancient, the origin of the magic square is up for debate. Magic squares appear as early as 4,000 years ago in Chinese folklore. According to legend, a turtle adorned with a representation

of the Lo Shu 3×3 magic square from Figure 1.1 emerged from the Yellow River, forever imbibing magic squares into Chinese culture.

$$\begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix}$$

Figure 1.1: The Lo Shu square

Benjamin Franklin is most well known for his colorful proverbs, but he also constructed magic squares. After constructing an incredible 16×16 magic square, he proclaimed it to be, “the most magically magical of any magic square ever made by any magician [34].” He was possibly referring to other properties that his square possessed such as; all the entries in every half row or half column have the same sum, all the entries in every 4×4 subsquare have the same sum, and all entries in each bent diagonal have constant sum.

The idea of constructing square arrays with constant row and column sum can be generalized to rectangles. A *magic rectangle* is an $m \times n$ array containing the first mn natural numbers such that the sum of all the entries in each row is ρ and the sum of all the entries in each column is σ , for some ρ and σ . It is an easy exercise to show that if such an $m \times n$ magic rectangle exists, then $\rho = \frac{n(mn+1)}{2}$ and $\sigma = \frac{m(mn+1)}{2}$, since $1 + 2 + \dots + mn = \frac{mn(mn+1)}{2}$.

Magic squares and magic rectangles can be constructed using another popular and prevalent combinatorial object. A *Latin square* of order n is an $n \times n$ array containing

n symbols such that each symbol appears exactly once in every row and every column.

For example, a completed Sudoku puzzle is a Latin square of order 9. Also, the Cayley table of any finite group of order n is a Latin square of order n . In Chapter 5, Latin

$$\begin{bmatrix} A & D & C & B \\ B & A & D & C \\ C & B & A & D \\ D & C & B & A \end{bmatrix}$$

Figure 1.2: A Latin square of order 4

squares will play a key role in labeling the vertices of a graph.

We now turn our attention to graphs. A *graph*, $G = (V, E)$ is an ordered pair where V is a set of elements called *vertices* and E is a collection of two element subsets of V called *edges*. V and E are called the *vertex set* and *edge set*, respectively and sometimes the notation $V(G)$ and $E(G)$ are used, respectively. If $|V| = n$, we say the *order* of G is n . An assignment of a direction to the edges may also be considered. An *orientation* of a graph $G = (V, E)$ is an assignment of a direction to each edge, turning G into a *directed graph* $\vec{G} = (V, A)$. A graph can be visualized by drawing a point for each vertex and connecting two points with a line segment whenever the two points represent an edge. A directed edge may be visualized by using an arrow instead of a line segment.

Example 1.0.1. The graph $G = (V, E)$, where $V = \{0, 1, 2, 3\}$ and $E = \{\{0, 1\}, \{1, 2\}, \{2, 3\}, \{0, 2\}\}$ is shown in Figure 1.3. Figure 1.4 shows one possible orientation of G .

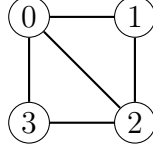


Figure 1.3: A visual representation of graph G

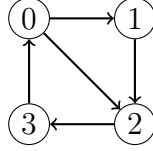


Figure 1.4: An orientation of G

We will use the standard notations xy to denote the edge $\{x, y\}$ and $x \sim y$ to denote the phrase, “ x is adjacent to y .” For a vertex x , the set of all vertices y such that $x \sim y$ is called the *neighborhood* of x and we denote it by $N(x)$. For an oriented graph \vec{G} , let $N^+(x) = \{y \in V : \vec{xy} \in A\}$ and $N^-(x) = \{z \in V : \vec{zx} \in A\}$ for all $x \in V(G)$. A graph is *simple* if it contains no edge of the form vv for some $v \in V$ and if E contains no edge more than once. Let $G = (V, E)$ be a simple graph and let C denote all possible two element subsets of V . Then the graph $H = (V, C \setminus E)$ is called the *complement* of G and we use the notation $H = \overline{G}$. For two graphs $G = (V, E)$ and $H = (V, E')$ with the same vertex set, by $G + H$ we mean the graph with vertex set V and edge set $E \cup E'$. Similarly, $G - H$ will denote the graph with vertex set V and edge set $E \cap \overline{E'}$, where $\overline{E'}$ is the set complement of E' . Two graphs G and H are *isomorphic*, and we write $G \cong H$, if there is a bijection $f : V(G) \rightarrow V(H)$ such that $xy \in E(G)$ if and only if $f(x)f(y) \in E(H)$. We leave it as an exercise for the reader to verify that the graph shown in Figure 1.5 is isomorphic to its complement.

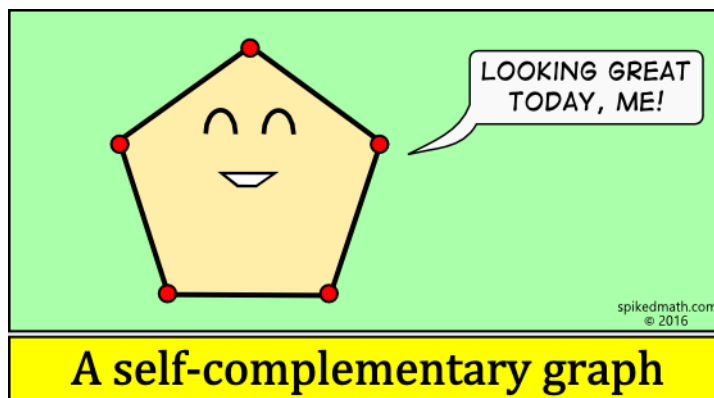


Figure 1.5: spikedmath.com/580.html

Some graphs are used so commonly they warrant special notation. The *complete graph* of order n , denoted K_n is the unique simple graph containing every possible edge. The *complete multipartite graph*, K_{n_1, n_2, \dots, n_k} is a graph on $n = n_1 + n_2 + \dots + n_k$ vertices which have been partitioned into k partite sets of size n_i for $i = 1, 2, \dots, k$ such that $xy \in E(K_{n_1, n_2, \dots, n_k})$ if and only if x and y are in different partite sets. A *path* on n vertices, P_n is a sequence of $n - 1$ edges such that every pair of consecutive edges in the sequence share exactly one vertex. Figure 1.6 shows a representation of P_4 .

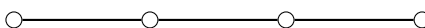


Figure 1.6: A path, P_4

A *cycle* of length n , C_n is the simple graph of order n with vertex set $V = \{x_1, x_2, \dots, x_n\}$ and edge set $E = \{x_i x_{i+1} : 1 \leq i \leq n\}$, where the addition in the indices is performed modulo n . For example, the graph in Figure 1.5 is isomorphic to C_5 . Let $0 \leq d_1 < d_2 < \dots < d_m \leq \lfloor n/2 \rfloor$. The *circulant graph* $C_n(d_1, d_2, \dots, d_m)$ is a graph with vertex set $V = \{x_0, x_1, \dots, x_{n-1}\}$ and edge set

$E = \{x_i x_{i+d_j} : 1 \leq i \leq n, 1 \leq j \leq m\}$ with addition in the indices performed modulo n . Figure 1.7 shows the circulant graph $C_8(1, 4)$.

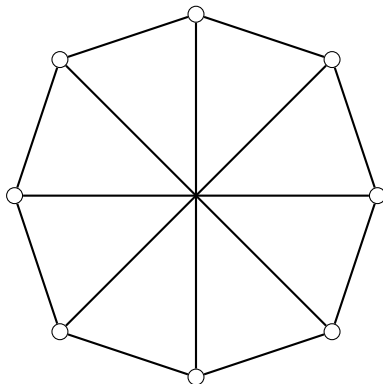


Figure 1.7: The circulant graph, $C_8(1, 4)$

A graph H is a *subgraph* of the graph G if $V(H) \subseteq V(G)$ and $E(H) \subset E(G)$. We say H is a *spanning subgraph* of G if $V(H) = V(G)$. A k -*factor* of a graph G is a spanning k -regular subgraph of G . A k -*factorization* of G is a partitioning of $E(G)$ into disjoint k -factors. If the graph G admits a k -factorization, we say G is k -*factorable*. When $k = 1$, we call the 1-factorization a *perfect matching*. Figure 1.8 shows a perfect matching of K_4 . Each color class is a 1-factor.

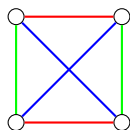


Figure 1.8: A perfect matching of K_4

Graphs may be combined using a variety of products. In the chapters that follow we will use four products, all of which are presented in [32]. All four, the *Cartesian*

product $G \square H$, direct product $G \times H$, the strong product $G \boxtimes H$, and the lexicographic product $G \circ H$, are graphs with the vertex set $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent in:

- $G \times H$ if $g \sim g'$ in G and $h \sim h'$ in H ;
- $G \square H$ if $g = g'$ and $h \sim h'$ in H , or $h = h'$ and $g \sim g'$ in G ;
- $G \boxtimes H$ if either $g = g'$ and $h \sim h'$ in H , or $h = h'$ and $g \sim g'$ in G , or $g \sim g'$ in G and $h \sim h'$ in H ;
- $G \circ H$ if and only if either $g \sim g'$ in G or $g = g'$ and $h \sim h'$ in H .

For a fixed vertex g of G , the subgraph of either of the above products induced by the set $\{(g, h) : h \in V(H)\}$ is called an H -layer and is denoted H^g . Similarly, if $h \in V(H)$ is fixed, then G^h , the subgraph induced by $\{(g, h) : g \in V(G)\}$, is called a G -layer. Interestingly, when the vertices of the product of two paths are arranged in the natural grid layout, the edges form a pattern akin to the symbol used to denote the product, with the exception of the lexicographic product. Figure 1.9 shows the Cartesian product $P_2 \square C_4$. Notice the graph is composed of 4 isomorphic P_2 -layers running vertically and 3 isomorphic C_4 -layers running horizontally.

Observe that of the products defined above, only the lexicographic product is not necessarily commutative. The lexicographic product $G \circ H$ is sometimes also referred

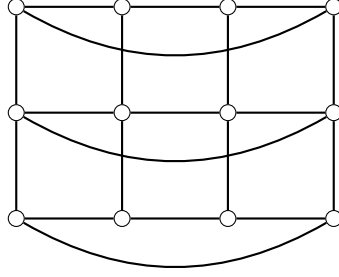


Figure 1.9: $P_2 \square C_4$

to as graph composition and denoted $G[H]$. The following informal description may be helpful to construct the graph $G \circ H$. Replace each vertex of G with an isomorphic copy of H . Then for every $xy \in E(G)$, construct the complete bipartite graph $K_{|V(H)|, |V(H)|}$ between the corresponding copies of H . Through this construction, one can see how a graph may be expressed via the lexicographic product. For example, $K_{3,3,3,3} \cong K_4 \circ \overline{K_3}$.

Let $G = (V, E)$ be given. A *graph labeling* of G is an assignment of integers or group elements to the elements of V , E , or $V \cup E$ satisfying certain prescribed properties. Credit for introducing the concept of graph labeling is most often given to Alex Rosa who in 1967 used it as a means for decomposing complete graphs into isomorphic subgraphs [42]. For a graph G on m edges, Rosa defined a β -valuation as an injection from the vertices of G to the set $\{0, 1, \dots, m-1\}$ such that for every edge xy , the induced edge labels $|f(x) - f(y)|$ are all distinct. The term *graceful* labeling is now almost exclusively used for this kind of labeling. Determining which graphs allow a graceful labeling provided the jumping off point for graph labeling research that

continues fifty years later. Gallian maintains an online survey of results in graph labeling containing over 1,200 references [31].

Since Rosa applied graceful labelings to decompose graphs, many applications of graph labeling have been found, fueling a staggering amount of research. Some application areas include coding theory, communication networks, radar, and x-ray crystallography [39]. One application we will consider in Chapter 2 is tournament design.

The focus of this manuscript is on a family of graph labelings called distance magic/antimagic-type labelings. Let G be a simple, undirected graph on n vertices. Let ℓ be a bijection $\ell : V(G) \rightarrow \{1, 2, \dots, n\}$, and define for every vertex $x \in V(G)$, the *weight* of x , $w(x) = \sum_{y \in N(x)} \ell(y)$. If the weight of every vertex is equal to the same number k , called the *magic constant*, then we say ℓ is a *distance magic labeling* of G . If such a labeling can be found, we say that G is *distance magic*. If instead it is required that all weights differ, then we say ℓ is a *distance antimagic labeling* of G . If such a labeling can be found, we say that G is *distance antimagic*. Figure 1.10 shows a distance magic labeling of the circulant graph $C_8(1, 3)$. Notice the weight of every vertex is 18.

A *d-handicap distance antimagic labeling* (or *d-handicap labeling* for short) of a graph

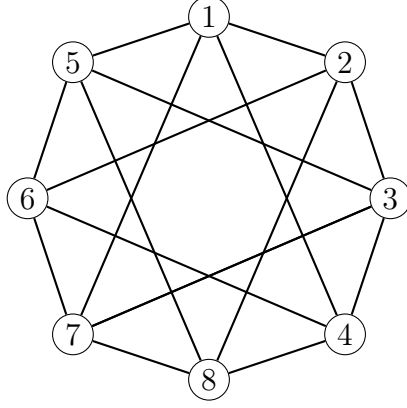


Figure 1.10: Distance magic labeling of $C_8(1, 3)$.

$G = (V, E)$ of order n is a bijection $\ell : V \rightarrow \{1, 2, \dots, n\}$ with induced *weight function*

$$w(x_i) = \sum_{x_j \in N(x_i)} \ell(x_j),$$

such that $\ell(x_i) = i$ and the sequence of weights $w(x_1), w(x_2), \dots, w(x_n)$ forms an arithmetic sequence with constant difference $d \geq 1$. If a graph G admits a d -handicap labeling, we say G is a *d-handicap graph*.

Let Γ be an abelian group of order n with operation $+$. For two elements $g, h \in \Gamma$, we use the notation $g - h$ to mean $g + h^{-1}$, where h^{-1} is the additive inverse of h . Also, for repeated addition $g + g + \dots + g$, where g appears k times, we use the notation kg . Let $G = (V, E)$ be a simple graph of order n . Let ℓ be a bijection $\ell : V(G) \rightarrow \Gamma$. If there exists $\mu \in \Gamma$ such that

$$w(x) = \sum_{y \in N(x)} \ell(y) = \mu,$$

for all vertices $x \in V(G)$, then we say G is Γ -distance magic. Clearly, if G is distance magic, then it is also \mathbb{Z}_n -distance magic (where \mathbb{Z}_n denotes the cyclic group of order n), but the converse is not necessarily true.

A *directed Γ -distance magic labeling* of an oriented graph $\vec{G}(V, A)$ of order n is a bijection $\vec{\ell} : V \rightarrow \Gamma$ with the property that there is a $\gamma \in \Gamma$, such that

$$w(x) = \sum_{y \in N_G^+(x)} \vec{\ell}(y) - \sum_{y \in N_G^-(x)} \vec{\ell}(y) = \gamma \text{ for every } x \in V.$$

If a graph G admits an orientation \vec{G} for which a directed Γ -distance magic labeling $\vec{\ell}$ exists, we say that G is *orientable Γ -distance magic* and we call the directed Γ -distance magic labeling $\vec{\ell}$ an *orientable Γ -distance magic labeling*.

One of the most interesting applications of distance magic/antimagic-type labelings is designing tournaments on ranked teams. Suppose we wish to construct a tournament of n teams ranked with the first n natural numbers. In practice, this ranking is usually based on previous performance. Let 1 be the weakest team, 2 be the second weakest team, and so on, so that n is the strongest team. The tournament may be modeled with a graph G in the most natural way; each team is represented by a vertex and $xy \in E(G)$ if and only if team x plays team y in the tournament. Define the *strength of schedule* of each team x as $S(x) = \sum_{y \in N(x)} f(y)$, where $f(y)$ is the ranking of team y .

An *equalized incomplete tournament*, $\text{EIT}[n, k]$ is a tournament in which every team

plays exactly $k < n - 1$ games and $S(x) = \mu$, for some constant μ and every team x . Therefore, constructing an $\text{EIT}[n, k]$ is equivalent to finding a k -regular distance magic graph of order n . In an equalized incomplete tournament, the strongest teams should fare better than the weaker teams, since the strength of schedule is the same for all teams. This observation motivates the next tournament type, designed to assist the weaker teams.

A *d-handicap incomplete tournament*, $H(n, k, d)$ is a tournament in which every team plays exactly $k < n - 1$ games and the strength of schedule of the i^{th} ranked team is d more than the $(i + 1)^{\text{st}}$ ranked team. That is, strength of schedule increases d -arithmetically with strength of team. Observe that finding an $H(n, k, d)$ is equivalent to finding a k -regular d -handicap graph, G of order n . We now have all the tools and basic notions required to proceed to Chapter 2.

Chapter 2

d-Handicap tournaments[★]

2.1 Motivation

When scheduling a tournament, it is common practice to use the rankings of the teams from the previous season, or some other source to determine the list of opponents for each team. A tournament may be modeled with a graph in the most natural way; each team is represented with a vertex and two vertices are adjacent if and only if the corresponding two teams play each other.

Suppose we have n teams ranked with the first n natural numbers and let i be the team ranked i . For $i \in \{1, 2, \dots, n\}$, let $w(i)$ represent the sum of the rankings of all

[★]The material in this chapter has been submitted to Discrete Mathematics

opponents of team i . We call w the *strength of schedule*. In a round-robin tournament (a tournament in which every team plays all other teams), $w(i) = \frac{n(n+1)}{2} - i$ for each i . Since $\frac{\Delta w}{\Delta i} = -1$, the strengths of schedules form an arithmetic progression with difference -1 in a round-robin tournament. Therefore, the weakest team has the most difficult strength of schedule while the strongest team has the weakest strength of schedule. Clearly, the strongest team is most likely to benefit from this kind of tournament. This motivates the following questions that are of interest for both tournament scheduling reasons and purely graph theoretic reasons.

- Can we design a tournament with less games, but maintain the same weight structure as the round-robin tournament?
- Can we design a tournament so each team has the same strength of schedule?
- Can we turn things around so that the weakest team has the weakest strength of schedule?

Clearly, to address these questions, one must consider only *incomplete* tournaments, that is tournaments in which each team plays exactly $k < n - 1$ other teams (unless otherwise noted, it is assumed that all the tournaments discussed here are regular tournaments). A *fair incomplete tournament* is an incomplete tournament where $\frac{\Delta w}{\Delta i} = -1$ for every team i . These tournaments address the first question above. See [22, 29] for results regarding fair incomplete tournaments.

Equalized incomplete tournaments address the second question above. An *equalized incomplete tournament* is an incomplete tournament such that $w(i) = \mu$, for some constant μ , for every team i . The corresponding graph is called a *distance magic graph*. A *distance magic labeling* of a simple graph $G = (V, E)$ of order n is a bijection $f : V \rightarrow \{1, 2, \dots, n\}$ such that there exists an integer μ called the *magic constant*, so that $w(x) = \sum_{y \in N(x)} f(y) = \mu$ for all $x \in V$. Here $N(x) = \{y | xy \in E\}$ represents the *open neighborhood* of x .

The last question is addressed by a d -handicap tournament. A *d -handicap distance antimagic labeling* (or *d -handicap labeling* for short) of a graph $G = (V, E)$ of order n is a bijection $\ell : V \rightarrow \{1, 2, \dots, n\}$ with induced *weight function*

$$w(x_i) = \sum_{x_j \in N(x_i)} \ell(x_j),$$

such that $\ell(x_i) = i$ and the sequence of weights $w(x_1), w(x_2), \dots, w(x_n)$ forms an arithmetic sequence with constant difference $d \geq 1$. If a graph G admits a d -handicap labeling, we say G is a *d -handicap graph*. If G is k -regular, then we say G corresponds to a *k -regular d -handicap tournament*, and we denote it by $H(n, k, d)$. For $d = 1$, the existence for n even has recently been settled for every pair (n, k) (see Theorem 2.5.1) [30]. For $d = 1$ and n odd, the existence of a 1-handicap tournament with at least one value of k is settled [28]. For $d = 2$, one class of n has been completely settled and one class partially settled (see Theorems 2.4.1, 2.4.2) [20, 21, 26].

A similar but less restrictive labeling has been considered by Arumugam and Kamatchi in [6]. An (a, d) -distance antimagic labeling of a graph $G = (V, E)$ of order n is a bijection $l : V \rightarrow \{1, 2, \dots, n\}$ with induced weight function

$$w(x_i) = \sum_{x_j \in N(x_i)} l(x_j),$$

such that the weights form the set $\{a, a + d, a + 2d, \dots, a + (n - 1)d\}$ for some fixed integers a and $d \geq 0$. Therefore, a d -handicap distance antimagic labeling is an (a, d) -distance magic labeling, but the converse is not necessarily true.

In this chapter we provide necessary conditions for the existence of d -handicap tournaments, $H(n, k, d)$, and construct such tournaments for large classes of n and a wide range of regularities k , for every $d \geq 1$. Corollaries of our main result include complete characterizations of 1-handicap tournaments for $n \equiv 0 \pmod{8}$ and 2-handicap tournaments for $n \equiv 0 \pmod{16}$, although both results were known [20, 43]. For larger d , our construction provides a nearly complete characterization for appropriate classes of n .

For a survey of distance magic and antimagic labelings, see [5]. The survey also provides a summary of the results regarding the tournaments we have discussed in this section.

2.2 Definitions, notation, and tools

All graphs in this chapter are simple, finite graphs. We use the notation $V(G)$ to denote the vertex set of G and the notation $E(G)$ to denote the edge set of G . If $|V(G)| = n$, we say the graph G has *order* n . The *neighborhood* of a vertex $x \in V(G)$, denoted $N(x)$ or $N_G(x)$, is the set of all vertices in $V(G)$ adjacent to x .

We use the notation xy to denote an edge between vertex x and vertex y and the notation $x \sim y$ to mean x is adjacent to y . Let $K_{n,m} = [x_1, x_2, \dots, x_n \mid y_1, y_2, \dots, y_m]$ denote the complete bipartite graph $K_{n,m}$ with partite sets $\{x_1, x_2, \dots, x_n\}$ and $\{y_1, y_2, \dots, y_m\}$. Let $K_{(n;d)}$ denote the complete equipartite graph with d partite sets of size n . For two graphs G and H , we use the notation $G + H$ to denote the union of graphs G and H . That is, $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H)$. The *complement* of G is denoted by \overline{G} .

The constructions used in this chapter utilize two graph products. Given two graphs G and H , both products, the *lexicographic product*, $G \circ H$, and the *Cartesian product*, $G \square H$, have vertex set $V(G) \times V(H)$ and two vertices (g_1, h_1) and (g_2, h_2) are adjacent in

- $G \circ H$ if and only if $g_1 \sim g_2$ in G or $g_1 = g_2$ and $h_1 \sim h_2$ in H ,
- $G \square H$ if and only if $g_1 = g_2$ and $h_1 \sim h_2$ in H , or $h_1 = h_2$ and $g_1 \sim g_2$ in G .

The lexicographic product $G \circ H$ has sometimes been called graph composition and has also been denoted $G(H)$. To construct the graph $G \circ H$, the following informal description may be helpful. First, replace each vertex of G with an isomorphic copy of H . Then for every $xy \in E(G)$, construct the complete bipartite graph $K_{|V(H)|, |V(H)|}$ between the corresponding copies of H . For the graph $G \circ \overline{K_2}$, we will refer to each pair of isolated vertices which replace a vertex of G as *blown-up* vertices. For a fixed vertex g of G , the subgraph of either of the above products induced by the set $\{(g, h) : h \in V(H)\}$ is called an H -layer and is denoted H^g . Similarly, if $h \in V(H)$ is fixed, then G^h , the subgraph induced by $\{(g, h) : g \in V(G)\}$, is a G -layer.

Circulant graphs are nice candidates for constructing tournaments since they are vertex-transitive, regular, and can easily be manipulated to be more or less dense. Let $S = \{d_1, d_2, \dots, d_m\}$ and $1 \leq d_1 < d_2 < \dots < d_m \leq \lfloor \frac{m}{2} \rfloor$. We call S the *connection set*. Then the *circulant graph* $G = C_n(S)$ is a graph with vertex set $V(G) = \{x_0, x_1, \dots, x_{n-1}\}$ and two vertices x_i and x_j are adjacent in G if and only if $i - j \equiv d_k \pmod{n}$ for some $k \in \{1, 2, \dots, m\}$. Froncek and Cichacz in [15] showed certain classes of circulant graphs are distance magic.

A *1-factor* or *perfect matching* of a graph G is a union of disjoint edges $xy \in E(G)$

such that every vertex $v \in V(G)$ appears exactly once in the union. If for a graph G the edge set $E(G)$ can be partitioned into a disjoint union of 1-factors, then we say G is 1-*factorable*.

Let $S = \{a, a+1, a+2, \dots, b\}$ be a set of consecutive integers for integers a, b such that $a \leq b$. If $\alpha, \beta \in S$ such that $\alpha + \beta = a + b$, we will refer to the numbers α and β as S -*complements*, or simply complements if the set is clear from the context. It is obvious that if $|S|$ is even, then S can be partitioned into complement pairs.

We finish this section by introducing the main mechanism for increasing the density of a d -handicap graph. To the authors' knowledge, this derived graph was first used by Shepanik in his master's thesis in which he coined it a, "bubble graph [43] ."

Let G be a simple k -regular graph of even order n . Let a bijective labeling $l : V(G) \rightarrow \{1, 2, \dots, n\}$ be given and define the weight of a vertex, $w(i) = \sum_{y \in N(i)} l(y)$ for all $i \in V(G)$. Then, the *bubble graph of G* , denoted $B(G)$, is a simple graph with vertex set

$$V(B(G)) = \{(a, A) : a, A \in V(G), l(a) + l(A) = n + 1, l(a) < l(A)\}$$

and $(a, A)(b, B) \in E(B(G))$ if and only if $\{ab, aB, Ab, AB\} \cap E(G)$ is non-empty.

Observe that every edge $(a, A)(b, B) \in E(\overline{B(G)})$ represents the $K_{2,2} = [a, A \mid b, B]$

which may be added to $E(G)$ so that $\deg_G(i) = k + 2$ and $w_G(i)$ is increased by $n + 1$, for all $i \in \{a, A, b, B\}$. Therefore, each 1-factor of $\overline{B(G)}$ (if one exists) gives rise to a 2-regular distance magic factor, H which may be added to G , increasing the regularity of G by two while adding the same weight to every vertex.

In order to simplify notation, we may sometimes refer to a vertex by its label. This should not cause any confusion since the labelings considered in this chapter are bijections.

2.3 Necessary conditions and lemmas

We begin with some necessary conditions.

Theorem 2.3.1. *If an $H(n, k, d)$ exists, then*

1. $w(x_i) = di + \frac{(k-d)(n+1)}{2}$, for all $i \in \{1, 2, \dots, n\}$.
2. If n is even, then $k \equiv d \pmod{2}$.
3. If n is odd, then $k \equiv 0 \pmod{2}$.
4. $n \geq \left\lceil 2 \left(d + 1 + \sqrt{d(d+1)} \right) \right\rceil$.
5. $\left\lceil \frac{n-2-\sqrt{D}}{2} \right\rceil \leq k \leq \left\lfloor \frac{n-2+\sqrt{D}}{2} \right\rfloor$, where $D = (n-2)^2 - 4d(n-1)$.

Proof. Let $G \cong H(n, k, d)$ be given for some n , k , and d with d -handicap distance antimagic labeling $l(x_i) = i$. Then $w(x_i) = \mu + di$ for all $i \in \{1, 2, \dots, n\}$ and some integer μ . Then summing the weights, we have

$$w(G) = \sum_{i=1}^n w(x_i) = \mu n + d \sum_{i=1}^n i$$

and

$$w(G) = k \sum_{i=1}^n i,$$

since G is k -regular. Therefore,

$$\mu = \frac{k-d}{n} \sum_{i=1}^n i = \frac{(k-d)(n+1)}{2}.$$

If n is even, then $k-d$ must be even (recall μ is an integer), which implies $k \equiv d \pmod{2}$. If n is odd, then obviously $k \equiv 0 \pmod{2}$, so we have proven 1, 2 and 3.

The weight of 1 is at least as large as the sum of the smallest k possible neighbors.

Therefore,

$$w(x_1) = d + \frac{(k-d)(n+1)}{2} \geq \sum_{i=1}^k (i+1) = k + \frac{k(k+1)}{2}.$$

Hence,

$$-k^2 + k(n-2) - d(n-1) \geq 0.$$

The discriminant, $D = (n - 2)^2 - 4d(n - 1) \geq 0$ which implies

$$n \geq \left\lceil 2 \left(d + 1 + \sqrt{d(d + 1)} \right) \right\rceil$$

by the quadratic formula and the fact that n is an integer, proving 4. Finally, we have

$$\frac{2 - n - \sqrt{D}}{2} \leq k \leq \frac{2 - n + \sqrt{D}}{2}$$

again by the quadratic formula, which gives

$$\left\lceil \frac{n - 2 - \sqrt{D}}{2} \right\rceil \leq k \leq \left\lfloor \frac{n - 2 + \sqrt{D}}{2} \right\rfloor$$

after multiplying through by -1 and acknowledging that k is an integer. Therefore, 5 is true and we are done. \square

The following theorem was proved by Anderson and Lipman in [2].

Theorem 2.3.2. [2] *Let G be a k -regular graph which is 1-factorable and let H be any r -regular graph. Then the lexicographic product $G \circ H$ is 1-factorable.*

We will now prove some factorization lemmas to be used in the main theorems.

Lemma 2.3.3. *For every integer $n \geq 2$, the graph $C_n \circ \overline{K_2}$ is 1-factorable.*

Proof. Let $G = C_n \circ \overline{K_2}$ with vertex set $V(G) = \{x_i^j : i = 0, 1, \dots, n - 1, j = 0, 1\}$ and

edge set $E(G) = \{x_i^j x_{i+1}^p : i = 0, 1, \dots, n-1, j, p \in \{0, 1\}\}$, where the arithmetic is performed modulo n in the subscript. If n is even, then C_n is obviously 1-factorable, so we are done by Theorem 2.3.2. If n is odd, let

$$\begin{aligned} F_0 &= \{x_i^0 x_{i+1}^1 : i = 0, 1, \dots, m-1\}, \\ F_1 &= \{x_i^1 x_{i+1}^0 : i = 0, 1, \dots, m-1\}, \\ F_2 &= \{x_{m-1}^0 x_0^1\} \cup \{x_i^0 x_{i+1}^0, x_{i+1}^1 x_{i+2}^1 : i = 0, 2, \dots, m-3\}, \\ F_3 &= \{x_{m-1}^1 x_0^0\} \cup \{x_i^1 x_{i+1}^1, x_{i+1}^0 x_{i+2}^0 : i = 0, 2, \dots, m-3\}. \end{aligned}$$

Then it is easy to see that each F_i is a 1-factor. Since it is also clear that the 1-factors are disjoint and partition $E(G)$, we have found a 1-factorization of G , proving the lemma. \square

Lemma 2.3.4. *For every integer $n \geq 2$, the graph $C_n(S) \circ \overline{K_2}$ is 1-factorable for any connection set S .*

Proof. Let $n \geq 2$ and let $G \cong C_n(S) \circ \overline{K_2}$ for some connection set S . Let $d \in S$. Then it is easy to see that d induces the spanning subgraph $\frac{n}{m}(C_m \circ \overline{K_2})$ of G where $m = \text{ord}_{\mathbb{Z}_n} d = \frac{n}{\gcd(d, n)}$. Therefore, it suffices to show that for any $m \geq 2$, the graph $C_m \circ \overline{K_2}$ is 1-factorable, so Lemma 2.3.3 gives the result. \square

The next lemma follows easily.

