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## Hamilton decompositions of 6-regular abelian Cayley graphs

Erik E. Westlund

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HAMILTON DECOMPOSITIONS OF 6-REGULAR ABELIAN CAYLEY GRAPHS

By

ERIK E. WESTLUND

A DISSERTATION

Submitted in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

(Mathematical Sciences)

MICHIGAN TECHNOLOGICAL UNIVERSITY

2010

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This dissertation, “Hamilton Decompositions of 6-Regular Abelian Cayley Graphs”, is hereby approved in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in the field of Mathematical Sciences.

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To my family.



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## Abstract

In 1969, Lovász asked whether every connected, vertex-transitive graph has a Hamilton path. This question has generated a considerable amount of interest, yet remains vastly open. To date, there exist no known connected, vertex-transitive graph that does not possess a Hamilton path. For the Cayley graphs, a subclass of vertex-transitive graphs, the following conjecture was made:

**Weak Lovász Conjecture:** Every nontrivial, finite, connected Cayley graph is hamiltonian.

The Chen-Quimpo Theorem proves that Cayley graphs on abelian groups flourish with Hamilton cycles, thus prompting Alspach to make the following conjecture:

**Alspach Conjecture:** Every  $2k$ -regular, connected Cayley graph on a finite abelian group has a Hamilton decomposition.

Alspach's conjecture is true for  $k = 1$  and  $2$ , but even the case  $k = 3$  is still open. It is this case that this thesis addresses.

Chapters 1–3 give introductory material and past work on the conjecture. Chapter 3 investigates the relationship between 6-regular Cayley graphs and associated quotient graphs. A proof of Alspach's conjecture is given for the odd order case when  $k = 3$ . Chapter 4 provides a proof of the conjecture for even order graphs with 3-element connection sets that have an element generating a subgroup of index 2, and having a linear dependency among the other generators.

Chapter 5 shows that if  $\Gamma = \text{CAY}(A, \{s_1, s_2, s_3\})$  is a connected, 6-regular, abelian Cayley graph of even order, and for some  $1 \leq i \leq 3$ ,  $\Delta_i = \text{CAY}(A/\langle s_i \rangle, \{\overline{s_{j_1}}, \overline{s_{j_2}}\})$  is 4-regular, and  $\Delta_i \not\cong \text{CAY}(\mathbb{Z}_3, \{1, 1\})$ , then  $\Gamma$  has a Hamilton decomposition. Alternatively stated, if  $\Gamma = \text{CAY}(A, S)$  is a connected, 6-regular, abelian Cayley graph of even order, then  $\Gamma$  has a Hamilton decomposition if  $S$  has no involutions, and for some  $s \in S$ ,  $\text{CAY}(A/\langle s \rangle, \overline{S})$  is 4-regular, and of order at least 4.

Finally, the Appendices give computational data resulting from C and MAGMA programs used to generate Hamilton decompositions of certain non-isomorphic Cayley graphs on low order abelian groups.



# Chapter 1

## Cayley Graphs and Hamilton Cycles

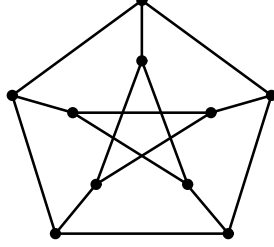
### 1.1 Overview

This thesis discusses the classical problem in graph theory of finding Hamilton cycles, a cycle that visits each vertex once, in finite simple graphs. The search for necessary and sufficient conditions for the existence of Hamilton paths, cycles, and decompositions in graphs, digraphs, and hypergraphs, is as old as graph theory itself, and many questions yet remain. As far as the author is aware, there is considerable dissension among graph theorists as to whether there even exist good characterizations for hamiltonicity in general, and to complicate things, this problem falls in the set of NP-complete problems of computational complexity theory. We consider the problem of obtaining Hamilton decompositions, partitions of the edge set of a graph into Hamilton cycles, by restricting attention to graphs that are based on an algebraic group, i.e., the family of Cayley graphs.

We begin with an introduction to terminology, definitions, and notation, which can be found in standard textbooks on graph theory, such as [52] and [17]. We then proceed to develop a framework of theorems that answer, in the affirmative, a subcase of a conjecture of Alspach, stemming from questions of Lovász, Parsons, and many others, on Hamilton cycles and decompositions of Cayley graphs on finite abelian groups.

### 1.2 Preliminaries

Graphs are a vast class of combinatorial structures and are ubiquitous in that they are used to describe relationships. Graphs are used to model ecosystems, phylogenetic trees, and protein-protein interactions in biology; network flows, routing problems, and data structures in computer science and engineering; molecular structure in organic chemistry; countless problems from combinatorics, abstract algebra, matrix algebra, recreational mathematics, probability theory, and statistics.



**Figure 1.1:** The famous Petersen graph is a 3-regular vertex-transitive graph.

A *graph*  $\Gamma$  is a pair of sets  $(V, E)$ , where  $V = V(\Gamma)$  is called the *vertex set* (or node set or point set) and  $E = E(\Gamma) = \{\{x, y\} : x, y \in V(\Gamma)\}$  is called the *edge set*. Elements of  $E$  are called *edges* of  $\Gamma$ . For example, a vertex set could be the set of all airports located in U.S. cities, where vertices  $x$  and  $y$  have an edge between them, denoted  $\{x, y\}$ , if one city has a direct flight to the other. The cardinality of  $V$  is called the *order* of the graph, and the cardinality of  $E$  is called the *size* of the graph.

If  $\{x, y\} \in E$ , it is common to write  $xy$  in place of  $\{x, y\}$  when the context is clear, and we say  $x$  is *adjacent* to  $y$ . If  $e = \{x, y\}$ , we say  $x$  and  $y$  are *incident* to  $e$ . If  $E$  is a multiset, we say  $\Gamma$  is a *multigraph*. Any edge of the form  $\{x, x\}$  for some  $x \in V(\Gamma)$ , is called a *loop*. A *simple* graph is a graph without multiple edges or loops. Most of the graphs in this thesis will be simple. Two graphs  $\Gamma_1$  and  $\Gamma_2$  are *equal* if and only if  $V(\Gamma_1) = V(\Gamma_2)$  and  $E(\Gamma_1) = E(\Gamma_2)$ . Given a graph  $\Gamma = (V, E)$ , the *complement* of  $\Gamma$ , denoted  $\bar{\Gamma}$  is the graph with vertex  $V(\Gamma)$ , and edge  $xy$  if and only if  $xy \notin E(\Gamma)$ .

More commonly, we are interested in when two graphs *behave essentially the same*. The graphs  $\Gamma_1$  and  $\Gamma_2$  are said to be *homomorphic*, if there exists a mapping  $\varphi : V(\Gamma_1) \rightarrow V(\Gamma_2)$ , such that  $\{x, y\} \in E(\Gamma_1)$  if and only if  $\{\varphi(x), \varphi(y)\} \in E(\Gamma_2)$ . I.e., any homomorphism maps edges to edges and nonedges to nonedges. A graph homomorphism  $\varphi$  that is a bijective map is called an *isomorphism*, denoted  $\Gamma_1 \simeq \Gamma_2$ . Order, size, degree sequence, cycle structure, and many other parameters are invariant under isomorphism. An isomorphism from a graph to itself is called an *automorphism* of  $\Gamma$ . The set of all automorphisms of  $\Gamma$  form an algebraic group, called it *automorphism group*, denoted  $\text{AUT}(\Gamma)$ . For example,  $\text{AUT}(K_n) \cong S_n$ , the group of all permutations of  $n$  elements. It is usually a very difficult problem to determine the automorphism group of a graph, though it pays large dividends in terms of understanding the structure of the graph. It is readily seen that  $\text{AUT}(\Gamma)$  permutes the set of vertices of common degree  $r$  among themselves.

A graph  $\Gamma$  is called *vertex-transitive* if for any  $x, y \in V(\Gamma)$ , there exists  $\varphi \in \text{AUT}(\Gamma)$  such that  $\varphi(x) = y$ . Vertex-transitive graphs are necessarily regular. For example, the well-known Petersen graph shown in Figure 1.1 is a vertex-transitive graph with automorphism group  $S_5$ . An important class of vertex-transitive graphs, called Cayley graphs, are introduced in Section 1.4. Cayley graphs, named after the mathematician Arthur Cayley, encode group-theoretic structure, are used extensively in combinatorial group theory.

The *union* of the graphs  $\Gamma_1$  and  $\Gamma_2$ , denoted  $\Gamma_1 \cup \Gamma_2$  is the graph with vertex set  $V(\Gamma_1 \cup \Gamma_2) = V(\Gamma_1) \cup V(\Gamma_2)$  and edge set  $E(\Gamma_1 \cup \Gamma_2) = E(\Gamma_1) \cup E(\Gamma_2)$ . Similarly, the *intersection* of the graphs  $\Gamma_1$  and  $\Gamma_2$ , denoted  $\Gamma_1 \cap \Gamma_2$  is the graph with vertex set  $V(\Gamma_1 \cap \Gamma_2) = V(\Gamma_1) \cap V(\Gamma_2)$  and edge set

$E(\Gamma_1 \cap \Gamma_2) = E(\Gamma_1) \cap E(\Gamma_2)$ . If  $E(\Gamma_1 \cap \Gamma_2) = \emptyset$ , then  $\Gamma_1$  and  $\Gamma_2$  are said to be *edge-disjoint*. If  $V(\Gamma_1 \cap \Gamma_2) = \emptyset$ , then  $\Gamma_1$  and  $\Gamma_2$  are said to be *vertex-disjoint* (any two vertex-disjoint graphs are also edge-disjoint). If  $\Gamma_1 \cap \Gamma_2 = \emptyset$ , the empty graph, then  $\Gamma_1$  and  $\Gamma_2$  are said to be *disjoint*.

