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# Distribution of permutation statistics across pattern avoidance classes, and the search for a Denert-associated condition equivalent to pattern avoidance

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DISTRIBUTION OF PERMUTATION STATISTICS ACROSS PATTERN  
AVOIDANCE CLASSES, AND THE SEARCH FOR A DENERT-ASSOCIATED  
CONDITION EQUIVALENT TO PATTERN AVOIDANCE

By

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## Abstract

We begin with a discussion of the symmetricity of MAJ over  $\text{des}$  in pattern avoidance classes, and its relationship to MAJ-Wilf equivalence. From this, we explore the distribution of permutation statistics across pattern avoidance for patterns of length 3 and 4.

We then begin discussion of Han's bijection, a bijection on permutations which sends the major index to Denert's statistic and the descent number to the (strong) excedance number. We show the existence of several infinite families of fixed points for Han's bijection.

Finally, we discuss the image of pattern avoidance classes under Han's bijection, for the purpose of finding a condition which has the same distribution of DEN over  $\text{exc}$  as pattern avoidance does of MAJ over  $\text{des}$ .





# Chapter 1

## Introduction

A *permutation*  $\sigma$  of  $n$  is an arrangement of the numbers 1 to  $n$  in some order; it can alternatively be viewed as a bijection  $\sigma : [n] \rightarrow [n]$ , where  $[n] = \{1, 2, 3, \dots, n\}$ . In this sense, permutations form a group under composition: this is known as the *symmetric group* of order  $n$ , and denoted  $\mathcal{S}_n$ . We can also consider a permutation  $\sigma$  to be a list  $\sigma_1\sigma_2\dots\sigma_n$ , containing exactly the numbers 1 to  $n$ .

A permutation can be written simply as a list, known as the *one-line notation*, e.g.  $\tau = 7246135$ ; the *two-line notation*, an array with one row showing the input and the other the output, e.g.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 2 & 4 & 6 & 1 & 3 & 5 \end{pmatrix};$$

or the *cycle notation*, a product of cycles representing its orbits, e.g.

$$(2)(634)(751).$$

While there are many ways in which to write a permutation in cycle notation, the *standard form*[Sta11] of a permutation lists the cycles such that each cycle's largest element is first, and the cycles are placed in increasing order of the largest element. Note that, as in our example, cycle notation includes any 1-cycles (fixed points) of the permutation.

A *descent* in a permutation  $\sigma$  is an index  $i$  such that  $\sigma_i > \sigma_{i+1}$ : in other words, it is an index in the permutation where the value decreases. Every position which is not a descent is an *ascent*. In our sample permutation  $\tau$ , we have descents at 1 ( $7 > 2$ ) and 4 ( $6 > 1$ ).

An *excedance* (also called a *strong excedance*) in a permutation  $\sigma$  is any index  $i$  such that  $\sigma_i > i$ . The case where  $\sigma_i \geq i$  is known as the *weak excedances*; every excedance is a weak excedance, but not every weak excedance is an excedance. (For instance, the permutation 1234 has four weak excedances, but no excedances!) In our sample permutation, we have strong excedances at 1, 3, and 4, and a weak excedance at 2.

An *inversion* in a permutation  $\sigma$  is a pair of indices  $(i, j)$  such that  $i < j$  and  $\sigma_i > \sigma_j$ .

Our sample permutation  $\tau$  has twelve inversions:  $(7, 2)$ ,  $(7, 4)$ ,  $(7, 6)$ ,  $(7, 1)$ ,  $(7, 3)$ ,  $(7, 5)$ ,  $(2, 1)$ ,  $(4, 1)$ ,  $(4, 3)$ ,  $(6, 1)$ ,  $(6, 3)$ , and  $(6, 5)$ .

A *permutation statistic* is some function  $f : \mathcal{S}_n \rightarrow \mathbb{N}$ , which typically relates to some property of the permutation. Statistics which are relevant to this thesis include:

† *Inversion number* (denoted  $\text{INV}$ ): The number of inversions. As above,  $\text{INV } \tau = 12$ .

† *Descent number* (denoted  $\text{des}$ ): The number of descents. As above,  $\text{des } \tau = 2$ .

† *Major index* (denoted  $\text{MAJ}$ ): The sums of the indices of the descents. The descent set of  $\tau$  is  $\{1, 4\}$ , so  $\text{MAJ } \tau = 1 + 4 = 5$ .

† *Excedance number* (denoted  $\text{exc}$ ): The number of (strong) excedances. As above,  $\text{exc } \tau = 3$ .

† *Excedance bottom number* (denoted  $\text{Ebot}$ ): The sum of the indices of the (strong) excedances. The indices of the excedances of  $\tau$  are  $\{1, 3, 4\}$ , so  $\text{Ebot } \tau = 1 + 3 + 4 = 8$ .

† *Inversion bottom number* (denoted  $\text{Ibot}$ ): The number of inversions where the lesser value is an excedance. For  $\tau$ , the values of our excedances are 7, 4, and 6; of the inversions,  $(7, 4)$  and  $(7, 6)$  are the only ones with any of these values as the lesser, thus  $\text{Ibot } \tau = 2$ .

† *Inversion top number* (denoted  $\text{Itop}$ ): The number of inversions where the greater value is a nonexcedance.  $\tau$ 's nonexcedance values are 2, 1, 3, and 5; of the inversions, (2,1) is the only one with 2, 1, 3, or 5 at the earlier index. Thus  $\text{Itop } \tau = 1$ .

† *Denert's statistic* (denoted  $\text{DEN}$ ): The sum of the excedance bottom number, inversion bottom number, and inversion top number. For  $\tau$ , we have  $\text{DEN } \tau = \text{Ebot } \tau + \text{Ibot } \tau + \text{Itop } \tau = 8 + 2 + 1 = 11$ .

We define the distribution of a permutation statistic  $st$  as the coefficients of the polynomial

$$F_n^{st}(q) = \sum_{\sigma \in \mathcal{S}_n} q^{\text{st}(\sigma)}.$$

We say that a permutation statistic  $st$  is *Mahonian* if

$$F_n^{\text{MAJ}}(q) = F_n^{st}(q),$$

i.e., if it has the same distribution as the major index, for all  $n$ . This family of statistics was named for MacMahon's proof that the major index and inversion number were equidistributed; the generating functions  $F_n^{\text{MAJ}}$  are the *Mahonian polynomials*

$$F_n^{\text{INV}}(q) = F_n^{\text{MAJ}}(q) = \prod_{i=0}^{n-1} (1 + q + \dots + q^i)$$

Using the notation  $[m]_q = 1 + q + q^2 + \dots + q^{m-1}$  for the  $q$ -analogue of  $m$ , we can also

write this as

$$F_n^{\text{INV}} = F_n^{\text{MAJ}} = \prod_{i=0}^{n-1} [m]_q,$$

i.e., the direct  $q$ -analogue of  $m!$ . [BB13]

Likewise, we say that  $st$  is *Eulerian* if

$$F_n^{\text{des}}(q) = F_n^{\text{st}}(q),$$

i.e., if it has the same distribution as the descent number, for all  $n$ . The generating functions of the descents are closely related to the *Eulerian polynomials* [BB13]  $A_n(q)$ ,

$$A_n(q) = \sum_{\sigma \in \mathcal{S}_n} q^{1+\text{des } \sigma}$$

which are given by the identity

$$\sum_{k \geq 0} (k+1)^n t^{k+1} = \frac{A_n(t)}{(1-t)^{n+1}}.$$

The major index, inversion number, and Denert's statistic are all known to be Mahonian, while the descent and excedence numbers are Eulerian.

A *bistatistic* is an ordered pair of statistics  $(\text{sta}, \text{stb})$ .