Lemma 2.3.5. *For every integer $n \geq 2$, the graph $C_n(S) \circ K_2$ is 1-factorable for any connection set S .*

An equipartite graph is a multipartite graph in which all partite sets have the same cardinality. We conclude this section by considering regular equipartite graphs. It is well known that both the even-ordered complete graph, K_{2n} and every regular bipartite graph allow 1-factorizations. Thus we have the following result.

Lemma 2.3.6. *Let G be an equipartite graph with an even number of partite sets. If the edges between every pair of partite sets form an r -regular subgraph of G for some fixed r , then G is 1-factorable.*

Alspach and Gavlas proved that the graph $K_{2n} - I$, where I is a 1-factor, may be decomposed into cycles of length m where m divides the number of edges in G [1]. The next theorem follows easily since $K_{2n} - I$ contains $m(n-1)$ edges where $m = 2n$.

Theorem 2.3.7. [1] *Let $G = K_{2n} - I$ where I is any 1-factor. The graph G allows a 1-factorization.*

The next result follows in the same way as Lemma 2.3.6.

Lemma 2.3.8. *Let positive integers n, r be given and let G be an equipartite graph with partite sets P_1, P_2, \dots, P_{2n} . If for each P_i , there exists exactly one P_j , $i \neq j$, such that there are no edges between P_i and P_j and the edges between P_i and P_k , $k \neq j$ form an r -regular graph, then G is 1-factorable.*

2.4 Even d

2.4.1 Known results

For $d = 2$, the only work that has been done is by Froncek who used magic rectangle sets to obtain the following results.

Theorem 2.4.1. [20] *If $n \equiv 0 \pmod{16}$, then an $H(n, k, 2)$ exists if and only if k is even and $4 \leq k \leq n - 6$.*

When $n \equiv 8 \pmod{16}$, he obtained the following partial results.

Theorem 2.4.2. [26] *If $n \equiv 8 \pmod{16}$ and $n \geq 56$, then an $H(n, k, 2)$ exists if k is even and $6 \leq k \leq n - 50$.*

One of the primary ingredients for the constructions given in this section are distance magic graphs. Froncek et. al. proved the following in [29].

Theorem 2.4.3. [29] *For n even, an r -regular distance magic graph of order n exists if and only if $2 \leq r \leq n - 2$, $r \equiv 0 \pmod{2}$, and either $n \equiv 0 \pmod{4}$ or $r \equiv 0 \pmod{4}$.*

For regular graphs of odd order, the existence question is partially answered by the following result proved by Froncek in [22].

Theorem 2.4.4. [22] *Let n be an odd integer and $r = 2^s q$ with $q > 1$ odd and $s \geq 1$. Then an r -regular distance magic graph of order n exists whenever $r \leq \frac{2}{7}(n - 2)$.*

2.4.2 New results

The first construction in this section uses distance magic graphs to produce classes of $2d$ -regular d -handicap graphs for any even $d \geq 2$.

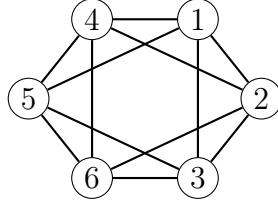


Figure 2.1: Distance magic labeling of $C_6(1, 2)$.

Theorem 2.4.5. *Let $d \geq 2$ be an even integer and let G be any d -regular distance magic graph of order $v \geq d + 2$. Let $n = vt$ for any even integer $t \geq d + 2$. If $d \equiv 0 \pmod{4}$ or $t \equiv 0 \pmod{4}$, then there exists an $H(n, 2d, d)$.*

Proof. Let G be a d -regular distance magic graph on $v \geq d + 2$ vertices with vertex set $V(G) = \{g_0, g_1, \dots, g_{v-1}\}$ and distance magic labeling f . Such a graph exists by Theorem 2.4.3. We may assume $f(g_i) = i + 1$ for $i = 0, 1, \dots, v - 1$. Then since G is

distance magic and d -regular,

$$\begin{aligned}
\sum_{g_p \in N(g_i)} f(g_p) &= \sum_{g_p \in N(g_i)} (p + 1) \\
&= d + \sum_{g_p \in N(g_i)} p \\
&= \mu,
\end{aligned}$$

where μ is the magic constant of G . In particular, we will use the identity $\sum_{g_p \in N(g_i)} p = \mu - d$ later. Then $\mu = \frac{d(v+1)}{2}$ since

$$\begin{aligned}
\sum_{g \in V(G)} w(g) &= d \sum_{i=1}^v i \\
&= \frac{dv(v+1)}{2} \\
&= v\mu.
\end{aligned}$$

Let $H = C_{\frac{t}{2}}(1, 2, \dots, \frac{d}{4}) \circ \overline{K_2}$ if $d \equiv 0 \pmod{4}$, otherwise let $H = C_{\frac{t}{2}}(1, 2, \dots, \frac{d-2}{4}, \frac{t}{4}) \circ \overline{K_2}$. Let the vertex set of H be $V(H) = \{h_0, h_1, \dots, h_{t-1}\}$ where each pair (h_j, h_{j+1}) for $j = 0, 2, \dots, t-2$ forms the blown-up vertices of H . Let T be the Cartesian product $T = G \square H$. For ease of notation, let $x_i^j = (g_i, h_j) \in V(T)$ for $i = 0, 1, \dots, v-1$, $j = 0, 1, \dots, t-1$.

Let $l : V(T) \rightarrow \{1, 2, \dots, n\}$ be defined as

$$\begin{aligned} l(x_i^j) &= ti + \frac{j}{2} + 1, \\ l(x_i^{j+1}) &= t(i+1) - \frac{j}{2}, \end{aligned}$$

for all $i = 0, 1, \dots, v-1, j = 0, 2, \dots, t-2$. Clearly, l is a bijection. Notice that $l(x_i^j) + l(x_i^{j+1}) = t(2i+1) + 1$. Therefore, since H is d -regular, the weight induced on every vertex by each H -layer is

$$w_H(x_i^j) = \frac{d}{2} (t(2i+1) + 1),$$

for all $x_i^j \in V(T)$. Now for $j = 0, 2, \dots, t-d$, we have

$$N_{G^{h_j}}(x_i^j) = \{x_p^j : g_p \in N_G(g_i)\}.$$

Then for $i = 0, 1, \dots, v-1$, the weight induced on every vertex by each G -layer is

$$\begin{aligned} w_G(x_i^j) &= \sum_{x_p^j \in N_T(x_i^j)} l(x_p^j) \\ &= d\left(\frac{j}{2} + 1\right) + t \sum_{g_p \in N_G(g_i)} p \\ &= d\left(\frac{j}{2} + 1\right) + t(\mu - d) \\ &= d\left(\frac{j}{2} + 1 - t\right) + t\mu, \end{aligned}$$

and

$$\begin{aligned}
w_G(x_i^{j+1}) &= \sum_{x_p^{j+1} \in N_T(x_i^{j+1})} l(x_p^{j+1}) \\
&= d\left(t - \frac{j}{2}\right) + t \sum_{g_p \in N_G(g_i)} p \\
&= -\frac{jd}{2} + t\mu.
\end{aligned}$$

Summing the weights, we express the weight of every vertex $v \in V(T)$ by

$$\begin{aligned}
w(x_i^j) &= w_G(x_i^j) + w_H(x_i^j) \\
&= \frac{d(j+t(2i-1)+3)}{2} + t\mu,
\end{aligned}$$

and

$$\begin{aligned}
w(x_i^{j+1}) &= w_G(x_i^{j+1}) + w_H(x_i^{j+1}) \\
&= \frac{d(-j+t(2i+1)+1)}{2} + t\mu,
\end{aligned}$$

for $i = 0, 1, \dots, v-1$ and $j = 0, 2, \dots, t-2$. Now we will show that l is a d -handicap labeling. Let $x_i^j \in V(T)$ be given.

Case 1. $j = 0, 2, \dots, t-4$. Then $l(x_i^{j+2}) - l(x_i^j) = [ti + \frac{j+2}{2} + 1] - [ti + \frac{j}{2} + 1] = 1$, and $w(x_i^{j+2}) - w(x_i^j) = \frac{d(j+2+t(2i-1)+3)}{2} + t\mu - \left(\frac{d(j+t(2i-1)+3)}{2} + t\mu\right) = d$.

Case 2. $j = t-2$. Then $l(x_i^{t-1}) - l(x_i^{t-2}) = [t(i+1) - \frac{t-2}{2}] - [ti + \frac{t-2}{2} + 1] = 1$, and $w(x_i^{t-1}) - w(x_i^{t-2}) = \frac{d(-(t-2)+t(2i+1)+1)}{2} + t\mu - \left(\frac{d(t-2+t(2i-1)+3)}{2} + t\mu\right) = d$.

Case 3. $j = 3, 5, \dots, t-1$. Then $l(x_i^{j-2}) - l(x_i^j) = t(i+1) - \frac{j-3}{2} - [t(i+1) - \frac{j-1}{2}] = 1$, and $w(x_i^{j-2}) - w(x_i^j) = \frac{d(-(j-3)+t(2i+1)+1)}{2} + t\mu - \left(\frac{d(-(j-1)+t(2i+1)+1)}{2} + t\mu\right) = d$.

Therefore, the sequence $s_i = l(x_i^0), l(x_i^2), \dots, l(x_i^{t-2}), l(x_i^{t-1}), l(x_i^{t-3}), \dots, l(x_i^3), l(x_i^1)$ is 1-arithmetic and the corresponding sequence of weights $w_i = w(x_i^0), w(x_i^2), \dots, w(x_i^{t-2}), w(x_i^{t-1}), \dots, w(x_i^3), w(x_i^1)$ is d -arithmetic. Now consider the set $S = \{w(x_0^0), w(x_1^0), \dots, w(x_{v-1}^0)\}$. We have $w(x_i^0) = \frac{d(t(2i-1)+3)}{2} + t\mu$. Therefore $S = \{w(x_0^0), w(x_0^0) + td, \dots, w(x_0^0) + (d-1)td\}$ since $i \in \{0, 1, \dots, v-1\}$. Hence, l is a d -handicap labeling and we have proven the theorem. \square

If we impose some additional restrictions on the distance magic graph G in the previous theorem, we can provide a large range of regularities for each class of d -handicap graphs produced.

Theorem 2.4.6. *Let $d \geq 2$ and $t, v \geq d + 2$ be even integers and let $n = vt$. Let $G = (V, E)$ be a d -regular distance magic graph of even order v with vertex set $V = \{g_0, g_1, \dots, g_{v-1}\}$ and the following additional properties.*

- $g_i g_j \in E$ if and only if $g_{v-1-i} g_{v-1-j} \in E$ and
- \overline{G} is 1-factorable.

If $d \equiv 0 \pmod{4}$ or $t \equiv 0 \pmod{4}$, then there exists an $H(n, k, d)$ for all even k such that $2d \leq k \leq n - 2d - 2$.

Proof. Let $T = G \square H$ be the $H(n, 2d, d)$ constructed in Theorem 2.4.5 with associated d -handicap labeling l . Observe that $l(x_i^j) + l(x_{v-1-i}^{j+1}) = l(x_i^{j+1}) + l(x_{v-1-i}^j) = n + 1$ for

$i = 0, 1, \dots, \frac{v}{2} - 1$ and $j = 0, 2, \dots, t - 2$. Now consider the bubble graph, $B(T)$. We have

$$V(B(T)) = \left\{ (x_i^j, x_{v-1-i}^{j+1}), (x_i^{j+1}, x_{v-1-i}^j), 0 \leq i \leq \frac{v}{2} - 1, j = 0, 2, \dots, t - 2 \right\}.$$

For ease of notation, let $u_i^j = (x_i^j, x_{v-1-i}^{j+1})$ and $u_i^{j+1} = (x_i^{j+1}, x_{v-1-i}^j)$ for $0 \leq i \leq \frac{v}{2} - 1$, and $j = 0, 2, \dots, t - 2$.

In T , every pair of H -layers, $(H^{g_i}, H^{g_{v-1-i}})$ form a subgraph of $B(T)$ isomorphic to H^{g_i} for all $i = 0, 1, \dots, \frac{v-1}{2}$. Indeed, let $[x_i^j, x_i^{j+1} \mid x_i^{j+2s}, x_i^{j+1+2s}] \subseteq H^{g_i}$ for some s belonging to the connection set of H . Then $[x_{v-1-i}^j, x_{v-1-i}^{j+1} \mid x_{v-1-i}^{j+2s}, x_{v-1-i}^{j+1+2s}] \subseteq H^{g_{v-1-i}}$. Therefore, $[u_i^j, u_i^{j+1} \mid u_i^{j+2s}, u_i^{j+1+2s}] \subseteq B(T)$. Hence, the H -layers of T induce a subgraph of $B(T)$ isomorphic to $\frac{v}{2}H$.

Similarly, every pair of G -layers $(G^{h_j}, G^{h_{j+1}})$ forms a subgraph of $B(T)$ isomorphic to G^{h_j} for all $j = 0, 2, \dots, t - 2$. To see this is true, let $x_i^j x_{i+s}^j \in G^{h_j}$ for j even and some s . Then recalling that $g_i g_{i+s} \in E(G)$ if and only if $g_{v-1-i} g_{v-1-(i+s)} \in E(G)$, we obtain that $x_{v-1-i}^{j+1} x_{v-1-(i+s)}^{j+1} \in G^{h_{j+1}}$. Therefore, if $s \leq \frac{v}{2} - 1 - i$, then $u_i^j u_{i+s}^j \in E(B(T))$ and if $s > \frac{v}{2} - 1 - i$, then $u_i^j u_{v-1-(i+s)}^{j+1} \in E(B(T))$. Hence, the G -layers of T induce a subgraph of $B(T)$ isomorphic to $\frac{t}{2}G$.

We have shown that $B(T)$ is $2d$ -regular (and consequently $\overline{B(T)}$ is $\frac{n}{2} - 1 - 2d$ -regular) and $B(T) \cong \frac{v}{2}H + \frac{t}{2}G$. We proceed to find 1-factors of $\overline{B(T)}$, the complement of $B(T)$.

Edges of the form $u_i^j u_p^j$ or $u_i^j u_q^{j+1}$ (with $i = q \iff q = v-1-i$) in $\overline{B(T)}$ form the graph $\frac{t}{2}\overline{G}$ which is 1-factorable since \overline{G} is 1-factorable. So far, we count $v-1-d$ 1-factors of $\overline{B(T)}$. Observe that \overline{H} contains only edges of the form $u_i^j u_i^p$. Let $\overline{S} = \{\frac{d}{4}+1, \frac{d}{4}+2, \dots, \frac{t}{4}\}$ when $d \equiv 0 \pmod{4}$, otherwise let $\overline{S} = \{\frac{d-2}{4}+1, \frac{d-2}{4}+2, \dots, \frac{t}{4}-1\}$. Then $C_{\frac{t}{2}}(\overline{S}) \circ \overline{K_2}$ is a spanning subgraph of \overline{H} and it allows a 1-factorization by Lemma 2.3.4. We have counted $t-d-2$ more 1-factors of $\overline{B(T)}$.

Now the remaining edges of $\overline{B(T)}$ form an equipartite graph with partite sets $P_j = \{u_i^j, u_i^{j+1} : 0 \leq i \leq \frac{v}{2}-1\}$ for $j = 0, 2, \dots, t-2$, and edge set $\{u_i^j u_p^s, u_i^{j+1} u_p^s, u_i^j u_p^{s+1}, u_i^{j+1} u_p^{s+1} : i \neq p\}$ between any two partite sets P_j and P_s . If $t \equiv 0 \pmod{4}$, these edges allow a 1-factorization (into $(\frac{t}{2}-1)(v-2)$ 1-factors) by Lemma 2.3.6. Otherwise, if $t \equiv 2 \pmod{4}$, we may partition each P_j into two partite sets $P_j^1 = \{u_i^j : 0 \leq i \leq \frac{v}{2}-1\}$ and $P_j^2 = \{u_i^{j+1} : 0 \leq i \leq \frac{v}{2}-1\}$, so these edges form an equipartite graph of the type from Lemma 2.3.8. Thus, the edges allow a 1-factorization into $(\frac{v}{2}-1)(t-2)$ 1-factors. Since $(\frac{t}{2}-1)(v-2) = (\frac{v}{2}-1)(t-2)$, $\overline{B(T)}$ allows a 1-factorization into $(v-1-d) + (t-d-2) + (\frac{tv}{2} - t - v + 2) = \frac{n}{2} - 2d - 1$ 1-factors.

Let $\overline{B(T)}$ have 1-factorization, $\overline{B(T)} \cong I_1 + I_2 + \dots + I_{\frac{n}{2}-1-2d}$. Now define a graph B with vertex set $V(B) = V(T)$ and

$$B = \{[u, v \mid x, y] : (u, v)(x, y) \in E(I_i), \forall i\}.$$

Then construct the graph $T + B$ with vertex set $V(T + B) = V(T)$, and labeling l as in the construction of T . Since $l(u) + l(v) = l(x) + l(y) = n + 1$, the labeling l remains a d -handicap labeling and we have proven that a $H(n, k, d)$ exists for all even k such that $2d \leq k \leq 2d + 2(\frac{n}{2} - 1 - 2d) = n - 2d - 2$. \square

In the proof of Theorem 2.4.3 (which appears as Theorem 3 in [29]), Froncek et. al. built r -regular distance magic graphs of the form $H \circ \overline{K_2}$. In particular, they allow H to be any $\frac{r}{2}$ -regular spanning subgraph of $K_{\frac{n}{2}}$ when $n \equiv 0 \pmod{4}$, or any $\frac{r}{2}$ -regular spanning subgraph of $K_{\frac{n}{2}}$ consisting of $\frac{r}{4}$ Hamiltonian cycles when $n \equiv 2 \pmod{4}$. Therefore, if $n \equiv 0 \pmod{4}$ and $r \equiv 0 \pmod{4}$, we may choose $H = C_{\frac{n}{2}}(1, 2, \dots, \frac{r}{4})$. If $n \equiv 0 \pmod{4}$ and $r \equiv 2 \pmod{4}$, we may choose $H = C_{\frac{n}{2}}(1, 2, \dots, \frac{r-2}{4}, \frac{n}{4})$. If $n \equiv 2 \pmod{4}$, we may choose $H = C_{\frac{n}{2}}(S)$ where $S = \{1, 2, \dots, \frac{r}{4} + 2\} \setminus \{3, \frac{n}{6}\}$ when $n \equiv 0 \pmod{6}$ and $S = \{1, 2, \dots, \frac{r}{4}\}$ when $n \not\equiv 0 \pmod{6}$. We exclude $\{3, \frac{n}{6}\}$ from S when $n \equiv 0 \pmod{6}$ because the corresponding cycles in H are not Hamiltonian as these numbers divide $\frac{n}{2}$ while all other members of S are relatively prime with $\frac{n}{2}$. Since it is clear that graphs of the form $H \circ \overline{K_2}$ for our choice of H exhibit the necessary edge property of Theorem 2.4.6, the next result follows easily from Lemma 2.3.5.

Corollary 2.4.7. *Let $d \geq 2$ and $t, v \geq d + 2$ be even integers and let $n = vt$. If $d \equiv 0 \pmod{4}$ or $v \equiv t \equiv 0 \pmod{4}$, then there exists an $H(n, k, d)$ for all even k such that $2d \leq k \leq n - 2d - 2$.*

It should be noted that if a graph G from Theorem 2.4.5 does not meet either the edge requirement or the 1-factorability requirements necessary to apply Theorem 2.4.6, it is still possible to provide a range of regularities, k such that $2d \leq k \leq r$ for some $r < n - 2d - 2$ using the techniques given in the proof of Theorem 2.4.6.

Example 2.4.8. Suppose we want a 4-handicap tournament with 36 teams in which each team plays 10 games. We know such a tournament exists by Corollary 2.4.7. Figure 2.1 shows a graph $G = C_6(1, 2)$ and its distance magic labeling. Since G satisfies the necessary edge property of Theorem 2.4.6, the graph G can be used to construct 4-handicap tournaments in which each team plays any even number of games k such that $8 \leq k \leq 26$. We will use the construction in Theorem 2.4.5 to build the 8-regular tournament. The 8-regular graph is shown in separate Figures 2.2 and 2.4 for clarity. Figures 2.3 and 2.5 show the corresponding layers in $B(T)$. To obtain the 1-regular tournament, we need only add one distance magic $K_{2,2}$ -factor to complete the construction. We leave it to the reader to accomplish this by finding a 1-factor in the complement of the bubble graph.

One may wonder how close Corollary 2.4.7 comes to providing all feasible regularities given a class of n . To obtain a partial answer, let $v = 2d$ in Corollary 2.4.7. The necessary conditions in this case are as follows.

Theorem 2.4.9. *If an $H(n, k, d)$ exists where d is even, $n \equiv 0 \pmod{4d}$, and $n \geq 2d(d + 2)$, then $d + 2 \leq k \leq n - d - 4$.*

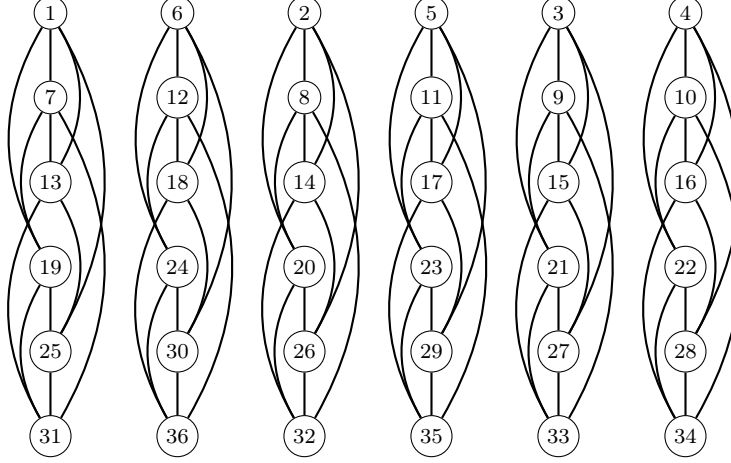


Figure 2.2: G -layers of an $H(36, 8, 4)$.

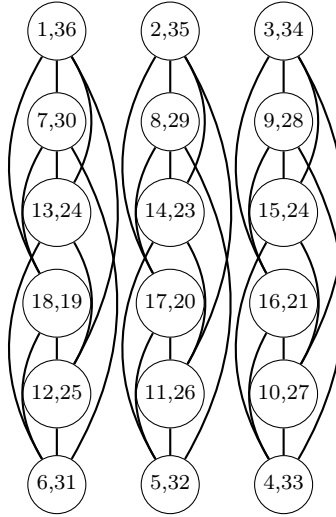


Figure 2.3: Edges of the bubble graph induced by the G -layers.

Proof. Let $n = 4dc$ for some integer $c \geq \frac{d}{2} + 1$ and let $D = (n - 2)^2 - 4d(n - 1)$. Let $b(c) = \frac{n-2-\sqrt{D}}{2}$. Then we have $b(c) = 2dc - 1 - \sqrt{p(c)}$, where $p(c) = 4d^2c^2 - 4dc(d + 1) + (d + 1)$. Then it is clear that $p(c) < (2dc - (d + 1))^2$. Therefore, $b(c) > d$. On the other hand, we have $p(c) > (2dc - (d + 2))^2$ since $c \geq \frac{d}{2} + 1 > \frac{d^2 + 3d + 3}{4d}$. Thus, $d < b(c) < d + 1$ which gives $\lceil b(c) \rceil = d + 1$. Then since k must be even, we have $k \geq d + 2$ by 2 and 5 of Theorem 2.3.1.

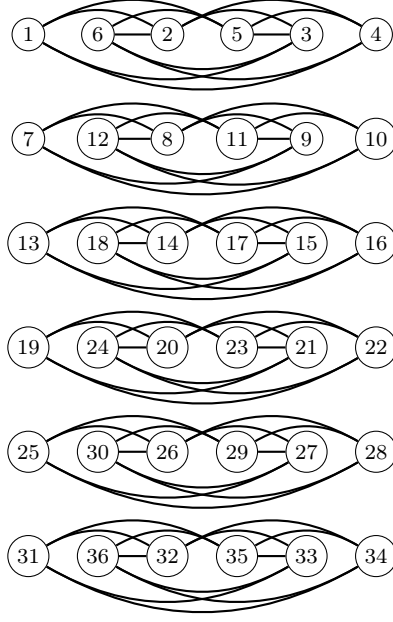


Figure 2.4: H -layers of an $H(36, 8, 4)$.

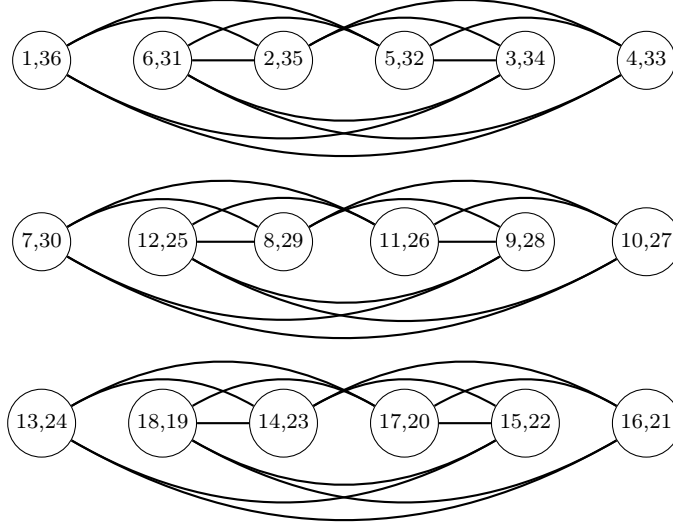


Figure 2.5: Edges of the bubble graph induced by the H -layers.

Let $\overline{b(c)} = 2dc - 1 + \sqrt{p(c)}$. It follows from the above bound on $p(c)$ that $n - d - 3 < \overline{b(c)} < n - d - 2$. Then we have $k \leq n - d - 4$ by 2 and 5 of Theorem 2.3.1. \square

By letting $v = 2d$ in Corollary 2.4.7, we obtain the following result.

Corollary 2.4.10. *For every even d and even integer k such that $2d \leq k \leq n - 2d - 2$, there exists an $H(n, k, d)$ if*

- $n \equiv 0 \pmod{8d}$, $n \geq 2d(d + 4)$, and $d \equiv 0 \pmod{4}$ or
- $n \equiv 4d \pmod{8d}$, $n \geq 2d(d + 2)$, and $d \equiv 0 \pmod{4}$ or
- $n \equiv 0 \pmod{8d}$, $n \geq 2d(d + 2)$, and $d \equiv 2 \pmod{4}$.

Therefore, Theorem 2.4.1 follows from Corollary 2.4.7 and we observe that for $d = 2c$, Corollary 2.4.7 misses only $c - 1$ of the smallest, and $c - 1$ of the largest feasible regularities, k .

2.5 Odd d

2.5.1 Known results

For n even, the question of when an $H(n, k, 1)$ exists has recently been completely settled for every pair (n, k) [30].

Theorem 2.5.1. [30] *Let $H(n, k, 1)$ be a k -regular 1-handicap graph on n vertices. For $n \equiv 0 \pmod{4}$, an $H(n, k, 1)$ exists if and only if k is odd and $3 \leq k \leq n-5$. For $n \equiv 2 \pmod{4}$, an $H(n, k, 1)$ exists if and only if $k \equiv 3 \pmod{4}$, and $3 \leq k \leq n-7$ except for $k = 3$ and $n \leq 26$.*

For n odd, an $H(n, k, 1)$ is known to exist for every feasible n and some k [28].

Theorem 2.5.2. [28] *Let n be an odd positive integer. Then an $H(n, k, 1)$ exists for at least one value of k if and only if $n = 9$ or $n \geq 13$.*

2.5.2 New results

The following theorem indicates why the construction for even d will not work when d is odd. It was originally proved by Vilfred in his Ph.D. thesis in 1999.

Theorem 2.5.3. [44] *Let $d \geq 1$ be an odd integer. No d -regular graph is distance magic.*

The construction in this section is a generalization of the class of 1-handicap graphs constructed by Shepanik in [43]. As was the case for even d , our approach will be to first construct a class of d -handicap graphs for a small regularity k , and then use the bubble graph to add distance magic layers to increase k until the bound is met.

Theorem 2.5.4. *For every odd d , there exists an $H(n, 2d + 1, d)$ provided*

- $n \equiv 0 \pmod{4d + 4}$, $n \geq (d + 1)(d + 3)$, and $d \equiv 1 \pmod{4}$ or
- $n \equiv 0 \pmod{4d + 4}$, $n \geq (d + 1)(d + 5)$, and $d \equiv 3 \pmod{4}$ or
- $n \equiv 2d + 2 \pmod{4d + 4}$, $n \geq (d + 1)(d + 3)$, and $d \equiv 3 \pmod{4}$.