If the vertex  $x$  of  $\Gamma$  is adjacent to  $r$  other vertices, we say  $x$  has *degree*  $r$ , denoted  $\deg_\Gamma(x) = r$ . If every vertex of  $\Gamma$  has common degree  $r$ , then  $\Gamma$  is said to be  *$r$ -regular*. Furthermore,  $\Gamma$  is simply said to be *regular* if it is  $r$ -regular for some  $r \geq 0$ . A finite simple graph  $\Gamma$  of order  $n$  in which every pair of vertices are joined by an edge, i.e.,  $|V(\Gamma)| = n$  and  $|E(\Gamma)| = \binom{n}{2}$ , is called a *complete* graph. This is denoted  $\Gamma = K_n$ , and note that complete graphs are  $(n-1)$ -regular. 3-regular graphs are called *cubic* graphs and 4-regular graphs are sometimes called *quartic* graphs. The *minimum degree* of  $\Gamma$  is  $\delta(\Gamma) = \min\{\deg(x) : x \in \Gamma\}$  and the *maximum degree* of  $\Gamma$  is  $\Delta(\Gamma) = \max\{\deg(x) : x \in \Gamma\}$ .

Any alternating sequence  $W$  of (not necessarily distinct) vertices and edges in  $\Gamma$ ,

$$W := x_0 e_0 x_1 e_1 \cdots x_{k-1} e_{k-1} x_k, \quad e_i = \{x_i, x_{i+1}\} \quad 0 \leq i \leq k-1,$$

is called a *walk* (of length  $k$ ).  $x_0$  is called the *initial vertex* and  $x_k$ , the *terminal vertex*. Any walk with distinct vertices is called a *path*. If  $x_0 = x_k$ , the walk is said to be *closed*. A closed path of length  $k \geq 3$  is called a  *$k$ -cycle*, usually denoted  $C_k$ . A graph is *connected* if there exists a path between any two vertices. Determining how connected a graph is is a central topic in graph theory. We say  $\Gamma$  is  *$k$ -connected*, provided that we may delete any  $k-1$  or fewer vertices from  $\Gamma$  and it will remain connected. The greatest integer  $k$ , such that  $\Gamma$  is  $k$ -connected, is called the *vertex-connectivity*, or just *connectivity*, of  $\Gamma$ , and is denoted  $\kappa(\Gamma)$ . Any cycle is a connected 2-regular graph. The smallest cycle a graph can have is  $K_3$ , the triangle. The length of the smallest cycle in a graph  $\Gamma$  is called its *girth*, denoted  $g(\Gamma)$ .

A *subgraph*  $\Delta$  of a graph  $\Gamma$  is a graph where  $V(\Delta) \subseteq V(\Gamma)$  and  $E(\Delta) \subseteq E(\Gamma)$ . If  $V(\Delta) = V(\Gamma)$ ,  $\Delta$  is a *spanning subgraph* of  $\Gamma$ . A  *$k$ -factor* of  $\Gamma$  is a  $k$ -regular spanning subgraph of  $\Gamma$ . If  $V_0 \subseteq V(\Gamma)$ , then the subgraph of  $\Gamma$  *induced* by  $V_0$ , is defined to be the graph  $\Gamma[V_0]$  with vertex set  $V_0$ , and all edges  $xy \in E(\Gamma)$  with  $x, y \in V_0$ . The *independence number* of a graph  $\Gamma$ , denoted  $\alpha(\Gamma)$ , is the largest number of vertices, such that their induced subgraph forms a *clique*, i.e., a graph with no edges.

If  $E_0 \subseteq E(\Gamma)$ , and  $E_0 \neq \emptyset$ , then the subgraph of  $\Gamma$  *edge-induced* by  $E_0$ , is denoted  $\langle E_0 \rangle$ , and is defined to be the graph having as vertex set all vertices of  $\Gamma$  which are incident with at least one edge in  $E_0$ , and whose edge set is  $E_0$ . Given a graph  $\Gamma$ , a partition of  $E(\Gamma)$  into subsets so that

$$E(\Gamma) = E_1 \cup E_2 \cup \cdots \cup E_t$$

where  $|E_i| = |E_j|$  and  $E_i \cap E_j = \emptyset$  for all  $i \neq j$  is called an *isomorphic factorization* of  $\Gamma$  provided the  $t$  subgraphs edge-induced on the sets  $\langle E_i \rangle$ ,  $i = 1, 2, \dots, t$  are pairwise isomorphic. If  $\Gamma$  has an isomorphic factorization into subgraphs with  $k$  edges (each being isomorphic to the graph  $H$ ), we say  $\Gamma$  has a  *$k$ -isofactorization into the graph  $H$* . See Alspach, Dyer, Kreher [5] and Kreher-Westlund [26], for recent results in isofactorizations of circulant graphs.

Let  $\Gamma_1 = (V_1, E_1)$  and  $\Gamma_2 = (V_2, E_2)$  be graphs. The *cartesian product* (or *box product*) of  $\Gamma_1$  and  $\Gamma_2$ , denoted,  $\Gamma = \Gamma_1 \square \Gamma_2$  is the graph with vertex set  $V(\Gamma) = V_1 \times V_2 = \{(v_1, v_2) : v_1 \in V_1 \text{ and } v_2 \in V_2\}$ , and edge set

$$E(\Gamma) = \{\{(u_1, u_2), (u_1, v_2)\} : \{u_2, v_2\} \in E_2\} \cup \{\{(u_1, u_2), (v_1, u_2)\} : \{u_1, v_1\} \in E_1\}.$$

### 1.3 Hamilton Cycles

If  $\Gamma$  has order  $n$ , then any path of length  $n - 1$  (a path using every vertex once) is called a *Hamilton path*. A graph is called *hamilton-connected* (or *strongly hamiltonian*) if every two vertices are joined by a Hamilton path. Any closed walk using every edge once is called a *Hamilton cycle*. Any graph that contains at least one Hamilton cycle is said to be *hamiltonian*. Clearly, a graph must be connected to possibly be hamiltonian and any Hamilton cycle is a connected 2-factor. A bipartite graph with bipartition  $A$  and  $B$ , where  $|A| = |B|$ , is said to be *hamilton-laceable* if, for all  $a \in A$ ,  $b \in B$ , there exists a Hamilton path from  $a$  to  $b$ .

Hamilton cycles are named after Sir William Rowan Hamilton, an Irish mathematician who invented a puzzle called the *Icosian game*, that involves finding Hamilton cycles on the edge graph of the dodecahedron (see Figure 1.2). In general, there is no known good characterization for graphs that are hamiltonian, in part due to the fact that the decision problem of determining the existence of a Hamilton path (or cycle) in a graph is one of the classical NP-complete problems of computational complexity theory. A simple sufficiency condition for the existence of Hamilton cycles was proved by Dirac in 1952.

**Theorem 1.3.1** (Dirac [18]). *Every graph  $\Gamma$  of order  $n \geq 3$  where  $\delta(\Gamma) \geq n/2$  is hamiltonian.*

Dirac's theorem was generalized by Ore in 1960.

**Theorem 1.3.2** (Ore [46]). *Every graph,  $\Gamma$ , of order  $n \geq 3$  is hamiltonian if for all  $x, y \in V(\Gamma)$ ,*

$$\deg(x) + \deg(y) \geq n.$$

Given a graph  $\Gamma$  of order  $n$ , the *closure* of  $\Gamma$  is the graph with the same vertex set as  $\Gamma$  that is obtained from  $\Gamma$  by iteratively adding an edge  $\{x, y\}$  for all nonadjacent pairs of vertices  $x$  and  $y$  satisfying  $\deg(x) + \deg(y) \geq n$ , until no such pair remains. The closure is well-defined, as the order in which edges are added does not matter.

**Theorem 1.3.3** (Bondy-Chvátal [10]). *A simple graph of order  $n$  is hamiltonian if and only if its closure is also hamiltonian.*

Another, well-known result, proved in 1972 by Chvátal and Erdős, relates the independence number and vertex-connectivity to determine when a graph is hamiltonian.

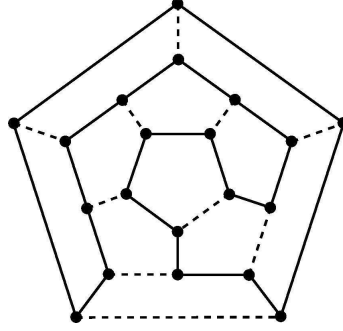
**Theorem 1.3.4** (Chvátal-Erdős [13]). *Every graph,  $\Gamma$ , of order at least three, is hamiltonian if*

$$\alpha(\Gamma) \leq \kappa(\Gamma).$$

Given a graph  $\Gamma = (V, E)$ , define the graph  $\Gamma^d = (V', E')$  to be the graph where  $V = V'$  and  $\{x, y\} \in E'$  if and only if the shortest path between  $x$  and  $y$  has length  $d$ .

**Theorem 1.3.5** (Fleischner [22, 23]). *If  $\Gamma$  is a 2-connected graph, then  $\Gamma^2$  is hamiltonian.*

An excellent survey of recent developments on the Hamilton Problem is Gould [24].



**Figure 1.2:** One solution to Hamilton's puzzle.

### 1.3.1 Hamilton decompositions

If a graph  $\Gamma$  is  $2k$ -regular, any partition of  $E(\Gamma)$  into  $k$  Hamilton cycles (if one exists) is called a *Hamilton decomposition*. If  $\Gamma$  is  $(2k + 1)$ -regular, a Hamilton decomposition is defined to be any partition of  $E(\Gamma)$  into  $k$  Hamilton cycles and a 1-factor (a 1-regular spanning subgraph of  $\Gamma$  or perfect matching). A well-studied general conjecture on sufficient conditions for a Hamilton decomposition is the Nash-Williams Conjecture.

**Nash-Williams Conjecture** [45]. Every  $k$ -regular graph with at most  $2k + 1$  vertices has a Hamilton decomposition.

In 1974, Kotzig proved that every cartesian product of two cycles,  $C_a \square C_b$ , has a Hamilton decomposition. The following 1982 theorem generalized this result.

**Theorem 1.3.6** (Aubert-Schneider [7]). *If  $\Gamma$  can be decomposed into two Hamilton cycles and  $C$  is a cycle, then  $\Gamma \square C$  can be decomposed into three Hamilton cycles.*

In 1990, Alspach, Bermond, and Sotteau generalized Kotzig's result to the cartesian product of any finite number of cycles.