**Definition 1.** For a given bistatistic  $(\text{sta}, \text{stb})$ , we say that  $\text{sta}$  is *symmetrically*

*distributed* over stb across class  $\mathcal{C}$  if, given any  $k$ , then

$$\sum_{\tau \in \mathcal{C}: \text{stb } \tau = k} q^{\text{sta } \tau} = q^j f(q),$$

where  $f(q)$  is a symmetric polynomial: i.e., if  $f$  is of degree  $n$ , then  $f(q) = q^n f(\frac{1}{q})$ .

For instance, MAJ is symmetric over des over the class of 1234-avoiding permutations:

For example, with  $k = 3$ , we have  $j = 6$  and  $f(q) = 10 + 35q + 66q^2 + 80q^3 + 66q^4 + 35q^5 + 10q^6$ . We will prove this property holds for several avoidance classes of length 3 and 4.

We say that a given bivariate statistic  $(\text{STA}, \text{stb})$  is *Euler-Mahonian* if

$$\sum_{\sigma \in \mathcal{S}_n} q^{\text{MAJ } \sigma} t^{\text{des } \sigma} = \sum_{\sigma \in \mathcal{S}_n} q^{\text{STA } \sigma} t^{\text{stb } \sigma}$$

These generating functions are known as the *Euler-Mahonian polynomials* [BB13], and are given by the following  $q$ -analogue of the formula for the Eulerian polynomial identity:

$$\sum_{k \geq 0} [k+1]_q^n t^k = \frac{\sum_{\pi \in \mathcal{S}_n} t^{\text{des } \pi} q^{\text{MAJ } \pi}}{\prod_{j=0}^n (1 - tq^j)}$$

It is clear that it is a necessary condition that STA be Mahonian and stb be Eulerian, but these conditions are not sufficient;  $(\text{MAJ}, \text{des})$  has a different bivariate generating function than  $(\text{MAJ}, \text{exc})$ ; the latter is not Euler-Mahonian. It is thus conventional

to, given a particular Eulerian (resp. Mahonian) statistic  $STA$  (resp.  $stb$ ), attempt to find a Mahonian (resp. Eulerian) statistic  $stb$  (resp.  $STA$ ) such that the bivariate statistic  $(STA, stb)$  is Euler-Mahonian. Denert's statistic was originally conjectured in [Den90] to be such a partner for the excedance number; this was then shown to be true by Foata and Zeilberger in [FZ90].

Consider two sequences of distinct values  $a_1a_2\dots a_n$  and  $b_1b_2\dots b_n$ . We say that  $a$  and  $b$  are *order isomorphic* if  $b_i < b_j$  if and only if  $a_i < a_j$ . We say that a permutation  $\sigma$  contains a permutation  $\pi$  if there is a subsequence of  $\sigma$  which is order isomorphic to  $\pi$ ; if  $\sigma$  contains no such subsequence, we say that it *avoids*  $\pi$ . In this context we refer to  $\pi$  as a *pattern*.

The set of permutations of order  $n$  which avoid a pattern  $\pi$  is denoted  $\mathcal{S}_n(\pi)$ . The study of pattern avoidance developed out of Knuth's work on stack-sorting [Knu68], and has become a major topic in combinatorics.

Two permutations  $\pi$  and  $\pi^*$  are said to be *Wilf equivalent* if  $|\mathcal{S}_n(\pi)| = |\mathcal{S}_n(\pi^*)|$ . It is well-known that for all  $\pi \in \mathcal{S}_3$ ,  $|\mathcal{S}_n(\pi)| = C_n$ , where  $C_n$  is the  $n$ th Catalan number; thus, all permutations of length 3 are Wilf equivalent. However, we have  $|\mathcal{S}_6(1234)| = 513$  but  $|\mathcal{S}_6(1342)| = 512$ ; thus, not all permutations of the same length are Wilf equivalent.

In [DDJ<sup>+</sup>12], Dokos *et al.* consider a  $q$ -analogue of Wilf equivalence, which they refer



to as *st*-Wilf equivalence, and which they define as follows:

Suppose  $\text{st} : \mathcal{S}_n \rightarrow \mathbb{N}$  is a permutation statistic; let  $F_n^{\text{st}}(\pi, q)$  be the generating function of  $\text{st}$  over the permutations avoiding  $\pi$ ; i.e.,

$$F_n^{\text{st}}(\pi, q) = \sum_{\theta \in \mathcal{S}_n(\pi)} q^{\text{st } \theta}$$

We say that  $\pi$  and  $\pi^*$  are *st*-Wilf equivalent if

$$F_n^{\text{st}}(\pi, q) = F_n^{\text{st}}(\pi^*, q)$$

for all  $n$ .

Note that if we set  $q = 1$ , we get

$$F_n^{\text{st}}(\pi, 1) = |\mathcal{S}_n(\pi)|,$$

and thus *st*-Wilf equivalence implies Wilf equivalence.

Many of the conjectures posited and questions posed by [DDJ<sup>+</sup>12] have already been proven, disproven, or otherwise addressed, more extensively than we shall address here.

† In [CEKS13a], Cheng *et al.* show a generalisation of the conjecture that the

distribution of INV over  $S_n(321)$  is given by

$$f_n(q)f_{n-1}(q) + \sum_{k=0}^{n-2} q^{k+1} f_k(q)f_{n-1-k}(q)$$

They also answer the questions posed by [DDJ<sup>+</sup>12] regarding recursions for bivariate polynomials. In a later addendum [CEKS13b], they prove the symmetry of MAJ over des for  $S_n(321)$ . In [MS14], Mansour and Shattuck offer an algebraic proof for this result in addition to the earlier combinatorial one.

† In [Blo14], Bloom proves combinatorially the conjecture that 1423, 2413, and 3214 are MAJ-Wilf equivalent.

† In [Kil12], Killpatrick relates MAJ-Wilf equivalence to charge-Wilf equivalence, and uses this to prove the conjecture on the parity of the coefficients of  $F_{2^k-1}^{\text{MAJ}}(321, q)$ .

† In [Tro15], Trongsirawat demonstrates a means to construct non-trivial examples of INV-Wilf equivalence for pairs of sets of two avoided permutations, disproving the conjecture that none exist.

† In [Cha15], Chan strengthens the result of [Tro15] by constructing a family of INV-Wilf equivalent pairs of single permutations.

† In [YGZ15], Yan *et al.* prove part of a conjecture that  $12 \dots k(k+m+1) \dots (k+2)(k+1)$  and  $(m+1)(m+2) \dots (k+m+1)m \dots 21$  are MAJ-Wilf equivalent;

specifically, they show it is true for the subcase  $k \geq 1$  and  $m = 1$ .

While many of these conjectures were proven in more detail than we address, this notion of st-Wilf equivalence, specifically MAJ-Wilf equivalence, provided the inspiration for our work here.

Given a permutation  $\sigma \in \mathcal{S}_n$ , we define its *complement*  $C(\sigma)$  as the permutation given by

$$C(\sigma)_i = n + 1 - \sigma_i$$

for  $1 \leq i \leq n$ . We also define its *reverse*  $R(\sigma)$  as the permutation given by

$$R(\sigma)_i = \sigma_{n+1-i}$$

and its *reverse complement*  $\sigma' = R(C(\sigma))$ .

It can be simply shown that  $R(C(\sigma)) = C(R(\sigma))$ , and it should be clear that  $R(\sigma)$ ,  $C(\sigma)$ , and  $\sigma'$  are all involutions.

In Chapter 2, we will show that if a permutation  $\sigma$  avoids a pattern  $\pi$ , then  $\sigma'$  avoids  $\pi'$ . It can be shown that  $\text{des } \sigma' = \text{des } \sigma$ , while  $\text{MAJ } \sigma' = n \text{des } \sigma - \text{MAJ } \sigma$ . From this, we can observe that if  $(\text{MAJ}, \text{des})$  is symmetrically distributed in  $\mathcal{S}_n(\pi)$ , then  $\pi$  is MAJ-Wilf equivalent to  $\pi'$ : since the reverse complement preserves descents and sends the major index to  $n \text{des } \sigma - \text{MAJ } \sigma$ , if the polynomial is symmetric, the same

numbers simply exchange, giving the same distribution.