Proof. Let $G = K_{d+1}$ with vertex set $V(G) = \{g_0, g_1, \dots, g_d\}$. Define $t = \frac{n}{d+1}$ and if $d \equiv 1 \pmod{4}$, let $H = C_{\frac{t}{2}}(\frac{t}{4}, 1, 2, \dots, \frac{d-1}{4}) \circ \overline{K_2}$, otherwise let $H = C_{\frac{t}{2}}(1, 2, \dots, \frac{d+1}{4}) \circ \overline{K_2}$. Let $V(H) = \{h_0, h_1, \dots, h_{t-1}\}$ where each pair of isolated vertices, (h_j, h_{j+1}) for $j = 0, 2, \dots, t - 2$, corresponds to the blown-up vertices in H . Notice that if $d \equiv 1 \pmod{4}$, then $n \geq (d + 1)(d + 3)$ implies $\frac{d-1}{4} < \frac{t}{4}$. If $d \equiv 3 \pmod{4}$ and $n \equiv 0 \pmod{4d + 4}$, then $n \geq (d + 1)(d + 5)$ implies $\frac{d+1}{4} < \frac{t}{4}$. Finally, if $d \equiv 3 \pmod{4}$ and $n \equiv 2d + 2 \pmod{4d + 4}$, then $n \geq (d + 1)(d + 3)$ implies $\frac{d+1}{4} \leq \frac{t-2}{4}$. Now let $T = G \square H$ and denote by x_i^j each vertex $(g_i, h_j) \in V(T)$ for all $i = 0, 1, \dots, d$ and $j = 0, 1, \dots, t - 1$. Define $l : V(T) \rightarrow \{1, 2, \dots, n\}$ by

$$l(x_i^j) = \begin{cases} ti + \frac{j+2}{2}, & j = 0, 2, \dots, t - 2 \\ t(i + 1) - \frac{j-1}{2}, & j = 1, 3, \dots, t - 1 \end{cases}$$

for all $i = 0, 1, \dots, d$. Clearly, l is a bijection.

Notice that $l(x_i^j) + l(x_i^{j+1}) = t(2i + 1) + 1$ for $j = 0, 2, \dots, t - 2$, so each pair

$(l(x_i^j), l(x_i^{j+1}))$ are S_{i+1} -complements. For every vertex $x_i^j \in V(T)$, we have

$$\begin{aligned}
w(x_i^j) &= \sum_{p \neq i} l(x_p^j) + \frac{d+1}{2}[t(2i+1)+1] \\
&= \sum_{p=0,1,\dots,d} l(x_p^j) - l(x_i^j) + \frac{d+1}{2}[t(2i+1)+1] \\
&= \begin{cases} \frac{n(d+2i+1)+(d+1)(j+3)}{2} - l(x_i^j), & j = 0, 2, \dots, t-2 \\ \frac{n(d+2i+3)-(d+1)(j-2)}{2} - l(x_i^j), & j = 1, 3, \dots, t-1 \end{cases}.
\end{aligned}$$

For every $i = 0, 1, \dots, d$, define the sequences

$$s_i = l(x_i^0), l(x_i^2), \dots, l(x_i^{t-2}), l(x_i^{t-1}), l(x_i^{t-3}), \dots, l(x_i^3), l(x_i^1)$$

and

$$w_i = w(x_i^0), w(x_i^2), \dots, w(x_i^{t-2}), w(x_i^{t-1}), w(x_i^{t-3}), \dots, w(x_i^3), w(x_i^1).$$

Observe that $l(x_i^{j+2}) - l(x_i^j) = 1$ for $j = 0, 2, \dots, t-4$, $l(x_i^j) - l(x_i^{j+2}) = 1$ for $j = 1, 3, \dots, t-3$, and $l(x_i^{t-1}) - l(x_i^{t-2}) = 1$. Similarly, we have $w(x_i^{j+2}) - w(x_i^j) = d$ for $j = 0, 2, \dots, t-4$, $w(x_i^j) - w(x_i^{j+2}) = d$ for $j = 1, 3, \dots, t-3$, and $w(x_i^{t-1}) - w(x_i^{t-2}) = d$. Therefore, $s_i = ti+1, ti+2, \dots, ti+t$ and $w_i = w(x_i^0), d+w(x_i^0), 2d+w(x_i^0), \dots, (t-1)d+w(x_i^0)$. Then since $l(x_{i+1}^j) = l(x_i^j) + t$ and $w(x_{i+1}^j) = w(x_i^j) + td$, for all $i = 0, 2, \dots, d-1$, $j = 0, 1, \dots, t-1$, we conclude that the sequence $s_0, s_1, \dots, s_d = 1, 2, 3, \dots, n$ and the sequence $w_0, w_1, \dots, w_d = w_0, d+w_0, 2d+w_0, \dots, (n-1)d+w_0$, proving that T is a d -handicap graph. Since T is $2d+1$ -regular, we have constructed an $H(n, 2d+1, d)$. \square

We will now employ the bubble graph to give a wide range of possible densities given n and d .

Theorem 2.5.5. *For every odd d , there exists an $H(n, k, d)$ for every odd k such that $2d + 1 \leq k \leq n - (2d + 3)$ provided*

- $n \equiv 0 \pmod{4d + 4}$, $n \geq (d + 1)(d + 3)$, and $d \equiv 1 \pmod{4}$ or
- $n \equiv 0 \pmod{4d + 4}$, $n \geq (d + 1)(d + 5)$, and $d \equiv 3 \pmod{4}$ or
- $n \equiv 2d + 2 \pmod{4d + 4}$, $n \geq (d + 1)(d + 3)$, and $d \equiv 3 \pmod{4}$.

Proof. Let $T = G \square H$ be the $H(n, 2d + 1, d)$ produced in Theorem 2.5.4 with d -handicap labeling l and recall that $t = \frac{n}{d+1}$. Observe that $l(x_i^j) + l(x_{d-i}^{j+1}) = l(x_{d-i}^j) + l(x_i^{j+1}) = n + 1$ for $i = 0, 1, \dots, \frac{d-1}{2}$ and $j = 0, 2, \dots, t - 2$. So the pairs $(l(x_i^j), l(x_{d-i}^{j+1})), (l(x_{d-i}^j), l(x_i^{j+1}))$ partition $\{1, 2, \dots, n\}$ into complements. Consider now $B(T)$. For $i = 0, 1, \dots, \frac{d-1}{2}$, $j = 0, 2, \dots, t - 2$, let

$$V(B(T)) = \{u_i^j = (x_i^j, x_{d-i}^{j+1}), u_i^{j+1} = (x_i^{j+1}, x_{d-i}^j)\}.$$

From here, the proof is essentially the same as the proof of Theorem 2.4.6 since the graph $G = K_{d+1}$ meets the edge requirements of Theorem 2.4.6 and the graph H allows a 1-factorization. Therefore, we omit the details. \square

One may wonder how close Theorem 2.5.5 comes to providing tournaments for all feasible regularities given a class of n . We provide the appropriate corollary to Theorem 2.3.1 below.

Corollary 2.5.6. *If an $H(n, k, d)$ exists where d is odd and $n \equiv 0 \pmod{2d+2}$, $n \geq (d+1)(d+5)$, then $d+2 \leq k \leq n-d-4$.*

Proof. Let $n = (d+1)2c$ for some integer $c \geq \frac{1}{2}(d+5)$ and let $D = (n-2)^2 - 4d(n-1)$. Let $b(c) = \frac{n-2-\sqrt{D}}{2}$ and let $u = d+1$. Then we have $b(c) = cu - 1 - \sqrt{u}\sqrt{c(c-2)u+1}$ in terms of c and u . Then since $d+1 > 1$, it follows that $(c-1)^2u > c(c-2)u+1$. Therefore, $b(c) > d$. On the other hand, we have $(u(c-1)-1)^2 < u^2c(c-2)+u$ since $c \geq \frac{1}{2}(d+5) > \frac{1}{2}u+2 > \frac{u^2+u+1}{2u}$. Thus, $d < b(c) < d+1$ which gives $\lceil b(c) \rceil = d+1$. Then since r must be odd, we have $r \geq d+2$ by 2 and 5 of Theorem 2.3.1.

Let $\overline{b(c)} = \frac{n-2+\sqrt{D}}{2}$. It follows from the above bound on $b(c)$ that $n-d-3 < \overline{b(c)} < n-d-2$. Then we have $\lfloor \overline{b(c)} \rfloor = n-d-3$ $r \leq n-d-4$ by 2 and 5 of Theorem 2.3.1. □

Therefore, if $d = 2c+1$, Theorem 2.5.5 gives all feasible regularities with the exception of the c smallest and c largest values of k . This leads to the following corollary of Theorems 2.3.1 and 2.5.5 first proved by Shepanik in [43].

Corollary 2.5.7. [43] *Let $n \equiv 0 \pmod{8}$, $n \geq 8$ be given. Then an $H(n, k, 1)$ exists if and only if k is an odd number such that $3 \leq k \leq n-5$.*

2.6 Conclusion

We have constructed many classes of k -regular d -handicap tournaments for every $d \geq 1$, addressing the spectrum question for all three parameters; number of teams, number of games, and handicap number. Also, it is an easy observation that the complement of a d -handicap graph is an antimagic graph in which the weights form an arithmetic sequence with difference $-1 - d$. Therefore, in combination with results on distance magic graphs (“ $d = 0$ ”), we have provided infinite classes of graphs which can be labeled $f(x_i) = i$ with the first n natural numbers such that the induced weights $w(x_1), w(x_2), \dots, w(x_n)$ form a d -arithmetic progression for any *integer* d .

One direction forward is to find constructions for the extreme values of k missed by Theorems 2.4.6 and 2.5.5 for $d \geq 3$. We conjecture that d -handicap graphs can be found for these missing parameters, but it will take a new approach perhaps not considered here. Another direction forward is to find classes of d -handicap graphs for the missing classes of n , for example $d = 2$, $n \equiv 4 \pmod{8}$.

Chapter 3

Orientable \mathbb{Z}_n -distance magic graphs[★]

3.1 Introduction

All graphs considered in this chapter are simple finite graphs. Consider a simple graph G . We denote by $V(G)$ the vertex set and $E(G)$ the edge set of G . We denote the order of G by $|V(G)| = n$. The *open neighborhood* $N(x)$ of a vertex x is the set of vertices adjacent to x , and the degree $d(x)$ of x is $|N(x)|$, the size of the neighborhood of x . By C_n we denote a cycle on n vertices.

In this chapter we investigate distance magic labelings, which belong to a large family of magic-type labelings. Generally speaking, a magic-type labeling of a graph $G =$

[★]The material in this chapter has been submitted to *Discussiones Mathematicae Graph Theory*

(V, E) is a mapping from V, E , or $V \cup E$ to a set of labels which most often is a set of integers or group elements. Then the weight of a graph element is typically the sum of labels of the neighboring elements of one or both types. If the weight of each element is required to be equal, then we speak about magic-type labeling; when the weights are all different (or even form an arithmetic progression), then we speak about an antimagic-type labeling. Probably the best known problem in this area is the *antimagic conjecture* by Hartsfield and Ringel [33], which claims that the edges of every graph except K_2 can be labeled by integers $1, 2, \dots, |E|$ so that the weight of each vertex is different. A comprehensive dynamic survey of graph labelings is maintained by Gallian [31]. A more detailed survey related to our topic by Arumugam et al. [5] was published recently.

A *distance magic labeling* (also called *sigma labeling*) of a graph $G = (V, E)$ of order n is a bijection $\ell: V \rightarrow \{1, 2, \dots, n\}$ with the property that there is a positive integer k (called the *magic constant*) such that

$$w(x) = \sum_{y \in N_G(x)} \ell(y) = k \text{ for every } x \in V(G),$$

where $w(x)$ is the *weight* of vertex x . If a graph G admits a distance magic labeling, then we say that G is a *distance magic graph*.

The following observations were proved independently:

Observation 3.1.1 ([35], [38], [40], [44]). *Let G be an r -regular distance magic graph on n vertices. Then $k = \frac{r(n+1)}{2}$.*

Observation 3.1.2 ([35], [38], [40], [44]). *There is no distance magic r -regular graph with r odd.*

The notion of group distance magic labeling of graphs was introduced in [23]. A Γ -distance magic labeling of a graph $G = (V, E)$ with $|V| = n$ is an injection from V to an abelian group Γ of order n such that the weight of every vertex evaluated under group operation $x \in V$ is equal to the same element $\mu \in \Gamma$. Some families of graphs that are Γ -distance magic were studied in [10, 12, 15, 23].

An *orientation* of an undirected graph $G = (V, E)$ is an assignment of a direction to each edge, turning the initial graph into a directed graph $\vec{G} = (V, A)$. An arc \vec{xy} is considered to be directed from x to y , moreover y is called the *head* and x is called the *tail* of the arc. For a vertex x , the set of head endpoints adjacent to x is denoted by $N^-(x)$, and the set of tail endpoints adjacent to x denoted by $N^+(x)$. Let $\deg^-(x) = |N^-(x)|$, $\deg^+(x) = |N^+(x)|$ and $\deg(x) = \deg^-(x) + \deg^+(x)$.

Bloom and Hsu defined graceful labelings on directed graphs [7]. Bloom et al. also defined magic labelings on directed graphs [8]. Probably the biggest challenge (among directed graphs) are Tutte's flow conjectures. An H -flow on D is an assignment of values of H to the edges of D , such that for each vertex v , the

sum of the values on the edges going in is the same as the sum of the values on the edges going out of v . The 3-flow conjecture says that every 4-edge-connected graph has a nowhere-zero 3-flow (which is equivalent that it has an orientation such that each vertex has the same outdegree and indegree modulo 3). In this chapter we ask when we can assign n distinct labels from the set $\{1, 2, \dots, n\}$ to the vertices of a graph G of order n such that the the sum of the labels on heads minus the sum of the labels on tails is constant modulo n for each vertex of G . Therefore we introduce a generalization of distance magic labeling on directed graphs.

Assume Γ is an abelian group of order n with the operation denoted by $+$. For convenience we will write ka to denote $a + a + \dots + a$ (where the element a appears k times), $-a$ to denote the inverse of a and we will use $a - b$ instead of $a + (-b)$. A *directed Γ -distance magic labeling* of an oriented graph $\vec{G} = (V, A)$ of order n is a bijection $\vec{\ell} : V \rightarrow \Gamma$ with the property that there is $\mu \in \Gamma$ (called the *magic constant*) such that

$$w(x) = \sum_{y \in N_G^+(x)} \vec{\ell}(y) - \sum_{y \in N_G^-(x)} \vec{\ell}(y) = \mu \text{ for every } x \in V(G).$$

If for a graph G there exists an orientation \vec{G} such that there is a directed Γ -distance magic labeling $\vec{\ell}$ for \vec{G} , we say that G is *orientable Γ -distance magic* and the directed

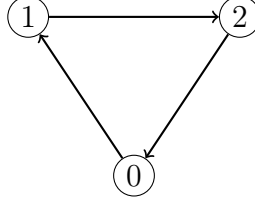


Figure 3.1: An orientable \mathbb{Z}_3 -distance magic labeling of C_3 .

Γ -distance magic labeling $\vec{\ell}$ we call an *orientable Γ -distance magic labeling*.

The following cycle-related result was proved by Miller, Rodger, and Simanjuntak.

Theorem 3.1.3 ([38]). *The cycle C_n of length n is distance magic if and only if $n = 4$.*

One can check that C_n is distance magic if and only if $n = 4$, however it is not longer true for the case of orientable \mathbb{Z}_n -distance magic labeling (see Fig.3.1).

Circulant graphs are an interesting family of vertex-transitive graphs. These graphs arise in various settings; for instance, they are the Cayley graphs over the cyclic group of order n . The circulant graph $C_n(s_1, s_2, \dots, s_k)$ for $0 \leq s_1 < s_2 < \dots < s_k \leq n/2$ is the graph on the vertex set $V = \{x_0, x_1, \dots, x_{n-1}\}$ with edges (x_i, x_{i+s_j}) for $i = 0, \dots, n-1, j = 1, \dots, k$ where $i + s_j$ is taken modulo n .

We recall three graph products (see [32]). All three, the *Cartesian product* $G \square H$, *lexicographic product* $G \circ H$, *direct product* $G \times H$ are graphs with the vertex set $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent in:

- $G \square H$ if and only if $g = g'$ and $h \sim h'$ in H , or $h = h'$ and $g \sim g'$ in G ;
- $G \times H$ if $g \sim g'$ in G and $h \sim h'$ in H ;
- $G \circ H$ if and only if either $g \sim g'$ in G or $g = g'$ and $h \sim h'$ in H .

For a fixed vertex g of G , the subgraph of any of the above products induced by the set $\{(g, h) : h \in V(H)\}$ is called an H -layer and is denoted H^g . Similarly, if $h \in H$ is fixed, then G^h , the subgraph induced by $\{(g, h) : g \in V(G)\}$, is a G -layer.

In this chapter we show some families of orientable \mathbb{Z}_n -distance magic graphs.

3.2 Circulant graphs and their products

We start by proving a general theorem for orientable Γ -distance magic labeling similar to Theorem 3.1.2.

Theorem 3.2.1. *Let G have order $n \equiv 2 \pmod{4}$ and all vertices of odd degree. There does not exist an orientable Γ -distance magic labeling of G for any abelian group Γ of order n .*

Proof. Suppose to the contrary that G is orientable Γ -distance magic with orientation \vec{G} , orientable Γ -distance magic labeling $\vec{\ell}$, and magic constant μ . Since

$n \equiv 2 \pmod{4}$, we may assume that Γ is a direct product of cyclic groups containing \mathbb{Z}_2 . For all $g \in \Gamma$, let g_0 denote the \mathbb{Z}_2 component of g . Similarly, for all $x \in V(G)$, let $w_0(x)$ and $\vec{\ell}_0(x)$ denote the \mathbb{Z}_2 component of $w(x)$ and $\vec{\ell}(x)$ respectively. Observe that

$$w_0(x) = \sum_{y \in N_G^+(x)} \vec{\ell}_0(y) - \sum_{y \in N_G^-(x)} \vec{\ell}_0(y) = \sum_{y \in N_G(x)} \vec{\ell}_0(y) \text{ for every } x \in V(G).$$

Let $w_0(\vec{G}) = \sum_{x \in V(G)} w_0(x)$. Then clearly $w_0(\vec{G}) = n\mu_0 = 0$. However, since each vertex has odd degree and $\frac{n}{2}$ is odd, we have $w_0(\vec{G}) = \sum_{x \in V(G)} \sum_{y \in N_G(x)} \vec{\ell}_0(y) = 1$, a contradiction. \square

Notice that the above proof also shows that there exists no abelian group Γ of order $n \equiv 2 \pmod{4}$ such that G is Γ -distance magic. From the above Theorem 3.2.1 the below observation easily follows:

Observation 3.2.2. *Let G be an r -regular graph on $n \equiv 2 \pmod{4}$ vertices, where r is odd. There does not exist an orientable \mathbb{Z}_n -distance magic labeling for the graph G .*

However, there are $(2k+1)$ -regular graphs on $n \equiv 0 \pmod{4}$ vertices that are not orientable \mathbb{Z}_n -distance magic (see Theorem 3.3.1), the converse of the above Theorem 3.2.1 is not true in general for $n \equiv 0 \pmod{4}$. Consider the graph $G = K_{3,3,3,3}$ with the partition vertex sets $A^1 = \{x_0^1, x_1^1, x_2^1\}$, $A^2 = \{x_0^2, x_1^2, x_2^2\}$, $A^3 = \{x_0^3, x_1^3, x_2^3\}$

and $A^4 = \{x_0^4, x_1^4, x_2^4\}$. Let $o(uv)$ be the orientation for the edge $uv \in E(G)$ such that:

$$o(x_i^j x_k^p) = \begin{cases} \overrightarrow{x_i^2 x_0^1} & \text{for } i = 0, 1, 2, \\ \overrightarrow{x_i^1 x_k^2} & \text{for } i = 1, 2, k = 0, 1, 2, \\ \overrightarrow{x_i^1 x_k^p} & \text{for } i = 0, 1, 2, k = 0, 1, 2, p = 3, 4, \\ \overrightarrow{x_i^j x_k^p} & \text{for } i, k = 0, 1, 2, 2 \leq j < p \leq 4. \end{cases}$$

Let now:

$$\begin{aligned} \overrightarrow{\ell}(x_0^1) &= 3, & \overrightarrow{\ell}(x_0^2) &= 6, & \overrightarrow{\ell}(x_0^3) &= 1, & \overrightarrow{\ell}(x_0^4) &= 11, \\ \overrightarrow{\ell}(x_1^1) &= 9, & \overrightarrow{\ell}(x_1^2) &= 2, & \overrightarrow{\ell}(x_1^3) &= 4, & \overrightarrow{\ell}(x_1^4) &= 8, \\ \overrightarrow{\ell}(x_2^1) &= 0, & \overrightarrow{\ell}(x_2^2) &= 10, & \overrightarrow{\ell}(x_2^3) &= 7, & \overrightarrow{\ell}(x_2^4) &= 5. \end{aligned}$$

Obviously $w(x) = 6$ for any $x \in V(G)$.

Theorem 3.2.3. *If $G = C_n(s_1, s_2, \dots, s_k)$ is a circulant graph such that $s_k < n/2$, then p copies of G is orientable \mathbb{Z}_{np} -distance magic for any $p \geq 1$.*

Proof. Note that G is a $2k$ -regular graph, because $s_k < n/2$. Let $V^i = x_0^i, x_1^i, \dots, x_{n-1}^i$ be the set of vertices of i th copy G^i of the graph G , $i = 0, 1, \dots, p-1$. It is easy to see that we can partition G into disjoint cycles $x_j, x_{j+s_h}, x_{j+2s_h}, \dots, x_j$ of length of order of the subgroup $\langle s_h \rangle$ for $h \in \{1, 2, \dots, k\}$ and $j = 0, 1, \dots, s_h - 1$. Orient each copy of G such that the orientation is clockwise (in which order the subscripts go) around

each cycle $x_j, x_{j+s_h}, x_{j+2s_h}, \dots, x_j$ for $h \in \{1, 2, \dots, k\}$ and $j = 0, 1, \dots, s_h - 1$. Set now $\vec{\ell}(x_m^i) = mp + i$ for $m = 0, 1, \dots, n - 1$, $i = 0, 1, \dots, p - 1$. Obviously $\vec{\ell}$ is a bijection. Moreover $w(x) = \sum_{y \in N^+(x)} \vec{\ell}(y) - \sum_{y \in N^-(x)} \vec{\ell}(y) = -2p \sum_{j=1}^k s_j$ for any $x \in V(pG)$. \square

From the above proof of Theorem 3.2.3 it is easy to conclude that in general the magic constant for orientable \mathbb{Z}_n -distance magic graphs is not unique (just take counterclockwise orientation in each cycle).

Theorem 3.2.4. *If $G = C_n(s_1, s_2, \dots, s_k)$ and $H = C_m(s'_1, s'_2, \dots, s'_p)$ are circulant graphs such that $s_k < n/2$, $s'_p < m/2$ and $\gcd(m, n) = 1$, then the Cartesian product $G \square H$ is orientable \mathbb{Z}_{nm} -distance magic.*

Proof. Let G be a graph with the vertex set $V(G) = \{g_0, g_1, \dots, g_{n-1}\}$, whereas $V(H) = \{x_0, x_1, \dots, x_{m-1}\}$. As in the proof of Theorem 3.2.3 we orient each copy of H (i.e H^g -layer for any $g \in V(G)$) such that the orientation is clockwise around each cycle $(g_i, x_j), (g_i, x_{j+s'_a}), (g_i, x_{j+2s'_a}), \dots, (g_i, x_j)$ for $a = 1, 2, \dots, p$, $j = 0, 1, \dots, s'_a - 1$ and $i = 0, 1, \dots, n - 1$, whereas each copy of G (i.e G^h -layer for any $h \in V(H)$) such that the orientation is clockwise around each cycle $(g_i, x_j), (g_{i+s_b}, x_j), (g_{i+2s_b}, x_j), \dots, (g_i, x_j)$ for $b = 1, 2, \dots, k, i = 0, 1, \dots, s_b - 1$ and $j = 0, 1, \dots, m - 1$.

Recall that $\mathbb{Z}_n \times \mathbb{Z}_m \cong \mathbb{Z}_{nm}$ because $\gcd(n, m) = 1$. Define $\vec{\ell} : V(G \square H) \rightarrow \mathbb{Z}_n \times \mathbb{Z}_m$

as $\vec{\ell}(g_i, x_j) = (i, j)$ for $i = 0, 1, \dots, n-1$, $j = 0, 1, \dots, m-1$. Obviously $\vec{\ell}$ is a bijection. Notice that $w(g_i, x_j) = \sum_{y \in N^+(g_i, x_j)} \vec{\ell}(y) - \sum_{y \in N^-(g_i, x_j)} \vec{\ell}(y) = (-2 \sum_{i=1}^k s_i, -2 \sum_{j=1}^p s'_j)$, thus we obtain that $G \square H$ is orientable \mathbb{Z}_{nm} -distance magic. \square

Note that the above Theorem 3.2.4 is not “if and only if” since the Cartesian product $C_n \square C_m$ is orientable \mathbb{Z}_{nm} -distance magic for any $n, m \geq 3$, see [19].

We will show now some sufficient conditions for the lexicographic product to be orientable \mathbb{Z}_n -distance magic.

Theorem 3.2.5. *Let $H = C_{2n}(s_1, s_2, \dots, s_k)$ be a circulant graph such that $s_k < n$ and G be a graph of order t . The lexicographic product $G \circ H$ is orientable \mathbb{Z}_{2tn} -distance magic, if one of the following holds:*

- *graph G has all degrees of vertices of the same parity,*
- *n is even.*

Proof. Let G be a graph with the vertex set $V(G) = \{g_0, g_1, \dots, g_{t-1}\}$, whereas $V(H) = \{x_0, x_1, \dots, x_{2n-1}\}$. Let now $(g_i, x_j) = x_j^i$. As in the proof of Theorem 3.2.3 we orient each copy of H (i.e H^g -layer for any $g \in V(G)$) such that the orientation is clockwise around each cycle $x_j^i, x_{j+s_a}^i, x_{j+2s_a}^i, \dots, x_j^i$ for $a = 1, 2, \dots, k$,

$j = 0, 1, \dots, s_a - 1$ and $i = 0, 1, \dots, t - 1$. If $g_i g_p \in E(G)$ ($i < p$), then the orientation $o(x_j^i x_b^p)$ for an edge $x_j^i x_b^p \in E(G \circ H)$ is given in the following way:

$$o(x_j^i x_b^p) = \begin{cases} \overrightarrow{x_j^i x_b^p}, & \text{for } j, b < n \text{ or } j, b \geq n, \\ \overrightarrow{x_b^p x_j^i}, & \text{otherwise.} \end{cases}$$

Set now $\overrightarrow{\ell}(x_m^i) = mt + i$ for $m = 0, 1, \dots, 2n - 1$, $i = 0, 1, \dots, t - 1$. Obviously $\overrightarrow{\ell}$ is a bijection. Notice that $w(x_j^i) = \sum_{y \in N^+(x_j^i)} \overrightarrow{\ell}(y) - \sum_{y \in N^-(x_j^i)} \overrightarrow{\ell}(y) = -2t \sum_{j=1}^k s_j + \deg(g_i)n(tn)$. If now $\deg(g_i) \equiv c \pmod{2}$ then we are done. If n is even, then $n(tn) \equiv 0 \pmod{2tn}$, thus we obtain that $G \circ H$ is orientable \mathbb{Z}_{2tn} -distance magic. \square

One can ask if $G \circ H$ of order n is still orientable \mathbb{Z}_n -distance magic if the circulant graph H has an odd number of vertices. The partial answer is given in the Theorems 3.2.7, 3.2.8 and 3.2.9. Before we proceed, we will need the following theorem.

Theorem 3.2.6 ([36]). *Let $n = r_1 + r_2 + \dots + r_q$ be a partition of the positive integer n , where $r_i \geq 2$ for $i = 1, 2, \dots, q$. Let $A = \{1, 2, \dots, n\}$. Then the set A can be partitioned into pairwise disjoint subsets A_1, A_2, \dots, A_q such that for every $1 \leq i \leq q$, $|A_i| = r_i$ with $\sum_{a \in A_i} a \equiv 0 \pmod{n+1}$ if n is even and $\sum_{a \in A_i} a \equiv 0 \pmod{n}$ if n is odd.*

Theorem 3.2.7. *If G is a graph of odd order t , then the lexicographic product $G \circ \overline{K}_{2n+1}$ is orientable $\mathbb{Z}_{t(2n+1)}$ -distance magic for $n \geq 1$.*

Proof. Let G be a graph with the vertex set $V(G) = \{g_0, g_1, \dots, g_{t-1}\}$, whereas $V(\overline{K}_{2n+1}) = \{x_0, x_1, \dots, x_{2n}\}$. Give first to the graph G any orientation and now orient the graph $G \circ \overline{K}_{2n+1}$ such that each edge $(g_i, x_j)(g_p, x_h) \in E(G \circ \overline{K}_{2n+1})$ has the corresponding orientation to the edge $g_i g_p \in E(G)$.