**Theorem 1.3.7** (Alspach, Bermond, Sotteau [4]).  *$C_{\ell_1} \square C_{\ell_2} \square \cdots \square C_{\ell_t}$  has a Hamilton decomposition, for all integers  $\ell_i$ , where  $1 \leq i \leq t$ .*

In 1991, Stong provided the following generalization.

**Theorem 1.3.8** (Stong [50]). *If  $\Gamma_1$  and  $\Gamma_2$  are decomposable into  $n$  and  $m$  Hamilton cycles, respectively, with  $n \geq m$ , then  $\Gamma_1 \square \Gamma_2$  has a Hamilton decomposition if one of the following holds:*

1.  $m \leq 3n$ ,
2.  $n \geq 3$ ,
3.  $|V(\Gamma_1)|$  is even, or
4.  $|V(\Gamma_2)| \geq 6\lceil m/n \rceil - 3$



## 1.4 Cayley Graphs

Much of the material in this section can be found in [8]. A Cayley graph,  $\Gamma = (V, E)$ , is a graph whose vertex set is a group  $G$ . Although it is perfectly reasonable to take  $G$  to be of infinite order, we are primarily interested in a finite number of vertices. Let  $S \subseteq G \setminus \{1\}$  be a subset of non-identity elements of  $G$ , such that  $S = S^{-1}$ .  $S$  is called the *connection set* of  $\Gamma$ , and for brevity, we usually write  $S = \{s_1, s_2, \dots, s_k\}$  in place of  $S = \{s_1, s_1^{-1}, s_2, s_2^{-1}, \dots, s_k, s_k^{-1}\}$ . The edge set of  $\Gamma$  is defined to be

$$E(\Gamma) = \{\{x, y\} : yx^{-1} \in S\}.$$

The requirement that  $1 \notin S$  ensures there are no loops in  $\Gamma$ , and the requirement that  $S^{-1} = S$  ensures that  $\Gamma$  is undirected. A *Cayley digraph*, denoted  $\overrightarrow{\text{CAY}}(G, S)$ , is a graph with vertex set  $G$ , and arc set  $E = \{(x, y) : yx^{-1} \in S\}$ . If  $yx^{-1} = s \in S$ , the edge  $\{x, y\}$  is said to be *generated by*  $s$ . A subgraph  $H$  is generated by  $s$  if every edge in  $H$  is generated by  $s$ .  $\Gamma$  is a connected graph if and only if  $S$  generates  $G$ . Let  $|s|$  denote the additive order of the element  $s$ . Any involution  $s$ , an element of order 2, generates a 1-factor (or 1-regular spanning subgraph) and if  $|s| > 2$ ,  $s$  generates a 2-factor of  $G$ . If  $S = \{s_1, \dots, s_k\}$  and  $|s_i| > 2$  for all  $1 \leq i \leq k$ , then  $\Gamma$  is  $2k$ -regular. If  $G$  is a cyclic group, i.e.  $G \cong \mathbb{Z}_n$ , then  $\Gamma$  is called a *circulant graph*. The adjacency matrix of a circulant graph is a circulant matrix. Regarding the isomorphism problem for circulant graphs, the following is known:

**Theorem 1.4.1** (Turner [51]). *For any prime  $p$ ,  $\text{CAY}(\mathbb{Z}_p, S) \cong \text{CAY}(\mathbb{Z}_p, S') \Leftrightarrow S' = aS$ , for some  $a \in \mathbb{Z}_p^*$ .*

The family of  $k$ -bit binary cubes, or  $k$ -cubes used in the construction of reflected Grey codes are Cayley graphs. The underlying group is the elementary abelian 2-group  $\mathbb{Z}_2^k$  and  $S$  is the standard generating set,  $S = \{e_1, e_2, \dots, e_k\}$  where  $e_i := (e'_1, \dots, e'_k)$ ,  $e'_i = 1$ , and  $e'_j = 0$  for all  $j \neq i$ . Figure 1.3 illustrates the circulant graph  $\text{CAY}(\mathbb{Z}_{18}, \{2, 6, 9\})$  and a 4-bit binary cube  $\text{CAY}(\mathbb{Z}_2^4, \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\})$ .

The following well-known result places Cayley graphs inside the family of vertex-transitive graphs.

**Theorem 1.4.2.** *Every Cayley graph is vertex-transitive.*

*Proof.* For each  $g \in G$ , the map  $\varphi_g : G \rightarrow G$  defined by  $\varphi_g : x \mapsto xg$  is bijection on  $G$ . Also  $\varphi_g \in \text{AUT}(\text{CAY}(G, S))$  as

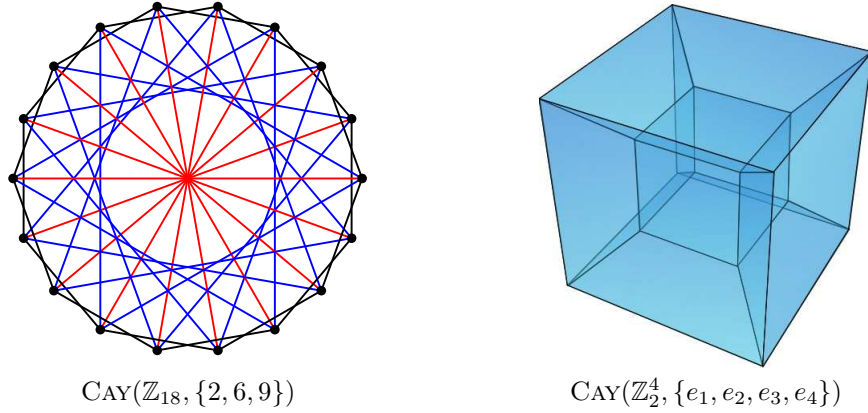
$$\{x, y\} \in E(\Gamma) \Leftrightarrow yx^{-1} = s \Leftrightarrow ygg^{-1}x = (yg)(xg)^{-1} = s \Leftrightarrow \{\varphi_g(x), \varphi_g(y)\} \in E(\Gamma)$$

The group  $G_R = \{\varphi_g : g \in G\}$  is a transitive subgroup of  $\text{AUT}(\Gamma)$  as  $\varphi_{x^{-1}y}(x) = y \forall x, y \in G$ . ■

The Petersen graph, shown in Figure 1.1, is vertex-transitive graph that is not the Cayley graph of any group. This is a consequence of the following result that characterizes all Cayley graphs.

**Theorem 1.4.3** (Sabidussi [48]). *A graph  $\Gamma$  is a Cayley graph if and only if  $\text{AUT}(\Gamma)$  contains a regular subgroup, i.e., a sharply 1-transitive subgroup.*

*Proof.* (Sketch). If  $\Gamma = \text{CAY}(G, S)$ , then  $R = \{\varphi_g : g \in G\} \leq \text{AUT}(\Gamma)$  is regular, for  $|G| = |R| \Rightarrow |R_g| = 1$ . Conversely, if  $G$  is a regular subgroup of  $\text{AUT}(\Gamma)$ , then label an arbitrary  $x \in V(\Gamma)$  with  $1 = 1_G$  and label  $y \in V(\Gamma)$  with  $g \in G$  such that  $x^g = y$ . Clearly,  $\Gamma$  is a Cayley graph on  $G$ . ■



**Figure 1.3:** A circulant graph and the 4-bit binary cube (hypercube)\*.

The Petersen graph has automorphism group  $\text{AUT}(P) \cong S_5$ , which possesses no transitive subgroup of order 10. See McKay-Praeger [44] for other vertex-transitive graphs that are not Cayley graphs. If  $\varphi$  is any automorphism of the group  $G$ , then  $\text{CAY}(G, S)$  and  $\text{CAY}(G, \varphi(S))$  are isomorphic. Additionally, the following is known.

**Theorem 1.4.4** (Sabidussi [49]). *Every vertex-transitive graph is the homomorphic image of a Cayley graph.*

**Theorem 1.4.5** (Folklore, Marušič [40]). *For any prime  $p$ , all vertex transitive graphs of order  $p$ ,  $p^2$ , and  $p^3$ , or  $2p$  (where  $p \equiv 3 \pmod{4}$ ) are Cayley graphs.*

### 1.4.1 Lovász Conjecture

Almost forty years ago, Lovász [36], Parsons, and others, posed the following research problem (though originally phrased in the negative): Which connected, vertex-transitive graphs have a Hamilton path?

**Lovász Conjecture.** *Every finite, connected, vertex-transitive graph has a Hamilton path.*

**Theorem 1.4.6** ([1, 12, 27, 29, 40, 41, 42, 43, 39, 51]). *Every connected, vertex-transitive graph of order  $kp$ ,  $p^k$ , or  $2p^2$ , where  $k \leq 4$ , and  $p$  a prime, is either hamiltonian, the Petersen graph, or the Coxeter graph. Furthermore, every connected, vertex-transitive graph of order  $5p$  or  $6p$  has a Hamilton path.*

This conjecture has generated a huge body of literature, and to date, there exists no known connected, vertex-transitive graph that does not possess a Hamilton path. Furthermore, of these, only four nontrivial vertex-transitive graphs exist that do not have a Hamilton cycle. They are the Petersen graph, the Coxeter graph, and the truncated Petersen and Coxeter graphs, all of which are hypohamiltonian. However, none of these are Cayley graphs. Thus, we have the following conjecture:

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\*The circulant graph of Figure 1.3 was drawn using *Mathematica*. The hypercube image is available for free use from *Wikipedia Commons* under the *GNU Free Documentation License*.

**Weak Lovász Conjecture** [36]. *Every non-trivial, connected, Cayley graph is hamiltonian.*

There are no known Cayley graphs which are non-hamiltonian. The Lovász conjecture is true for Cayley graphs on finite abelian groups, proved in [38], though it appeared earlier as a problem in [37]. We offer a proof of this important result to aid the reader.