This provides one way of proving MAJ-Wilf equivalence for some permutations, which inspired the beginning of the investigation. We then show that if  $\pi$  is equal to  $\pi'$ , then MAJ is necessarily symmetrically distributed over  $\text{des}$  on  $\mathcal{S}_n(\pi)$ .

In Chapter 3, we move from the results of Chapter 2 to consider finding a subset of  $\mathcal{S}_n$  on which the bivariate statistic  $(\text{DEN}, \text{exc})$  has the same distribution as  $(\text{MAJ}, \text{des})$  does over  $\mathcal{S}_n(\pi)$ . To this end we employ a bijection of Han [Han90], prove the existence of several families of fixed points, and set out some avenues for further work in identifying a condition not reliant on pattern avoidance and Han's bijection.

# Chapter 2

## Symmetry of the $(\text{maj}, \text{des})$

### Bistatistic Across Pattern

### Avoidance Classes

We consider now the problem of the distribution of permutation bistatistics across avoidance classes. Stanley[Sta] produced a proof that  $(\text{MAJ}, \text{des})$  is symmetrically distributed over  $\mathcal{S}_n(123 \cdots p)$ , which we here recreate and extend to cover all permutations  $\pi$  which are preserved by reverse complement.

**Lemma 1.** *Let  $\sigma$  be a permutation of length  $n$  and  $\pi$  be a pattern of length  $m \leq n$ , and let  $R(\sigma)$  and  $R(\pi)$  denote their respective reverses. Then  $\sigma$  avoids  $\pi$  if and only if  $R(\sigma)$  avoids  $R(\pi)$ .*

*Proof.* If  $\sigma$  contains  $\pi$ , then by definition we can pick  $a_1, \dots, a_m$  such that  $a_i < a_j$  if  $i < j$  and the sequence of  $\sigma_{a_i}$  matches exactly the pattern  $\pi$ .

We then let  $b_i = n - a_i$ , so  $R(\sigma)_{b_i} = \sigma_{a_i}$ . Since  $\sigma_{a_i}$  matches the pattern  $\pi$ , and  $R(\sigma)_{b_i}$  is simply that sequence reversed,  $R(\sigma)_{b_i}$  matches the pattern  $R(\pi)$ . Thus, if  $\sigma$  contains  $\pi$ ,  $R(\sigma)$  contains  $R(\pi)$ ; by the contrapositive, if  $R(\sigma)$  avoids  $R(\pi)$ ,  $\sigma$  avoids  $\pi$ .

This same logic can be used, save for starting with  $R(\sigma)$  containing  $R(\pi)$ , to show that if  $\sigma$  avoids  $\pi$ ,  $R(\sigma)$  avoids  $R(\pi)$ . □

**Lemma 2.** *Let  $\sigma$  be a permutation of length  $n$  and  $\pi$  be a pattern of length  $m \leq n$ , and let  $C(\sigma)$  and  $C(\pi)$  denote their respective complements. Then  $\sigma$  avoids  $\pi$  if and only if  $C(\sigma)$  avoids  $C(\pi)$ .*

*Proof.* If  $\sigma$  contains  $\pi$ , then by then by definition we can pick  $a_1, \dots, a_m$  such that  $a_i < a_j$  if  $i < j$  and the sequence of  $\sigma_{a_i}$  matches exactly the pattern  $\pi$ .

By the definition of the complement,  $C(\sigma)_{a_i} = n + 1 - \sigma_{a_i}$ . Thus, the larger an element  $\sigma_{a_i}$ , the smaller an element  $C(\sigma)_{a_i}$  and vice versa. This sends the pattern  $\pi$  to the pattern  $C(\pi)$ .

This same logic can be used, save for starting with  $C(\sigma)$  containing  $C(\pi)$ , to show that if  $\sigma$  avoids  $\pi$ ,  $C(\sigma)$  avoids  $C(\pi)$ . □

**Corollary 1.** *Let  $\sigma$  be a permutation of length  $n$  and  $\pi$  be a pattern of length  $m \leq n$ , and let  $\sigma'$  and  $\pi'$  be their respective reverse complements. Then  $\sigma$  avoids  $\pi$  if and only if  $\sigma'$  avoids  $\pi'$ .*

Using 1, we can extend Stanley's result.

**Theorem 1.** *Let  $\pi$  be a permutation in  $\mathcal{S}_p$ , and let  $\pi'$  denote the reverse complement of  $\pi$ . Then if  $\pi = \pi'$ , MAJ is symmetrically distributed over des across  $\mathcal{S}_n(\pi)$ .*

*Proof.* Let  $\sigma \in \mathcal{S}_n(\pi)$ . By Corollary 1,  $\sigma' \in \mathcal{S}_n(\pi')$ . Since  $\pi = \pi'$ ,  $\sigma' \in \mathcal{S}_n(\pi)$ .

Given  $k$  descents, the smallest value that the major index can take on is

$$1 + 2 + \dots + k = \binom{k+1}{2}$$

(having every descent in the earliest possible position), while the largest possible value is

$$n - 1 + n - 2 + \dots + n - k = kn - \binom{k+1}{2}$$

(having every descent in the last possible position).

Since  $\text{MAJ } \sigma' = n \text{ des } \sigma - \text{MAJ } \sigma$  and  $\text{des } \sigma' = \text{des } \sigma$ , this is sufficient to show that MAJ is symmetrically distributed over des in  $\mathcal{S}_n(\pi)$ . □

## 2.1 Special Case for $k = 4$

In almost every case for  $n = 4$ , the only time where  $(\text{MAJ}, \text{des})$  is symmetrically distributed across an avoidance class  $\mathcal{S}_n(\pi)$  is when  $\pi$  is preserved by reverse complement. There are, however, four outliers: 1423, 2314, 3241, and 4132; these are the only patterns we have found which are symmetric without being preserved by reverse-complement.

In [Blo14], Bloom shows that the major index is equidistributed among 1423-avoiding, 2413-avoiding, and 2314-avoiding permutations by way of a pair of bijections  $\Theta : \mathcal{S}_n(1423) \rightarrow \mathcal{S}_n(2413)$  and  $\Omega : \mathcal{S}_n(2314) \rightarrow \mathcal{S}_n(2413)$ . Bloom's bijections preserve not only major index, but descent set, and thus this proves that  $(\text{maj}, \text{des})$  is equidistributed across these avoidance classes.

We will now show that Bloom's method also applies to  $\mathcal{S}_n(3241)$ ,  $\mathcal{S}_n(3142)$ , and  $\mathcal{S}_n(4132)$ . Since 2413 and 3142 are preserved by reverse-complement, these results thus imply the symmetry of MAJ over des for these permutation avoidance classes.

**Lemma 3.** *Let  $\sigma$  be a permutation of length  $n$ , and let  $R(\sigma)$  denote its reverse. Then  $\text{des } R(\sigma) = n - 1 - \text{des } \sigma$ .*

*Proof.* Let  $i$  be the position of an ascent in  $\sigma$ ; i.e.,  $\sigma_i < \sigma_{i+1}$ . Since  $R(\sigma)_{n+1-i} = \sigma_i$ ,



this becomes  $R(\sigma)_{n+1-i} < R(\sigma)_{n-i}$ : i.e.,  $n - i$  is a descent in  $R(\sigma)$ . Similarly, we can show that if  $i$  is a descent in  $\sigma$ ,  $n - i$  is an ascent in  $R(\sigma)$ . As there are  $n - 1$  places which must be either ascents or descents, there are  $n - 1 - \text{des } \sigma$  ascents in  $\sigma$ , so we have  $\text{des } R(\sigma) = n - 1 - \text{des } \sigma$ .  $\square$

**Lemma 4.** *Let  $\sigma$  be a permutation of length  $n$ , and let  $R(\sigma)$  denote its reverse. Then  $\text{MAJ } R(\sigma) = \binom{n}{2} - n \text{des } \sigma + \text{MAJ } \sigma$ .*