Since $t, 2n+1$ are odd, there exists a partition A_1, A_2, \dots, A_t of the set $\{1, 2, \dots, (2n+1)t\}$ such that for every $1 \leq i \leq t$, $|A_i| = 2n+1$ with $\sum_{a \in A_i} a \equiv 0 \pmod{(2n+1)t}$ by Theorem 3.2.6. Label the vertices of the i th copy of \overline{K}_{2n+1} using elements from the set A_i for $i = 1, 2, \dots, t$. Notice that $\sum_{j=1}^{2n+1} \vec{\ell}(g_i, x_j) = 0$ for $i = 1, 2, \dots, t$. Therefore $w(g_i, x_j) = \sum_{y \in N^+(g_i, x_j)} \vec{\ell}(y) - \sum_{y \in N^-(g_i, x_j)} \vec{\ell}(y) = 0$. \square

Theorem 3.2.8. *If $G = C_n(s_1, s_2, \dots, s_k)$ and $H = C_m(s'_1, s'_2, \dots, s'_p)$ are circulant graph such that $s_k < n/2$, $s'_p < m/2$ and $\gcd(m, n) = 1$, then lexicographic product $G \circ H$ is orientable \mathbb{Z}_{nm} -distance magic.*

Proof. Let G be a graph with the vertex set $V(G) = \{g_0, g_1, \dots, g_{n-1}\}$, whereas $V(H) = \{x_0, x_1, \dots, x_{m-1}\}$. Give first to the graph G the orientation as in the proof of Theorem 3.2.3, i.e. $g_i, g_{i+s_b}, g_{i+2s_b}, \dots, g_i$ for $b = 1, 2, \dots, k$, $i = 0, 1, \dots, s_b - 1$. For $i \neq p$ orient now each edge $(g_i, x_j)(g_p, x_h) \in E(G \circ H)$ such that it has the corresponding orientation to the edge $g_i g_p \in E(G)$. Recall that for each vertex $g \in V(G)$ we have $\deg^+(g) = \deg^-(g)$. Each copy of H (i.e H^g -layer for any $g \in V(G)$) we orient such that the orientation is clockwise around each cycle $(g_i, x_j), (g_i, x_{j+s'_a}), (g_i, x_{j+2s'_a}), \dots, (g_i, x_j)$ for $a = 1, 2, \dots, p, j = 0, 1, \dots, s'_a - 1$ and

$i = 0, 1, \dots, n-1$. Recall that $\mathbb{Z}_n \times \mathbb{Z}_m \cong \mathbb{Z}_{nm}$ because $\gcd(n, m) = 1$. Then define $\vec{\ell}: V(G \circ H) \rightarrow \mathbb{Z}_n \times \mathbb{Z}_m$ as $\vec{\ell}(g_i, x_j) = (i, j)$ for $i = 0, 1, \dots, n-1$, $j = 0, 1, \dots, m-1$. Obviously $\vec{\ell}$ is a bijection. Notice that $w(g_i, x_j) = \sum_{y \in N^+(g_i, x_j)} \vec{\ell}(y) - \sum_{y \in N^-(g_i, x_j)} \vec{\ell}(y) = (-2m \sum_{i=1}^k s_i, -2 \sum_{j=1}^p s'_j)$, thus we obtain that $G \circ H$ is orientable \mathbb{Z}_{nm} -distance magic. \square

Theorem 3.2.9. *The lexicographic product $C_n \circ C_m$ is orientable \mathbb{Z}_{nm} -distance magic for all $n, m \geq 3$.*

Proof. Let $G = C_n, H = C_m$ be graphs with the vertex sets $V(G) = \{g_0, g_1, \dots, g_{n-1}\}$, $V(H) = \{x_0, x_1, \dots, x_{m-1}\}$. Give first to the graph G the orientation counter-clockwise around the cycle $g_0, g_1, g_2, \dots, g_0$. For each i orient now each edge $(g_i, x_j)(g_{i+1}, x_h) \in E(G \circ H)$ such that it has the corresponding orientation to the edge $g_i g_{i+1} \in E(G)$. Each copy of H (i.e H^g -layer for any $g \in V(G)$) we orient such that the orientation is counter-clockwise around each cycle $(g_i, x_0), (g_i, x_1), (g_i, x_2), \dots, (g_i, x_0)$ for $i = 0, 1, \dots, n-1$. Define $\vec{\ell}: V(G \circ H) \rightarrow \mathbb{Z}_{mn}$ as $\vec{\ell}(g_i, x_j) = jn + i$ for $i = 0, 1, \dots, n-1, j = 0, 1, \dots, m-1$.

$$\begin{aligned} w(g_i, x_j) &= \sum_{h=0}^{m-1} \left(\vec{\ell}(g_{i+1}, x_h) - \vec{\ell}(g_{i-1}, x_h) \right) \\ &+ \vec{\ell}(g_i, x_{j+1}) - \vec{\ell}(g_i, x_{j-1}) \\ &= 2n + 2m, \end{aligned}$$

proving that $G \circ H$ is orientable \mathbb{Z}_{nm} -distance magic. \square

The analogous theorem is also true for a direct product of cycles shown in the following theorem.

Theorem 3.2.10. *The direct product $C_n \times C_m$ is orientable \mathbb{Z}_{nm} -distance magic for all $n, m \geq 3$.*

Proof. Let $G = C_n = g_0, g_1, \dots, g_{n-1}$ and $H = C_m = x_0, x_1, \dots, x_{m-1}$. For all i and j , orient counter-clockwise with respect to j each cycle of the form $(g_i, x_j), (g_{i-1}, x_{j+1}), (g_{i-2}, x_{j+2}), \dots, (g_i, x_j)$ and each cycle of the form $(g_i, x_j), (g_{i+1}, x_{j+1}), (g_{i+2}, x_{j+2}), \dots, (g_i, x_j)$, where the arithmetic in the indices is performed modulo n and m respectively. Then define $\vec{\ell}: V(G \times H) \rightarrow \mathbb{Z}_{nm}$ as $\vec{\ell}(g_i, x_j) = jn + i$ for $i = 0, 1, \dots, n-1$, $j = 0, 1, \dots, m-1$. Therefore for all i and j we have,

$$\begin{aligned} w(g_i, x_j) &= \vec{\ell}(g_{i-1}, x_{j+1}) + \vec{\ell}(g_{i+1}, x_{j+1}) - \vec{\ell}(g_{i-1}, x_{j-1}) - \vec{\ell}(g_{i+1}, x_{j-1}) \\ &= 4n. \end{aligned}$$

Since $\vec{\ell}$ is obviously a bijection, we have proved that $G \times H$ is orientable \mathbb{Z}_{nm} -distance magic. □

Theorem 3.2.11. *Let $H = C_{2n}(1, 3, 5, \dots, 2 \lceil \frac{n}{2} \rceil - 1)$. If G is an Eulerian graph of order t , then the direct product $G \times H$ is orientable \mathbb{Z}_{2nt} -distance magic.*

Proof. Let G be a graph with the vertex set $V(G) = \{g_0, g_1, \dots, g_{t-1}\}$, whereas

$V(H) = \{x_0, x_1, \dots, x_{2n-1}\}$. Give first to the graph G the orientation according to Fleury's Algorithm for finding an Eulerian trail in G and now orient the graph $G \times H$ such that each edge $(g_i, x_j)(g_p, x_h) \in E(G \times H)$ has the corresponding orientation to the edge $g_i g_p \in E(G)$. Recall that for each vertex $g \in V(G)$ we have $\deg^+(g) = \deg^-(g)$. Observe that $H \cong K_{n,n}$ with the partition sets $A = \{x_0, x_2, \dots, x_{2n-2}\}$ and $B = \{x_1, x_3, \dots, x_{2n-1}\}$.

Define

$$\vec{\ell}(g_i, x_j) = \begin{cases} ti + j & \text{for } j = 0, 2, \dots, 2n-2, \\ 2tn - 1 - \vec{\ell}(g_i, x_{j-1}) & \text{for } j = 1, 3, \dots, 2n-1, \end{cases}$$

for $i = 0, 1, \dots, t-1$.

Notice that $\vec{\ell}(g_i, x_j) + \vec{\ell}(g_i, x_{j-1}) = 2tn - 1$ for $i = 0, 1, \dots, t-1, j = 1, 3, \dots, 2n-1$.

Therefore $w(g_i, x_j) = \sum_{y \in N^+(g_i, x_j)} \vec{\ell}(y) - \sum_{y \in N^-(g_i, x_j)} \vec{\ell}(y) = \frac{\deg^+(g_i)}{2} 2n(2nt - 1) - \frac{\deg^-(g_i)}{2} 2n(2nt - 1) = 0$. □

3.3 Complete t -partite graphs

Theorem 3.3.1. K_n is orientable \mathbb{Z}_n -distance magic if and only if n is odd.

Proof. Suppose first that n is odd. Then $K_n \cong C_n(1, 2, \dots, (n-1)/2)$ and thus it is

orientable \mathbb{Z}_n -distance magic by Theorem 3.2.3. By Theorem 3.2.1 we can consider now only the case when $n \equiv 0 \pmod{4}$. Suppose that K_n is orientable \mathbb{Z}_n -distance magic. Let $\vec{\ell}(x) = 1$, $\vec{\ell}(u) = 0$, then it is easy to see that $w(x) = \sum_{y \in N^+(x)} \vec{\ell}(y) - \sum_{y \in N^-(x)} \vec{\ell}(y) \equiv 1 \pmod{2}$, whereas $w(u) = \sum_{y \in N^+(u)} \vec{\ell}(y) - \sum_{y \in N^-(u)} \vec{\ell}(y) \equiv 0 \pmod{2}$, a contradiction. \square

Observation 3.3.2. *Let $G = K_{n_1, n_2, n_3, \dots, n_k}$ be a complete k -partite graph such that $1 \leq n_1 \leq n_2 \leq \dots \leq n_k$ and $n = n_1 + n_2 + \dots + n_k$ is odd. The graph G is orientable \mathbb{Z}_n -distance magic graph if $n_2 \geq 2$.*

Proof. Give first to the graph G an orientation such that all arcs from the set of lower index go to the set of higher index. Since n is odd, there exists a partition A_0, A_1, \dots, A_{k-1} of $\{1, 2, \dots, n\}$ such that for every $0 \leq i \leq k-1$, $|A_i| = n_i$ with $\sum_{a \in A_i} a \equiv 0 \pmod{n}$ by Theorem 3.2.6. Label the vertices from i th partition set of G using elements from the set A_i for $i = 0, 1, \dots, k-1$.

Notice that $w(x) = 0$ for any $x \in V(G)$. \square

Notice that the above Observation 3.3.2 is not “if and only if” since $K_{1,1,1} \cong C_3(1)$ is orientable \mathbb{Z}_3 -distance magic graph by Theorem 3.2.3.

Observation 3.3.3. *$K_{n,n}$ is orientable \mathbb{Z}_{2n} -distance magic if and only if n is even.*

Proof. Suppose first that n is even then $K_{n,n} \cong C_{2n}(1, 3, 5, \dots, n-1)$ thus is orientable

\mathbb{Z}_{2n} -distance magic by Theorem 3.2.3. If n is odd, then because $2n \equiv 2 \pmod{4}$, then $K_{n,n}$ is not orientable \mathbb{Z}_{2n} -distance magic by Theorem 3.2.1. \square

Recall that if $n = n_1 + n_2 \equiv 2 \pmod{4}$ and n_1, n_2 are both odd, then K_{n_1, n_2} is not orientable \mathbb{Z}_n -distance magic by Theorem 3.2.1. It was proved in [17] that if K_{n_1, n_2} is orientable \mathbb{Z}_n -distance magic, then $n \not\equiv 2 \pmod{4}$. The next theorem proves the converse is also true.

Theorem 3.3.4. *Let $G = K_{n_1, n_2}$ and $n = n_1 + n_2$. If $n \not\equiv 2 \pmod{4}$, then G is orientable \mathbb{Z}_n -distance magic.*

Proof. Let $G = K_{n_1, n_2}$ with the partition vertex sets $A^i = \{x_0^i, x_1^i, \dots, x_{n_i-1}^i\}$ for $i = 1, 2$. Without loss of generality we can assume that $n_1 \geq n_2$.

Let $\mathbb{Z}_n = \{a_0, a_1, a_2, \dots, a_{n-1}\}$ such that $a_0 = 0$, $a_1 = n/4$, $a_2 = n/2$, $a_3 = 3n/4$ and $a_{i+1} = -a_i$ for $i = 4, 6, 8, \dots, n-2$. Let $o(uv)$ be the orientation for the edge $uv \in E(G)$ such that:

$$o(x_i^j x_k^p) = \begin{cases} \overrightarrow{x_i^2 x_0^1} & \text{for } i = 0, 1, \dots, n_2 - 1, \\ \overrightarrow{x_i^1 x_k^2} & \text{for } i = 1, 2, \dots, n_1 - 1, k = 0, 1, \dots, n_2 - 1. \end{cases}.$$

Case 1. n_1, n_2 are both odd.

$\overrightarrow{\ell}(x_0^1) = a_1$, $\overrightarrow{\ell}(x_1^1) = a_3$, $\overrightarrow{\ell}(x_2^1) = a_0$ and $\overrightarrow{\ell}(x_i^1) = a_{1+i}$ for $i = 3, 4, \dots, n_1 - 1$.

$\overrightarrow{\ell}(x_0^2) = a_2$ and $\overrightarrow{\ell}(x_i^2) = a_{n_1+i}$ for $i = 1, 2, \dots, n_2 - 1$.

Case 2. n_1, n_2 are both even.

$\overrightarrow{\ell}(x_0^1) = a_1$, $\overrightarrow{\ell}(x_1^1) = a_3$ and $\overrightarrow{\ell}(x_i^1) = a_{2+i}$ for $i = 2, 3, \dots, n_1 - 1$.

$\overrightarrow{\ell}(x_0^2) = a_2$, $\overrightarrow{\ell}(x_1^2) = a_0$ and $\overrightarrow{\ell}(x_i^2) = a_{n_1+i}$ for $i = 2, 3, \dots, n_2 - 1$.

Note that in both cases $w(x) = n/2$ for any $x \in V(G)$. \square

Theorem 3.3.5. *Let $G = K_{n_1, n_2, n_3}$ and $n = n_1 + n_2 + n_3$. Then G is orientable \mathbb{Z}_n -distance magic for all n_1, n_2, n_3 .*

Proof. Let $G = K_{n_1, n_2, n_3}$ with the partition vertex sets $A^i = \{x_0^i, x_1^i, \dots, x_{n_i-1}^i\}$ for $i = 1, 2, 3$.

Assume first that n is odd. We have to consider only the case $n_1 = n_2 = 1$ by Observation 16. If $n_3 = 1$, then $G \cong C_3$ is orientable \mathbb{Z}_n -distance magic, so assume $n_3 \geq 3$ is odd. Set the orientation $o(uv)$ for the edge $uv \in E(G)$ such that:

$$o(x_i^j x_k^p) = \begin{cases} \overrightarrow{x_0^1 x_0^2}, \\ \overrightarrow{x_i^3 x_0^2} & i = 0, 1, \dots, n_3 - 1 \end{cases}.$$

We will orient the remaining edges of the form $x_0^1 x_i^3$ for $i = 0, 1, \dots, n_3 - 1$ later. Now let $\overrightarrow{\ell}(x_0^1) = 0$, $\overrightarrow{\ell}(x_0^2) = n - 1$, and $\overrightarrow{\ell}(x_i^3) = i + 1$ for $i = 0, 1, \dots, n_3 - 1$. Notice that $\sum_{i=0}^{n_3-1} \overrightarrow{\ell}(x_i^3) = 1$. Observe now that $w(x_0^2)$ and $w(x_i^3)$ for $i = 0, 1, \dots, n_3 - 1$ are independent of the yet-to-be oriented edges and hence $w(x_0^2) = w(x_i^3) = 1$. So all that remains is to orient the edges of the form $x_0^1 x_i^3$ for $i = 0, 1, \dots, n_3 - 1$ so that $w(x_0^1) = 1$. It is easy to see that this is equivalent to finding $a, b \in \{1, 2, \dots, n - 2\} \subseteq \mathbb{Z}_n$ such

that $a + b = \frac{n+1}{2}$, $a \neq b$. Clearly such a and b exist for all odd $n \geq 5$ since the group table for \mathbb{Z}_n is a latin square. Therefore, set the orientation

$$o(x_i^j x_k^p) = \begin{cases} \overrightarrow{x_0^1 x_i^3}, & i = a - 1, b - 1, \\ \overrightarrow{x_i^3 x_0^1}, & \text{otherwise,} \end{cases}$$

which implies that $w(v) = 1$ for any $v \in V(G)$.

From now on n is even. Without loss of generality we assume that n_1 is even. Let $\mathbb{Z}_n = \{a_0, a_1, a_2, \dots, a_{n-1}\}$. We will consider now two cases:

Case 1. $n \equiv 0 \pmod{4}$.

Let $a_0 = 0$, $a_1 = n/4$, $a_2 = n/2$, $a_3 = 3n/4$ and $a_{i+1} = -a_i$ for $i = 4, 6, 8, \dots, n - 2$.

Set the orientation $o(uv)$ for the edge $uv \in E(G)$ such that:

$$o(x_i^j x_k^p) = \begin{cases} \overrightarrow{x_i^2 x_0^1} & \text{for } i = 0, 1, \dots, n_2 - 1, \\ \overrightarrow{x_i^1 x_k^2} & \text{for } i = 1, 2, \dots, n_1 - 1, k = 0, 1, \dots, n_2 - 1, \\ \overrightarrow{x_i^1 x_k^3} & \text{for } i = 0, 1, \dots, n_1 - 1, k = 0, 1, \dots, n_3 - 1, \\ \overrightarrow{x_i^2 x_k^3} & \text{for } i = 0, 1, \dots, n_2 - 1, k = 0, 1, \dots, n_3 - 1. \end{cases}$$

Let now $\overrightarrow{\ell}(x_0^1) = a_1$, $\overrightarrow{\ell}(x_1^1) = a_3$ and $\overrightarrow{\ell}(x_i^1) = a_{i+2}$ for $i = 2, 3, \dots, n_1 - 1$.

Case 1.1 n_2, n_3 are both odd.

$\overrightarrow{\ell}(x_0^2) = a_2$ and $\overrightarrow{\ell}(x_i^2) = a_{n_1+1+i}$ for $i = 1, 2, \dots, n_2 - 1$.

$\overrightarrow{\ell}(x_0^3) = a_0$ and $\overrightarrow{\ell}(x_i^3) = a_{n_1+n_2+i}$ for $i = 1, 2, \dots, n_3 - 1$.

Case 1.2. n_2, n_3 are both even.

$$\overrightarrow{\ell}(x_0^2) = a_0, \overrightarrow{\ell}(x_1^2) = a_2 \text{ and } \overrightarrow{\ell}(x_i^2) = a_{n_1+i} \text{ for } i = 2, 3, \dots, n_2 - 1.$$

$$\overrightarrow{\ell}(x_i^3) = a_{n_1+n_2+i} \text{ for } i = 0, 1, \dots, n_3 - 1.$$

Note that in both subcases $w(v) = n/2$ for any $v \in V(G)$.

Case 2. $n \equiv 2 \pmod{4}$.

Without loss of generality we can assume that $n_2 \geq n_3$.

Let $a_0 = 0$, $a_1 = n/2$, $a_2 = 1$, $a_3 = n/2 - 1$, $a_4 = n - 1$, $a_5 = n/2 + 1$ and $a_{i+1} = -a_i$ for $i = 6, 8, 10, \dots, n - 2$. Set the orientation $o(uv)$ for the edge $uv \in E(G)$ such that:

$$o(x_i^j x_k^p) = \begin{cases} \overrightarrow{x_i^j x_k^p} & \text{for } j < p. \end{cases}$$

Let now $\overrightarrow{\ell}(x_0^1) = a_2$, $\overrightarrow{\ell}(x_1^1) = a_3$ and $\overrightarrow{\ell}(x_i^1) = a_{i+4}$ for $i = 2, 3, \dots, n_1 - 1$.

Case 2.1. n_2, n_3 are both even.

$$\overrightarrow{\ell}(x_0^2) = a_4, \overrightarrow{\ell}(x_1^2) = a_5 \text{ and } \overrightarrow{\ell}(x_i^2) = a_{n_1+2+i} \text{ for } i = 2, 3, \dots, n_2 - 1.$$

$$\overrightarrow{\ell}(x_0^3) = a_0, \overrightarrow{\ell}(x_1^3) = a_1 \text{ and } \overrightarrow{\ell}(x_i^3) = a_{n_1+n_2+i} \text{ for } i = 2, 3, \dots, n_3 - 1.$$

Note that $\sum_{x \in A^i} \overrightarrow{\ell}(x) = n/2$ for $i = 1, 2, 3$ thus $w(v) = 0$ for any $v \in V(G)$.

Case 2.2 n_2, n_3 are both odd.

Assume first that $n_2 \geq 3$. Set $\overrightarrow{\ell}(x_0^2) = a_0$, $\overrightarrow{\ell}(x_1^2) = a_4$, $\overrightarrow{\ell}(x_2^2) = a_5$ and

$$\overrightarrow{\ell}(x_i^1) = a_{n_1+1+i} \text{ for } i = 3, 4, \dots, n_2 - 1.$$

$\overrightarrow{\ell}(x_0^3) = a_1$ and $\overrightarrow{\ell}(x_i^3) = a_{n_1+n_2+i}$ for $i = 1, 2, \dots, n_3 - 1$. As in Case 2.1

$\sum_{x \in A^i} \overrightarrow{\ell}(x) = n/2$ for $i = 1, 2, 3$ thus $w(v) = 0$ for any $v \in V(G)$.

Let now $n_2 = n_3 = 1$, then $n_1 \equiv 0 \pmod{4}$. Set the orientation $o(uv)$ for the edge $uv \in E(G)$ such that:

$$o(x_i^j x_k^p) = \begin{cases} \overrightarrow{x_0^2 x_i^1}, & i \text{ even} \\ \overrightarrow{x_i^1 x_0^2}, & i \text{ odd} \\ \overrightarrow{x_0^3 x_i^1}, & i = 0, 1, \dots, n_1 - 1 \\ \overrightarrow{x_0^3 x_0^2}. \end{cases}$$

Then let $\overrightarrow{\ell}(x_0^2) = \frac{n}{2}$, $\overrightarrow{\ell}(x_0^3) = \frac{n}{2} + 2$, $\overrightarrow{\ell}(x_{n/2}^1) = \frac{n}{2} + 1$, and

$$\overrightarrow{\ell}(x_i^1) = \begin{cases} i, & i = 0, 1, \dots, \frac{n}{2} - 1, \\ i + 2, & i = \frac{n}{2} + 1, \frac{n}{2} + 2, \dots, n_1 - 1. \end{cases}$$

Observe that $\sum_{g \in \mathbb{Z}_n} g = \frac{n}{2}$ since $n \equiv 2 \pmod{4}$, and also $\sum_{i \text{ odd}} \overrightarrow{\ell}(x_i^1) - \sum_{i \text{ even}} \overrightarrow{\ell}(x_i^1) = \frac{n}{2}$, so $w(v) = 2$ for any $v \in V(G)$. □

We finish this section with the following conjecture.

Conjecture 3.3.6. *If G is a $2r$ -regular graph of order n , then G is orientable \mathbb{Z}_n -distance magic.*

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Chapter 4

Orientable \mathbb{Z}_n -distance magic graphs via products^{*}

4.1 Introduction

In this chapter we study a generalization of distance magic graphs introduced recently in [14]. Let G be a simple, undirected graph on n vertices. Let ℓ be a bijection $\ell : V(G) \rightarrow \{1, 2, \dots, n\}$, and define for every vertex $x \in V(G)$, the *weight* of x , $w(x) = \sum_{y \in N(x)} \ell(y)$. If the weight of every vertex is equal to the same number k , called the *magic constant*, then we say ℓ is a *distance magic labeling* of G . If such a labeling

^{*}The material in this chapter has been submitted to Australasian Journal of Combinatorics

can be found, we say that G is *distance magic*. For a survey of distance magic graphs see [5]. If one uses group element as labels, the following generalization of distance magic labeling is possible.

Let Γ be an abelian group of order n with operation $+$. Let f be a bijection $f : V(G) \rightarrow \Gamma$. If there exists $\mu \in \Gamma$ such that $w(x) = \sum_{y \in N(x)} f(y) = \mu$, for all vertices $x \in V(G)$, then we say G is Γ -*distance magic*. Clearly if G is distance magic, then it is also \mathbb{Z}_n -distance magic, but the converse is not necessarily true.

We now consider the analogous labeling in the setting of directed graphs first introduced in [14]. We begin with an undirected graph $G = (V, E)$. A replacement of each edge in E with an arc, having a head at one vertex and tail at the other is called an *orientation* of G , denoted $\vec{G}(V, A)$. For a vertex x , let $N^+(x) = \{y \in V : \vec{yx} \in A\}$ and $N^-(x) = \{z \in V : \vec{xz} \in A\}$. Let $\text{indeg}(x) = |N^+(x)|$ and $\text{outdeg}(x) = |N^-(x)|$. Let Γ be an abelian group of order n with operation $+$. For two elements $g, h \in \Gamma$, we use the notation $g - h$ to mean $g + h^{-1}$, where h^{-1} is the additive inverse of h . Also, for repeated addition $g + g + \dots + g$, where g appears k times, we use the notation kg . A *directed Γ -distance magic labeling* of an oriented graph $\vec{G}(V, A)$ of order n is a bijection $\vec{\ell} : V \rightarrow \Gamma$ with the property that there is a $\mu \in \Gamma$, called the *magic constant*, such that

$$w(x) = \sum_{y \in N_G^+(x)} \vec{\ell}(y) - \sum_{y \in N_G^-(x)} \vec{\ell}(y) = \mu \text{ for every } x \in V(G).$$

It should be emphasized that the arithmetic takes place in Γ . If a graph G admits an orientation \vec{G} for which a directed Γ -distance magic labeling $\vec{\ell}$ exists, we say that G is *orientable Γ -distance magic* and we call the directed Γ -distance magic labeling $\vec{\ell}$ an *orientable Γ -distance magic labeling*.

Let \mathbb{Z}_n be the cyclic group of order n . With regards to orientable \mathbb{Z}_n -distance magic labeling, Cichacz et. al characterized complete graphs, complete bipartite graphs, complete tripartite graphs, circulant graphs, and certain products of graphs in [14]. They also showed that some graphs are not orientable \mathbb{Z}_n -distance magic. In particular, they proved the following.

Theorem 4.1.1. [14] *Let G have order $n \equiv 2 \pmod{4}$ and all vertices of odd degree. Then G is not orientable \mathbb{Z}_n -distance magic.*

The motivation for our study is a conjecture stated in their concluding remarks [14].

Conjecture 4.1.2. *If G is a $2r$ -regular graph of order n , then G is orientable \mathbb{Z}_n -distance magic.*

Determining whether an arbitrary graph is orientable \mathbb{Z}_n -distance magic is not practical. One strategy for building classes of graphs that are more fruitful to study is to combine common families of graphs via graph products. The three graph products we will use in this chapter are recalled in [32]. All three, the *direct product* $G \times H$, the *strong product* $G \boxtimes H$, and the *lexicographic product* $G \circ H$, are graphs with the

vertex set $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent in:

- $G \times H$ if $g \sim g'$ in G and $h \sim h'$ in H ;
- $G \boxtimes H$ if either $g = g'$ and $h \sim h'$ in H , or $h = h'$ and $g \sim g'$ in G , or $g \sim g'$ in G and $h \sim h'$ in H ;
- $G \circ H$ if and only if either $g \sim g'$ in G or $g = g'$ and $h \sim h'$ in H .

If $V(G) = \{g_0, g_1, \dots, g_{m-1}\}$ and $V(H) = \{h_0, h_1, \dots, h_{n-1}\}$ for some m and n , respectively, we use the notation (i, j) to denote the vertex (g_i, h_j) in any of the above products. If (i, j) appears as an argument in a function f , for easier reading and more transparent typesetting, we will use the notation $f(i, j)$ rather than $f((i, j))$. We will also use (i, j) to refer both to the vertex and the label of the vertex. Since the labelings considered in this chapter are bijections, this should cause no ambiguity.

Observe that of the products defined above, only the lexicographic product is not necessarily commutative. The lexicographic product $G \circ H$ is sometimes also referred to as graph composition and denoted $G[H]$.

Let $[n] = \{0, 1, \dots, n-1\}$ for a natural number n . Furthermore, for a given $i \in [n]$ and any integer j , let $\mathbf{i} + \mathbf{j}$ denote the smallest integer in $[n]$ such that $i + j \equiv \mathbf{i} + \mathbf{j} \pmod{n}$.

Let $C_n = x_0, x_1, \dots, x_{n-1}, x_0$ denote a cycle of length n . If the edges in C_n are oriented such that every arc has the form $\overrightarrow{x_i x_{i+1}}$ for all $i \in [n]$, then we say the cycle is oriented *clockwise*. On the other hand, if all the edges of the cycle are oriented such that every arc has the form $\overrightarrow{x_i x_{i-1}}$ for all $i \in [n]$, then we say the cycle is oriented *counter-clockwise*. In addition, we use the notation \bar{G} to denote the graph complement of G and the notation pG to denote p disjoint copies of the graph G .

4.2 Direct product of cycles

In this section, we turn our attention to the direct product. The direct product of two cycles is a four-regular graph. Anholcer et. al obtained the following result in [3].

Theorem 4.2.1. [3] *The direct product $C_m \times C_n$ is distance magic if and only if $m = 4$ or $n = 4$ or $m, n \equiv 0 \pmod{4}$.*

If instead the elements of \mathbb{Z}_{mn} are used as labels, Anholcer et. al obtained the following similar result in [4].

Theorem 4.2.2. [4] *The direct product $C_m \times C_n$ is \mathbb{Z}_{mn} -distance magic if and only if $m \in \{4, 8\}$ or $n \in \{4, 8\}$, or $m, n \equiv 0 \pmod{4}$.*

Allowing an orientation of the edges yields the following result.

Theorem 4.2.3. *For any $p \geq 1$, $m \geq 3$, and $n \geq 3$, p copies of the direct product $C_m \times C_n$ is orientable \mathbb{Z}_{pmn} -distance magic.*

Proof. Let $G = C_m = g_0, g_1, \dots, g_{m-1}, g_0$ and $H = C_n = h_0, h_1, \dots, h_{n-1}, h_0$. Then orient each copy of $G \times H$ as follows. For each $i \in [m]$ and $j \in [n]$, orient counter-clockwise with respect to j each cycle of the form $\{(i, j), (\mathbf{i} + \mathbf{1}, \mathbf{j} + \mathbf{1}), (\mathbf{i} + \mathbf{2}, \mathbf{j} + \mathbf{2}), \dots, (i, j)\}$. Similarly, orient counter-clockwise with respect to j each cycle of the form $\{(i, j), (\mathbf{i} - \mathbf{1}, \mathbf{j} + \mathbf{1}), (\mathbf{i} - \mathbf{2}, \mathbf{j} + \mathbf{2}), \dots, (i, j)\}$. Since the graph $G \times H$ can be edge-decomposed into cycles of those two forms, we have oriented every edge in each copy of $G \times H$. Now let (i, j, k) denote the vertex (i, j) of the k^{th} copy of $G \times H$ for $i \in [m]$, $j \in [n]$, and $k = 1, 2, \dots, p$. Now for all such i, j , and k , let $\vec{\ell} : V(p(G \times H)) \rightarrow \mathbb{Z}_{pmn}$ where

$$\vec{\ell}(i, j, k) = pmj + i + m(k - 1),$$

where the arithmetic is performed in \mathbb{Z}_{pmn} . To show that $\vec{\ell}$ is injective, we have

$\vec{\ell}(i, j, k) = \vec{\ell}(i', j', k')$ if and only if

$$pmj + i + m(k - 1) \equiv pmj' + i' + m(k' - 1) \pmod{pmn}.$$

Reducing this equation modulo m gives $i \equiv i' \pmod{m}$, which implies $i = i'$ since $i, i' \in [m]$. Then we have $m(pj + k - 1) \equiv m(pj' + k' - 1) \pmod{pmn}$, which implies

$p(j - j') \equiv k' - k \pmod{pn}$. Thus $j = j'$ and $k = k'$ since $0 \leq |j - j'| \leq n - 1$ and $0 \leq |k - k'| \leq p - 1$. Therefore $\vec{\ell}$ is injective, hence bijective. Finally, recalling that $w(i, j, k) \in \mathbb{Z}_{pmn}$, for any i, j , and k we have,

$$\begin{aligned}
w(i, j, k) &= \sum_{y \in N^+(i, j, k)} \vec{\ell}(y) - \sum_{y \in N^-(i, j, k)} \vec{\ell}(y) \\
&= \vec{\ell}(\mathbf{i} + \mathbf{1}, \mathbf{j} + \mathbf{1}, k) + \vec{\ell}(\mathbf{i} - \mathbf{1}, \mathbf{j} + \mathbf{1}, k) \\
&\quad - \vec{\ell}(\mathbf{i} + \mathbf{1}, \mathbf{j} - \mathbf{1}, k) - \vec{\ell}(\mathbf{i} - \mathbf{1}, \mathbf{j} - \mathbf{1}, k) \\
&= 2pm[(\mathbf{j} + \mathbf{1}) - (\mathbf{j} - \mathbf{1})].
\end{aligned}$$

If $j \in \{0, n - 1\}$, then

$$\begin{aligned}
w(i, j, k) &= 2pm(2 - n) \\
&= 4pm - 2pmn \\
&= 4pm,
\end{aligned}$$

while if $0 < j < n - 1$, we have

$$\begin{aligned}
w(i, j, k) &= 2pm \cdot 2 \\
&= 4pm.
\end{aligned}$$

Hence, p copies of the direct product $C_m \times C_n$ is orientable \mathbb{Z}_{pmn} -distance magic. \square

4.3 Lexicographic product

We begin this section by considering the lexicographic product of two cycles. The graph $C_m \circ C_n$ is a $(2n + 2)$ -regular graph. The following was proved by Anholcer et. al in [3].