**Theorem 1.4.7** (Lovász [37]). *If  $\Gamma$  is a connected graph, and  $\text{AUT}(\Gamma)$  contains an abelian, transitive subgroup, then  $\Gamma$  is hamiltonian.*

*Proof.* Let  $H \leq \text{AUT}(\Gamma)$  be abelian, and act transitively on  $V(\Gamma)$ . If  $h(x) = x$ , for some  $x \in V(\Gamma)$ , and  $h \in H$ , then  $\forall y \in V(\Gamma)$ , there exists  $g \in H$  such that  $g(x) = y$ , and so

$$h(y) = h(g(x)) = g(h(x)) = g(x) = y \Rightarrow h = 1.$$

Hence,  $|V(\Gamma)| = |H(x)| = [H : H_x] = |H|$ , and the action is regular. Consider all subgroups  $H' \leq H$  with the property that  $H'(x)$  forms a cycle. In particular, at least one such  $H'$  exists: take  $h' \in H$  with  $\{x, h'(x)\} \in E$ . Then  $H = \langle h' \rangle$  is a cycle. Let  $K \leq H$  be maximal with respect to this property,

$$K(x) := x, k_1(x), k_2(x), \dots, k_{d-1}(x), x.$$

We claim  $d = |V(\Gamma)|$ . If  $d < |V(\Gamma)|$  then there exists  $y \in V(\Gamma)$  such that  $h(x) = y$ , for some  $h \in H \setminus K$ , where  $\{k^i(x), y\} \in E(\Gamma)$ . Hence,  $\{x, k^{-i}(y)\} \in E$ . Let  $\ell \in H$  satisfy  $\ell(x) = k^{-i}(y)$ . Clearly, for all  $z \in V(\Gamma)$ ,  $\{z, \ell(z)\} \in E$ . Let  $\alpha$  be the minimum integer such that  $\ell^\alpha \in K$ . As  $H$  is abelian,  $L$  is a subgroup of  $H$ , where

$$L = K \cup \ell K \cup \ell^2 K \cup \dots \cup \ell^{\alpha-1} K.$$

Now,  $P := x, \ell(x), \ell^2(x), \dots, \ell^{\alpha-1}(x)$  is an  $\alpha$ -path, and  $P \sqcup K(x) \simeq P \sqcup C_d$  is subgraph of  $\Gamma$ , spanning  $L(x)$ , and is easily seen to be hamiltonian. This is a contradiction to  $K$  being maximal, thus  $\Gamma$  is hamiltonian. ■

The result for abelian groups is also a corollary of the well-known Chen-Quimpo Theorem, for every edge lies on a Hamilton cycle.

**Theorem 1.4.8** (Chen-Quimpo [11]). *Every connected Cayley graph, of degree at least three, on a finite abelian group, is hamilton-connected if not bipartite, or hamilton-laceable if bipartite.*

**Theorem 1.4.9** (Durnberger [20], Marušič [38], Keating-Witte [25]). *Every Cayley graph on a group  $G$  is hamiltonian if it is prime-power order, or the commutator subgroup  $[G, G] = \langle g^{-1}h^{-1}gh : g, h \in G \rangle$  is cyclic of prime-power order.*

**Theorem 1.4.10** (Dobson et al. [19]). *Every connected graph of order at least 3, with a transitive group of automorphisms, whose commutator subgroup is cyclic of prime-power order, has a Hamilton cycle, or is the Petersen graph.*

However, there do exist infinite families of Cayley digraphs that do not have a directed Hamilton cycle. For example, there even exist infinite families of circulant digraphs that are non-hamiltonian.

**Theorem 1.4.11** (Locke-Witte [35]). *There exists no Hamilton cycle in  $\overrightarrow{\text{CAY}}(\mathbb{Z}_{12k}, \{6k, 6k+2, 6k+3\})$  or in  $\overrightarrow{\text{CAY}}(\mathbb{Z}_{2k}, \{a, a+1, a+k\})$  if  $a+k$  is even, and  $\gcd(2k, a), \gcd(2k, a+k) > 1$ .*

The following result of Witte answers the question for Cayley digraphs of prime power order:

**Theorem 1.4.12** (Witte [54]). *Every connected Cayley digraph on a group of order  $p^\alpha$ , where  $p$  is a prime, and  $\alpha \geq 1$ , has a directed Hamilton cycle.*

For more recent developments on the Lovász conjecture, see Kutnar-Marušič [28] and Pak-Radoičić [47]. For surveys on hamiltonicity properties of Cayley graphs, see Curran-Gallian [14] and Witte-Gallian [55].

## 1.5 Alspach Conjecture

One implication of the Chen-Quimpo Theorem is that Cayley graphs on abelian groups are not only hamiltonian, but *flourish* with Hamilton cycles. Thus, Alspach posed the following question:

*Does every connected Cayley graph on an abelian group admit a Hamilton decomposition?*

In [2], Alspach made the following conjecture.

**Alspach Conjecture** [2]. *Every  $2k$ -regular connected Cayley graph on a finite abelian group has a decomposition into  $k$  edge-disjoint Hamilton cycles.*

This is trivially true if  $k = 1$ , for any such graph is already a Hamilton cycle. Bermond et al. proved the conjecture true when  $k = 2$ .

**Theorem 1.5.1** (Bermond et al. [9]). *Every 4-regular connected Cayley graph on a finite abelian group is decomposable into two Hamilton cycles.*

There are no known counterexamples to Alspach's conjecture, but even the case  $k = 3$  is still open. In 2006, Dean proved the following for circulant graphs.

**Theorem 1.5.2** (Dean [15, 16]). *If  $\Gamma = \text{CAY}(A, S)$  is a connected 6-regular circulant graph then  $\Gamma$  has a Hamilton decomposition if one of the following holds:*

1.  $|A|$  is odd, or
2. there exists  $s \in S$ , such that  $\langle s \rangle = A$ .

Let  $S := \{s_1, s_2, \dots, s_k\}$  generate a finite group  $A$ .  $S$  is said to be *minimal generating set* if for each  $1 \leq i \leq k$ , the element  $s_i \notin \langle s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_k \rangle$ . Likewise, the set  $S$  is said to be *strongly minimal* if for all  $1 \leq i \leq k$ , the element  $2s_i \notin \langle s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_k \rangle$ . Clearly, every strongly minimal generating set is also a minimal generating set. Liu has used induction on the order of a quotient group to prove the following general result.

**Theorem 1.5.3** (Liu [33, 34]). *Let  $S = \{s_1, \dots, s_k\}$  generate a finite abelian group  $A$ . Then  $\text{CAY}(A, S)$  has a Hamilton decomposition if one of the following holds:*

1. the order of  $A$  is odd, and  $S$  is a minimal generating set, or
2. the order of  $A$  is even, and  $S$  is a strongly minimal generating set.

**Corollary 1.5.4** (Liu [34]). *Let  $S$  be a minimal connection set of  $A$  having even order at least four. If  $\langle s \rangle$  has odd index, for all  $s \in S$ , then  $\text{CAY}(A, S)$  has a Hamilton decomposition.*

**Theorem 1.5.5** (Li et al. [30]). *Any connected Cayley graph on an abelian group of order  $p^2$  or  $pq$ , where  $p$  and  $q$  are odd primes, has a Hamilton decomposition.*

## 1.6 New Results

In Chapter 3, the following result is shown.

**Theorem 3.2.1** (Kreher, Liu, Westlund [53]). *Every connected, 6-regular, abelian Cayley graph of odd order has a Hamilton decomposition.*

In Chapter 4, the following results are shown.

**Theorem 4.1.5.** *If  $A = \langle s_2, s_3 \rangle$  is an abelian group of even order,  $|s_3| \geq 3$ , and  $|A|/2 = |s_1| \geq |s_2| \geq |s_3|$ , where at least one inequality is strict, then  $\text{CAY}(A, \{s_1, s_2, s_3\})$  has a Hamilton decomposition.*

**Corollary 4.1.6.** *Let  $\Delta = \text{CAY}(A/\langle s_3 \rangle, \{\overline{s_1}, \overline{s_2}\})$  be a quotient of  $\Gamma = \text{CAY}(A, \{s_1, s_2, s_3\})$  of order at least three. If  $\overline{s_2}$  generates a Hamilton cycle in  $\Delta$ , and  $\langle s_1 \rangle$  has index 2 in  $A$ , then  $\Gamma$  has a Hamilton decomposition.*

**Corollary 4.1.7.** *If  $\Gamma = \text{CAY}(\mathbb{Z}_{2m}, \{a, b, c\})$  is connected, 6-regular,  $|a| = m$ , and  $\gcd(2m, b, c) = 1$ , then  $\Gamma$  has a Hamilton decomposition.*

In Chapter 5, combining the computational results of the Appendix, the following results are shown.

**Theorem 5.4.1.** *If  $\Gamma = \text{CAY}(A, \{s_1, s_2, s_3\})$  is a connected, 6-regular, abelian Cayley graph of even order, and for some  $1 \leq i \leq 3$ ,  $\Delta_i = \text{CAY}(A/\langle s_i \rangle, \{\overline{s_{j_1}}, \overline{s_{j_2}}\})$  is 4-regular, and  $\Delta_i \not\cong \text{CAY}(\mathbb{Z}_3, \{1, 1\})$ , then  $\Gamma$  has a Hamilton decomposition.*

Alternatively, Theorem 5.4.1 may be stated as follows.

**Theorem 5.4.2** *If  $\Gamma = \text{CAY}(A, S)$  is a connected, 6-regular, abelian Cayley graph of even order, then  $\Gamma$  has a Hamilton decomposition if  $S$  has no involutions, and for some  $s \in S$ ,  $\text{CAY}(A/\langle s \rangle, \overline{S})$  is 4-regular, and of order at least 4.*

**Corollary 5.4.3.** *If  $\Gamma = \text{CAY}(A, \{s_1, s_2, s_3\})$  is a connected, 6-regular, abelian Cayley graph of even order, then  $\Gamma$  has a Hamilton decomposition if one of the following holds:*

- (a)  $s_1 \in \langle s_2, s_3 \rangle$ ,  $s_2 \in \langle s_1, s_3 \rangle$ , and  $[A : \langle s_3 \rangle] \geq 4$ , or
- (b)  $|s_1| \geq |s_2| > 2|s_3|$ , or
- (c)  $|s_1| \geq |s_2| > |s_3|$ , and either
  - i.  $|A| = (2k + 1)|s_3|$ , with  $k \geq 2$ , or
  - ii.  $|A| \geq 4|s_3|$  and  $|s_1|$  and  $|s_2|$  are odd.