*Proof.* Let  $S$  be the descent set of  $\sigma$ . Then  $R(A) = \{n - i \mid i \in S\}$  is the ascent set of  $R(\sigma)$ ; it follows that  $R(S) = [n - 1] \setminus R(A)$  is the descent set of  $R(\sigma)$ . Then by the definition of major index,

$$\begin{aligned}
\text{MAJ } R(\sigma) &= \sum_{i \in R(S)} i \\
&= \sum_{i \in [n-1]} i - \sum_{i \in R(A)} i \\
&= \binom{n}{2} - \sum_{i \in S} (n - i) \\
&= \binom{n}{2} - \sum_{i \in S} n + \sum_{i \in S} i \\
&= \binom{n}{2} - n \text{des } \sigma + \text{MAJ } \sigma.
\end{aligned}$$

Combined, these show that the major index and descent number of a permutation are determined exactly by the major index and descent number of its reverse. (It is

an exercise in simple arithmetic to show that when these rules are applied twice, they result in the original values.) □

**Theorem 2.** *The bivariate statistic  $(\text{MAJ}, \text{des})$  is equidistributed among 3241-avoiding, 3142-avoiding, and 4132-avoiding permutations.*

*Proof.* Let  $\sigma$  be a permutation of length  $n \geq 4$  that avoids 3241, and let  $R(\sigma)$  denote its reverse. Then by Lemma 1,  $R(\sigma)$  avoids 1423; by Lemmas 3 and 4, it has descent number  $\text{des}(R(\sigma)) = n - 1 - \text{des } \sigma$  and major index  $\text{MAJ}(R(\sigma)) = \binom{n}{2} - n \text{des } \sigma + \text{MAJ } \sigma$ .

Then  $\tau = \Theta(R(\sigma))$  is a 2413-avoiding permutation with descent number equal to  $\text{des}(R(\sigma))$  and major index equal to  $\text{MAJ}(R(\sigma))$ . Then  $R(\tau)$ , by Lemmas 1, 3, and 4 is a 3142-avoiding permutation with descent number  $\text{des}(\sigma)$  and major index  $\text{MAJ}(\sigma)$ . Thus, we have a bijection between 3241-avoiding and 3142-avoiding permutations given by  $f(\sigma) = R(\Theta(R(\sigma)))$  which preserves both the major index and descent number, and thus  $(\text{MAJ}, \text{des})$ .

Similarly, we can construct  $g(\sigma) = R(\Omega(R(\sigma)))$  as a bijection between 4132-avoiding permutations and 3142-avoiding permutations which also preserves  $(\text{MAJ}, \text{des})$ . □

# Chapter 3

## Finding an Equivalent Condition to Pattern Avoidance for $(\text{den}, \text{exc})$

### 3.1 Other Bistatistics

In our investigation, we examined all bistatistics involving two of the major index, descent number, excedance number, weak excedance number, inversion number, and Denert's statistic. In all cases except for  $(\text{MAJ}, \text{des})$ , the bistatistics did not show a symmetric distribution across pattern avoidance classes for general  $n$ . This motivated us to ask our next question:

If  $(\text{MAJ}, \text{des})$  can produce a symmetric distribution over pattern avoidance, can we

construct some other condition, equivalent to pattern avoidance, which permits symmetric distribution of other bistatistics?

In order to find a pattern avoidance counterpart for  $(\text{DEN}, \text{exc})$ , we looked to a bijection which sends MAJ to DEN and des to exc, and thus  $\mathcal{S}_n(\pi)$  to its equivalent under such a condition. In order to construct this bijection, we must consider the *coding* of permutations.

We will say that a sequence  $a_0 a_1 a_2 \dots a_{n-1}$  is a *code sequence* if, for every  $0 \leq i < n$ ,  $0 \leq a_i \leq i$ , and denote the set of code sequences of length  $n$  as  $\mathcal{CS}_n$ . It should be obvious that  $|\mathcal{CS}_n| = n! = |\mathcal{S}_n|$  for all  $n$ . A *permutation code* is some bijection  $f : \mathcal{S}_n \rightarrow \mathcal{CS}_n$ . Such bijections are not difficult to construct in general; some with particularly desirable properties exist, such as the two we now consider.

## 3.2 The Major Index Code

**Definition 2.** The *major-index code* or MAJ code is a bijection  $MC : \mathcal{S}_n \rightarrow \mathcal{CS}_n$ . Given some permutation  $\sigma$  we construct the major-index code  $MC(\sigma)$  as follows: Let  $\sigma_i$  denote  $\sigma$  with elements 1 to  $i$  deleted, and the remaining elements all reduced by  $i$ . Then  $a_{n-i} = \text{MAJ } \sigma_{i-1} - \text{MAJ } \sigma_i$ ; i.e.,  $a_{n-i}$  is the change in major index when 1 is deleted from  $\sigma_{i-1}$  to yield  $\sigma_i$ .

**Example 3.2.1.** To return to our example permutation  $\tau$ , we find  $MC(\tau)$  as follows:

$i$	$\tau_i$	MAJ $\tau_i$	$a_i$
7	7 2 4 6 1 3 5	5	—
6	6 1 3 5 2 4	5	0
5	5 2 4 1 3	4	1
4	4 1 3 2	4	0
3	3 2 1	3	1
2	2 1	1	2
1	1	0	1
0		0	0

Thus we have  $MC(\tau) = 0121010$ .

To decode a major-index code from a given code sequence  $c$ , we define a decoding function  $\Phi(\sigma_i, a_i) : \mathcal{S}_i \times \mathbb{Z}_i \rightarrow \mathcal{S}_{i+1}$ . First,  $\Phi$  increases the value of every element of  $\sigma_i$  by 1.

† If  $a_i = \text{des } \sigma_i$ , insert 1 at the beginning of  $\sigma_i$ ; this moves every descent by 1, and thus increases the major index by  $\text{des } \sigma_i = a_i$ .

† If  $a_i < \text{des } \sigma_i$ , insert 1 at the bottom of the  $(a_i + 1)$ st descent from the right; this moves the  $a_i$  descents to the right of that index to the right by 1, and thus

increases the major index by  $a_i$ .

† If  $a_i > \text{des } \sigma_i$ , we cannot increase  $\text{MAJ } \sigma_i$  by  $a_i$  by simply moving descents. Thus, we must add a descent. Suppose we have  $0 \leq j \leq i$ , and let  $D(j)$  be the number of descents in the substring starting at index  $j$ . Then  $j + D(j)$  is a weakly increasing function, since each step always increases  $j$  by 1, and  $D(j)$  decreases by either 1 or 0. The smallest value is  $\text{des } \sigma_i + 1$  and the largest is  $i$ ; thus, there is some run of numbers such that  $j + D(j) = a_i$ . Of these numbers, the very last (the one after which the value increases) is necessarily a non-descent (since moving past a descent does not change  $j + D(j)$ ). Insert 1 after this  $j$ , which adds a new descent at index  $j$ , and moves the remaining  $D(j)$  descents to the right, increasing the major index by  $j + D(j) = a_i$ .

We can then construct  $MC^{-1}(\sigma)$  by repeatedly applying  $\Phi(\sigma_i, a_i)$  until every  $a_i$  has been used.

As an example, we can decode the code string  $c = 00213116$  as follows:

0.  $\Phi(\emptyset, 0) = 1$ . Since  $0 = \text{des } \emptyset$ , we insert 1 at the beginning.
1.  $\Phi(1, 0) = 12$ . Since  $0 = \text{des } 1$ , we insert 1 at the beginning.
2.  $\Phi(12, 2) = 231$ . Since  $2 > \text{des } 12$ , we need to find an ascent  $j$  such that  $j + D(j) = 2$ ; since there are no descents, we just pick  $j = 2$  and insert 1 after

index 2.