Theorem 4.3.1. [3] *The lexicographic product of two cycles $C_m \circ C_n$, $m, n \geq 3$ is distance magic if and only if $n = 4$.*

For oriented graphs, every cycle length is allowed.

Theorem 4.3.2. *For any $p \geq 1$, $m \geq 3$, and $n \geq 3$, p copies of the lexicographic product of two cycles $C_m \circ C_n$ is orientable \mathbb{Z}_{pmn} -distance magic.*

Proof. Let $G = C_m = g_0, g_1, \dots, g_{m-1}, g_0$ and $H = C_n = h_0, h_1, \dots, h_{n-1}, h_0$. Let (i, j, k) denote the vertex (i, j) of the k^{th} copy of $G \circ H$ for $i \in [m]$, $j \in [n]$, and $k \in \{1, 2, \dots, p\}$. For each copy of $G \circ H$, orient the edges so that $N^+(i, j, k) = \{(i, \mathbf{j} + \mathbf{1}, k)\} \cup \{(\mathbf{i} + \mathbf{1}, t, k) : t \in [n]\}$ and $N^-(i, j, k) = \{(i, \mathbf{j} - \mathbf{1}, k)\} \cup \{(\mathbf{i} - \mathbf{1}, t, k) : t \in [n]\}$. Now for all such i, j , and k , let $\vec{\ell} : V(p(G \circ H)) \rightarrow \mathbb{Z}_{pmn}$ where

$$\vec{\ell}(i, j, k) = p(mj + i) + (k - 1),$$

where the arithmetic is performed modulo pmn . Clearly, $\vec{\ell}$ is bijective.

For any i, j , and k we now have,

$$\begin{aligned}
w(i, j, k) &= \sum_{y \in N^+(i, j, k)} \vec{\ell}(y) - \sum_{y \in N^-(i, j, k)} \vec{\ell}(y) \\
&= \vec{\ell}(i, \mathbf{j} + \mathbf{1}, k) + \sum_{t \in [n]} \vec{\ell}(\mathbf{i} + \mathbf{1}, t, k) \\
&\quad - \vec{\ell}(i, \mathbf{j} - \mathbf{1}, k) - \sum_{t \in [n]} \vec{\ell}(\mathbf{i} - \mathbf{1}, t, k) \\
&= pm[(\mathbf{j} + \mathbf{1}) - (\mathbf{j} - \mathbf{1})] + pn[(\mathbf{i} + \mathbf{1}) - (\mathbf{i} - \mathbf{1})] \\
&= pma + pnb,
\end{aligned}$$

where $a = \begin{cases} 2, & \text{if } j \in \{0, n-1\} \\ 2-n, & \text{otherwise} \end{cases}$, and $b = \begin{cases} 2, & \text{if } i \in \{0, m-1\} \\ 2-m, & \text{otherwise} \end{cases}$. Then since $pma + pnb \equiv 2pm + 2pn \pmod{pmn}$, we have that $w(i, j, k) = 2pm + 2pn$ for all $(i, j, k) \in V(p(G \circ H))$, proving the result. \square

Next we turn our attention to complete multipartite graphs. Cichacz et. al proved the following in [14].

Theorem 4.3.3. [14] *The complete graph K_n is orientable \mathbb{Z}_n -distance magic if and only if n is odd.*

For complete bipartite graphs, they proved the following sufficient conditions.

Theorem 4.3.4. [14] *Let $G = K_{n_1, n_2}$ and $n_1 + n_2 = n$. If $n \not\equiv 2 \pmod{4}$, then G is orientable \mathbb{Z}_n -distance magic.*

Combined with Theorem 4.3.4, the next result settles the spectrum for complete bipartite graphs and more.

Theorem 4.3.5. *Let $n_1 + n_2 + \dots + n_p = n$. If $n \equiv 2 \pmod{4}$ and $p = 1$ or p is even, then K_{n_1, n_2, \dots, n_p} is not orientable \mathbb{Z}_n -distance magic.*

Proof. Let $G = K_{n_1, n_2, \dots, n_p}$. If $p = 1$, then $G \cong K_n$ is an odd regular graph on $n \equiv 2 \pmod{4}$ vertices, so it is not orientable \mathbb{Z}_n -distance magic by Theorem 4.1.1. So assume p is even. For the sake of contradiction, suppose $n \equiv 2 \pmod{4}$ and G is orientable \mathbb{Z}_n -distance magic with orientation \vec{G} and orientable \mathbb{Z}_n -distance magic labeling $\vec{\ell} : V(G) \rightarrow \mathbb{Z}_n$. Observe that $\mathbb{Z}_n \cong \mathbb{Z}_2 \times \mathbb{Z}_{n/2}$ by the Fundamental Theorem of Abelian Groups since $\gcd(2, \frac{n}{2}) = 1$. Therefore, there exists a labeling $\vec{f} : V(G) \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_{n/2}$ and element $(a, b) \in \mathbb{Z}_2 \times \mathbb{Z}_{n/2}$ such that $w(v) = (a, b)$ for all $v \in V(G)$. Let $\vec{f}_2(v)$ represent the \mathbb{Z}_2 component of $\vec{f}(v)$ for all $v \in V(G)$. Therefore,

$$a = \sum_{y \in N^+(v)} \vec{f}_2(y) - \sum_{y \in N^-(v)} \vec{f}_2(y) = \sum_{y \in N(v)} \vec{f}_2(y).$$

Since \vec{f} is bijective, there exists an odd number of $v \in V(G)$ such that $\vec{f}_2(v) = 1$.

Since p is even, it follows that there exists an odd number of partite sets A such that

$\sum_{v \in A} \vec{f}_2(v) = 1$ and an odd number of partite sets B such that $\sum_{v \in B} \vec{f}_2(v) = 0$. But this

leads to a contradiction, since for $x \in A$ and $y \in B$,

$$w(x) = (0, b) \neq (1, b) = w(y).$$

□

Corollary 4.3.6. *Let $G = K_{n_1, n_2}$ be a complete bipartite graph such that $n_1 + n_2 = n$. Then G is orientable \mathbb{Z}_n -distance magic if and only if $n \not\equiv 2 \pmod{4}$.*

The next result serves as a cautionary tale for the lexicographic product.

Corollary 4.3.7. *Let $n = n_1 + n_2 \equiv 1, 3 \pmod{4}$ and $k \equiv 2 \pmod{4}$. Then K_{n_1, n_2} is orientable \mathbb{Z}_n -distance magic, but $K_{n_1, n_2} \circ \overline{K_k}$ is not orientable \mathbb{Z}_{nk} -distance magic.*

Proof. The proof is clear since $K_{n_1, n_2} \circ \overline{K_k} \cong K_{kn_1, kn_2}$ and $kn_1 + kn_2 \equiv 2 \pmod{4}$. □

Next we recall a theorem regarding distance magic labelings proved by Miller, Rodger, and Simanjuntak in [38].

Theorem 4.3.8. [38] *If H is an r -regular graph, then $G = H \circ \overline{K_{2k}}$ is distance magic.*

Now we prove an analogous theorem in the setting of oriented graphs.

Theorem 4.3.9. *If H is an orientable \mathbb{Z}_n -distance magic graph of order n , then the lexicographic product $G = H \circ \overline{K_k}$ is orientable \mathbb{Z}_{nk} -distance magic except possibly when $k \equiv 2 \pmod{4}$ and H contains a vertex x such that $\text{indeg}(x) \not\equiv \text{outdeg}(x) \pmod{2}$.*

Proof. Let H be an orientable \mathbb{Z}_n -distance magic graph on n vertices with directed \mathbb{Z}_n -distance magic labeling $\vec{f} : V(H) \rightarrow \mathbb{Z}_n$, orientation \vec{H} , and magic constant

$\mu \in \mathbb{Z}_n$. Then construct the graph $G = H \circ \overline{K_k}$ with vertex set $V(G) = \{(i, j) : i \in V(H), j = 1, 2, \dots, k\}$ by replacing each vertex i of H with k isolated vertices such that two of the new vertices are adjacent whenever their counterparts in H are adjacent. We will orient the edges of G later. For all $i \in V(H)$, let B_i represent the set of k vertices which have replaced the vertex i . We will now label the vertices in each set B_i . For $i \in [n]$, define cosets $A_i = \{i + \langle n \rangle\} \subseteq \mathbb{Z}_{nk}$, where $\langle n \rangle$ is the subgroup generated by n . Clearly, $\mathbb{Z}_{nk} = A_0 \cup A_1 \cup \dots \cup A_{n-1}$ and $A_i \cap A_j = \emptyset \Leftrightarrow i \neq j$. For all $i \in [n]$, let $\vec{l} : A_i \rightarrow B_i$ be an arbitrary bijection.

Case 1. k is odd or $\text{indeg}(x)$ and $\text{outdeg}(x)$ have the same parity for all $x \in V(H)$.

Orient the edges of G so that each edge $(i, j)(p, q)$ for $(i, j) \in B_i$ and $(p, q) \in B_p$ has the same orientation as its counterpart ip in \vec{H} . Let

$$\begin{aligned}
S_i &= \sum_{x \in B_i} \vec{l}(x) \\
&= \sum_{a \in A_i} a \\
&= i + (n + i) + \dots + ((k - 1)n + i) \\
&= \frac{k[2i + (k - 1)n]}{2},
\end{aligned}$$

with all arithmetic performed modulo nk . Let $x \in B_i$ and let $N_H^+(i) = \{a_1, \dots, a_p\}$ and $N_H^-(i) = \{b_1, \dots, b_q\}$ where $p = \text{indeg}(i)$ and $q = \text{outdeg}(i)$. If we write $\sum_{i=1}^p a_i = a$ and $\sum_{i=1}^q b_i = b$, then $a - b = \mu$, (with all arithmetic performed in \mathbb{Z}_n) since H is orientable

\mathbb{Z}_n -distance magic. Then recalling k is odd or $p \equiv q \pmod{2}$, we have

$$\begin{aligned}
w(x) &= \sum_{i=1}^p S_{a_i} - \sum_{i=1}^q S_{b_i} \\
&= \frac{k[2(a_1+\dots+a_p)+p(k-1)n]}{2} - \frac{k[2(b_1+\dots+b_q)+q(k-1)n]}{2} \\
&= S_{a_1+\dots+a_p} + \frac{k(k-1)(p-1)n}{2} - S_{b_1+\dots+b_q} - \frac{k(k-1)(q-1)n}{2} \\
&= S_a - S_b + \frac{(k-1)(p-q)}{2}nk \\
&\equiv S_a - S_b \pmod{nk} \\
&\equiv k(a-b) \pmod{nk} \\
&\equiv k\mu \pmod{nk},
\end{aligned}$$

which shows \vec{l} is a directed \mathbb{Z}_{nk} -distance magic labeling of G .

Case 2. $k \equiv 0 \pmod{4}$.

Notice that every vertex in B_i can be expressed uniquely as $i + tn$ for some $t \in [k]$.

For every edge $ij \in E(H)$, orient the edges in G between B_i and B_j as follows. For

all $a, b \in [k]$, orient the edges between $i + an \in B_i$ and $j + bn \in B_j$ such that if

$a \equiv 0, 3 \pmod{4}$,

$$N_G^+(i + an) = \{j + bn : b \equiv 0, 3 \pmod{4}\},$$

$$N_G^-(i + an) = \{j + bn : b \equiv 1, 2 \pmod{4}\},$$

and if $a \equiv 1, 2 \pmod{4}$

$$\begin{aligned} N_G^+(i + an) &= \{j + bn : b \equiv 1, 2 \pmod{4}\}, \\ N_G^-(i + an) &= \{j + bn : b \equiv 0, 3 \pmod{4}\}. \end{aligned}$$

Let $i + an \in B_i$ for some $a \in [k]$ and let $ij \in E(H)$. Denote $w_{ij}(i + an)$ as the weight of $i + an$ in G induced by the edge $ij \in E(H)$. If $a \equiv 0, 3 \pmod{4}$, then

$$\begin{aligned} w_{ij}(i + an) &= \sum_{b \equiv 0, 3 \pmod{4}} (j + bn) - \sum_{b \equiv 1, 2 \pmod{4}} (j + bn) \\ &= [(0n + 3n) - (1n + 2n)] + \dots + [(4n + 7n) - (5n + 6n)] \\ &+ [((k - 4)n + (k - 1)n) - ((k - 3)n + (k - 2)n)] \\ &= 0. \end{aligned}$$

If $a \equiv 1, 2 \pmod{4}$, essentially the same calculation shows $w_{ij}(i + an) = 0$. Therefore, each edge ij in H induces 0 weight in G , so the graph G is orientable \mathbb{Z}_{nk} -distance magic. □

4.4 Strong product of cycles

The strong product of two cycles, $C_m \boxtimes C_n$ is an eight-regular graph which contains the direct product $C_m \times C_n$ as a spanning subgraph. For all of the theorems in this section, let $m, n \geq 3$ be given and let $G = C_m = g_0, g_1, \dots, g_{m-1}, g_0$ and

$H = C_n = h_0, h_1, \dots, h_{n-1}, h_0$. Form the strong product $G \boxtimes H$ with vertex set $V(G \boxtimes H) = \{(i, j) : i \in [m], j \in [n]\}$. The graph $G \boxtimes H$ can be edge-decomposed into cycles of the following four types:

- I. $(i, j), (i, \mathbf{j} + \mathbf{1}), (i, \mathbf{j} + \mathbf{2}), \dots, (i, j),$
- II. $(i, j), (\mathbf{i} - \mathbf{1}, \mathbf{j} + \mathbf{1}), (\mathbf{i} - \mathbf{2}, \mathbf{j} + \mathbf{2}), \dots, (i, j),$
- III. $(i, j), (\mathbf{i} + \mathbf{1}, \mathbf{j} + \mathbf{1}), (\mathbf{i} + \mathbf{2}, \mathbf{j} + \mathbf{2}), \dots, (i, j),$
- IV. $(i, j), (\mathbf{i} + \mathbf{1}, j), (\mathbf{i} + \mathbf{2}, j), \dots, (i, j).$

Orient each cycle of types I, II, and III counter-clockwise with respect to j , and orient each cycle of type IV counter-clockwise with respect to i . For a given bijection $\vec{\ell} : V(G \boxtimes H) \mapsto \mathbb{Z}_{mn}$ and any vertex $(i, j) \in V(G \boxtimes H)$, we have

$$\begin{aligned}
w(i, j) &= \sum_{x \in N^+(i, j)} \vec{\ell}(x) - \sum_{x \in N^-(i, j)} \vec{\ell}(x) \\
&= \vec{\ell}(\mathbf{i} + \mathbf{1}, j) + \vec{\ell}(\mathbf{i} + \mathbf{1}, \mathbf{j} + \mathbf{1}) + \vec{\ell}(i, \mathbf{j} + \mathbf{1}) + \vec{\ell}(\mathbf{i} - \mathbf{1}, \mathbf{j} + \mathbf{1}) \\
&\quad - \left[\vec{\ell}(\mathbf{i} - \mathbf{1}, j) + \vec{\ell}(\mathbf{i} - \mathbf{1}, \mathbf{j} - \mathbf{1}) + \vec{\ell}(i, \mathbf{j} - \mathbf{1}) + \vec{\ell}(\mathbf{i} + \mathbf{1}, \mathbf{j} - \mathbf{1}) \right].
\end{aligned}$$

The first set of constructions is based on the greatest common divisor of the cycle lengths.

Theorem 4.4.1. *If $\gcd(m, n) = 1$, then the strong product $C_m \boxtimes C_n$ is orientable \mathbb{Z}_{mn} -distance magic.*

Proof. If $\gcd(m, n) = 1$, then $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$ by the Fundamental Theorem of

Abelian Groups. Define $\vec{\ell} : V(C_m \boxtimes C_n) \rightarrow \mathbb{Z}_m \times \mathbb{Z}_n$ where

$$\vec{\ell}(i, j) = (i, j).$$

For all $(i, j) \in V(C_m \boxtimes C_n)$, we have

$$\begin{aligned} w(i, j) &= (4i + 1, 4j + 3) - (4i - 1, 4j - 3) \\ &= (2, 6), \end{aligned}$$

proving the theorem. □

To prove the next theorem, we use a labeling similar to the one used by Froncek on the Cartesian product of two cycles in [23]. The Cartesian product $C_m \square C_n$ is a spanning subgraph of $C_m \boxtimes C_n$.

Theorem 4.4.2. *If $\gcd(m, n) = 2$ or 4 , then $C_m \boxtimes C_n$ is orientable \mathbb{Z}_{mn} -distance magic.*

Proof. Assume $m \leq n$ and let a *diagonal* of $C_m \boxtimes C_n$ be a sequence of vertices $(i, 0), (i + 1, 1), \dots, (m - 1, m - 1), (0, m), (1, m + 1), \dots, (i - 1, n - 1)$ of length l . Let $g = \gcd(m, n)$. Clearly, $l = \text{lcm}(m, n)$ and there are $\frac{mn}{l} = g$ diagonals. For ease of notation, we will denote the diagonal $D^i = (d_0^i, d_1^i, \dots, d_{l-1}^i)$ for $i = 0, 1, \dots, g - 1$. Similarly, we define a *back diagonal* as a sequence of vertices $(i, 0), (i - 1, 1), \dots, (0, i), (m - 1, i + 1), \dots, (i + 1, n - 1)$ and denote it by $B^i = (b_0^i, b_1^i, \dots, b_{l-1}^i)$ for $i = 0, 1, \dots, g - 1$.

Let $H \cong \langle g \rangle$, the subgroup of \mathbb{Z}_{mn} generated by g . Define $\vec{\ell} : V(C_m \boxtimes C_n) \rightarrow \mathbb{Z}_{mn}$ by labeling the vertices of the diagonal D^i with the elements of the coset $H + i$ in increasing order for $i = 0, 1, \dots, g-1$. Divide n by m and write $n = km + r$ for $0 \leq r \leq m-1$.

Case 1. $g = 2$.

Because there are only two diagonals, $b_1^i = d_h^i$ for some h . Counting steps through the lattice, it is not difficult to see that $h = \frac{(m-2)n}{r} + 1$. Therefore the two sequences, $(b_0^i, b_1^i, \dots, b_{l-1}^i)$ and $(d_0^i, d_h^i, d_{2h}^i, \dots)$ are equal since $|B^i| = |D^i|$. Notice for any vertex $(i, j) = d_a^t$ on D^t , we have $N^+(i, j) = \{d_{a+1}^t, b_{c+2}^t, d_{p+1}^{t+1}, d_{q+1}^{t+1}\}$ and $N^-(i, j) = \{d_{a-1}^t, b_c^t, d_p^{t+1}, d_q^{t+1}\}$ for some numbers c, p, q , and $t+1$ is performed modulo 2. Therefore,

$$\begin{aligned}
w(i, j) &= (d_{a+1}^t - d_{a-1}^t) + (d_{p+1}^{t+1} - d_p^{t+1}) \\
&+ (d_{q+1}^{t+1} - d_q^{t+1}) + (b_{c+2}^t - b_c^t) \\
&= 2g + g + g + 2gh \\
&= 2g(2g + h).
\end{aligned}$$

Case 2. $g = 4$.

Since the graph contains exactly four diagonals, $b_2^i = d_{h'}^i$ for some h' . Along the lines of the previous case, we obtain $h' = \frac{(m-4)n}{r} + 2$ and consequently $b_c^i - b_{c-2}^i = gh$.

For any vertex $(i, j) = d_a^t$ on D^t , we have $N^+(i, j) = \{d_{a+1}^t, b_{c+2}^{t'}, d_{p+1}^{t+1}, d_{q+1}^{t-1}\}$ and $N^-(i, j) = \{d_{a-1}^t, b_c^{t'}, d_p^{t+1}, d_q^{t-1}\}$ for some numbers c, p, q , and $t+1$ and $t-1$ are performed modulo 4. Therefore,

$$\begin{aligned}
w(i, j) &= (d_{a+1}^t - d_{a-1}^t) + (d_{p+1}^{t+1} - d_p^{t+1}) \\
&+ (d_{q+1}^{t-1} - d_q^{t-1}) + (b_{c+2}^{t'} - b_c^{t'}) \\
&= 2g + g + g + gh \\
&= g(4 + h).
\end{aligned}$$

Since h is independent of i and j , we have proven the result. \square

The next theorem uses an isomorphic group to provide the labeling.

Theorem 4.4.3. *Let $\gcd(m, n) = d$ where $d = 3, 5$, or 6 . If $d^2 \nmid m$ and $d^2 \nmid n$, then $C_m \boxtimes C_n$ is orientable \mathbb{Z}_{mn} -distance magic.*

Proof. Let $\gcd(m, n) = d \in \{3, 5, 6\}$, $d^2 \nmid m$, and $d^2 \nmid n$. Therefore, $\gcd(\frac{m}{d}, d) = \gcd(\frac{n}{d}, d) = 1$ which implies $\gcd(\frac{m}{d}, \frac{n}{d}) = \gcd(\frac{m}{d}, d^2) = \gcd(\frac{n}{d}, d^2) = 1$ and $\mathbb{Z}_{mn} \cong \mathbb{Z}_{d^2} \times \mathbb{Z}_{\frac{m}{d}} \times \mathbb{Z}_{\frac{n}{d}}$. Let \bar{i} and \bar{j} represent the remainder of i divided by d and j divided by d , respectively.

Define $\vec{\ell} : V(C_m \boxtimes C_n) \rightarrow \mathbb{Z}_{d^2} \times \mathbb{Z}_{\frac{m}{d}} \times \mathbb{Z}_{\frac{n}{d}}$ as

$$\vec{\ell}(i, j) = (\alpha_d, \beta, \gamma),$$

where $0 \leq \beta \leq \frac{m}{d}$, $\beta \equiv i \pmod{\frac{m}{d}}$, $0 \leq \gamma \leq \frac{n}{d}$, $\gamma \equiv j \pmod{\frac{n}{d}}$, and α_d is defined as follows. If $d = 3$, let

$$\alpha_3 = 3\bar{i} + \bar{j}.$$

If $d = 5$, let

$$\alpha_5 = 5\bar{j} + \overline{i - 2j}.$$

If $d = 6$, let

$$\alpha_6 = \begin{cases} 6\bar{i} + 2\bar{j}, & i \text{ even} \\ 6\overline{(i-1)} + 2\bar{j} + 1, & i \text{ odd} \end{cases},$$

where the arithmetic is performed modulo d^2 .

To show $\vec{\ell}$ is injective, we have $\vec{\ell}(i, j) = (\alpha_d, \beta, \gamma) = (\alpha'_d, \beta', \gamma') = \vec{\ell}(i', j')$ if and only if $\alpha_d \equiv \alpha'_d \pmod{d^2}$ and $\beta \equiv \beta' \pmod{\frac{m}{d}}$ and $\gamma \equiv \gamma' \pmod{\frac{n}{d}}$. Consequently,

$$\begin{aligned} \beta = \beta' &\iff i - i' \equiv 0 \pmod{\frac{m}{d}} \\ \gamma = \gamma' &\iff j - j' \equiv 0 \pmod{\frac{n}{d}} \end{aligned}.$$

Therefore, $i - i' = a\frac{m}{d}$ for some integer $a : |a| < d$, and $j - j' = b\frac{n}{d}$ for some integer $b : |b| < d$.

If $d = 3$, then

$$\begin{aligned}
\alpha_3 \equiv \alpha'_3 \pmod{3^2} &\implies 3\bar{i} + \bar{j} \equiv 3\bar{i}' + \bar{j}' \pmod{3^2} \\
&\implies \bar{j} \equiv \bar{j}' \pmod{3} \\
&\implies \bar{j} - \bar{j}' \equiv 0 \pmod{3} \\
&\implies \bar{j} - \bar{j}' \equiv 0 \pmod{n} \\
&\implies j = j',
\end{aligned}$$

since $\bar{j} - \bar{j}' \equiv 0 \pmod{\frac{n}{3}}$ and $\gcd(3, \frac{n}{3}) = 1$ by assumption. But,

$$\begin{aligned}
j = j' &\implies 3\bar{i} \equiv 3\bar{i}' \pmod{3^2} \\
&\implies 3(\bar{i} - \bar{i}') \equiv 0 \pmod{3^2} \\
&\implies \bar{i} - \bar{i}' \equiv 0 \pmod{3} \\
&\implies \bar{i} - \bar{i}' \equiv 0 \pmod{n} \\
&\implies i = i'.
\end{aligned}$$

Therefore, $\overrightarrow{\ell}$ is bijective when $d = 3$. A similar argument can be made to show $\overrightarrow{\ell}$ is bijective when $d = 5, 6$.

Let $(i, j) \in V(C_m \boxtimes C_n)$. We calculate $w(i, j)$ component-wise. Let

$$w(i, j) = (w_1, w_2, w_3),$$

where $w_1 \in \mathbb{Z}_{d^2}$, $w_2 \in \mathbb{Z}_{\frac{m}{d}}$, and $w_3 \in \mathbb{Z}_{\frac{n}{d}}$. First we determine w_1 for each $d \in \{3, 5, 6\}$.

Consequently, the arithmetic will be performed in \mathbb{Z}_{d^2} .

If $d = 3$,

$$\begin{aligned}
w_1 &= (3(\overline{i+1}) + \overline{j}) + (3(\overline{i+1}) + \overline{j+1}) \\
&+ (3\overline{i} + \overline{j+1}) + (3(\overline{i-1}) + \overline{j+1}) \\
&- [(3(\overline{i-1}) + \overline{j}) + (3(\overline{i+1}) + \overline{j-1})] \\
&- [(3\overline{i} + \overline{j-1}) + (3(\overline{i+1}) + \overline{j-1})] \\
&= 3.
\end{aligned}$$

If $d = 5$,

$$\begin{aligned}
w_1 &= (5\overline{j} + \overline{i+1-2j}) + (5(\overline{j+1}) + \overline{i+1-2j-2}) \\
&+ (5(\overline{j+1}) + \overline{i-2j-2}) + (5(\overline{j+1}) + \overline{i-1-2j-2}) \\
&- [(5\overline{j} + \overline{i-1-2j}) + (5(\overline{j-1}) + \overline{i-1-2j+2})] \\
&- (5(\overline{j-1}) + \overline{i-2j+2}) + (5(\overline{j-1}) + \overline{i+1-2j+2}) \\
&= 15[(\overline{j+1}) - (\overline{j-1})] + \overline{i-2j-2} - \overline{i-2j+3} \\
&+ \overline{i-2j+2} - \overline{i-2j-3} \\
&= 15[(\overline{j+1}) - (\overline{j-1})],
\end{aligned}$$

since $\overline{i-2j-2} = \overline{i-2j+3}$ and $\overline{i-2j+2} = \overline{i-2j-3}$. Then for $\overline{j} \in \{1, 2, 3\}$, we have $w_1 = 15 \cdot 2 \equiv 5 \pmod{25}$, and for $\overline{j} \in \{0, 4\}$, we have $w_1 = 15 \cdot (-3) \equiv 5 \pmod{25}$.

Therefore, $w_1 = 5$.

Finally, if $d = 6$ and i is even we have,

$$\begin{aligned}
w_1 &= (6\bar{i} + 2\bar{j} + 1) + (6\bar{i} + 2(\overline{j+1}) + 1) \\
&+ (6\bar{i} + 2(\overline{j+1})) + (6(\overline{i-2}) + 2(\overline{j+1}) + 1) \\
&- [(6(\overline{i-2}) + 2\bar{j} + 1) + (6(\overline{i-2}) + 2(\overline{j-1}) + 1)] \\
&- [(6\bar{i} + 2(\overline{j-1})) + (6\bar{i} + 2(\overline{j-1}) + 1)] \\
&= 6(\bar{i} - (\overline{i-2})) + 6((\overline{j+1}) - (\overline{j-1})).
\end{aligned}$$

Notice, $6(\bar{i} - (\overline{i-2})) \equiv 12 \pmod{36}$ for all i and $6((\overline{j+1}) - (\overline{j-1})) \equiv 12 \pmod{36}$

for all j , so $w_1 = 24$.

If $d = 6$ and i is odd,

$$\begin{aligned}
w_1 &= (6(\overline{i+1}) + 2\bar{j}) + (6(\overline{i+1}) + 2(\overline{j+1})) \\
&+ (6(\overline{i-1}) + 2(\overline{j+1}) + 1) + (6(\overline{i-1}) + 2(\overline{j+1})) \\
&- [(6(\overline{i-1}) + 2\bar{j}) + (6(\overline{i-1}) + 2(\overline{j-1}))] \\
&- [(6(\overline{i-1}) + 2(\overline{j-1}) + 1) + (6(\overline{i+1}) + 2(\overline{j-1}))] \\
&= 6(\overline{(i+1)} - (\overline{i-1})) + 6((\overline{j+1}) - (\overline{j-1})).
\end{aligned}$$

But, $6(\overline{(i+1)} - (\overline{i-1})) \equiv 12 \pmod{36}$ for all i and $6((\overline{j+1}) - (\overline{j-1})) \equiv 12 \pmod{36}$ for all j , so $w_1 = 24$.

Next we calculate w_2 and w_3 which are each independent of d .

Recall $w_2 \in \mathbb{Z}_{\frac{m}{d}}$ and $w_3 \in \mathbb{Z}_{\frac{n}{d}}$. We have

$$\begin{aligned} w_2 &= 2(i-1) + i + (i+1) \\ &- [2(i+1) + i + i-1] \\ &= -2, \end{aligned}$$

and

$$\begin{aligned} w_3 &= j + 3(j+1) \\ &- [j + 3(j-1)] \\ &= 6. \end{aligned}$$

Hence,

$$w(i, j) = (w_1, -2, 6),$$

where

$$w_1 = \begin{cases} 3, & \text{for } d = 3 \\ 5, & \text{for } d = 5 \\ 12, & \text{for } d = 6 \end{cases},$$

proving the theorem. □

The next set of constructions is based on the modulo 4 congruence class of the cycle lengths.