## Chapter 2

# Pseudo-Cartesian Products

### 2.1 The Pseudo-Cartesian Product of Cycles

Let  $A_n = a_1 a_2 \cdots a_n a_1$  and  $B_m = b_1 b_2 \cdots b_m b_1$  denote cycles of length  $n$  and  $m$  respectively, where all subscripts are expressed modulo  $n$  and  $m$  respectively. The  $r$ -pseudo-cartesian product of two cycles is a central tool in finding Hamilton decompositions.

**Definition 2.1.1** (Liu [32]). For an integer  $0 \leq r < m$ , the  $r$ -pseudo cartesian product of  $A_n$  and  $B_m$ , denoted  $A_n \square_r B_m$ , is the simple graph with vertex set  $\{(a_i, b_j) : 1 \leq i \leq n, 1 \leq j \leq m\}$  and edge set consisting of horizontal and vertical edges.

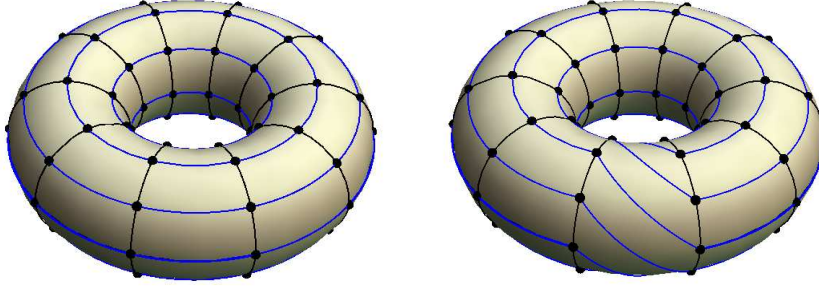
Horizontal edges:  $\{(a_i, b_j), (a_{i+1}, b_j)\}, \{(a_n, b_j), (a_1, b_{j+r})\} : 1 \leq i < n, 1 \leq j \leq m\}$

Vertical edges:  $\{(a_i, b_j), (a_i, b_{j+1})\}, \{(a_i, b_1), (a_i, b_m)\} : 1 \leq i \leq n, 1 \leq j \leq m-1\}$

The following is Remark 3.2 in [32]:

*Remark 2.1.2* (Liu [32]). If  $\gcd(r, m) = t$ , in  $A_n \square_r B_m$ , then the horizontal edges form a 2-factor  $H$  which consists of  $t$  cycles of length  $mn/t$  and any consecutive  $t$  rows of  $A_n \square_r B_m$  are on  $t$  different cycles of  $H$ . In other words, the horizontal edges in the  $b_i$  and  $b_j$ -rows are in the same cycle if and only if  $i \equiv j \pmod{t}$ . If  $H$  is given an orientation, so that each cycle in  $H$  becomes a directed cycle, then all horizontal edges in the rows contained in a particular cycle have the same direction.

The graph  $A_n \square_r B_m$  is embedded on a torus and drawn so that vertex  $(a_i, b_j)$  is in the  $i$ th column (i.e.  $\{(a_i, b_j) : 1 \leq j \leq m\}$ ) and  $j$ th row ( $\{(a_i, b_j) : 1 \leq i \leq n\}$ ). The parameter  $r$  is called the *jump number* for the graph because edges in the  $a_n$ -column jump down  $r$  rows (modulo  $m$ ) to connect to vertices in the  $a_1$ -column. Call the  $b_j$ -row *even* if  $j$  is even, and *odd* if  $j$  is odd, where  $1 \leq j \leq m$ . Similarly, call the  $a_i$ -column *even* or *odd*, depending on the parity of  $i$ , where  $1 \leq i \leq n$ . An example is shown in Figure 2.1. The following establishes a well-known connection between  $r$ -pseudo cartesian products and connected 4-regular Cayley graphs. We first recall concepts from



**Figure 2.1:**  $A_{10} \square B_8$  (left) and  $A_{10} \square_2 B_8$  (right)

Section 2 of [9].

**Theorem 2.1.3** (Bermond et al. [9]). *If  $\Gamma = \text{CAY}(A, \{s_1, s_2\})$  is a connected abelian Cayley graph,  $|s_1| \geq |s_2| = m \geq 3$ , and  $[A : \langle s_2 \rangle] = n \geq 3$ , then  $\Gamma \simeq A_n \square_r B_m$ , where  $ns_1 = rs_2$ .*

*Proof.* Without loss of generality,  $|s_1| \geq |s_2| = m > 2$ ,  $s_1 \neq \pm s_2$ , and if  $J := \langle s_2 \rangle$ , then  $\langle \overline{s_1} \rangle = A/J \Rightarrow |\overline{s_1}| = n > 2$ . Hence,  $ns_1 = rs_2$  for some  $0 \leq r < m$ . We claim  $\varphi : V(A_n \square_r B_m) \rightarrow \Gamma$  is an isomorphism between  $\Gamma$  and  $A_n \square_r B_m$ , where

$$\varphi : (a_i, b_j) \mapsto (i-1)s_1 + (j-1)s_2,$$

and  $1 \leq i \leq n$ ,  $1 \leq j \leq m$ . The vertical edges  $\{(a_i, b_j), (a_i, b_{j+1})\} \in E(A_n \square_r B_m)$  are preserved, for

$$\{\varphi((a_i, b_j)), \varphi((a_i, b_{j+1}))\} = \{(i-1)s_1 + (j-1)s_2, (i-1)s_1 + js_2\} \in E(\Gamma).$$

Likewise, for  $1 \leq i < n-1$  the horizontal edges  $\{(a_i, b_j), (a_{i+1}, b_j)\}$  are preserved, for

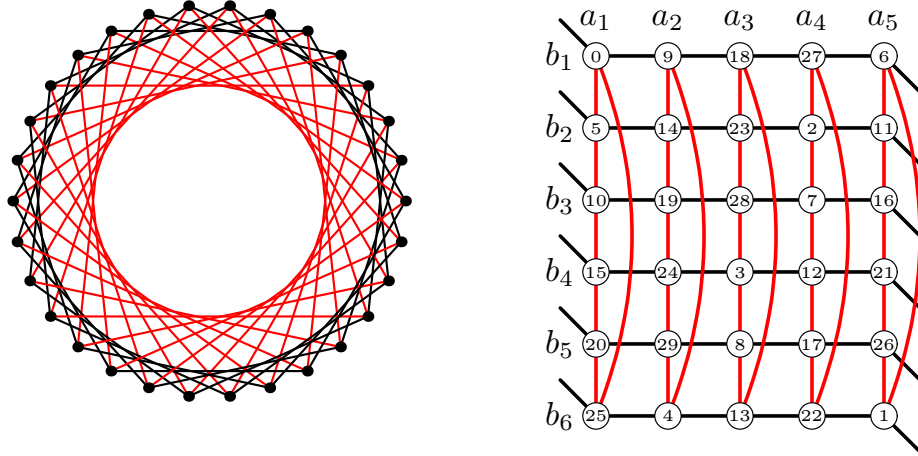
$$\{\varphi((a_i, b_j)), \varphi((a_{i+1}, b_j))\} = \{(i-1)s_1 + (j-1)s_2, is_1 + (j-1)s_2\} \in E(\Gamma).$$

Finally, the *jump-edges*,  $\{(a_n, b_j), (a_1, b_{j+r})\}$  are also preserved because

$$\{\varphi((a_n, b_j)), \varphi((a_1, b_{j+r}))\} = \{(n-1)s_1 + (j-1)s_2, (j+r-1)s_2\},$$

and  $(j+r-1)s_2 = ns_1 + (j-1)s_2 \Rightarrow ns_1 + (j-1)s_2 - ((n-1)s_1 + (j-1)s_2) = s_1$ . The edges generated by  $s_1$  form the horizontal 2-factor  $H$  in Remark 2.1.2, and so  $t = \gcd(r, m) = [G : \langle s_1 \rangle]$ . ■

**Example 2.1.4.** Let  $\Gamma = \text{CAY}(\mathbb{Z}_{30}, \{s_1 = 9, s_2 = 5\})$ . As  $\gcd(30, 5, 9) = 1$ ,  $|s_2| = 30/\gcd(5, 30) = 6$ , and  $|s_1| = 30/\gcd(9, 30) = 10$ ,  $\Gamma$  is connected and 4-regular. Let  $J = \langle 5 \rangle$ . Now,  $n = |\mathbb{Z}_{30} : J| = 6$  and  $r$  satisfies  $5r = 5(9) = 45 \equiv 15 \pmod{30} \Rightarrow r = 3$ . By Theorem 2.1.3,  $\Gamma \simeq A_6 \square_3 B_6$  where the isomorphism is  $f : (a_i, b_j) \mapsto 9(i-1) + 5(j-1) \pmod{30}$ , for  $1 \leq i \leq 6$  and  $1 \leq j \leq 6$ . The two graphs are shown in Figure 2.2.

Figure 2.2:  $\text{Cay}(\mathbb{Z}_{30}, \{5, 9\}) \simeq A_5 \square_3 B_6$ .