3.  $\Phi(231, 1) = 1342$ ; since  $1 = \text{des } 231$ , we insert 1 at the beginning.
4.  $\Phi(1342, 3) = 24153$ ; since  $3 > \text{des } 1342$ , we need to find an ascent  $j$  such that  $j + D(j) = 3$ . There is one descent, at index 3, so  $2 + D(2) = 3$ . We thus insert 1 after index 2.
5.  $\Phi(24153, 1) = 351264$ ; since  $1 < \text{des } 24153$ , we insert 1 in the middle of the second descent from the right, which is at position 2.
6.  $\Phi(351264, 6) = 4623751$ ; since  $6 > \text{des } 351264$ , we need to find an ascent  $j$  such that  $j + D(j) = 6$ . Constructing a table of  $j + D(j)$ , we get

$\sigma_i$	3	5	1	2	6	4
$j$	1	2	3	4	5	6
$D(j)$	2	2	1	1	1	0
$j + D(j)$	3	4	4	5	6	6

so we place 1 after position 6.

Thus,  $MC^{-1}(00213116) = 4623751$ .

There are two key properties of the major-index code. The first, that

$$\sum_{i=1}^{n-1} a_i = \text{MAJ } \sigma,$$

follows directly from the definition; from our explanation of  $\Phi$ , we see that  $\text{des } \sigma_{i-1} \leq a_i$  iff  $\text{des } \sigma_i = \text{des } \sigma + 1$ ; i.e., whenever  $a_i$  is greater than the descent number of  $\sigma_{i-1}$ , an additional descent is added when reconstructing  $\sigma_i$ . Together, these make it possible to show that a bivariate statistic  $(\text{STA}, \text{stb})$  is Euler-Mahonian by showing the existence of a bijection  $\Psi : \mathcal{S}_{n-1} \times [n] \rightarrow \mathcal{S}_n$  such that the properties

$$\begin{aligned} \text{STA } \Psi(\sigma, a) &= \text{STA } \sigma + a; \\ \text{stb } \Psi(\sigma, a) &= \begin{cases} \text{stb } \sigma & \text{if } a \leq \text{stb } \sigma; \\ \text{stb } \sigma + 1 & \text{if } a > \text{stb } \sigma \end{cases} \end{aligned}$$

hold true. This is equivalent to showing the existence of an STA code which corresponds to the major-index code.

### 3.3 Han's Denert Code

In [Han90], Han constructs a bijection  $f : \mathcal{S}_n \rightarrow \mathcal{S}_n$  such that  $\text{MAJ } \sigma = \text{DEN } f(\sigma)$  and  $\text{des } \sigma = \text{exc } f(\sigma)$  as a combinatorial proof that the bivariate statistic  $(\text{DEN}, \text{exc})$  is Euler-Mahonian. He does so by constructing a DEN decoder which behaves similarly to the MAJ decoder: that is, he constructs a function

$$\Psi : \mathcal{S}_{n-1} \times [0, n-1] \rightarrow \mathcal{S}_n$$



such that

$$\begin{aligned} \text{DEN } \Psi(\sigma, a) &= \text{DEN } \sigma + a; \\ \text{exc } \Psi(\sigma, a) &= \begin{cases} \text{exc } \sigma & \text{if } a \leq \text{exc } \sigma; \\ \text{exc } \sigma + 1 & \text{if } a > \text{exc } \sigma. \end{cases} \end{aligned}$$

Han's  $\Psi$  works as follows. Let a permutation  $\sigma \in \mathcal{S}_n$  and some  $a$  such that  $0 \leq a \leq n$  be given.

We define an *enumeration* of  $\sigma$  as a permutation  $\tau \in \mathcal{S}_n$  given by

$$\tau_i = \begin{cases} \#\{j \text{ such that } \sigma_j > j \text{ and } \sigma_j \geq \sigma_i\} & \text{if } \sigma_i > i; \\ \text{exc } \sigma + \#\{j \text{ such that } \sigma_j \leq j \text{ and } \sigma_j \leq \sigma_i\} & \text{if } \sigma_i \leq i. \end{cases}$$

In other words, if  $i$  is the index of an excedance of  $\sigma$ , then  $\tau_i$  is the number of excedances with value greater than or equal to  $\sigma_i$ ; if  $i$  is not an index of an excedance of  $\sigma$ , then  $\tau_i$  is the number of all excedances of  $\sigma$  plus the number of nonexcedances with value less than or equal to  $\sigma_i$ . It should be clear that this formula will always produce a permutation of  $[n]$ : the first  $\text{exc } \sigma$  values will correspond to the excedances, in decreasing order of their value, while the remaining values will correspond to the nonexcedances in increasing order.

Given  $\tau$ , we define the function  $\nu : [n] \rightarrow [n]$  such that  $\nu(\sigma_i) = \tau_i$ . It should be clear that  $\nu$  is a bijection, since both are permutations of  $[n]$ ; thus we can also define  $\nu^{-1}$ .

We then construct an ordered list  $\text{Repl}(\sigma, s)$  as follows:  $\text{Repl}[0] = n + 1$ , and then  $\{\text{Repl}_{i=\nu^{-1}(i)}\}$  for  $1 \leq i \leq \text{exc } \sigma$  and  $\nu^{-1}(s) < \nu^{-1}(i)$ . (For this purpose, we define  $\nu^{-1}(0) = n + 1$ .) We then replace the values in the permutation such that  $\text{Repl}(n)$  replaces  $\text{Repl}(n + 1)$  in the permutation; finally, the last element of  $\text{Repl}$  is inserted into the permutation at index  $\nu^{-1}(s)$ .

With  $\Psi$  so constructed, we can define  $HC^{-1}(c)$ , where  $c$  is some code sequence, by repeatedly applying  $\Psi$  as we did for  $MC^{-1}(c)$ . In order to construct the encoding function  $HC(\sigma)$ , we must first define an index  $j$  as *strongly-fixed*<sup>1</sup> if it is either a fixed point, or satisfies the property

$$\{\sigma_i : \sigma_i > i, \sigma_j > \sigma_i \geq j\} = \emptyset.$$

In other words, it is either a nonexcedance, or there is no excedance value which fits between  $\sigma_j$  and  $j$ .

Let  $G(\sigma)$  be the set of values of strongly-fixed indices in  $\sigma$ . Then remove  $\max G(\sigma)$  from the permutation, and replace the first excedance to have a greater value than the removed value with it. Repeat this until  $n$  has been removed, giving the next

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<sup>1</sup>Han uses *place grande-fixe* in the original French; “strongly-fixed” is an attempt at a literal English translation, though it is a weaker condition than being a fixed point.

permutation in the stage; the difference in the Denert code gives the value of the Han code at  $n - 1$ .

We will henceforth refer to this code as the *Han Denert code* or simply *Han code*, given by  $HC(\sigma)$  and  $HC^{-1}(c)$ .

With the major index code and Han code, we can construct the bijection as

$$\sigma' = HC^{-1}(MC(\sigma)),$$

which sends MAJ to DEN and des to exc. This bijection can be reversed as

$$\sigma = MC^{-1}(HC(\sigma')).$$

### 3.4 Fixed Points of Han's Bijection

In the case of pattern avoidance, no permutation avoids itself: that is,  $\sigma \notin \mathcal{S}_n(\sigma)$  for all  $\sigma \in \mathcal{S}_n$ . However, for the vast majority of permutations,  $\sigma \in HC^{-1}(MC(\mathcal{S}_n(\sigma)))$ ; the only cases where this is not the case is when  $\sigma = HC^{-1}(MC(\sigma))$ , i.e., when  $\sigma$  is fixed by Han's bijection.

It is trivial to see that  $\text{MAJ}\sigma = \text{DEN}\sigma$  and  $\text{des}\sigma = \text{exc}\sigma$  is a necessary

condition for this, but it is far from sufficient: for instance, we determined computationally that there are 116,929,919 permutations of length 13 such that  $\text{MAJ}\sigma = \text{DEN}\sigma$  and  $\text{des}\sigma = \text{exc}\sigma$ , of which seven were fixed by Han's bijection:  $123456789ABCD$ ,  $86D2134579ABC$ ,  $95D2134678ABC$ ,  $A4D21356789BC$ ,  $B3D21456789AC$ ,  $915B86C2D437A$ , and  $A471C95D3268B$  (using hexadecimal notation for the values 10-13).