Theorem 4.4.4. *If $mn \equiv 2 \pmod{4}$, then $C_m \boxtimes C_n$ is orientable \mathbb{Z}_{mn} -distance magic.*

Proof. If $mn \equiv 2 \pmod{4}$, then 2 divides exactly one of m or n . Without loss of generality, we may assume $2 \mid m$. By the Fundamental Theorem of Abelian Groups, we have $\mathbb{Z}_{mn} \cong \mathbb{Z}_2 \times \mathbb{Z}_{\frac{mn}{2}}$. Define $\vec{\ell} : V(C_m \boxtimes C_n) \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_{\frac{mn}{2}}$ where

$$\vec{\ell}(i, j) = \begin{cases} (0, \frac{i}{2}n + j), & i \text{ even} \\ (1, \frac{i-1}{2}n + j), & i \text{ odd} \end{cases}.$$

Clearly $\vec{\ell}$ is a bijection, and if i is even we have,

$$\begin{aligned} w(i, j) &= (1, \frac{i}{2}n + j) + (1, \frac{i}{2}n + j + 1) \\ &+ (1, \frac{i}{2}n + j) + (1, \frac{i}{2}n + j + 1) \\ &- (1, \frac{i-2}{2}n + j) - (1, \frac{i-2}{2}n + j - 1) \\ &- (0, \frac{i}{2}n + j - 1) - (1, \frac{i}{2}n + j - 1) \\ &= (0, n + 6). \end{aligned}$$

While if i is odd we have,

$$\begin{aligned} w(i, j) &= (0, \frac{i+1}{2}n + j) + (0, \frac{i+1}{2}n + j + 1) \\ &+ (1, \frac{i-1}{2}n + j + 1) + (1, \frac{i-1}{2}n + j + 1) \\ &- (0, \frac{i-1}{2}n + j) - (0, \frac{i-1}{2}n + j - 1) \\ &- (1, \frac{i-1}{2}n + j - 1) - (0, \frac{i+1}{2}n + j - 1) \\ &= (0, n + 6), \end{aligned}$$

proving the theorem. □

One may ask what can be said of $C_m \boxtimes C_n$ if $mn \equiv 0 \pmod{4}$. A partial answer is given in the next theorem.

Theorem 4.4.5. *If $m \equiv n \equiv 2 \pmod{4}$, then $C_m \boxtimes C_n$ is orientable \mathbb{Z}_{mn} -distance magic.*

Proof. Let $\gcd(\frac{m}{2}, \frac{n}{2}) = \delta$. We establish two cases based on δ .

Case 1. $\delta = 1$.

Since $\gcd(\frac{m}{2}, \frac{n}{2}) = 1$ and both $\frac{m}{2}$ and $\frac{n}{2}$ are odd, we have $\mathbb{Z}_{mn} \cong \mathbb{Z}_4 \times \mathbb{Z}_{\frac{m}{2}} \times \mathbb{Z}_{\frac{n}{2}}$ by the Fundamental Theorem of Abelian Groups. Define $\vec{\ell} : V(C_m \boxtimes C_n) \rightarrow \mathbb{Z}_4 \times \mathbb{Z}_{\frac{m}{2}} \times \mathbb{Z}_{\frac{n}{2}}$ where

$$\vec{\ell}(i, j) = (r, \bar{i}, \bar{j}),$$

where

$$r = \begin{cases} 0, & i \text{ even}, & j \text{ even} \\ 1, & i \text{ even}, & j \text{ odd} \\ 2, & i \text{ odd}, & j \text{ even} \\ 3, & i \text{ odd}, & j \text{ odd} \end{cases},$$

$\bar{i} \in \mathbb{Z}_{\frac{m}{2}}$ and $\bar{i} \equiv i \pmod{\frac{m}{2}}$, and $\bar{j} \in \mathbb{Z}_{\frac{n}{2}}$ and $\bar{j} \equiv j \pmod{\frac{n}{2}}$. To show $\vec{\ell}$ is injective, we observe that

$$\vec{\ell}(i, j) = (r, \bar{i}, \bar{j}) = (r, \bar{i}', \bar{j}') = \vec{\ell}(i', j')$$

implies $i \equiv i' \pmod{\frac{m}{2}}$. Say $i = a\frac{m}{2} + b$ and $i' = c\frac{m}{2} + b$ for some $a, c \in \{0, 1\}$ and some $0 \leq b < \frac{m}{2}$. Since $\frac{m}{2}$ is odd, and $r = r'$ implies that i and i' have the same parity, it must be the case that $a = c$, and hence $i = i'$. Essentially the same argument shows that $j = j'$. Therefore, $\overrightarrow{\ell}$ is injective, and hence bijective. We calculate $w(i, j)$ component-wise. Let

$$w(i, j) = (w_1, w_2, w_3),$$

where $w_1 \in \mathbb{Z}_4$, $w_2 \in \mathbb{Z}_{\frac{m}{2}}$, and $w_3 \in \mathbb{Z}_{\frac{n}{2}}$. We leave it to the reader to show $w_1 = 0$.

Then

$$\begin{aligned} w_2 &= 2(i+1) + i + (i-1) \\ &\quad - [2(i-1) + i + (i+1)] \\ &= 2, \end{aligned}$$

and

$$\begin{aligned} w_3 &= j + 3(j+1) \\ &\quad - [j + 3(j-1)] \\ &= 6. \end{aligned}$$

We conclude that for any $(i, j) \in V(C_m \boxtimes C_n)$, we have

$$w(i, j) = (0, 2, 6).$$

Case 2. $\delta > 1$.

Since $\frac{mn}{4}$ is odd we have $\mathbb{Z}_{mn} \cong \mathbb{Z}_4 \times \mathbb{Z}_{\frac{mn}{4}}$. Define $\vec{\ell} : V(C_m \boxtimes C_n) \rightarrow \mathbb{Z}_4 \times \mathbb{Z}_{\frac{mn}{4}}$

where

$$\vec{\ell}(x_i^j) = \begin{cases} (r, \frac{in}{4} + j) & i \text{ even} \\ (r, \frac{(i-1)n}{4} + j) & i \text{ odd} \end{cases},$$

where r is as in the previous case. To show $\vec{\ell}$ is injective, assume that $\vec{\ell}(i, j) = \vec{\ell}(i', j')$. As in the previous case, $r = r'$ implies that i and i' have the same parity.

Suppose i and i' are both even. Then $\frac{i}{2} \frac{n}{2} + j \equiv \frac{i'}{2} \frac{n}{2} + j' \pmod{\frac{mn}{4}}$ implies $j \equiv j' \pmod{\frac{n}{2}}$. By the same argument used in the previous case, it follows that $j = j'$.

But then we have $\frac{i}{2} \frac{n}{2} \equiv \frac{i'}{2} \frac{n}{2} \pmod{\frac{mn}{4}}$ which implies $i \equiv i' \pmod{\frac{mn}{4}}$ since n is even and $\frac{mn}{4}$ is odd. But this along with the fact that $0 \leq |i - i'| < m$ implies that $i = i'$.

The argument proving $i = i'$ and $j = j'$ if i and i' are both odd is essentially the same and is left to the reader. Therefore, $\vec{\ell}$ is injective, hence bijective and we proceed to determine the weights. As in the previous case, we calculate $w(i, j)$ for any $(i, j) \in V(C_m \boxtimes C_n)$, component-wise. Let

$$w(i, j) = (w_1, w_2)$$

where $w_1 = 0 \in \mathbb{Z}_4$ (from Case 1) and $w_2 \in \mathbb{Z}_{\frac{mn}{4}}$. If i is even,

$$\begin{aligned} w_2 &= \left(\frac{i}{2} \frac{n}{2} + j\right) + \left(\frac{i}{2} \frac{n}{2} + j + 1\right) + \left(\frac{i}{2} \frac{n}{2} + j + 1\right) + \left(\frac{i-2}{2} \frac{n}{2} + j + 1\right) \\ &\quad - \left[\left(\frac{i-2}{2} \frac{n}{2} + j\right) + \left(\frac{i-2}{2} \frac{n}{2} + j - 1\right) + \left(\frac{i}{2} \frac{n}{2} + j - 1\right) + \left(\frac{i}{2} \frac{n}{2} + j - 1\right)\right] \\ &= \frac{n}{2} + 6, \end{aligned}$$

while if i is odd,

$$\begin{aligned}
w_2 &= \left(\frac{i+1}{2} \frac{n}{2} + j\right) + \left(\frac{i+1}{2} \frac{n}{2} + j + 1\right) + \left(\frac{i-1}{2} \frac{n}{2} + j + 1\right) + \left(\frac{i-1}{2} \frac{n}{2} + j + 1\right) \\
&- \left[\left(\frac{i-1}{2} \frac{n}{2} + j\right) + \left(\frac{i-1}{2} \frac{n}{2} + j - 1\right) + \left(\frac{i-1}{2} \frac{n}{2} + j - 1\right) + \left(\frac{i+1}{2} \frac{n}{2} + j - 1\right)\right] \\
&= \frac{n}{2} + 6.
\end{aligned}$$

Hence,

$$w(i, j) = \left(0, \frac{n}{2} + 6\right),$$

proving the theorem. □

We conclude this section with a labeling over the group $\mathbb{Z}_n \times \mathbb{Z}_n$.

Theorem 4.4.6. *For any $n \geq 3$, the graph $C_n \boxtimes C_n$ is orientable $\mathbb{Z}_n \times \mathbb{Z}_n$ -distance magic.*

Proof. Define $\vec{\ell} : V(C_n \boxtimes C_n) \rightarrow \mathbb{Z}_n \times \mathbb{Z}_n$ where

$$\vec{\ell}(i, j) = (\overline{j-i}, j),$$

and $\overline{j-i}$ represents the remainder of $j-i$ divided by n . Clearly, $\vec{\ell}$ is a bijection

and for any $(i, j) \in V(C_n \boxtimes C_n)$, we have

$$\begin{aligned}
w(i, j) &= (j - i - 1, j) + (j - i, j + 1) + (j - i + 1, j + 1) + (j - i, j + 1) \\
&- [(j - i + 1, j) + (j - i, j - 1) + (j - i - 1, j - 1) + (j - i, j - 1)] \\
&= (0, 6),
\end{aligned}$$

which proves the theorem. □

4.5 Conclusion

We have proven that for two cycles C_m and C_n , both the lexicographic product and the direct product are orientable \mathbb{Z}_{mn} -distance magic. We have also provided some constructions for labeling lexicographic products of regular graphs. With regards to the strong product of two cycles, we have settled some cases. Hence, we conclude with the following open problem.

Problem 4.5.1. Find an orientable \mathbb{Z}_{mn} -distance magic labeling for $C_m \boxtimes C_n$ for all $m, n \geq 3$.

Chapter 5

Orientable \mathbb{Z}_n -distance magic labeling of Cartesian product of two cycles[★]

5.1 Definitions and known results

A *distance magic labeling* of a graph $G = (V, E)$ of order n is a bijection $f: V \rightarrow \{1, 2, \dots, n\}$ with the property that there is a positive integer k (called the *magic constant*) such that

$$w(x) = \sum_{y \in N(x)} f(y) = k \text{ for every } x \in V(G),$$

[★]The material in this chapter has been submitted to Australasian Journal of Combinatorics

where $N(x) = \{y | xy \in E\}$ is the *open neighborhood* of vertex x . We call $w(x)$ the *weight* of vertex x . See [5] for a survey of results regarding distance magic graphs. Froncek adapted distance magic labeling by using the elements from an abelian group as labels rather than integers in [23]. Let $G = (V, E)$ be a graph of order n and let Γ be an abelian group of order n . Then if there exists a bijection $\ell : V \rightarrow \Gamma$ with the property that there is an element $\mu \in \Gamma$ such that

$$w(x) = \sum_{y \in N(x)} \ell(y) = \mu \text{ for every } x \in V(G),$$

we say the labeling ℓ is a Γ -*distance magic labeling* and we say the graph G is Γ -*distance magic*. If such a labeling exists for every abelian group of order n , then we say G is *group distance magic*.

The Cartesian product $G \square H$ of two graphs G and H is a graph with vertex set $V(G) \times V(H)$ and two vertices (g, h) and (g', h') are adjacent in $G \square H$ if and only if $g = g'$ and h is adjacent to h' in H , or $h = h'$ and g is adjacent to g' in G . Let $C_n = x_0, x_1, \dots, x_{n-1}, x_0$ denote a cycle of length n . Froncek proved the following result in [23].

Theorem 5.1.1. [23] *The Cartesian product $C_m \square C_n$ is \mathbb{Z}_{mn} -distance magic if and only if mn is even.*

Cichacz made progress towards settling when $C_m \square C_n$ is group distance magic by proving the following in [9].

Theorem 5.1.2. [9] *Let $l = \text{lcm}(m, n)$. If m or n is even, then $C_m \square C_n$ is $Z_\alpha \times \Gamma$ -distance magic for any $\alpha \equiv 0 \pmod{l}$ and any abelian group Γ of order $\frac{mn}{\alpha}$.*

Cichacz and Froncek proved the following non-existence result in [15].

Theorem 5.1.3. [15] *If G is an r -regular graph of order n and r is odd, then G is not \mathbb{Z}_n -distance magic.*

The following analog of group distance magic labeling for directed graphs was introduced in [14]. Let $G = (V, E)$ be a graph. Assigning a direction to the edges of G gives an *oriented graph* $\vec{G}(V, A)$. In this chapter, we will use the notation \overrightarrow{xy} to denote an edge directed from vertex x to vertex y . Let $N^+(x) = \{y | \overrightarrow{xy} \in A\}$ and $N^-(x) = \{z | \overrightarrow{zx} \in A\}$. A *directed Γ -distance magic labeling* of an oriented graph $\vec{G} = (V, A)$ of order n is a bijection $\vec{\ell} : V \rightarrow \Gamma$ with the property that there is a $\mu \in \Gamma$ (called the *magic constant*) such that

$$w(x) = \sum_{y \in N^+(x)} \vec{\ell}(y) - \sum_{y \in N^-(x)} \vec{\ell}(y) = \mu \text{ for every } x \in V(G).$$

If for a graph G there exists an orientation \vec{G} such that there is a directed Γ -distance magic labeling $\vec{\ell}$ for \vec{G} , we say that G is *orientable Γ -distance magic*.

In this chapter, we focus on orientable \mathbb{Z}_n -distance magic labeling, where \mathbb{Z}_n is the cyclic group of order n . For the sake of orienting a cycle C_n , if the edges are oriented such that every arc has the form $\overrightarrow{x_i x_{i+1}}$ for all $i \in \{0, 1, \dots, n-1\}$ (where the addition in the subscript is taken modulo n), then we say the cycle is oriented *clockwise*. On the other hand, if all the edges of the cycle are oriented such that every arc has the form $\overrightarrow{x_i x_{i-1}}$ for all $i \in \{0, 1, \dots, n-1\}$, then we say the cycle is oriented *counter-clockwise*.

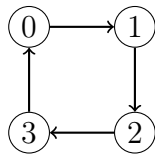


Figure 5.1: Orientable \mathbb{Z}_4 -distance magic labeling of C_4

It is an easy observation that C_n is orientable \mathbb{Z}_n -distance magic for all $n \geq 3$ (orient all the edges in the same direction around the cycle and label the vertices consecutively $\{0, 1, \dots, n-1\}$).

The following theorem was proved by Cichacz et. al in [14] .

Theorem 5.1.4. [14] *Let G be a graph of order n in which every vertex has odd degree. If $n \equiv 2 \pmod{4}$, then G is not orientable \mathbb{Z}_n -distance magic.*

Regarding the Cartesian product of two cycles, they obtained the following partial result.

Theorem 5.1.5. [14] *If $\gcd(m, n) = 1$, then the Cartesian product $C_m \square C_n$ is orientable distance magic.*

In Section 5.2 we prove that Theorem 5.1.5 is true without the assumption $\gcd(m, n) = 1$.

5.2 Cartesian product of two cycles

We begin with a construction for the case when one cycle length is a multiple of the other.

Theorem 5.2.1. *The Cartesian product $C_m \square C_{km}$ is orientable \mathbb{Z}_{km^2} -distance magic for all $m \geq 3$ and $k \geq 1$.*

Proof. Let $G = C_m = g_0, g_1, \dots, g_{m-1}, g_0$ and $H = C_{km} = h_0, h_1, \dots, h_{km-1}, h_0$. Then orient each copy of $G \square H$ as follows. Fix $j \in [km]$. Then for all $i \in [m]$, orient counter-clockwise each cycle of the form $(g_i, h_j), (g_{i+1}, h_j), \dots, (g_{i-1}, h_j), (g_i, h_j)$, where the arithmetic in the subscript is performed modulo m . Similarly, fix $i \in [m]$. Then for all $j \in [km]$, orient counter-clockwise each cycle of the form $(g_i, h_j), (g_i, h_{j+1}), \dots, (g_i, h_{j-1}), (g_i, h_j)$, where the arithmetic in the subscript is performed modulo km . Since the graph $G \square H$ can be edge-decomposed into cycles of those two forms, we have oriented every edge in $G \square H$. For a given a , let $0 \leq \mathcal{M}(a) < m$ be defined as the remainder of a divided by m . That is, $a = qm + \mathcal{M}(a)$ for some positive integer q . Now let x_i^j denote the vertex $(g_i, h_j) \in V(G \square H)$ for

$i \in [m], j \in [km]$. Now define $\vec{l}: V \rightarrow \mathbb{Z}_{km^2}$ by

$$\vec{l}(x_i^j) = mj + \mathcal{M}(i - j).$$

Expressing $\vec{l}(x_i^j)$ in the following alternative way,

$$\vec{l}(x_i^j) = \begin{cases} mj, & \text{for } i \equiv j \pmod{m} \\ mj + 1, & \text{for } i \equiv j + 1 \pmod{m} \\ mj + 2, & \text{for } i \equiv j + 2 \pmod{m} \\ \vdots & \vdots \\ mj + (m - 1), & \text{for } i \equiv j - 1 \pmod{m} \end{cases},$$

we see that \vec{l} is clearly bijective.

Then for all x_i^j we have $N^+(x_i^j) = \{x_i^{j+1}, x_{i+1}^j\}$ and $N^-(x_i^j) = \{x_i^{j-1}, x_{i-1}^j\}$ where the arithmetic is performed modulo km in the superscript and modulo m in the subscript.

Therefore,

$$\begin{aligned}
w(x_i^j) &= \vec{l}(x_i^{j+1}) + \vec{l}(x_{i+1}^j) - [\vec{l}(x_{i-1}^j) + \vec{l}(x_i^{j-1})] \\
&= m(j+1) + mj - mj - m(j-1) \\
&+ [\mathcal{M}(i-j-1) + \mathcal{M}(i-j+1) - \mathcal{M}(i-j-1) - \mathcal{M}(i-j+1)] \\
&= \begin{cases} m(2-km), j \in \{0, km-1\} \\ m \cdot 2, j \in \{1, \dots, km-2\} \end{cases} \\
&= 2m,
\end{aligned}$$

since $m(2-km) \equiv 2m \pmod{km^2}$.

Thus, \vec{l} is an orientable \mathbb{Z}_{km^2} -distance magic labeling. □

For a given integer a , let $0 \leq \mathcal{R}(a) < d$ represent the remainder when a is divided by d . That is, $a = qd + \mathcal{R}(a)$ for some positive integer q . For a given natural number p , let $[p]$ denote the set $\{0, 1, \dots, p-1\}$. For a set S and a number c , let $S + c = \{x + c : x \in S\}$. Each case in the proof of the next theorem uses a directed labeling which is shown to be a bijection from the vertex set of the graph to the appropriate group in the Appendix.

Theorem 5.2.2. *The Cartesian product $C_m \square C_n$ is orientable \mathbb{Z}_{mn} -distance magic for all $m, n \geq 3$.*

Proof. Let $m, n \geq 3$ be given and let $\gcd(m, n) = d$. Then define $\lambda = \frac{m+n}{d}$ and let

$\gcd(\frac{m}{d}, d) = \alpha$. By Theorem 5.1.5, we may assume $d \geq 2$. By Theorem 5.2.1 and the fact that the product is commutative, we may further assume $d < \min\{m, n\}$. Let $G = C_m = \{g_0, g_1, \dots, g_{m-1}, g_0\}$ and $H = C_n = \{h_0, h_1, \dots, h_{n-1}, h_0\}$. Then orient each copy of $G \square H$ as in Theorem 5.2.1. Now let x_i^j denote the vertex $(g_i, h_j) \in V(G \square H)$ for all $i \in [m]$ and $j \in [n]$. We proceed in three cases based on the parity of m and n . Since we will define a different directed labeling for each case, we first pause to make the following observation. For any directed labeling $\vec{l}: V(G \square H) \rightarrow \mathbb{Z}_{mn}$ of $\overrightarrow{G \square H}$, we have

$$\begin{aligned} w(x_i^j) &= \sum_{y \in N^+(x_i^j)} \vec{l}(y) - \sum_{y \in N^-(x_i^j)} \vec{l}(y) \\ &= \vec{l}(x_i^{j+1}) + \vec{l}(x_{i+1}^j) - [\vec{l}(x_{i-1}^j) + \vec{l}(x_i^{j-1})], \end{aligned}$$

for every vertex $x_i^j \in V(G \square H)$, where the arithmetic is performed modulo n in the superscript and modulo m in the subscript. However, it should be emphasized that the weight calculation is performed in the group \mathbb{Z}_{mn} .

Case 1.1. m and n both odd and $\gcd(\lambda, d) = 1$.

For all $x_i^j \in V(G \square H)$, define $\vec{l}: V(G \square H) \rightarrow \mathbb{Z}_{mn}$ where

$$\vec{l}(x_i^j) = jm + in + \mathcal{R}(j - i).$$

Then,

$$\begin{aligned}
w(x_i^j) &= (j+1)m + in + jm + (i+1)n \\
&- [jm + (i-1)n + (j-1)m + in] \\
&+ \mathcal{R}(j-i+1) - \mathcal{R}(j-i+1) + \mathcal{R}(j-i-1) - \mathcal{R}(j-i-1) \\
&= sm + rn,
\end{aligned}$$

where $s \in \{2, 2-n\}$ and $r \in \{2, 2-m\}$. But, $sm + rn \equiv 2m + 2n \pmod{mn}$, so

$$w(x_i^j) = 2m + 2n.$$

In the remaining cases we will omit the equality involving s and r above and only show the final congruence modulo mn .

Case 1.2. m and n both odd and $\gcd(\lambda, d) > 1$.

Let $k = 1$ when $\alpha^2 \nmid d$ and let $k = 2$ when $\alpha^2 \mid d$. Then for all $x_i^j \in V(G \square H)$, define $\vec{l}_{\alpha^k} : V(G \square H) \rightarrow \mathbb{Z}_{mn}$ where

$$\vec{l}_{\alpha^k}(x_i^j) = jm + in \frac{d}{\alpha^k} + \mathcal{R}(j-i).$$

Therefore,

$$\begin{aligned}
w(x_i^j) &= (j+1)m + in \frac{d}{\alpha^k} + jm + (i+1)n \frac{d}{\alpha^k} \\
&- [jm + (i-1)n \frac{d}{\alpha^k} + (j-1)m + in \frac{d}{\alpha^k}] \\
&+ \mathcal{R}(j-i+1) - \mathcal{R}(j-i+1) + \mathcal{R}(j-i-1) - \mathcal{R}(j-i-1) \\
&= 2m + 2n \frac{d}{\alpha^k}.
\end{aligned}$$

Suppose exactly one of m and n is odd. Since the Cartesian product is commutative, we may assume without loss of generality that m is even and n is odd. Then as in the previous case, $\gcd(\lambda, d)$ establishes two subcases.

Case 2.1. m even, n odd, and $\gcd(\lambda, d) = 1$.

For all $x_i^j \in V(G \square H)$, define $\vec{l}: V(G \square H) \rightarrow \mathbb{Z}_{mn}$ where

$$\vec{l}(x_i^j) = \begin{cases} jm + in + \mathcal{R}(j-i), & i \text{ even} \\ (j-1)m + (i-1)n + d + \mathcal{R}(j-i), & i \text{ odd} \end{cases}.$$

If i is even we have

$$\begin{aligned}
w(x_i^j) &= (j+1)m + in + (j-1)m + in + d \\
&- [(j-1)m + (i-2)n + d + (j-1)m + in] \\
&+ \mathcal{R}(j-i+1) - \mathcal{R}(j-i+1) + \mathcal{R}(j-i-1) - \mathcal{R}(j-i-1) \\
&= 2m + 2n.
\end{aligned}$$

While if i is odd we have

$$\begin{aligned}
w(x_i^j) &= jm + (i-1)n + d + jm + (i+1)n \\
&- [jm + (i-1)n + (j-2)m + (i-1)n + d] \\
&+ \mathcal{R}(j-i+1) - \mathcal{R}(j-i+1) + \mathcal{R}(j-i-1) - \mathcal{R}(j-i-1) \\
&= 2m + 2n.
\end{aligned}$$

Case 2.2. m even, n odd, and $\gcd(\lambda, d) > 1$.

As in Case 1.2, let $k = 1$ when $\alpha^2 \nmid d$ and let $k = 2$ when $\alpha^2 \mid d$. For all $x_i^j \in V(G \square H)$, define $l_{\alpha^k}^{\vec{}} : V(G \square H) \rightarrow \mathbb{Z}_{mn}$ where

$$l_{\alpha^k}^{\vec{}}(x_i^j) = \begin{cases} jm + in \frac{d}{\alpha^k} + \mathcal{R}(j-i), & i \text{ even} \\ (j-1)m + (i-1)n \frac{d}{\alpha^k} + d + \mathcal{R}(j-i), & i \text{ odd} \end{cases}.$$

If i is even we have

$$\begin{aligned}
w(x_i^j) &= (j+1)m + in \frac{d}{\alpha^k} + (j-1)m + in \frac{d}{\alpha^k} \\
&- [(j-1)m + (i-2)n \frac{d}{\alpha^k} + d + (j-1)m + in \frac{d}{\alpha^k}] \\
&+ \mathcal{R}(j-i+1) - \mathcal{R}(j-i+1) + \mathcal{R}(j-i-1) - \mathcal{R}(j-i-1) \\
&= 2m + 2n \frac{d}{\alpha^k}.
\end{aligned}$$

While if i is odd we have

$$\begin{aligned}
w(x_i^j) &= jm + (i-1)n\frac{d}{\alpha^k} + d + jm + (i+1)n\frac{d}{\alpha^k} \\
&- [jm + (i-1)n\frac{d}{\alpha^k} + (j-2)m + (i-1)n\frac{d}{\alpha^k} + d] \\
&+ \mathcal{R}(j-i+1) - \mathcal{R}(j-i+1) + \mathcal{R}(j-i-1) - \mathcal{R}(j-i-1) \\
&= 2m + 2n\frac{d}{\alpha^k}.
\end{aligned}$$

Suppose that both m and n are even. Since $\gcd(m, n) = d$, at most one of $\frac{m}{d}$ and $\frac{n}{d}$ is even. Since the Cartesian product is commutative, if one of $\frac{m}{d}$ and $\frac{n}{d}$ is even, we may assume without loss of generality that $\frac{n}{d}$ is odd. Then as in the previous cases, $\gcd(\lambda, d)$ establishes two subcases.

Case 3.1. $\frac{m}{d}$ even, $\frac{n}{d}$ odd, and $\gcd(\lambda, d) = 1$.

For all $x_i^j \in V(G \square H)$, define $\vec{f}: V(G \square H) \rightarrow \mathbb{Z}_{mn}$ and $\vec{l}: V(G \square H) \rightarrow \mathbb{Z}_{mn}$ where

$$\vec{f}(x_i^j) = \begin{cases} jm + in + \mathcal{R}(j-i), & \text{for } i \equiv j \pmod{2} \\ (j-1)m + (i-1)n + d + \mathcal{R}(j-i), & \text{for } i \equiv j \equiv 1 \pmod{2} \end{cases},$$

and

$$\vec{l}(x_i^j) = \begin{cases} \vec{f}(x_i^j), & \text{for } i \equiv j \pmod{2} \\ \vec{f}(x_i^{j-1}) + 1, & \text{for } i \not\equiv j \pmod{2} \end{cases}.$$

By Lemma 5.4.7, \vec{f} maps the vertices $\{x_i^j : i \equiv j \pmod{2}\}$ bijectively to $2\mathbb{Z}_{mn}$.

Then clearly $\vec{f} + 1$ maps the vertices $\{x_i^{j-1} : i \not\equiv j \pmod{2}\}$ bijectively to $2\mathbb{Z}_{mn} + 1$.

Therefore, $\vec{l} : V(G \square H) \rightarrow \mathbb{Z}_{mn}$ is a bijection since $\mathbb{Z}_{mn} \cong 2\mathbb{Z}_{mn} \cup 2\mathbb{Z}_{mn} + 1$. If $i \equiv j \pmod{2}$ we have

$$\begin{aligned}
w(x_i^j) &= \vec{l}(x_i^{j+1}) + \vec{l}(x_{i+1}^j) - [\vec{l}(x_{i-1}^j) + \vec{l}(x_i^{j-1})] \\
&= \vec{f}(x_i^j) + 1 + \vec{f}(x_{i+1}^{j-1}) + 1 - [\vec{f}(x_{i-1}^{j-1}) + 1 + \vec{f}(x_i^{j-2}) + 1] \\
&= \vec{f}'(x_i^j) + \vec{f}'(x_{i+1}^{j-1}) - [\vec{f}'(x_{i-1}^{j-1}) + \vec{f}'(x_i^{j-2})] \\
&+ \mathcal{R}(j-i) + \mathcal{R}(j-i-2) - \mathcal{R}(j-i) - \mathcal{R}(j-i-2) \\
&= 2m + 2n.
\end{aligned}$$

While if $i \not\equiv j \pmod{2}$ we have

$$\begin{aligned}
w(x_i^j) &= \vec{l}(x_i^{j+1}) + \vec{l}(x_{i+1}^j) - [\vec{l}(x_{i-1}^j) + \vec{l}(x_i^{j-1})] \\
&= \vec{f}(x_i^{j+1}) + \vec{f}(x_{i+1}^j) - [\vec{f}(x_{i-1}^j) + \vec{f}(x_i^{j-1})] \\
&= \vec{f}'(x_i^{j+1}) + \vec{f}'(x_{i+1}^j) - [\vec{f}'(x_{i-1}^j) + \vec{f}'(x_i^{j-1})] \\
&+ \mathcal{R}(j-i+1) + \mathcal{R}(j-i-1) - \mathcal{R}(j-i+1) - \mathcal{R}(j-i-1) \\
&= 2m + 2n.
\end{aligned}$$

Case 3.2. $\frac{m}{d}$ even, $\frac{n}{d}$ odd, and $\gcd(\lambda, d) > 1$.