## 2.2 Edge Color-Switches

**Definition 2.2.1.** Color the vertical edges of  $A_n \square_r B_m$  red, and the horizontal edges black. For integers  $i$  and  $j$ , where  $1 \leq i < n$  and  $1 \leq j \leq m$ , we define an  $\{a_i, a_{i+1}, b_j, b_{j+1}\}$ -color-switch as an operation that interchanges the color of the edges

$$\{\{(a_i, b_j), (a_{i+1}, b_j)\}, \{(a_i, b_{j+1}), (a_{i+1}, b_{j+1})\}\}$$

with the color of the edges

$$\{\{(a_i, b_j), (a_i, b_{j+1})\}, \{(a_{i+1}, b_j), (a_{i+1}, b_{j+1})\}\}.$$

Similarly, for an integer  $1 \leq j \leq m$ , an  $\{a_n, a_1, b_j, b_{j+1}\}$ -color-switch interchanges the color of the edges

$$\{\{(a_n, b_j), (a_1, b_{j+r})\}, \{(a_n, b_{j+1}), (a_1, b_{j+1+r})\}\}$$

with the color of the edges

$$\{\{(a_n, b_j), (a_n, b_{j+1})\}, \{(a_1, b_{j+r}), (a_1, b_{j+1+r})\}\}.$$

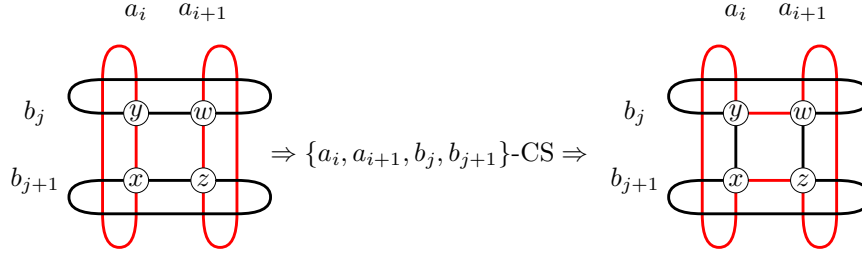
For brevity, we shall denote this as  $\{a_i, a_{i+1}, b_j, b_{j+1}\}$ -CS. A *color-switching configuration*, or *CS-configuration*, is a set  $\{X_i\}_{i=1}^d$ , of color-switches which are edge-disjoint.

The following remark combines Facts 3.10 and 3.11 in [32]:

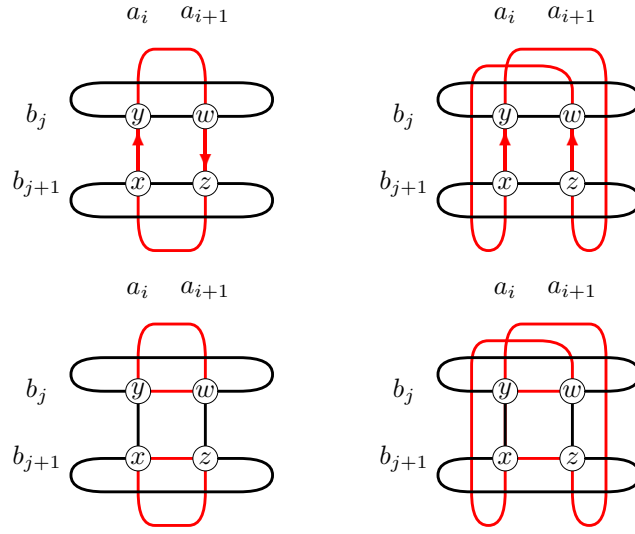
*Remark 2.2.2* (Liu [32]). If the edges  $\{x, y\}$  and  $\{z, w\}$  lie on vertex-disjoint cycles of color  $c$  in  $A_n \square_r B_m$ , then applying an  $\{a_i, a_{i+1}, b_j, b_{j+1}\}$ -CS will join the two cycles into a single cycle of color  $c$ . If  $\{x, y\}$  and  $\{z, w\}$  lie on a cycle  $C$ , of length  $\delta$  and color  $c$ , and are separated by at least two edges, then applying an  $\{a_i, a_{i+1}, b_j, b_{j+1}\}$ -CS will produce a cycle also having length  $\delta$  and color  $c$ , if and only if, upon making  $C$  a directed cycle,  $C$  has  $x, y, z, w$  as a subsequence of vertices. See Figure 2.4.

CS-configurations have been used to obtain the following result.





**Figure 2.3:** A red-black color-switch from Definition 2.2.1.



**Figure 2.4:** A red cycle-preserving color-switch from Remark 2.2.2.

**Theorem 2.2.3** (Fan et al. [21]). *For all  $n, m \geq 3$ , then graph  $A_n \square_r B_m$  has a Hamilton decomposition.*

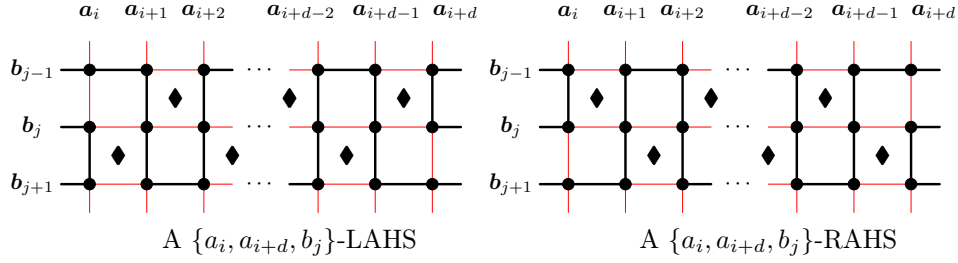
**Definition 2.2.4.** For  $d = 2d' \geq 2$ , a *left-alternating horizontal switch*, denoted  $\{a_i, a_{i+d}, b_j\}$ -LAHS, is the CS-configuration consisting of the  $d$  color-switches,

$$\{\{a_{i+2x}, a_{i+1+2x}, b_j, b_{j+1}\}\text{-CS}, \{a_{i+1+2x}, a_{i+2+2x}, b_{j-1}, b_j\}\text{-CS} : 0 \leq x \leq (d-2)/2\}.$$

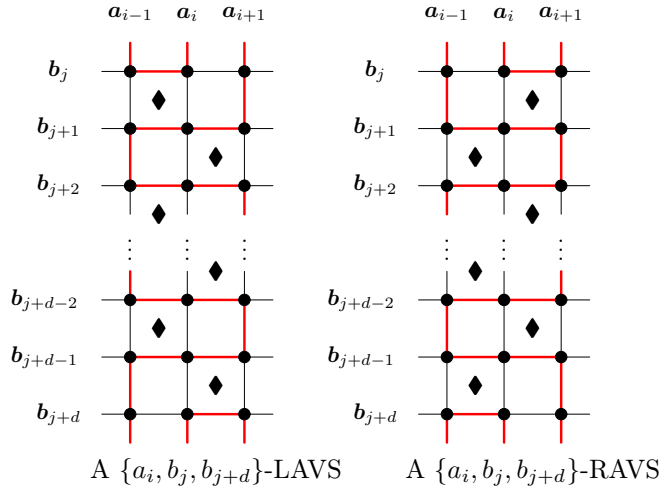
Likewise, a *right-alternating horizontal switch*, denoted  $\{a_i, a_{i+d}, b_j\}$ -RAHS, is the CS - configuration consisting of the  $d$  color-switches,

$$\{\{a_{i+2x}, a_{i+1+2x}, b_{j-1}, b_j\}\text{-CS}, \{a_{i+1+2x}, a_{i+2+2x}, b_j, b_{j+1}\}\text{-CS} : 0 \leq x \leq (d-2)/2\}.$$

Note that if the vertical edges in the  $a_j$ -columns for  $i \leq j \leq i+d$  are in different cycles, then applying  $\{a_i, a_{i+d}, b_j\}$ -LAHS or  $\{a_i, a_{i+d}, b_j\}$ -RAHS will join all those cycles together, by Remark 2.2.2. Furthermore, if the horizontal edges in the  $b_k$ -rows where  $j \leq k \leq j+1$  are in different cycles,



**Figure 2.5:** Left and right-alternating horizontal switches (LAHS and RAHS) of Definition 2.2.4.



**Figure 2.6:** Left and right-alternating vertical switches (LAVS and RAVS) of Definition 2.2.5.

then a left or right-alternating switch will join the three horizontal cycles together.

**Definition 2.2.5.** If  $d = 2d' \geq 2$ , then a *left-alternating vertical switch*, denoted  $\{a_i, b_j, b_{j+d}\}$ -LAVS, is the CS-configuration consisting of the  $d$  color-switches,

$$\{\{a_{i-1}, a_i, b_{j+2x}, b_{j+1+2x}\}\text{-CS}, \{a_i, a_{i+1}, b_{j+1+2x}, b_{j+2+2x}\}\text{-CS} : 0 \leq x \leq (d-2)/2\}.$$

Likewise, a *right-alternating vertical switch*, denoted  $\{a_i, b_j, b_{j+d}\}$ -RAVS, is the CS - configuration consisting of the  $d$  color-switches,

$$\{\{a_i, a_{i+1}, b_{j+2x}, b_{j+1+2x}\}\text{-CS}, \{a_{i-1}, a_i, b_{j+1+2x}, b_{j+2+2x}\}\text{-CS} : 0 \leq x \leq (d-2)/2\}.$$

**Definition 2.2.6.** Changing a RAVS (or RAHS) to a LAVS (or LAHS) or vice versa, for a fixed set of parameters, will be called a *color-switch reflect*.

The remainder of this chapter outlines color-switching configurations that will be used throughout

the rest of this thesis. Apply a 2-coloring to  $A_n \square_r B_m$ , with the vertical edges colored red, and the horizontal and jump edges colored black.