**Corollary 2.** *The identity permutation  $123\dots n$  is fixed by Han's bijection for all  $n$ .*

*Proof.* This follows directly from the above observation; the identity is the only one permutation with zero descents, and it is also the only permutation with zero excedances. Thus, the identity must be sent to itself by Han's bijection.  $\square$

**Proposition 1.** *The following permutations are fixed by Han's bijection for all values of  $n$ :*

$$\sigma = n + 1, 1, 2, 3, 4, \dots, n - 1, n, n + 2, n + 3, \dots, 2n \tag{i}$$

$$\tau = n + 1, 1, n + 2, 2, \dots, n + i, i, \dots, 2n, n \tag{ii}$$

*Proof.* For (i), when we apply the major-index code, we note that there is only one descent, at 1, with value  $n + 1$ ; deleting 1 does not change the major index for the first  $n - 1$  times the operation is applied. After  $n - 1$  operations, we have the permutation  $2, 1, 3, 4, \dots, n + 1$ ; removing 1 will decrease the major index by 1, and

give the permutation  $123 \dots n$ , which has major index 0. Thus, the major-index code of  $\sigma$  is  $n$  zeroes, a 1, and then  $n - 1$  zeros.

Applying the Han decoder  $HC^{-1}(c)$  to this string will produce  $123 \dots n$  after the first  $n$  steps, since there are no excedances added. The  $(n + 1)$ st operation will add a single excedance, and increase Denert's statistic by 1; the only way to do this is to add  $n + 1$  at the beginning. The remaining values are all 0, and thus will append  $i + 1$  to the end at each step; this gives exactly the permutation  $n + 1, 1, 2, 3, 4, \dots, n - 1, n, n + 2, n + 3, \dots, 2n$ .

Since the decodings are the same,  $\sigma$  is thus a fixed point of Han's bijection.

For (ii), when we apply  $MC(\tau)$ , when  $1 \leq i \leq n$  (the first  $n$  deletions), we remove 1 from the permutation. There are  $i$  consecutive values before 1, corresponding to  $n + 1 - i, n + 2 - i$ , all the way up to  $n + 1$ . When we delete 1,  $n + 1$  will be followed immediately by  $n + 2$ , so we remove a descent at position  $i$ . There are still  $n - i$  descents after that position, each of which moves back one index; thus the major index changes by 1. Once the first  $n$  deletions have been performed we are left with the permutation  $123 \dots n$ , and thus the remaining deletions all leave 0 change. Thus,  $MC(\tau) = 0 \dots 0n \dots n$ , with  $n$  zeroes followed by  $n$   $ns$ .

When we apply  $HC(\tau)$ , the permutation has  $n + 1$  strongly-fixed points: the  $n$  nonexcedances the smallest excedance, which is also the first excedance to occur. Since the

smallest value occurs at the smallest index, it is strongly-fixed, but could substitute for any excedance after it, so none of them are. When we remove the smallest excedance, it replaces the next-smallest excedance, which replaces the next-smallest excedance, all the way up to the removal of the length. This repeats each time, since the relative order of the excedances does not change; when the  $k$ th excedance from the start is removed, it leaves  $n - k$  remaining excedances, still in increasing order, with each value moved left one and decreased by 1; as the excedance at index  $2k - 1$  is removed, this decreases Denert's statistic by  $n$ , until every excedance is removed; this gives  $n$   $ns$ , and leaves the permutation  $12 \dots n$ , which gives  $n$  zeroes. Thus, the major-index code is  $n$  zeroes followed by  $n$   $ns$ .

Since the decodings are the same,  $\tau$  is thus a fixed point of Han's bijection. □

There are also some more esoteric families of permutations which are preserved by Han's bijection.

**Theorem 3.** *Let  $C$  be the set of code strings of length  $n \geq 6$  constructed as follows: beginning with  $01$ , followed by  $0 \leq k \leq \frac{1}{2}(n - 6)$  zeroes, followed by  $1$ , followed by  $n - 6 - 2k$  zeroes, followed by  $2$ , followed by  $k + 1$  zeroes, ending with  $4$ .*

*Then if  $MC(\sigma) \in C$ ,  $HC^{-1}(MC(\sigma)) = \sigma$ .*

*Proof.* Let  $n$  and  $k$  be given, and construct the corresponding code string  $c$ .

Apply  $\Phi$  to each part in turn. The initial 01 gives the permutation 21; the following  $k$  zeroes give the permutation  $k+2, 1, 2, \dots, k+1$ ; the next 1 pushes the single descent to the right, giving  $1, k+3, 2, 3, \dots, k+2$ . We then add the  $n-6-2k$  zeroes, which gives the permutation

$$n-5-2k, n-3-k, 1, 2, \dots, n-6-2k, n-4-2k, n-3-2k, \dots, n-4-k$$

The next 2 adds a descent at index 1, pushing the other descent to the right:

$$n-4-2k, 1, n-2-k, 2, 3, \dots, n-5-2k, n-3-2k, n-2-2k, \dots, n-3-k$$

The following  $k+1$  zeroes are placed after the second descent, giving

$$n-3-k, k+2, n-1, 1, 2, \dots, k+1, k+3, k+4, \dots, n-4-k, n-2-k, n-1-k, \dots, n-2$$

Finally, we insert a descent at index 4, giving

$$n-2-k, k+3, n, 2, 1, 3, 4, \dots, k+2, k+4, k+5, \dots, n-3-k, n-1-k, n-k, \dots, n-1$$

Notably, there are four “out-of-place” elements:  $n-2-k$ ,  $k+3$ ,  $n$ , and 2, at the beginning, and all other elements are in order.

Apply  $\Psi$  to each part in turn. As before, the initial 01 gives the permutation 21,

but the following zeroes instead yield  $2134 \dots k+2$ . Next,  $s = 1$ , so we place  $k+3$  in place of 2 (our sole excedance), and then insert 2 at position 2 (since  $nu^{-1}(1)$  yields the sole excedance, which is 2). This gives

$$k+3, 2, 1, 3, 4, \dots, k+2$$

The following  $n-6-2k$  zeroes make this

$$k+3, 2, 1, 3, 4, \dots, k+2, k+4, k+5, \dots, n-3-k$$

We now have  $s = 2$ ;  $n-2-k$  is put in place of our sole excedance,  $k+3$ , which is inserted at index  $nu^{-1}(2) = 1$ . This gives

$$k+3, n-2-k, 2, 1, 3, 4, \dots, k+2, k+4, k+5, \dots, n-3-k$$

We then go through the following  $k+1$  zeroes, which gives

$$k+3, n-2-k, 2, 1, 3, 4, \dots, k+2, k+4, k+5, \dots, n-3-k, n-1-k, n-k, \dots, n-1$$

Finally, we have the 4.  $n$  bumps  $n-2-k$ , which bumps  $k+3$ , which is inserted at index  $nu^{-1}(4) = 2$ ; this gives

$$n-2-k, k+3, n, 2, 1, 3, 4, \dots, k+2, k+4, k+5, \dots, n-3-k, n-1-k, n-k, \dots, n-1$$



As the decodings are the same, this permutation is preserved by Han's bijection.  $\square$

*Remark:* The smallest such code has  $n = 6$  and  $k = 0$ , giving 011204.