As in the previous cases, let $k = 1$ when $\alpha^2 \nmid d$, let $k = 2$ when $\alpha^2 \mid d$, and for all

$x_i^j \in V(G \square H)$, define $f_{\alpha^k}^{\vec{}} : V(G \square H) \rightarrow \mathbb{Z}_{mn}$ and $l_{\alpha^k}^{\vec{}} : V(G \square H) \rightarrow \mathbb{Z}_{mn}$ where

$$f_{\alpha^k}^{\vec{}}(x_i^j) = \begin{cases} jm + in \frac{d}{\alpha^k} + \mathcal{R}(j-i), & \text{for } i \equiv j \equiv 0 \pmod{2} \\ (j-1)m + (i-1)n \frac{d}{\alpha^k} + d + \mathcal{R}(j-i), & \text{for } i \equiv j \equiv 1 \pmod{2} \end{cases},$$

and

$$l_{\alpha^k}^{\vec{}}(x_i^j) = \begin{cases} f_{\alpha^k}^{\vec{}}(x_i^j), & \text{for } i \equiv j \pmod{2} \\ f_{\alpha^k}^{\vec{}}(x_i^{j-1}) + 1, & \text{for } i \not\equiv j \pmod{2} \end{cases}.$$

By Lemma 5.4.8 and essentially the same argument used in Case 3.1, we conclude that $l_{\alpha^k}^{\vec{}} : V(G \square H) \rightarrow \mathbb{Z}_{mn}$ is a bijection.

Finally, if $i \equiv j \pmod{2}$ we have

$$\begin{aligned} w(x_i^j) &= \vec{l}_{\alpha^k}(x_i^{j+1}) + \vec{l}_{\alpha^k}(x_{i+1}^j) - [\vec{l}_{\alpha^k}(x_{i-1}^j) + \vec{l}_{\alpha^k}(x_i^{j-1})] \\ &= \vec{f}_{\alpha^k}(x_i^j) + 1 + \vec{f}_{\alpha^k}(x_{i+1}^{j-1}) + 1 - [\vec{f}_{\alpha^k}(x_{i-1}^{j-1}) + 1 + \vec{f}_{\alpha^k}(x_i^{j-2}) + 1] \\ &= \vec{f}_{\alpha^k}'(x_i^j) + \vec{f}_{\alpha^k}'(x_{i+1}^{j-1}) - [\vec{f}_{\alpha^k}'(x_{i-1}^{j-1}) + \vec{f}_{\alpha^k}'(x_i^{j-2})] \\ &+ \mathcal{R}(j-i) + \mathcal{R}(j-i-2) - \mathcal{R}(j-i) - \mathcal{R}(j-i-2) \\ &= 2m + 2n \frac{d}{\alpha^k}. \end{aligned}$$

While if $i \not\equiv j \pmod{2}$ we have

$$\begin{aligned}
w(x_i^j) &= l_{\alpha^k}^{\vec{}}(x_i^{j+1}) + \vec{l}_{\alpha^k}(x_{i+1}^j) - [l_{\alpha^k}^{\vec{}}(x_{i-1}^j) + l_{\alpha^k}^{\vec{}}(x_i^{j-1})] \\
&= f_{\alpha^k}^{\vec{}}(x_i^{j+1}) + f_{\alpha^k}^{\vec{}}(x_{i+1}^j) - [f_{\alpha^k}^{\vec{}}(x_{i-1}^j) + f_{\alpha^k}^{\vec{}}(x_i^{j-1})] \\
&= f_{\alpha^k}^{\vec{}}(x_i^{j+1}) + f_{\alpha^k}^{\vec{}}(x_{i+1}^j) - [f_{\alpha^k}^{\vec{}}(x_{i-1}^j) + f_{\alpha^k}^{\vec{}}(x_i^{j-1})] \\
&+ \mathcal{R}(j-i+1) + \mathcal{R}(j-i-1) - \mathcal{R}(j-i+1) - \mathcal{R}(j-i-1) \\
&= 2m + 2n \frac{d}{\alpha^k}.
\end{aligned}$$

In every case, $w(x_i^j)$ is constant for all $x_i^j \in V(G \square H)$. Hence, $C_m \square C_n$ is orientable

\mathbb{Z}_{mn} -distance magic. □

It is worth mentioning that finding a directed \mathbb{Z}_{mn} -distance magic labeling of $C_m \square C_n$ oriented as in Theorem 5.2.2 is equivalent to constructing an $m \times n$ array $A = (a_{i,j})$ containing each element of \mathbb{Z}_{mn} exactly once with the property that

$$a_{i,j+1} - a_{i,j-1} + a_{i+1,j} - a_{i-1,j} \quad (+)$$

for every $a_{i,j}$. Accordingly, the $\mathcal{R}(j-i)$ term which appears in the labelings of the proof of Theorem 5.2.2 corresponds to a $d \times d$ sub-array which is a Latin square having property (+).

5.3 Future work

We have shown that the Cartesian product of any two cycles is orientable \mathbb{Z}_n -distance magic. In [18] we showed that the repeated Cartesian product of a cycle is orientable \mathbb{Z}_n -distance magic. Then a natural direction forward is to generalize to the Cartesian product of many cycles of various lengths. We pose the following problem.

Problem 5.3.1. *For what numbers n_1, n_2, \dots, n_k is the Cartesian product $C_{n_1} \square C_{n_2} \square \dots \square C_{n_k}$ orientable $\mathbb{Z}_{n_1, n_2, \dots, n_k}$ -distance magic?*

Another possibility for future work is to consider abelian groups other than the cyclic group. Therefore we pose the following.

Problem 5.3.2. *Determine all abelian groups Γ such that $C_m \square C_n$ is orientable Γ -distance magic.*

One may wonder whether an orientable distance magic graph G of order n is also \mathbb{Z}_n -distance magic, or vice versa. Since \mathbb{Z}_n -distance magic labeling is more restrictive than orientable \mathbb{Z}_n -distance magic labeling, intuitively it should not be the case that orientable \mathbb{Z}_n -distance magic implies \mathbb{Z}_n -distance magic. Indeed, Theorems 5.1.1 and 5.2.2 show that this is not the case.

But perhaps \mathbb{Z}_n -distance magic implies orientable \mathbb{Z}_n -distance magic. Theorems 5.1.3

and 5.1.4 indicate that the contrapositive checks for the case of odd regular graphs on $n \equiv 2 \pmod{4}$ vertices. Therefore, we end by posing the following conjecture.

Conjecture 5.3.3. *If a graph G of order n is \mathbb{Z}_n -distance magic, then it is orientable \mathbb{Z}_n -distance magic.*

5.4 Appendix

In this section, we prove a series of lemmas regarding the labelings used in the main theorem of Section 5.2. Recall for a given integer a , we let $0 \leq \mathcal{R}(a) < d$ represent the remainder when a is divided by d . Let $m, n \geq 3$ be given. Let $\gcd(m, n) = d$, $\gcd(\frac{m}{d}, d) = \alpha$, and define $\lambda = \frac{m+n}{d}$. We begin by establishing some relationships between m , n , d , and α . Assume $1 < d < \min\{m, n\}$ for the lemmas that follow.

Observation 5.4.1. *If $\alpha^2 \nmid d$, then $\gcd(\alpha \frac{m}{d}, d) = \alpha$ and $\gcd(\frac{m}{d}, \frac{n}{\alpha}) = 1$.*

Proof. By elementary properties of the greatest common divisor, $\gcd(\frac{m}{d}, \frac{n}{d}) = 1$ implies $\gcd(\frac{m}{d} \cdot \frac{n}{d}, \frac{n}{\alpha}) = \gcd(\frac{m}{d}, \frac{n}{\alpha}) \gcd(\frac{n}{d}, \frac{n}{\alpha})$. But $\gcd(\frac{m}{d} \cdot \frac{n}{d}, \frac{n}{\alpha}) = \frac{n}{d} \gcd(\frac{m}{d}, \frac{d}{\alpha})$, and $\gcd(\frac{n}{d}, \frac{n}{\alpha}) = \frac{n}{d} \gcd(1, \frac{d}{\alpha}) = \frac{n}{d}$. Therefore, $\gcd(\frac{m}{d}, \frac{d}{\alpha}) = \gcd(\frac{m}{d}, \frac{n}{\alpha})$. Multiplying both sides by α gives $\gcd(\alpha \frac{m}{d}, d) = \alpha \gcd(\frac{m}{d}, \frac{n}{\alpha})$. But since $\alpha^2 \nmid d$, we have $\gcd(\alpha \frac{m}{d}, d) = \alpha$ and hence, $\gcd(\frac{m}{d}, \frac{n}{\alpha}) = 1$. \square

Observation 5.4.2. *If $\alpha^2 \mid d$, then $\gcd(\alpha^2 \frac{m}{d}, d) = \alpha^2$ and $\gcd(\frac{m}{d}, \frac{n}{\alpha^2}) = 1$.*

Proof. Essentially the same argument as in the proof of Observation 5.4.1 gives $\gcd(\alpha \frac{m}{d}, \frac{d}{\alpha}) = \alpha \gcd(\frac{m}{d}, \frac{n}{\alpha^2})$. Since $\alpha^2 \mid d$, we have $\gcd(\alpha \frac{m}{d}, d) = \alpha^2$ and thus $\gcd(\alpha \frac{m}{d}, \frac{d}{\alpha}) = \alpha$. Hence, $\gcd(\frac{m}{d}, \frac{n}{\alpha^2}) = 1$. The fact that $\gcd(\alpha^2 \frac{m}{d}, d) = \alpha^2$ follows from $\gcd(\alpha \frac{m}{d}, \frac{d}{\alpha}) = \alpha$. \square

For the following lemmas, let \mathbb{Z}_{mn} be the cyclic group of order mn , let $V = \{(i, j) : i \in [m], j \in [n]\}$, and for a given function $g : V \mapsto \mathbb{Z}_{mn}$, define $g'(i, j) = g(i, j) - \mathcal{R}(j - i)$. For an element $g \in \mathbb{Z}_{mn}$, we denote by $\langle g \rangle$, the subgroup generated by g .

Lemma 5.4.3. *If $\gcd(\lambda, d) = 1$, then the mapping $g : V \mapsto \mathbb{Z}_{mn}$ given by $g(i, j) = jm + in + \mathcal{R}(j - i)$, is a bijection.*

Proof. To show that g is injective suppose that $g'(i, j) = g'(a, b)$ for some $(a, b), (i, j) \in V$. Therefore, we have

$$jm + in \equiv bm + an \pmod{mn}. \quad (5.1)$$

Rearranging this equation gives $(j - b)m + (i - a)n \equiv 0 \pmod{mn}$. For ease of notation, let $x = j - b$ and $y = i - a$. Then since $|x| \leq n - 1$, $|y| \leq m - 1$, and $xm + yn \equiv 0 \pmod{mn}$, we have that $xm + yn = kmn$ for some $k \in \{-1, 0, 1\}$.

Suppose $k = \pm 1$. Then $|y| = |i - a| = \frac{m(n-x)}{n} \in \mathbb{Z}$ if and only if $x = 0$ since $n \nmid m$ by assumption (recall $d < \min\{m, n\}$). But if $x = 0$, then $yn = \pm mn$, but this is impossible since $|y| < m$. Hence, $xm + yn = 0$. Then dividing by d , we have $x\frac{m}{d} + y\frac{n}{d} = 0$. Since $\frac{m}{d}$ and $\frac{n}{d}$ are relatively prime, the solutions have the form $(x, y) = (\frac{n}{d}r, -\frac{m}{d}r)$, $\forall r \in [d]$. We have now established that there are exactly d ordered pairs in V which have the same value under g' . This means that in order for g to be a bijection, we must show that $\{\mathcal{R}(y_r - x_r) : r \in [d]\} = [d]$. To this end, observe that $\mathcal{R}(y_r - x_r) \equiv (y_r - x_r) \equiv -\frac{m}{d}r - \frac{n}{d}r \equiv -r\lambda \pmod{d}$ for each $r \in [d]$. Since $\gcd(\lambda, d) = 1$, we have $\langle \lambda \rangle \cong \mathbb{Z}_d$, hence $\langle -\lambda \rangle \cong \mathbb{Z}_d$. Therefore, $\{\mathcal{R}(y_r - x_r) : r \in [d]\} = [d]$, so \vec{l} is an injection, hence bijection. \square

Lemma 5.4.4. *If $\gcd(\lambda, d) > 1$, let $k = 1$ if $\alpha^2 \nmid d$ and let $k = 2$ if $\alpha^2 \mid d$. Then the mapping $g_{\alpha^k} : V \mapsto \mathbb{Z}_{mn}$ given by $g_{\alpha^k}(i, j) = jm + in\frac{d}{\alpha^k} + \mathcal{R}(j - i)$, is a bijection.*

Proof. Suppose that $g'_{\alpha^k}(i, j) = g'_{\alpha^k}(a, b)$ for some $(i, j), (a, b) \in V$. Then we have $jm + in\frac{d}{\alpha^k} \equiv bm + an\frac{d}{\alpha^k} \pmod{mn}$. Letting $t = j - b$, and $u = i - a$, dividing by d , and observing that $\alpha^k \mid n$ gives

$$t\frac{m}{d} + u\frac{n}{\alpha^k} \equiv 0 \pmod{\frac{m}{d}n}. \quad (5.2)$$

Now observe that $\gcd(\frac{m}{d}, d) = \alpha \implies \alpha^2 \mid m$. Therefore, $\alpha \nmid \frac{n}{\alpha^k}$ since otherwise, $\alpha \mid \frac{n}{\alpha^k} \implies \alpha^{k+1} \mid n$. Then if $k = 1$, we have $\alpha^2 \mid n$ and $\alpha^2 \nmid d$ implies that $\alpha \mid \frac{n}{d}$ which in turn implies $\gcd(\frac{m}{d}, \frac{n}{d}) > 1$, contradicting the assumption, $\gcd(m, n) = d$. While

if $k = 2$, we have $\alpha^3 \mid n$ implies $\gcd(\frac{m}{d}, \frac{n}{\alpha^2}) > 1$, a contradiction of Observation 5.4.2.

Then since $\alpha \mid \frac{m}{d}$ but $\alpha \nmid \frac{n}{\alpha^k}$, we have that $\alpha \mid u$ from (5.2). But also, $\frac{m}{d} \mid u \frac{n}{\alpha^k}$.

By equation (5.2) and Observations 5.4.1 and 5.4.2, $\gcd(\frac{m}{d}, \frac{n}{\alpha^k}) = 1 \implies \frac{m}{d} \mid u$.

Therefore, both α and $\frac{m}{d}$ must divide u . Similarly, $\frac{n}{\alpha^k}$ must divide $t \frac{m}{d}$, which implies

that $\frac{n}{\alpha^k} \mid t$. This allows us to provide a full description of the pairs (u, t) satisfying

(5.2). Let S be the set of all such pairs. Then for all $p \in [\frac{d}{\alpha^k}]$, we have

$$\begin{aligned} S = & \{(\frac{m}{d}\alpha^k p, 0), (\frac{m}{d}\alpha^k p - \frac{m}{d}, \frac{n}{\alpha^k}), (\frac{m}{d}\alpha^k p - 2\frac{m}{d}, \frac{2n}{\alpha^k}), \\ & \dots, (\frac{m}{d}\alpha^k p - (\alpha^k - 1)\frac{m}{d}, \frac{(\alpha^k - 1)n}{\alpha^k})\}. \end{aligned}$$

Note that there are exactly $\alpha^k \cdot \frac{d}{\alpha^k} = d$ pairs in S . Therefore, we have established

that exactly d ordered pairs in V share the same value under g'_{α^k} . Now it remains to

show that these ordered pairs have distinct values under \mathcal{R} . For ease of notation, let

$x = \mathcal{R}(-\frac{m}{d}\alpha^k)$, $y = \mathcal{R}(\frac{n}{\alpha^k})$, and $z = \mathcal{R}(\frac{m}{d})$. Furthermore, let $H = \langle x \rangle \leq \mathbb{Z}_d$. Then,

$|H| = \text{ord}_{\mathbb{Z}_d}(x) = \frac{d}{\gcd(x, d)} = \frac{d}{\alpha^k}$, by Observations 5.4.1 and 5.4.2. Applying \mathcal{R} to each

member of S defines the multiset,

$$\mathcal{R}(S) = \{H + 0, H + (y + z), H + 2(y + z), \dots, H + (\alpha^k - 1)(y + z)\}.$$

It remains to show that the cosets of H in $\mathcal{R}(S)$ partition $[d]$. First observe that

$y + z \not\equiv 0 \pmod{d}$ since otherwise we have $\alpha \mid \frac{n}{\alpha^k}$ which we have already established

is a contradiction. Secondly, suppose $(y + z) \in H$. Then $\frac{n}{\alpha^k} + \frac{m}{d} \equiv -\frac{m}{d}\alpha^k q \pmod{d}$

for some $q \in [\frac{d}{\alpha^k}]$. But since $\alpha \mid \frac{m}{d}$, it must be the case that $\alpha \mid \frac{n}{\alpha^k}$, which leads to the same contradiction as before. Therefore, $(y + z) \notin H$. Hence $\mathcal{R}(S) = [d]$, and so g_{α^k} is an injection, hence bijection. \square

Lemma 5.4.5. *Let m be even and n be odd. If $\gcd(\lambda, d) = 1$, then the mapping $g : V \mapsto \mathbb{Z}_{mn}$ given by $g(i, j) = \begin{cases} jm + in + \mathcal{R}(j - i), & i \text{ even} \\ (j - 1)m + (i - 1)n + d + \mathcal{R}(j - i), & i \text{ odd} \end{cases}$ is a bijection.*

Proof. Suppose that $g'(i, j) = g'(a, b)$ for some $(i, j), (a, b) \in V$. It cannot be the case that i and a have different parities. For the sake of contradiction, suppose i is even and a is odd. Then we have $jm + in \equiv d + (b - 1)m + (a - 1)n \pmod{mn}$. Therefore, $(j - b + 1)m + (i - a + 1)n \equiv d \pmod{mn}$. But this is a contradiction since $(j - b + 1)m$ and $(i - a + 1)n$ are both even and d is necessarily odd. So it cannot be the case that i is even and a is odd. Essentially the same argument shows it cannot be the case that i is odd and a is even. Therefore, i and a must be of the same parity. If i and a are both even, then $g'(i, j) = g'(a, b)$ implies equation (5.1) from Lemma 5.4.3, while if i and a are both odd, then we have $d + (j - 1)m + (i - 1)n \equiv d + (b - 1)m + (a - 1)n \pmod{mn}$, which also is equivalent with (5.1). Thus g is a bijection by the same argument as in Lemma 5.4.3. \square

Lemma 5.4.6. *Let m be even and n be odd. If $\gcd(\lambda, d) > 1$, let $k = 1$ when $\alpha^2 \nmid d$,*

and let $k = 2$ when $\alpha^2 \mid d$. Then the mapping $g_{\alpha^k} : V \mapsto \mathbb{Z}_{mn}$ given by

$$g_{\alpha^k}(i, j) = \begin{cases} jm + in\frac{d}{\alpha^k} + \mathcal{R}(j - i), & i \text{ even} \\ (j - 1)m + (i - 1)n\frac{d}{\alpha^k} + d + \mathcal{R}(j - i), & i \text{ odd} \end{cases} \quad \text{is a bijection.}$$

Proof. Suppose that $g'_{\alpha^k}(i, j) = g'_{\alpha^k}(a, b)$ for some $(i, j), (a, b) \in V$. As in Lemma 5.4.5, i and a must be of the same parity. If i and a are both even, then necessarily $jm + in\frac{d}{\alpha^k} \equiv bm + an\frac{d}{\alpha^k} \pmod{mn}$. Whereas, if i and a are both odd, then we have that $d + (j - 1)m + (i - 1)n\frac{d}{\alpha^k} \equiv d + (b - 1)m + (a - 1)n\frac{d}{\alpha^k} \pmod{mn}$. However, letting $t = j - b$, $u = i - a$, dividing by d , and observing that $\alpha^k \mid n$, we see that both equations are equivalent to (5.2) from Lemma 5.4.4. Hence in either case, g_{α^k} is a bijection by the same argument used in Lemma 5.4.4. \square

In the next three lemmas, assume m and n are even. Then let $V_2 = \{(i, j) \in V : i \equiv j \pmod{2}\} \subseteq V$. Let $2\mathbb{Z}_{mn} = \{2h : h \in \mathbb{Z}_{mn}\}$ denote the subgroup of \mathbb{Z}_{mn} consisting of the even integers contained in \mathbb{Z}_{mn} . Similarly, let $2[d] = \{2h : h \in \mathbb{Z}_d\}$. Also note that since m and n are both even, then at most one of $\frac{m}{d}$ and $\frac{n}{d}$ may be even. So assume without loss of generality that $\frac{n}{d}$ is always odd.

Lemma 5.4.7. *Let m and n be even. If $\gcd(\lambda, d) = 1$, then the mapping $g : V_2 \mapsto$*

$2\mathbb{Z}_{mn}$ given by

$$g(i, j) = \begin{cases} jm + in + \mathcal{R}(j - i), & \text{for } i \equiv j \equiv 0 \pmod{2} \\ (j - 1)m + (i - 1)n + d + \mathcal{R}(j - i), & \text{for } i \equiv j \equiv 1 \pmod{2} \end{cases} \quad \text{is a bijection.}$$

Proof. Suppose $g'(i, j) = g'(a, b)$ for some $(i, j), (a, b) \in V$. Observe that it cannot be the case that i, j are both even and a, b are both odd, since otherwise $(j-b+1)m + (i-a+1)n = kmn + d$ for some integer k would imply $(j-b+1)\frac{m}{d} + (i-a+1)\frac{n}{d} = \frac{kmn}{d} + 1$, a contradiction since the left hand side of the equation is even and the right hand side is odd (recall that $\frac{n}{d}$ is odd). For the same reason, it cannot be the case that i, j are both odd while a, b are both even. Therefore, i, j, a , and b are all of the same parity. Consequently, $g'(i, j) = g'(a, b)$ implies equation (5.1) from Lemma 5.4.3. With no restriction on the parities of $x = j - b$ and $y = i - a$, this equation was found to have the d solutions $(x_r, y_r) = (\frac{n}{d}r, -\frac{m}{d}r)$ for each $r \in [d]$ in the proof of Lemma 5.4.3. However, in the present case we require that x and y both be even. Recall that $\frac{n}{d}$ is odd. Therefore, the $\frac{d}{2}$ solutions to (5.1) are $(x_r, y_r) = (\frac{n}{d}2r, -\frac{m}{d}2r)$ for each $r \in [\frac{d}{2}]$. We have now established that there are exactly $\frac{d}{2}$ ordered pairs in V_2 having the same value under g' . This means that in order for g to be a bijection, we must show that the set $\{\mathcal{R}(y_r - x_r) : r \in [\frac{d}{2}]\} = 2[d]$. To this end, observe $\mathcal{R}(y_r - x_r) \equiv (y_r - x_r) \equiv -\frac{m}{d}2r - \frac{n}{d}2r \equiv -2r\lambda \pmod{d}$ for each $r \in [\frac{d}{2}]$. Since $\gcd(\lambda, d) = 1$, we have $\langle \lambda \rangle \cong \mathbb{Z}_d$, hence $\langle -\lambda \rangle \cong \mathbb{Z}_d$. Therefore, $\{\mathcal{R}(x_r, y_r) : r \in [\frac{d}{2}]\} = 2[d]$. Therefore, the $\frac{d}{2}$ ordered pairs of V_2 having the same value under g' have distinct and even values under \mathcal{R} . Hence, $g : V_2 \mapsto 2\mathbb{Z}_{mn}$ is an injection, hence bijection. \square

Lemma 5.4.8. *Let m and n both be even. If $\gcd(\lambda, d) > 1$, let $k = 1$ when $\alpha^2 \nmid d$,*

and let $k = 2$ when $\alpha^2 \mid d$. Then the mapping $g_{\alpha^k} : V_2 \mapsto 2\mathbb{Z}_{mn}$ given by

$$g_{\alpha^k}(i, j) = \begin{cases} jm + in\frac{d}{\alpha^k} + \mathcal{R}(j - i), & \text{for } i \equiv j \equiv 0 \pmod{2} \\ (j - 1)m + (i - 1)n\frac{d}{\alpha^k} + d + \mathcal{R}(j - i), & \text{for } i \equiv j \equiv 1 \pmod{2} \end{cases} \quad \text{is a bi-}$$

jection.

Proof. Suppose $g'_{\alpha^k}(i, j) = g'_{\alpha^k}(a, b)$ for some $(i, j), (a, b) \in V$. As in Lemma 5.4.7, it must be the case that i, j, a , and b are all of the same parity. Then letting $u = i - a$, $t = j - b$, dividing by d , and observing that $\alpha^k \mid n$ we have that $g'_{\alpha^k}(i, j) = g'_{\alpha^k}(a, b)$ implies equation (5.2) from Lemma 5.4.4. With no restriction on the parities of u and t , we observed in the proof of Lemma 5.4.4 that a full description of the d pairs (u, t) satisfying (5.2) is given by

$$\begin{aligned} S = \{ & (\frac{m}{d}\alpha^k p, 0), (\frac{m}{d}\alpha^k p - \frac{m}{d}, \frac{n}{\alpha^k}), (\frac{m}{d}\alpha^k p - 2\frac{m}{d}, \frac{2n}{\alpha^k}), \\ & \dots, (\frac{m}{d}\alpha^k p - (\alpha^k - 1)\frac{m}{d}, \frac{(\alpha^k - 1)n}{\alpha^k}) \}, \end{aligned}$$

for all $p \in [\frac{d}{\alpha^k}]$. However, in this case we are restricted to the pairs in S such that u and t are both even.

If $\frac{m}{d}$ is odd, then $\gcd(\frac{m}{d}, d) = \alpha$ is odd, since d is even. So α^k is also odd, and hence $\frac{n}{\alpha^k}$ is even, since n is even. Then for all $p \in \{0, 2, \dots, \frac{d}{\alpha^k}\}$ and all $l \in \{1, 3, \dots, \frac{d}{\alpha^k} - 1\}$,

$S_1 \subset S$ where

$$\begin{aligned} S_1 = & \left\{ \left(\frac{m}{d} \alpha^k p, 0 \right), \left(\frac{m}{d} \alpha^k l - \frac{m}{d}, \frac{n}{\alpha^k} \right), \left(\frac{m}{d} \alpha^k p - 2 \frac{m}{d}, \frac{2n}{\alpha^k} \right), \right. \\ & \left. \dots, \left(\frac{m}{d} \alpha^k l - (\alpha^k - 2) \frac{m}{d}, \frac{(\alpha^k - 2)n}{\alpha^k} \right), \left(\frac{m}{d} \alpha^k p - (\alpha^k - 1) \frac{m}{d}, \frac{(\alpha^k - 1)n}{\alpha^k} \right) \right\}, \end{aligned}$$

is the full set of $\frac{d}{2}$ solutions to (5.2) in this case.

On the other hand, if $\frac{m}{d}$ is even we have $\gcd(\frac{m}{d}, d) = \alpha$ is even, so α^k is also even.

Then since $\gcd(\frac{m}{d}, \frac{d}{\alpha^k}) = 1$ by Observations 5.4.1 and 5.4.2, we have that $\frac{d}{\alpha^k}$ is odd and hence $\frac{n}{\alpha^k} = \frac{d}{\alpha^k} \cdot \frac{n}{d}$ is odd, since $\frac{n}{d}$ is odd. Then for all $p \in [\frac{d}{\alpha^k}]$, $S_2 \subset S$ where

$$\begin{aligned} S_2 = & \left\{ \left(\frac{m}{d} \alpha^k p, 0 \right), \left(\frac{m}{d} \alpha^k p - 2 \frac{m}{d}, \frac{2n}{\alpha^k} \right), \left(\frac{m}{d} \alpha^k p - 4 \frac{m}{d}, \frac{4n}{\alpha^k} \right), \right. \\ & \left. \dots, \left(\frac{m}{d} \alpha^k p - (\alpha^k - 2) \frac{m}{d}, \frac{(\alpha^k - 2)n}{\alpha^k} \right) \right\}, \end{aligned}$$

is the full set of $\frac{d}{2}$ solutions to (5.2) in this case. Therefore, in either case we have

established that exactly $\frac{d}{2}$ ordered pairs in V_2 share the same value under g'_{α^k} . Now

we will show that these ordered pairs have distinct values under \mathcal{R} . We have already

observed that $\mathcal{R}(S) = [d]$ in the proof of Lemma 5.4.4. Then since $S_1, S_2 \subseteq S$ and

$\frac{|S_1|}{|S|} = \frac{|S_2|}{|S|} = \frac{1}{2}$ and both S_1 and S_2 contain ordered pairs of the form (u, t) where u

and t are both even, we conclude that $\mathcal{R}(S_1) = \mathcal{R}(S_2) = 2[d]$. Hence, the mapping

$g_{\alpha^k} : V_2 \mapsto 2\mathbb{Z}_{mn}$ is an injection, hence bijection. \square

Chapter 6

Orientable \mathbb{Z}_n -distance magic labeling of Cartesian product of many cycles^{*}

6.1 Introduction

A *distance magic labeling* of a undirected graph G of order n is a bijection $\ell : V(G) \rightarrow \{1, 2, \dots, n\}$ such that the *weight* of every vertex, $w(x) = \sum_{y \in N(x)} \ell(y)$, is equal to some constant, μ (called the *magic constant*) for every $x \in V(G)$. For a comprehensive survey of distance magic labeling, we refer the reader to [5].

Let \mathbb{Z}_n be the cyclic group of order n . An orientable \mathbb{Z}_n -distance magic labeling of a

^{*}The material in this chapter was submitted to Electronic Journal of Graph Theory

graph, first introduced by Cichacz et. al in [14], is a generalization of distance magic labeling. Let $G = (V, E)$ be an undirected graph on n vertices. Assigning a direction to the edges of G gives an *oriented graph* $\vec{G}(V, A)$. In this chapter, we will use the notation \vec{xy} to denote an edge directed from vertex x to vertex y . That is, the *tail* of the arc is x and the *head* is y . For a vertex x , the set of head endpoints adjacent to x is denoted by $N^-(x)$, while the set of tail endpoints is denoted by $N^+(x)$. A *directed \mathbb{Z}_n -distance magic labeling* of an oriented graph $\vec{G}(V, A)$ of order n is a bijection $\vec{\ell} : V \rightarrow \mathbb{Z}_n$ with the property that there is a $\mu \in \mathbb{Z}_n$ (called the *magic constant*) such that

$$w(x) = \sum_{y \in N_G^+(x)} \vec{\ell}(y) - \sum_{y \in N_G^-(x)} \vec{\ell}(y) = \mu \text{ for every } x \in V(G).$$

If for a graph G there exists an orientation \vec{G} such that there is a directed \mathbb{Z}_n -distance magic labeling $\vec{\ell}$ for \vec{G} , we say that G is *orientable \mathbb{Z}_n -distance magic* and the directed \mathbb{Z}_n -distance magic labeling $\vec{\ell}$, we call an *orientable \mathbb{Z}_n -distance magic labeling*.