**Theorem 2.2.7** (Liu [32, 34]). *Suppose there are  $t = \gcd(r, m)$  horizontal cycles in  $A_n \square_r B_m$  with the horizontal cycles colored  $c_1$  and the vertical cycles colored  $c_2$ .*

(a) *If  $t = 2k + 1 \geq 3$ , then applying a  $\{a_i, b_{1+\ell}, b_{t+\ell}\}$ -LAVS or RAVS to  $A_n \square_r B_m$  for any  $2 \leq i \leq n-1$  and any integer  $0 \leq \ell \leq m-1$  will produce a Hamilton cycle  $C$  of color  $c_1$ , and join the vertical  $c_2$ -colored cycles in the  $a_{i-1}$ ,  $a_i$ , and  $a_{i+1}$ -columns into one cycle  $C'$ . Furthermore, applying an*

$$\{a_{i+1}, a_{i+2}, b_{t+\ell}, b_{t+1+\ell}\}\text{-CS}$$

*preserves the Hamilton cycle and joins the  $a_{i+2}$ -column to  $C'$ .*

(b) *If  $t = 2k \geq 4$ , then applying a  $\{a_i, b_{1+\ell}, b_{t-1+\ell}\}$ -LAVS or RAVS and an*

$$\{a_{i+1}, a_{i+2}, b_{t-1+\ell}, b_{t+\ell}\}\text{-CS}$$

*to  $A_n \square_r B_m$  for any  $2 \leq i \leq n-2$  and any integer  $0 \leq \ell \leq m-1$  will produce a Hamilton cycle of color  $c_1$ , and join the vertical  $c_2$ -colored cycles in the  $a_{i-1}$ ,  $a_i$ ,  $a_{i+1}$ , and  $a_{i+2}$ -columns into one cycle.*

The justification for Theorem 2.2.7(a) is that upon applying an  $\{a_i, b_1, b_t\}$ -RAVS or -LAVS to  $A_n \square_r B_m$ , we may orient the resulting Hamilton cycle so that it becomes a directed cycle. Now, all edges in a particular  $b_j$ -row have the same direction, so we may speak of the *direction of the  $b_j$ -row*. After the alternating switching configuration, the  $b_j$  and  $b_{j+1}$ -rows have opposite direction for  $1 \leq j \leq t-1$ . As  $t$  is odd, the  $b_1$  and  $b_t$ -rows have the same direction. Furthermore, the  $b_1, b_{t+1}, b_{2t+1}, \dots, b_{zt+1}$ -rows all have the same direction, as they were in the same cycle initially. For the same reason, the  $b_t, b_{2t}, \dots, b_{jt}, \dots, b_{kt} = b_m$ -rows, and by extension, the  $b_1$  and  $b_m$ -rows, have the same direction. By Remark 2.2.2, we may apply either an  $\{a_{i+1}, a_{i+2}, b_t, b_{t+1}\}$ -CS or an  $\{a_{i+1}, a_{i+2}, b_m, b_1\}$ -CS to preserve the Hamilton cycle. Thus, Theorem 2.2.7 generalizes as follows.

**Theorem 2.2.8.** *Suppose there are  $t = \gcd(r, m)$  horizontal cycles in  $A_n \square_r B_m$  colored  $c_1$  and  $n$  vertical cycles colored  $c_2$ . Let  $m = kt$ , and  $i, \ell, z, d$  be integers satisfying  $2 \leq i < n$ ,  $0 \leq \ell < m$ ,  $i < z \leq n$ , and  $1 \leq d \leq k$ .*

(a) *If  $t = 2k + 1 \geq 3$ , then the application of an  $\{a_i, b_{1+\ell}, b_{t+\ell}\}$ -LAVS or RAVS to  $A_n \square_r B_m$  will produce a Hamilton cycle,  $C$ , of color  $c_1$ , and join the vertical  $c_2$ -colored cycles in the  $a_{i-1}$ ,  $a_i$ , and  $a_{i+1}$ -columns into one cycle  $C'$ . Furthermore, applying an*

$$\{a_z, a_{z+1}, b_{dt+\ell}, b_{dt+1+\ell}\}\text{-CS}$$

*will preserve  $C$  and connect the vertical edges in the  $a_z$  and  $a_{z+1}$ -columns into one cycle.*

(b) *If  $t = 2k \geq 4$ , the application of an  $\{a_i, b_{1+\ell}, b_{t-1+\ell}\}$ -LAVS or -RAVS to  $A_n \square_r B_m$  and either an*

$$\{a_z, a_{z+1}, b_{dt-1+\ell}, b_{dt+\ell}\}\text{-CS or an } \{a_z, a_{z+1}, b_{m+\ell}, b_{1+\ell}\}\text{-CS}$$

*will produce a Hamilton cycle of color  $c_1$ .*

**Lemma 2.2.9** (Liu [34]). *If there are  $t = \gcd(r, m) = 2k \geq 2$  horizontal cycles in  $A_n \square_r B_m$  colored  $c_1$  and  $n$  vertical cycles colored  $c_2$ , then by switching the colors of the edge sets  $E_1$  and  $E_2$ , where*

$$E_1 = \{(a_1, b_{2j-1})(a_1, b_{2j}) : 1 \leq j \leq m/2\} \cup \{(a_n, b_{2j})(a_n, b_{2j+1}) : 1 \leq j \leq m/2\},$$

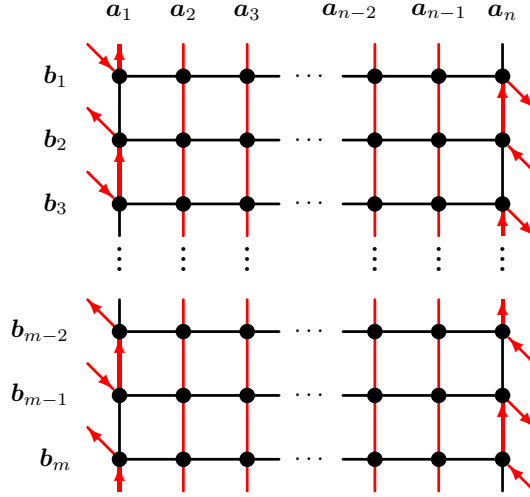


Figure 2.7: The CS-configuration of Lemma 2.2.9.

and  $E_2 = \{(a_n, b_i)(a_1, b_{i+r}) : 1 \leq i \leq m\}$  in  $A_n \square_r B_m$ , we obtain a Hamilton cycle of color  $c_1$  and a cycle, colored  $c_2$ , consisting of the vertices in the  $a_1$  and  $a_n$ -columns. Furthermore, if the  $c_2$ -colored cycle is given an orientation, all included vertical edges have the same direction. (See Figure 2.7.)

The following simple observation is formulated as a lemma.

**Lemma 2.2.10.** *If the CS-configuration of Lemma 2.2.9 is applied to  $A_n \square_r B_m$ , for  $t$  even, then upon applying an  $\{a_i, a_{i+d}, b_{2j+1}\}$ -RAHS, for some  $d \geq 2$  even, and  $j \geq 1$ , the  $c_1$ -colored Hamilton cycle is preserved.*

**Lemma 2.2.11** (Liu [34]). *If  $n \geq 6$  and  $t = \gcd(r, m) = 2k$ , apply an  $\{a_i, a_{i+1}, b_i, b_{i+1}\}$ -CS to  $A_n \square_r B_m$  where  $i = 1$  if  $k = 1$  and  $i = 1, 2, 3$  if  $k = 2$ . If  $k \geq 3$ , let  $i = 1, \dots, 6$  together with the CS-configuration*

$$\{\{a_1, a_2, b_j, b_{j+1}\}\text{-CS}, \{a_2, a_3, b_{j+1}, b_{j+2}\}\text{-CS}, \{a_3, a_4, b_{j+2}, b_{j+3}\}\text{-CS}, \{a_4, a_5, b_{j+3}, b_{j+4}\}\text{-CS}\},$$

where  $j \equiv 2 \pmod{4}$ , and

(a)  $6 \leq j \leq t - 4$ , if  $t \equiv 2 \pmod{4}$ , or

(b)  $6 \leq j \leq t - 6$ , if  $t \equiv 0 \pmod{4}$ . Additionally, apply an

$$\{a_1, a_2, b_{t-2}, b_{t-1}\}\text{-CS and an } \{a_2, a_3, b_{t-1}, b_t\}\text{-CS}.$$

The result is a black Hamilton cycle and a red cycle consisting of the vertical edges in the  $a_i$ -columns where  $1 \leq i \leq d$  and  $d = 2$  if  $k = 1$ ,  $d = 4$  if  $k = 2$ , and  $d = 6$  if  $k \geq 3$ .

**Definition 2.2.12.** If  $X := \{a_i, a_{i+1}, b_j, b_{j+1}\}$ -CS is a fixed color-switch in  $A_n \square_r B_m$ , we say  $X$  is  $c$ -incident to the  $a_i$  and  $a_{i+1}$ -columns and  $r$ -incident to the  $b_j$  and  $b_{j+1}$ -rows. Furthermore, applying  $X$  an even number of times leaves the edge-colors invariant. Thus, removing a color-switch means applying  $X$  any even number of times.

**Lemma 2.2.13.** *If  $n \geq 6$  and  $t = \gcd(r, m) = 2k \geq 6$ , then upon applying the CS-configuration of Lemma 2.2.11 to  $A_n \square_r B_m$ , we may remove the  $\{a_4, a_5, b_4, b_5\}$ -CS and  $\{a_5, a_6, b_5, b_6\}$ -CS, apply an  $\{a_4, a_5, b_5, b_6\}$ -CS and an  $\{a_5, a_6, b_4, b_5\}$ -CS, and preserve the black Hamilton cycle and the red cycle on the  $a_i$ -columns, where  $1 \leq i \leq 6$ .*

*Proof.* As the order in which any set of color-switches are applied is invariant, we may view the new CS-configuration as having first applied an  $\{\{a_i, a_{i+1}, b_i, b_{i+1}\}\text{-CS} : 1 \leq i \leq 3\}$ , and then applying an  $\{a_4, a_5, b_5, b_6\}$ -CS and an  $\{a_5, a_6, b_4, b_5\}$ -CS to  $A_n \square_r B_m$ . At this point, it is clear, by Remark 2.2.2, that a  $c_2$ -colored cycle is formed on the  $a_i$ -columns, where  $1 \leq i \leq 6$ . Each successive pair of color-switches as we move down the  $b_j$ -rows first breaks the  $c_2$ -colored cycle into two cycles, and then rejoins the two cycles again. By Remark 2.2.2, the result follows. ■

**Definition 2.2.14** (Fan et al. [21]). Let  $X_1, X_2, \dots, X_k$ , where  $k \geq 3$  be a set of edge-disjoint color switches in  $A_n \square_r B_m$ . Call  $\{X_{k-1}, X_k\}$  a *good pair* if  $X_{k-2}$  is the right of  $X_i$  for all  $i \leq k-3$ ,  $X_{k-1}$  is to the right of  $X_{k-2}$ ,  $X_k$  is to the right of  $X_{k-1}$ , and there exists a positive integer  $y$  such that  $X_{k-2}$  and  $X_k$  are r-incident to the  $b_y$  and  $b_{y+1}$ -rows, and  $X_{k-1}$  is r-incident to either the  $b_y$  and  $b_{y-1}$ -rows or the  $b_y$  and  $b_{y+1}$ -rows.