**Theorem 4.** *Let  $C$  be the set of code strings of length  $n \geq 11$  constructed as follows: beginning with 00213, followed by  $k$  zeroes, followed by 3, followed by  $k$  zeroes, followed by 53087. (These codes thus have length  $n = 11 + 2k$ .)*

*Then if  $MC(\sigma) \in C$ ,  $HC^{-1}(MC(\sigma)) = \sigma$ .*

*Proof.* Apply  $\Phi$  to each step in turn. The initial 00213 gives the permutation 24153; adding  $k$  zeroes makes this

$$k + 2, k + 4, k + 1, k + 5, 1, 2, \dots, k, k + 3.$$

There are two descents, at 2 and 4, so the 3 inserts a descent at 1 and bumps the other two right:

$$k + 3, 1, k + 5, k + 2, k + 6, 2, 3, \dots, k + 1, k + 4.$$

The next  $k$  zeroes make this

$$2k + 3, k + 1, 2k + 5, 2k + 2, 2k + 6, 1, 2, \dots, k, k + 2, k + 3, \dots, 2k + 1, 2k + 4.$$

and the descents remain at 1, 3, and 5. The 5 thus adds a descent at 4 and bumps the descent at 5 to the right:

$$2k + 4, k + 2, 2k + 6, 2k + 3, 1, 2k + 7, 2, 3, \dots, k + 1, k + 3, k + 4, \dots, 2k + 2, 2k + 5.$$

The descents are now at 1, 3, 4, and 6. The 3 simply bumps the last three descents to the right, giving

$$2k + 5, 1, k + 3, 2k + 7, 2k + 4, 2, 2k + 8, 3, 4, \dots, k + 2, k + 4, k + 5, \dots, 2k + 3, 2k + 6.$$

The zero inserts 1 at the bottom of the rightmost descent, giving

$$2k + 6, 2, k + 4, 2k + 8, 2k + 5, 3, 2k + 9, 1, 4, 5, \dots, k + 3, k + 5, k + 6, \dots, 2k + 4, 2k + 7.$$

The eight creates a new descent at place 8, giving

$$2k + 7, 3, k + 5, 2k + 9, 2k + 6, 4, 2k + 10, 2, 1, 5, 6, \dots, k + 4, k + 6, k + 7, \dots, 2k + 5, 2k + 8.$$

There are now descents at 1, 4, 5, 7, and 8. The seven thus inserts a 1 at place 3, giving

$$2k + 8, 4, k + 6, 1, 2k + 10, 2k + 7, 5, 2k + 11, 3, 2, 6, 7, \dots, k + 5, k + 7, k + 8, \dots, 2k + 6, 2k + 9.$$

Apply  $\Psi$  to each step in turn. The initial 00213 gives the permutation 14532; adding  $k$  zeroes makes this

$$1, 4, 5, 3, 2, 6, 7, \dots, k + 5.$$

There are two excedances at 2 and 3. The 3 adds one more excedance, which bumps both of these, and inserts 4 at  $\nu^{-1}(3) = 1$ :

$$4, 1, 5, k + 6, 3, 2, 6, 7, \dots, k + 5.$$

The next  $k$  zeroes make this

$$4, 1, 5, k + 6, 3, 2, 6, 7, \dots, k + 5, k + 7, k + 8, \dots, 2k + 6.$$

The 5 adds another excedance.  $2k + 7$  bumps  $k + 6$ , which bumps 5, which bumps 4, which is inserted at  $\nu^{-1}(5) = 2$ :

$$5, 4, 1, k + 6, 2k + 7, 3, 2, 6, 7, \dots, k + 5, k + 7, k + 8, \dots, 2k + 6.$$

The 3 does not add an excedance, but shuffles three of them.  $2k + 8$  bumps  $2k + 7$ , which bumps  $k + 6$ , which bumps 5, which is inserted at  $\nu^{-1}(3) = 5$ .

$$k + 6, 4, 1, 2k + 7, 5, 2k + 8, 3, 2, 6, 7, \dots, k + 5, k + 7, k + 8, \dots, 2k + 6.$$

The 0 inserts  $2k + 9$  at the end.

$$k + 6, 4, 1, 2k + 7, 5, 2k + 8, 3, 2, 6, 7, \dots, k + 5, k + 7, k + 8, \dots, 2k + 6, 2k + 9.$$

There are four excedances, at 1, 2, 4, and 6, so the 8 adds a new excedance. We have  $\nu^{-1}(8)$  as the fourth nonexcedance, 5.  $2k + 10$  bumps  $2k + 8$ , which bumps  $2k + 7$ , which bumps  $k + 6$ ; since 4 is less than  $\nu^{-1}(8)$ , it is left where it is, so  $k + 6$  is inserted at 5:

$$2k + 7, 4, 1, 2k + 8, k + 6, 5, 2k + 10, 3, 2, 6, 7, \dots, k + 5, k + 7, k + 8, \dots, 2k + 6, 2k + 9.$$

There are now excedances at 1, 2, 4, 5, and 7; the 7 at the end thus adds one more excedance;  $\nu^{-1}(7)$  is the second nonexcedance, 2, which is less than every value. Thus,  $2k + 11$  bumps  $2k + 10$ , which bumps  $2k + 8$ , which bumps  $2k + 7$ , which bumps  $k + 6$ , which bumps 4, which is inserted at 2. This gives

$$2k + 8, 4, k + 6, 1, 2k + 10, 2k + 7, 5, 2k + 11, 3, 2, 6, 7, \dots, k + 5, k + 7, k + 8, \dots, 2k + 6, 2k + 9.$$

Since both codes decode into the same permutation, it is a fixed point of Han's bijection. □

*Remark:* The smallest such code has  $k = 0$  and thus  $n = 11$ , giving 00213353087. A

code with a similar beginning, 002133063, occurs at length 9 but does not seem to split and recur for a larger  $n$ .

**Theorem 5.** *Let  $C$  be the set of code strings of  $n \geq 12$  constructed as follows: beginning with 000313, followed by  $k$  zeroes, followed by 440885.*

*Then if  $MC(\sigma) \in C$ ,  $HC^{-1}(MC(\sigma)) = \sigma$ .*

*Proof.* This can be shown with a similar argument to the above. □

We have shown through exhaustive search that there are no fixed points of length 13 other than those which match the patterns outlined above. Exhaustive search is the only means we have found to identify these fixed points; whether there is a better algorithm for identifying fixed points is an open question, as is the question of enumerating the fixed points of length  $n$ .

We also state the following conjecture.

**Conjecture 1.** *For every  $k \geq 0$ , there exists an infinite family of permutations  $\sigma$  such that the first nonzero entry of  $MC(\sigma)$  is  $k$ , and  $HC^{-1}(MC(\sigma)) = \sigma$ .*

The above families account for  $k = 0$  (the identity permutations, Corollary 2),  $k = 1$  (Theorem 3),  $k = 2$  (Theorem 4), and  $k = 3$  (Theorem 5).

### 3.5 Pattern Avoidance under Han's Bijection

To identify our equivalent condition, we must consider the image of pattern avoidance under Han's bijection: in other words, what happens when a given pattern is put through Han's bijection. We do not have any clear result to this effect, but have several conditions that such a result must meet.

While every permutation avoids itself ( $\sigma \notin \mathcal{S}_n(\sigma)$  for  $n = |\sigma|$ ), permutations almost always are in the image  $HC^{-1}(MC(\mathcal{S}_n(\sigma)))$ ; the exception is if they are fixed under Han's bijection.

One possibility is that the condition depends not directly on  $\pi$ , but on  $HC^{-1}(MC(\pi))$ , as this is the permutation absent from the resulting set.

**Theorem 6.** *Let  $\sigma_{i+1} = \Phi(\sigma_i, s)$ , where  $\Phi$  is a single step of  $MC^{-1}(c)$ . If  $\sigma_i$  contains a pattern  $\pi$ , then  $\sigma_{i+1}$  contains  $\pi$ .*

If  $\sigma_i$  contains  $\pi$ , there is some subsequence of  $\sigma_i$  which is order-isomorphic to  $\pi$ . In the first step of  $\Phi$ , all elements of  $\sigma_i$  are increased by 1; the subsequence remains order-isomorphic to  $\pi$ . Next, a 1 is inserted somewhere to yield  $\sigma_{i+1}$ ; since this does not change the relative order of any elements, the subsequence remains order-isomorphic to  $\pi$ .