Throughout this chapter we will use the notation $[n]$ to represent the set $\{0, 1, \dots, n-1\}$ for a natural number n . Furthermore, for a given $i \in [n]$ and any integer j , let $\mathbf{i} + \mathbf{j}$ denote the smallest integer in $[n]$ such that $i + j \equiv \mathbf{i} + \mathbf{j} \pmod{n}$. For a set S and a number a , let $S + a = \{s + a : s \in S\}$.

Let $C_n = x_0, x_1, \dots, x_{n-1}, x_0$ denote a cycle of length n . For the sake of orienting the cycle, if the edges are oriented such that every arc has the form $\overrightarrow{x_i x_{i+1}}$ for all $i \in [n]$, then we say the cycle is oriented *clockwise*. On the other hand, if all the edges of the cycle are oriented such that every arc has the form $\overrightarrow{x_i x_{i-1}}$ for all $i \in [n]$, then we say the cycle is oriented *counter-clockwise*.

6.2 Cartesian Product of Two Cycles

The *Cartesian product* $G \square H$ is a graph with the vertex set $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent in $G \square H$ if and only if $g = g'$ and h is adjacent with h' in H , or $h = h'$ and g is adjacent with g' in G . Hypercubes are interesting graphs which arise via the Cartesian product of cycles. The *hypercube* of order $2k$, Q_{2k} is equivalent to the graph $C_4 \square C_4 \square \dots \square C_4$, where C_4 appears k times in the product. This graph is $2k$ -regular on 4^k vertices. Labeling hypercubes has provided the motivation for the following theorems. Recall the following theorem proved in [19] (Chapter 5 of this dissertation).

Theorem 6.2.1. [19] *The Cartesian product of cycles, $C_m \square C_n$ is orientable \mathbb{Z}_{mn} -distance magic for all $m \geq 3$ and $n \geq 3$.*

The next theorem lays the groundwork for labeling hypercubes.

Theorem 6.2.2. *For any $p \geq 1$, and $n \geq 2$, p disjoint copies of the graph $C_m \square C_m$ is orientable \mathbb{Z}_{pm^2} -distance magic.*

Proof. Let $G = C_m = g_0, g_1, \dots, g_{m-1}, g_0$ and $H = C_m = h_0, h_1, \dots, h_{m-1}, h_0$. Then orient each copy of $G \square H$ as follows. Fix $j \in [m]$. Then for all $i \in [m]$, orient counter-clockwise each cycle of the form $(g_i, h_j), (g_{i+1}, h_j), \dots, (g_{i-1}, h_j), (g_i, h_j)$ in every copy of $G \square H$. Similarly, fix $i \in [m]$. Then for all $j \in [m]$, orient counter-clockwise each cycle of the form $(g_i, h_j), (g_i, h_{j+1}), \dots, (g_i, h_{j-1}), (g_i, h_j)$ in every copy of $G \square H$. Since the graph $G \square H$ can be edge-decomposed into cycles of those two forms, we have oriented every edge in each copy of $G \square H$. Now let ${}^k x_i^j$ denote the vertex (g_i, h_j) of the k^{th} copy of $G \square H$ for $i, j \in [m]$, and $k \in \{1, 2, \dots, p\}$. Then, for each $k \in \{1, 2, \dots, p\}$, define $\vec{\ell} : V \rightarrow \mathbb{Z}_{pm^2}$ by

$$\ell({}^k x_i^j) = pmj + (k-1)m + i - j,$$

where the arithmetic is done modulo pm^2 . Expressing $\vec{\ell}({}^k x_i^j)$ in the following alternative way

$$\ell({}^k x_i^j) = pmj + (k-1)m + \alpha,$$

for $i \equiv j \pmod{m}$, $\alpha \in [m]$ makes it clear that $\vec{\ell}$ is bijective. Then for any given k and for all $i, j \in [m]$ we have $N^+({}^k x_i^j) = \{{}^k x_i^{j+1}, {}^k x_{i+1}^j\}$ and $N^-({}^k x_i^j) = \{{}^k x_i^{j-1}, {}^k x_{i-1}^j\}$. Recalling that $w({}^k x_i^j) \in \mathbb{Z}_{pm^2}$, we have

$$\begin{aligned}
w(x_i^j) &= \vec{\ell}(x_i^{j+1}) + \vec{\ell}(x_{i+1}^j) - [\vec{\ell}(x_{i-1}^j) + \vec{\ell}(x_i^{j-1})] \\
&= [j + 1 + j - j - j - 1]pm \\
&+ (i - j - 1) - (i - j - 1) + (i - j + 1) - (i - j + 1) \\
&= [(j + 1) - (j - 1)]pm \\
&= \begin{cases} (2 - m)pm, j \in \{0, m - 1\} \\ 2pm, 0 < j < m - 1 \end{cases} \\
&= 2pm,
\end{aligned}$$

so $\vec{\ell}$ is an orientable \mathbb{Z}_{pm^2} -distance magic labeling, proving the result. \square

6.3 Cartesian Product of Many Cycles

In this section we consider the repeated Cartesian product of a cycle.

Theorem 6.3.1. *For any $m \geq 3$, the Cartesian product $C_m \square C_m \square \dots \square C_m$ is orientable \mathbb{Z}_{m^n} -distance magic.*

Proof. Let $G_n = C_m \square C_m \square \dots \square C_m$, the Cartesian product of n C_m 's. Then for $n \geq 2$ we may describe G_n recursively as $G_n \cong G_{n-1} \square C_m$. We also have $|V(G_n)| = m^n$, so the labeling will take place in \mathbb{Z}_{m^n} . The proof is by induction. For $n = 1$, we apply the labeling $\{x_0^0, x_0^1, \dots, x_0^{m-1}\} \mapsto \{0, 1, \dots, m - 1\}$ and orient the cycle counter-clockwise. Clearly for $j \in \{0, m - 1\}$, $w(x_0^j) = 2 - m \equiv 2 \pmod{m}$ and for $0 < j < m - 1$, we

have $w(x_0^j) = 2 \equiv 2 \pmod{m}$, so G_1 is orientable \mathbb{Z}_m -distance magic.

For $n = 2$, Theorem 6.2.2 gives that G_2 is orientable \mathbb{Z}_{m^2} -distance magic and using the nomenclature from Theorem 6.2.2, we have $w(x_i^j) = (2 - m)m \equiv 2m \pmod{m^2}$ for $j \in \{0, m - 1\}$, $i \in [m]$ and $w(x_i^j) = 2m$ for $0 < j < m - 1$, $i \in [m]$. Furthermore, for each fixed j , the labels of x_i^j belong to the set $[m] + jm$ for both base cases.

Now construct $G_n \cong G_{n-1} \square C_m$ as follows. Let $H_i \cong G_{n-1}$ for $i \in [m]$. Furthermore, for a given i , let $H_i^j \cong G_{n-2}$ for $j \in [m]$. Let x_i^j denote an arbitrary vertex in the subgraph H_i^j . Then for any integers a, b let x_{i+a}^{j+b} denote the corresponding vertex in the isomorphic subgraph H_{i+a}^{j+b} . Using this terminology, we have $N_{G_n}(x_i^j) = N_{G_{n-1}}(x_i^j) \cup \{x_{i+1}^j, x_{i-1}^j\}$. Let $w_{H_i}(x_i^j)$ denote the weight (in \mathbb{Z}_{m^n}) induced on x_i^j by the subgraph H_i and $w_{H_i^j}(x_i^j)$ denote the weight (again in \mathbb{Z}_{m^n}) induced on x_i^j by the subgraph H_i^j . Now partition $\mathbb{Z}_{m^{n-1}} = P_0 \cup P_1 \cup P_2 \cup \dots \cup P_{m-1}$ so that $P_j = [m^{n-2}] + jm^{n-2}$ for $j \in [m]$.

Assume G_{n-1} is orientable $\mathbb{Z}_{m^{n-1}}$ -distance magic with labeling $\vec{\ell}' : V(G_{n-1}) \rightarrow \mathbb{Z}_{m^{n-1}}$ and orientation $\overrightarrow{G_{n-1}}$. Then in G_n , apply $\vec{\ell}'$ and its corresponding orientation to $H_0 \cong G_{n-1}$. As in the base cases, we may assume that the labels of H_0^j belong to P_j for $j \in [m]$ and

$$w_{H_0}(x_0^j) = \begin{cases} (2 - m)m^{n-2}, & j \in \{0, m - 1\} \\ 2m^{n-2}, & j \in \{1, 2, \dots, m - 2\} \end{cases}.$$

Next, orient all the edges in each subgraph H_1, H_2, \dots, H_{m-1} as in H_0 . Then the only edges left to orient in G_n are cycles of the type $\{x_0^j, x_1^j, \dots, x_{m-1}^j\}$ for fixed j . Orient each of these cycles counter-clockwise. Now define $\vec{\ell} : V(G_n) \rightarrow \mathbb{Z}_{m^n}$ as follows.

$$\vec{\ell}(x_i^j) = \begin{cases} \vec{\ell}'(x_0^j) + i(m-1)m^{n-2} + m^{n-1}, & 0 \leq j < i \\ \vec{\ell}'(x_0^j) + i(m-1)m^{n-2}, & i \leq j \leq m-1 \end{cases}.$$

To show that $\vec{\ell}$ is a bijection, it suffices to show that for each fixed i , $\vec{\ell} : P_j \mapsto P_{j-i} + im^{n-1}$ for all j , since for each fixed i , $j - i$ runs through $[m]$ as j runs through $[m]$. Since the labels of H_0^j belong to P_j for $j \in [m]$, we have

$$\vec{\ell} : P_j \mapsto \begin{cases} P_j + i(m-1)m^{n-2} + m^{n-1}, & 0 \leq j < i \\ P_j + i(m-1)m^{n-2}, & i \leq j \leq m-1 \end{cases}.$$

Now, if $0 \leq j < i$, we have

$$\begin{aligned} P_j &\mapsto P_j + i(m-1)m^{n-2} + m^{n-1} \\ &= [m^{n-2}] + jm^{n-2} + i(m-1)m^{n-2} + m^{n-1} \\ &= [m^{n-2}] + (j-i)m^{n-2} + (i+1)m^{n-1} \\ &= [m^{n-2}] + (j-i)m^{n-2} + mm^{n-2} + im^{n-1} \\ &= [m^{n-2}] + (m+j-i)m^{n-2} + im^{n-1} \\ &= P_{j-i} + im^{n-1}. \end{aligned}$$

While if $i \leq j \leq m-1$, we have

$$\begin{aligned}
P_j &\mapsto P_j + i(m-1)m^{n-2} \\
&= [m^{n-2}] + jm^{n-2} + i(m-1)m^{n-2} \\
&= [m^{n-2}] + (j-i)m^{n-2} + im^{n-1} \\
&= P_{j-i} + im^{n-1}.
\end{aligned}$$

Therefore, it is clear that for each $i \in [m]$, the label set used on H_i is $i \cdot m^{n-1} + \{P_0 \cup P_1 \cup \dots \cup P_{m-1}\} = \mathbb{Z}_{m^{n-1}} + im^{n-1}$, so we see that $\vec{\ell} : V(G_n) \rightarrow \mathbb{Z}_{m^n}$ is bijective.

This completes the labeling and orientation of G_n .

Observe that $\vec{\ell}(x_i^j) \equiv \vec{\ell}(x_0^j) \pmod{m^{n-2}}$. Therefore, $w_{H_i^j}(x_i^j) = w_{H_0^j}(x_0^j)$ in \mathbb{Z}_{m^n} . Then we have,

$$\begin{aligned}
w_{H_i}(x_i^j) &= w_{H_i^j}(x_i^j) + \vec{\ell}(x_i^{j+1}) - \vec{\ell}(x_i^{j-1}) \\
&= w_{H_0^j}(x_0^j) + \vec{\ell}(x_i^{j+1}) - \vec{\ell}(x_i^{j-1})
\end{aligned}$$

But

$$w_{H_0^j}(x_0^j) = w_{H_0}(x_0^j) - [\vec{\ell}(x_0^{j+1}) - \vec{\ell}(x_0^{j-1})].$$

Therefore,

$$\begin{aligned}
w_{H_i}(x_i^j) &= w_{H_0}(x_0^j) - \vec{\ell}(x_0^{j+1}) + \vec{\ell}(x_0^{j-1}) + \vec{\ell}(x_i^{j+1}) - \vec{\ell}(x_i^{j-1}) \\
&= w_{H_0}(x_0^j) + [\vec{\ell}(x_i^{j+1}) - \vec{\ell}(x_0^{j+1})] - [\vec{\ell}(x_i^{j-1}) - \vec{\ell}(x_0^{j-1})] \\
&= a + b - c,
\end{aligned}$$

where $a = 2m^{n-2} - m^{n-1}\mathbb{I}\{j = 0 \text{ or } m-1\}$, $b = i(m-1)m^{n-2} + m^{n-1}\mathbb{I}\{0 \leq \mathbf{j} + \mathbf{1} \leq \mathbf{i} - \mathbf{1}\}$, and $c = i(m-1)m^{n-2} + m^{n-1}\mathbb{I}\{0 \leq \mathbf{j} - \mathbf{1} \leq \mathbf{i} - \mathbf{1}\}$, where \mathbb{I} is the indicator function. Then we can write

$$\begin{aligned} w_{H_i}(x_i^j) &= 2m^{n-2} + m^{n-1}[\mathbb{I}\{0 \leq \mathbf{j} + \mathbf{1} \leq \mathbf{i} - \mathbf{1}\} - \\ &\quad \mathbb{I}\{j = 0 \text{ or } m-1\} - \mathbb{I}\{0 \leq \mathbf{j} - \mathbf{1} \leq \mathbf{i} - \mathbf{1}\}] \end{aligned}$$

Let $I = \mathbb{I}\{0 \leq \mathbf{j} + \mathbf{1} \leq \mathbf{i} - \mathbf{1}\} - \mathbb{I}\{j = 0 \text{ or } m-1\} - \mathbb{I}\{0 \leq \mathbf{j} - \mathbf{1} \leq \mathbf{i} - \mathbf{1}\}$. We will now show that $I = -1$ when $j \equiv i$ or $i-1 \pmod{m}$ and $I = 0$ otherwise.

Case 1. $j \equiv i \pmod{m}$.

Since $j, i \in [m]$, we have $j = i$ and hence $I = 1 - 1 - 1 = -1$ when $j = 0$ or $j = m-1$.

Otherwise, if $1 \leq j \leq m-2$, we have $I = 0 - 0 - 1 = -1$.

Case 2. $j \equiv i-1 \pmod{m}$.

If $j = 0$, we have $I = 0 - 1 - 0 = -1$, while if $j = m-1$, we obtain $I = 1 - 1 - 1 = -1$.

If $1 \leq j \leq m-2$, we have $I = 0 - 0 - 1 = -1$.

Case 3. $j \not\equiv i, i-1 \pmod{m}$.

If $j = 0$, we have $I = 1 - 1 - 0 = 0$, while if $j = m-1$, we have $I = 1 - 1 - 0 = 0$.

Otherwise, if $1 \leq j \leq \mathbf{i} - \mathbf{2}$, we have $I = 1 - 0 - 1 = 0$, and if $\mathbf{i} + \mathbf{1} \leq j \leq m-2$,

we have $I = 0 - 0 - 0 = 0$.

In all cases, we have shown $I = -1$ when $j \equiv i$ or $i - 1 \pmod{m}$ and $I = 0$ otherwise.

We have now fully determined the weight induced by the subgraph H_i for each $i \in [m]$.

We have,

$$w_{H_i}(x_i^j) = \begin{cases} (2 - m)m^{n-2}, & j \equiv i \text{ or } i - 1 \pmod{m} \\ 2m^{n-2}, & \text{otherwise} \end{cases}.$$

We are ready to determine the weight of each vertex. To this end we have $w(x_i^j) = w_{H_i}(x_i^j) + \vec{\ell}(x_{i+1}^j) - \vec{\ell}(x_{i-1}^j)$ and we recall that the arithmetic is to be performed in \mathbb{Z}_{m^n} .

Suppose $j \equiv i$ or $i - 1 \pmod{m}$. Then we have

$$\begin{aligned} w(x_i^j) &= (2 - m)m^{n-2} + \begin{cases} 2(m - 1)m^{n-2} + m^{n-1}, & 1 \leq i \leq m - 2 \\ (2 - m)(m - 1)m^{n-2}, & i \in \{0, m - 1\} \end{cases} \\ &= \begin{cases} 2m^{n-1}, & 1 \leq i \leq m - 2 \\ 2m^{n-1} - m^n, & i \in \{0, m - 1\} \end{cases} \\ &\equiv 2m^{n-1} \pmod{m^n}, \end{aligned}$$

since $(i + 1) - (i - 1) \equiv 2 \pmod{m^n}$ when $1 \leq i \leq m - 2$ and $(i + 1) - (i - 1) \equiv 2 - m \pmod{m^n}$ when $i \in \{0, m - 1\}$.

On the other hand, suppose $j \not\equiv i, i-1 \pmod{m}$. We have

$$\begin{aligned}
w(x_i^j) &= 2m^{n-2} + \begin{cases} 2(m-1)m^{n-2}, & 1 \leq i \leq m-2 \\ (2-m)(m-1)m^{n-2} - m^{n-1}, & i \in \{0, m-1\} \end{cases} \\
&= \begin{cases} 2m^{n-1}, & 1 \leq i \leq m-2 \\ 2m^{n-1} - m^n, & i \in \{0, m-1\} \end{cases} \\
&\equiv 2m^{n-1} \pmod{m^n}.
\end{aligned}$$

Hence, $w(x_i^j) = 2m^{n-1}$ for all $i, j \in [m]$, so G_n is orientable \mathbb{Z}_{m^n} -distance magic for all $n \geq 1$. \square

Corollary 6.3.2. *The hypercube Q_{2k} is orientable \mathbb{Z}_{4^k} -distance magic for all $k \geq 1$.*

Proof. Since $Q_{2k} \cong C_4 \square C_4 \square \dots \square C_4$, the Cartesian product of k C_4 's, Theorem 6.3.1 gives the result. \square

One may wonder if the hypercube Q_{2k+1} is orientable distance magic. Since the graph is odd regular, a little pessimism is understandable. Indeed, Cichacz et. al proved in [14] that no odd regular graph on $n \equiv 2 \pmod{4}$ vertices is orientable \mathbb{Z}_n -distance magic. However, Q_{2k+1} contains 2^{2k+1} vertices, a number divisible by 4, so it is possible that Q_{2k+1} is orientable $\mathbb{Z}_{2^{2k+1}}$ -distance magic for some k . It can easily be checked that $Q_1 \cong K_2$ is not. The following theorem rules out Q_3 as well.

Theorem 6.3.3. *The hypercube Q_3 is not orientable \mathbb{Z}_8 -distance magic.*

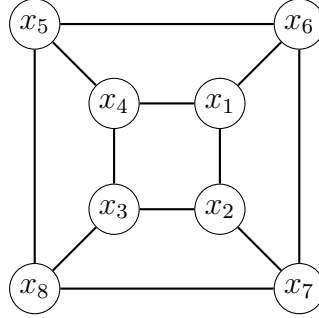


Figure 6.1: The hypercube, Q_3

Proof. Let $G \cong Q_3$ as shown in Figure 6.1. An important fact we will use is that regardless of the orientation of the edges, the (directed) weight of a given vertex has the same parity as the sum (performed in \mathbb{Z}_8 of course) of its neighbors. Suppose for the sake of contradiction that G is orientable \mathbb{Z}_8 -distance magic with orientable \mathbb{Z}_8 -distance magic labeling $\vec{\ell}: V(G) \rightarrow \mathbb{Z}_8$ and associated magic constant μ .

Suppose μ is even. Observe that $N(x_1) = \{x_2, x_4, x_6\}$. Then since μ is even, either all three of $\vec{\ell}(x_2), \vec{\ell}(x_4), \vec{\ell}(x_6)$ are even or exactly one is even. Suppose it is the case that all three of $\vec{\ell}(x_2), \vec{\ell}(x_4), \vec{\ell}(x_6)$ are even. That leaves but one other vertex with an even label. Since $N(x_3) = \{x_2, x_4, x_8\}$, and $w(x_3) = \mu$ is even, it must be the case that $\vec{\ell}(x_8)$ is even. Consequently, $\vec{\ell}(x_1), \vec{\ell}(x_3), \vec{\ell}(x_5)$ must all be odd. But $N(x_4) = \{x_1, x_3, x_5\}$, so $w(x_4) = \mu$ is odd, a contradiction. Therefore, it cannot be the case that all three of $\vec{\ell}(x_2), \vec{\ell}(x_4), \vec{\ell}(x_6)$ are even. In fact, because the graph is vertex transitive, we have shown that no vertex may be adjacent to three even labeled vertices. So it must be the case that every vertex is adjacent to exactly one even-labeled vertex and two

odd-labeled vertices. But this is impossible since there are an equal number of odd and even elements in \mathbb{Z}_8 .

If μ is odd, essentially the same argument leads to another contradiction. Hence, Q_3 is not orientable \mathbb{Z}_8 -distance magic. \square

We conclude this section with the following conjecture.

Conjecture 6.3.4. *The odd-ordered hypercube, Q_{2k+1} is not orientable $\mathbb{Z}_{2^{2k+1}}$ -distance magic for any k .*

6.4 Conclusion

We have proven that any number of disjoint copies of the Cartesian product of two cycles is orientable \mathbb{Z}_n -distance magic. We have also shown that the repeated Cartesian product of a cycle is orientable \mathbb{Z}_n -distance magic, a result which encompasses even-ordered hypercubes. Finally, we have shown that the two smallest odd-ordered hypercubes are not orientable \mathbb{Z}_n -distance magic graphs. One possible direction forward is to prove Conjecture 6.3.4. Another area for improvement is to consider the Cartesian product, $C_{n_1} \square C_{n_2} \square \dots \square C_{n_k}$ for integers n_1, n_2, \dots, n_k .

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Chapter 7

Conclusion and future work

My research on two problems are presented in this dissertation. The first problem asks which graphs allow a d -handicap labeling. That is, a labeling of the n vertices of a graph using the first n natural numbers is sought so that the weight of the vertex labeled i is d more than the vertex labeled $i - 1$. The second problem asks which graphs admit an orientable \mathbb{Z}_n -distance magic labeling. That is, for a simple graph G of order n , an orientation of the edges and a bijective labeling of the vertices using the elements of \mathbb{Z}_n is sought so that the weight of every vertex is equal to the same constant.

7.1 d -Handicap tournaments

7.1.1 Conclusions on d -handicap tournaments

In Chapter 2, the notion of handicap labeling is generalized to d -handicap labeling. Necessary conditions for the existence of a d -handicap graph, $H(n, k, d)$ are identified and families of d -handicap graphs for large classes of n and a wide range of regularities k , for every $d \geq 1$ are constructed. The following theorems provide a summary of the results from Chapter 2.

Theorem 7.1.1. *If an $H(n, k, d)$ exists, then*

1. $w(x_i) = di + \frac{(k-d)(n+1)}{2}$, for all $i \in \{1, 2, \dots, n\}$.
2. If n is even, then $k \equiv d \pmod{2}$.
3. If n is odd, then $k \equiv 0 \pmod{2}$.
4. $n \geq \left\lceil 2 \left(d + 1 + \sqrt{d(d+1)} \right) \right\rceil$.
5. $\left\lceil \frac{n-2-\sqrt{D}}{2} \right\rceil \leq k \leq \left\lfloor \frac{n-2+\sqrt{D}}{2} \right\rfloor$, where $D = (n-2)^2 - 4d(n-1)$.

Theorem 7.1.2. *Let $d \geq 2$ be an even integer and let G be any d -regular distance magic graph of order $v \geq d+2$. Let $n = vt$ for any even integer $t \geq d+2$. If $d \equiv 0 \pmod{4}$ or $t \equiv 0 \pmod{4}$, then there exists an $H(n, 2d, d)$.*

Theorem 7.1.3. *Let $d \geq 2$ and $t, v \geq d + 2$ be even integers and let $n = vt$. If $d \equiv 0 \pmod{4}$ or $v \equiv t \equiv 0 \pmod{4}$, then there exists an $H(n, k, d)$ for all even k such that $2d \leq k \leq n - 2d - 2$.*

Theorem 7.1.4. *For every odd d , there exists an $H(n, k, d)$ for every odd k such that $2d + 1 \leq k \leq n - (2d + 3)$ provided*

- $n \equiv 0 \pmod{4d + 4}$, $n \geq (d + 1)(d + 3)$, and $d \equiv 1 \pmod{4}$ or
- $n \equiv 0 \pmod{4d + 4}$, $n \geq (d + 1)(d + 5)$, and $d \equiv 3 \pmod{4}$ or
- $n \equiv 2d + 2 \pmod{4d + 4}$, $n \geq (d + 1)(d + 3)$, and $d \equiv 3 \pmod{4}$.

7.1.2 Future work regarding d -handicap tournaments

When compared with the necessary conditions, it was shown in Chapter 2 that Theorems 7.1.3 and 7.1.4 leave only a small number of feasible regularities k for which an $H(n, k, d)$ may exist. In fact, for $d = 1$ and $d = 2$, the necessary conditions are met for the appropriate class of n . Therefore, a natural direction forward is to "fill the holes" by finding a construction for these extreme values of k . The smallest open cases are presented in the following problems. Solutions to the problems would provide a full characterization of 3- and 4-handicap tournaments, respectively, for $n \equiv 0 \pmod{16}$.

Problem 7.1.5. *For all $n \equiv 0 \pmod{16}$, $n \geq 32$ construct an $H(n, k, 3)$ for $k \in \{5, n - 7\}$.*

Problem 7.1.6. *For all $n \equiv 0 \pmod{16}$, $n \geq 48$ construct an $H(n, k, 4)$ for $k \in \{6, n - 8\}$.*

Another more ambitious direction forward is to consider the following problem.

Problem 7.1.7. *For a given $d \geq 1$, characterize all n such that an $H(n, k, d)$ exists for some k .*

A solution to Problem 7.1.7 would provide a full list of the number of teams that could play a d -handicap tournament. In Chapter 2 it is shown that this problem has recently been solved for $d = 1$ and n even [30]. For d even, Theorem 7.1.2 ties possible n to the existence of distance magic graphs. For d odd, Theorem 7.1.4 provides a partial characterization of such n .

7.2 Orientable \mathbb{Z}_n -distance magic graphs

In Chapters 3 - 6 the notion of distance magic labeling is generalized to directed graphs.

7.2.1 Conclusions on orientable \mathbb{Z}_n -distance magic graphs

For the purposes of this subsection, for a given graph, G , let n be the order of G . A summary of the findings in Chapters 3 - 6 is provided by the following theorems.

Theorem 7.2.1. *The following graphs are not orientable \mathbb{Z}_n -distance magic.*

- Any graph G having all vertices of odd-degree and $n \equiv 2 \pmod{4}$.
- K_{n_1, n_2, \dots, n_p} , $n \equiv 2 \pmod{4}$, and p is even.
- The hypercube, Q_3 .

Theorem 7.2.2. *The following graphs are orientable \mathbb{Z}_n -distance magic.*

- Every cycle, C_n .
- The hypercube, Q_m , m even.
- Any number p copies of cycle related products; $C_{n_1} \square C_{n_2}$, $C_{n_1} \times C_{n_2}$, or $C_{n_1} \circ C_{n_2}$.
- Repeated Cartesian product of a cycle, $C_m \square C_m \square \dots \square C_m$.
- The strong product of cycles $C_{n_1} \boxtimes C_{n_2}$ if $\gcd(n_1, n_2) \in \{1, 2, 4\}$ or $\gcd(n_1, n_2) = d \in \{3, 5, 6\}$, $d^2 \nmid n_1$, and $d^2 \nmid n_2$.
- $C_{n_1} \boxtimes C_{n_2}$ if $n_1 n_2 \equiv 2 \pmod{4}$ or $n_1 \equiv n_2 \equiv 2 \pmod{4}$.

- Any number p copies of the circulant graph, $C_m(S)$, for any connection set S such that $\frac{m}{2} \notin S$.
- The products of circulants $C_{n_1}(S_1) \circ C_{n_2}(S_2)$ and $C_{n_1}(S_1) \square C_{n_2}(S_2)$, if $\gcd(n_1, n_2) = 1$, $\frac{n_1}{2} \notin S_1$, $\frac{n_2}{2} \notin S_2$.
- The lexicographic product $G \circ C_{2m}(S)$, $m \notin S$, for odd ordered graph G with all vertices of the same parity or m even.
- The lexicographic product $G \circ \overline{K}_{2m+1}$ for odd ordered graph G .
- The lexicographic product $G \circ \overline{K}_k$ for any orientable \mathbb{Z}_m -distance magic graph G except possibly when $k \equiv 2 \pmod{4}$ and G contains a vertex having indegree and outdegree of different parities.
- The direct product $G \times C_{2m}(1, 3, 5, \dots, 2 \lceil \frac{m}{2} \rceil - 1)$, G is an Eulerian graph.
- K_n if and only if n is odd.
- K_{n_1, n_2} , $n \not\equiv 2 \pmod{4}$.
- K_{n_1, n_2, n_3} , for all n .
- K_{n_1, n_2, \dots, n_k} , $1 \leq n_1 \leq n_2 \leq \dots \leq n_k$, $n_2 \geq 2$, n odd.

7.2.2 Future work regarding orientable \mathbb{Z}_n -distance magic graphs

The most obvious direction for future work is characterizing complete multipartite graphs. These graphs were characterized up to and including complete tripartite graphs in Chapter 3. Another more ambitious goal is to prove the conjecture stated at the end of Chapter 3 that all $2r$ -regular graphs admit an orientable \mathbb{Z}_n -distance magic labeling. Clearly, this cannot be proven by a construction. Rather, a novel approach must be found, perhaps using the adjacency matrix and linear algebra.

Another direction forward is to consider groups other than the cyclic groups, possibly classifying for a given graph G of order n all abelian groups Γ of order n which provide an orientable Γ -distance magic labeling of G . Alternatively, one may consider using the natural numbers as labels which would provide applications akin to tournament scheduling seen in Chapter 2.

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