This tells us that pattern avoidance, once present, cannot be removed by addition to the major-index code; thus, we should expect that the condition defining  $HC^{-1}(MC(\mathcal{S}_n(\sigma)))$  cannot be removed by addition to the Han code.

**Theorem 7.** *Let  $\sigma_{i+1} = \Psi(\sigma_i, s)$ , where  $\Psi$  is a single step of  $HC^{-1}(c)$ . Then the nonexcedances of  $\sigma_i$  are nonexcedances of  $\sigma_{i+1}$  and have the same ordering.*

*Proof.* Each iteration of Han's bijection does two things: it has some number of excedances replace smaller excedance values, and then inserts the last value replaced as either a new nonexcedance, at its own index, or as a new excedance.

If the value is inserted as a new excedance, it obviously does not change the relative order of the other nonexcedances.

If the value is inserted as a new nonexcedance, it is placed somewhere inside the string of nonexcedances, but does not reorder any of the nonexcedances already present.

Thus, the relative ordering of the nonexcedances is preserved. □

This is obviously an insufficient equivalent to pattern presence on its own, as the number of arrangements of nonexcedances is smaller than the number of strings which contain a given pattern.

In order to gain some insight into this condition, we considered the following special

case of 123- avoiding permutations.

**Theorem 8.** *If  $c \in \mathcal{CS}_n$  contains three consecutive instances of some value  $k$ , then  $MC^{-1}(c)$  is not 123-avoiding.*

*Proof.* Let  $\sigma$  be the permutation given by decoding all of  $c$  up to the first of the three consecutive  $k$ . Recall that from how the major-index code operates, lower values are inserted after higher ones.

† If  $k \leq \text{des } \sigma$ , we will insert 1 at the bottom of the  $(k + 1)$ st descent from the right (or at 1, if  $k = \text{des } \sigma$ ) for each  $k$ ; this does not move that descent, so this yields an explicit 123 after the function has been applied to all three  $k$ ; by Theorem 6,  $MC^{-1}(c)$  thus contains a 123-pattern.

† If  $k = \text{des } \sigma + 1$ , the first  $k$  will create a new descent somewhere (no further in the permutation than position  $k$ , depending on placement of the remaining descents). The remaining two  $k$  will both place a 1 at the beginning of the permutation; this gives 12 at the beginning and a third, greater value anywhere else, and thus we have a 123 after the  $k$  have been processed. As above, Theorem 6 states that  $MC^{-1}(c)$  thus contains a 123-pattern.

† If  $k = \text{des } \sigma + 2$ , the first two  $k$  will each add a descent. The first descent is at some position  $i$ , with  $k - i$  descents after at it; the second descent cannot be placed after  $i$ : as mentioned in the discussion of  $MC^{-1}$ ,  $i + D(i)$  (the change in



major index) is an increasing function, so it would increase the total by more than  $k$  if placed after  $i$ . It cannot be placed at  $i$  as that would increase the total by  $i + 1$ . Thus, it must be placed before  $i$ .

† If  $k \geq \text{des } \sigma + 3$ , each  $k$  will add a new descent. The first and second obey similar rules to the previous case, and the third likewise is placed before the second. Since lower values are inserted later, this gives a 123-pattern, and Theorem 6 again states that  $MC^{-1}(c)$  contains a 123-pattern.

□

We now consider what this translates to when  $HC^{-1}$  is applied. This should define a subset of the equivalent condition to containing the permutation 123.

**Theorem 9.** *Consider a permutation  $\sigma \in \mathcal{S}_n$  such that  $HC(\sigma)$  contains three consecutive instances of some number  $k$ . Then define  $\tau$  as the permutation given by numbering the excedances of  $\sigma$ , in increasing order of value, from 1 to  $\text{exc } \sigma$ , and the nonexcedences of  $\sigma$ , in increasing order of value, from  $\text{exc } \sigma + 1$  to  $n$ . Then  $\tau$  contains a 123-pattern.*

*Proof.* We start with  $c$ ,  $k$ , and  $\sigma$  exactly as they were in Theorem 8

† If  $k \leq \text{exc } \sigma$ , no new excedances will be added. The values will be substituted down until the  $k$ th excedance, which will be inserted at its own value to produce

a nonexcedance. These values will be increasing, so this ensures the presence of some 123-pattern of nonexcedances.

† If  $k = \text{exc } \sigma + 1$ , the first instance will add a new excedance: after excedance replacements are complete, the smallest excedance value will be moved to position 1, since  $\nu^{-1}(\text{exc } \sigma + 1)$  is always 1. The next instance will make the next-smallest excedance value will replace it, and it will become a nonexcedance at a position equal to itself. The last instance will push the third-smallest excedance value to position 1, and the second-smallest excedance will be inserted at a position equal to itself; necessarily later than the earlier insertion. Thus, we have a 312-pattern made up of the smallest excedance, at place 1, followed by two nonexcedances elsewhere in the permutation.

† If  $k = \text{exc } \sigma + 2$ , the first two instances each add an excedance. The first one will place the smallest excedance value greater than the second-smallest nonexcedance at the index given by the second-smallest nonexcedance. The second instance will place the smallest excedance value at 1, after pushing every excedance down. Finally, the smallest excedance value will be placed at its own index and become a nonexcedance. Since the smallest excedance could possibly be less than the second-smallest nonexcedance, this either gives us a 231 or 213 pattern of two excedances and a nonexcedance (where the 1 is always a nonexcedance). However, since the second case occurs when the smallest excedance is less than the second-smallest nonexcedance, and we know that the largest

excedance is placed at a position given by the second-smallest nonexcedance, we can construct a 231 from the same two excedances and that second-smallest nonexcedance instead.

† If  $k \geq \text{exc } \sigma + 3$ , each instance adds an excedance: one at the initial  $(k - \text{exc } \sigma)$ th nonexcedance, one at the initial  $(k - \text{exc } \sigma - 1)$ st nonexcedance, and one at the initial  $(k - \text{exc } \sigma - 2)$ nd nonexcedance. Since the Han code orders nonexcedances in increasing order, the first is greater than the second, which is greater than the third, so the excedances are added in decreasing order; further, since anything greater than the first is greater than the second, and anything greater than the second is greater than the third, the addition of the later excedances will definitely replace the earlier excedances. Thus, the first one added will be bumped down by the second, and both will be bumped down by the third, giving a 123 of excedances.

□

These conditions are sufficient to saying that there is a 123-pattern in  $\tau$ . However, this is a weak condition; many permutations which are not the image of a 123-avoiding permutation also possess this quality. Strengthening this result is a question for further work.

## 3.6 Further Work

The foremost further question to pose is this: What is, exactly, a condition  $C$  on permutations such that the distribution of  $(\text{DEN}, \text{exc})$  over the set of permutations that meet condition  $C$  is exactly the distribution of  $(\text{MAJ}, \text{des})$  over the set  $\mathcal{S}_n(\pi)$ ?

From our observations above, we can draw some conclusions to guide this problem:

† It may be useful to define the condition in terms of  $HC^{-1}(MC(\pi))$  rather than  $\pi$  itself, as no permutation is self-avoiding but almost every pattern  $\pi$  is in the image of the  $\pi$ -avoiding permutations.

† Once the equivalent of a pattern is present, it should remain present as the Han code is extended. The relative order of the nonexcedances is known to meet this condition, as are the values of strongly-fixed points.

† We define a permutation  $\tau$ , consisting of the excedances labelled in increasing order followed by the nonexcedances labelled in increasing order, which seems to be connected to our special case of 123-avoiding permutations. Strengthening this result could yield additional insights.

We also discuss the question of a family of families of fixed points of Han's bijection, which we formally ask as: Given some  $k \geq 0$  and sufficiently large  $n$ , is it possible

to construct a code string  $c$  of length  $n$  with first non-trivial entry  $k$ , such that  $HC^{-1}(c) = MC^{-1}(c)$  (i.e.,  $c$  corresponds to a fixed point of Han's bijection)?

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