**Lemma 2.2.15** (Fan et al. [21]). *If  $X_1, X_2, \dots, X_k$ , where  $k \geq 3$  is a set of edge-disjoint color-switches in  $A_n \square_r B_m$ , and  $\{X_{k-1}, X_k\}$  form a good pair, then if after applying*

$$X_1, \dots, X_{k-2},$$

*there exists a  $c_1$ -colored Hamilton cycle, applying  $X_{k-1}$  and  $X_k$  will preserve this cycle.*

The most common usage of Lemma 2.2.15 is the application of a LAHS or a RAHS, as these form color-switches that are pairwise, good pairs.

**Corollary 2.2.16.** *If a  $c_1$ -colored Hamilton cycle is created by applying a CS-configuration to  $A_n \square_r B_m$  such that, for some  $1 \leq i < n$  and  $1 \leq j \leq m$ ,  $\{a_i, a_{i+1}, b_j, b_{j+1}\}$ -CS is a rightmost color-switch, then applying an  $\{a_{i+1}, a_{i+1+d}, b_j\}$ -RAHS or an  $\{a_{i+1}, a_{i+1+d}, b_{j+1}\}$ -LAHS, for some  $d \geq 2$  even, will preserve the Hamilton cycle. Furthermore, removing any two consecutive color-switches in the RAHS or LAHS will preserve the Hamilton cycle.*

The following type of color-switch will be used heavily in Chapter 4.

**Definition 2.2.17.** If  $\Gamma$  is a graph that contains  $A_n \square_r B_m$  as a spanning subgraph, and for some integers

$$C := (a_i, b_j)(a_i, b_{j+1})(a_k, b_{\ell+1})(a_k, b_\ell)(a_i, b_j)$$

is a 4-cycle of  $\Gamma$  with  $(a_i, b_j)(a_i, b_{j+1})$  and  $(a_k, b_\ell)(a_k, b_{\ell+1})$  colored  $c_2$  and  $(a_i, b_j)(a_k, b_\ell)$  and  $(a_i, b_{j+1})(a_k, b_{\ell+1})$  colored  $c_1$ , then an  $\{a_i, a_k, b_j, b_\ell\}$ -vertical oblique color-switch, or -VOCS, is a color-switch that interchanges the colors of edges in  $C$ . Likewise, an  $\{a_i, a_k, b_j, b_\ell\}$ -horizontal oblique color-switch, or -HOCS, in  $A_n \square_r B_m$  is a color-switch that interchanges the colors of edges in the 4-cycle,

$$(a_i, b_j)(a_{i+1}, b_j)(a_k, b_\ell)(a_{k+1}, b_\ell)(a_i, b_j),$$

where  $1 \leq i \neq k \leq n$  and  $1 \leq j \neq \ell \leq m$ .

Note, if  $k = i + 1$  and  $\ell = j + 1$ , in Definition 2.2.17, an  $\{a_i, a_k, b_j, b_\ell\}$ -VOCS or -HOCS is an  $\{a_i, a_{i+1}, b_j, b_{j+1}\}$ -CS.

## Chapter 3

# Hamilton Decompositions for Graphs of Odd Order

### 3.1 Lifting to the 6-Regular Case

This chapter offers a proof of Alspach's conjecture for the odd order, 6-regular case, hence generalizing Theorem 1.5.2. A proof of this result was published by Westlund, Liu, Kreher [53] in 2009, though we re-present the proof (using more unifying notation) because the technique used will help to understand many of the constructions in Chapter 5. In this section, preliminary theorems and definitions from [21, 32, 33, 34] are presented, and where needed, proofs are supplied for clarity. We first recall the fundamental definition of a quotient graph.

**Definition 3.1.1.** If  $\Gamma = \text{CAY}(A, S)$  has connection set  $S = \{s_1, \dots, s_k\}$ , and  $J := \langle s_k \rangle$ , then the Cayley graph  $\Delta = \text{CAY}(A/J, \bar{S})$ , where  $\bar{S} = \{\bar{s}_i : i = 1, \dots, k-1\}$  and  $\bar{s}_i = s_i + J$ , is called a *quotient graph* of  $\Gamma$ . If  $\Gamma$  is connected, then  $\Delta$  is connected.

**Definition 3.1.2.** If  $\{\bar{x}, \bar{y}\} \in E(\Delta)$ , where  $\bar{x} - \bar{y} = \bar{s}_i$ , then the set

$$L_{\Delta}\{\bar{x}, \bar{y}\} = \{\{u, v\} : \bar{u} = \bar{x}, \bar{v} = \bar{y}, u - v = s_i\}$$

is called the *lift*  $\{\bar{x}, \bar{y}\}$ . Furthermore, given a subgraph  $\bar{F}$  of  $\Delta$ , let  $F$  be the subgraph of  $\Gamma$  that is induced on the lifts of all edges of  $\bar{F}$ . Then  $F$  is called the *lift of the subgraph  $\bar{F}$* , or we say  $F$  is the *subgraph that  $\bar{F}$  lifts to*.

Edge-disjoint subgraphs of  $\Delta$  lift to edge-disjoint subgraphs of  $\Gamma$ . Lifting Hamilton cycles in quotient graphs is by no means a new idea, and many approaches can be taken, e.g., Alspach [3]. For the

benefit of the reader, we now present proofs of the following two important results, which are Lemma 3.7 and Corollary 3.9 in [32], respectively.

**Lemma 3.1.3** (Liu [32]). *If  $\Gamma = \text{CAY}(A, \{s_1, s_2, s_3\})$  is a connected Cayley graph on an abelian group  $A$ , then any Hamilton cycle  $\overline{H}$  of  $\Delta = \text{CAY}(A/\langle s_3 \rangle, \{\overline{s_1}, \overline{s_2}\})$  lifts to a 2-factor  $H$  of  $\Gamma$ . Furthermore, the union of  $H$  and the 2-factor generated by  $s_3$  is an  $r$ -pseudo-cartesian product.*

*Proof.* Let  $|s_3| = m$ ,  $n = |A : \langle s_3 \rangle|$ ,  $A/\langle s_3 \rangle = \{\overline{g_1}, \dots, \overline{g_n}\}$ , where  $\overline{g_1} = \overline{0}$ . Suppose that  $\overline{H}$  is a Hamilton cycle of  $\Delta$ , where

$$\overline{H} = \overline{g_{\pi(1)}}, \overline{g_{\pi(2)}}, \dots, \overline{g_{\pi(n)}}, \overline{g_{\pi(1)}}, \text{ for some } \pi \in \text{SYM}(n).$$

For all  $1 \leq k \leq n$ , without loss of generality,  $\overline{g_{\pi(k)}} - \overline{g_{\pi(k+1)}} = \overline{s_j}$ , for some  $j \in \{1, 2\}$ . Thus, for  $1 \leq k \leq n-1$ , there exist integers  $x_k$  and  $y_k$  such that

$$(g_{\pi(k)} + x_k s_3) - (g_{\pi(k+1)} + y_k s_3) = s_j.$$

Let  $a_{\pi(k)} := g_{\pi(k)} + x_k s_3$  and  $a_{\pi(k+1)} := g_{\pi(k+1)} + y_k s_3$ . Hence,

$$L_{\Delta}\{\overline{g_{\pi(k)}}, \overline{g_{\pi(k+1)}}\} = \{\{a_{\pi(k)} + \ell s_3, a_{\pi(k+1)} + \ell s_3\} : 0 \leq \ell < m\}.$$

Next, as  $\overline{g_{\pi(1)}} = \overline{a_{\pi(1)}}$  and  $\overline{g_{\pi(n)}} = \overline{a_{\pi(n)}}$ , there exist integers  $q_1$  and  $q_2$ , for which

$$(a_{\pi(n)} + q_1 s_3) - (a_{\pi(1)} + q_2 s_3) = s_j \Rightarrow a_{\pi(n)} - (a_{\pi(1)} + (q_2 - q_1) s_3) = s_j.$$

Let  $r = q_2 - q_1$ , so that  $\{a_{\pi(n)}, a_{\pi(1)} + r s_3\} \in E(\Gamma)$ . Hence,

$$L_{\Delta}\{\overline{g_{\pi(n)}}, \overline{g_{\pi(1)}}\} = \{\{a_{\pi(n)} + \ell s_3, a_{\pi(1)} + (r + \ell) s_3\} : 0 \leq \ell < m\}.$$

Thus, a path of length  $n$  is formed:

$$P_n^j = a_{\pi(1)} + j r s_3, a_{\pi(2)} + j r s_3, \dots, a_{\pi(n)} + j r s_3, a_{\pi(1)} + (j+1) r s_3,$$

and the union of these paths

$$\bigcup_{j=0}^{d-1} P_n^j,$$

where  $d-1$  is the integer such that  $d r s_3 = 0$ , forms a cycle of length  $dn$ . Hence,  $dr = \text{lcm}(m, r)$  and so  $d = m / \gcd(r, m)$ . It follows that the lift of  $\overline{H}$ , denoted  $H$ , is a 2-factor of  $\Gamma$ , where

$$H = \bigcup_{j=1}^n L_{\Delta}\{\overline{g_{\pi(j)}}, \overline{g_{\pi(j+1)}}\}.$$

Futhermore,  $F_j := a_{\pi_i(j)}, a_{\pi_i(j)} + s_3, \dots, a_{\pi_i(j)} + (m-1)s_3, a_{\pi_i(j)}$  is a cycle for each  $j = 1, 2, \dots, n$ . Thus,  $F = \bigcup_{j=1}^n F_j$  is 2-factor of  $\Gamma$ . Viewing the edges of  $H$  as horizontal, and the edges of  $F$  as vertical, let  $\theta : V(H \cup F) \rightarrow V(A_n \square_r B_m)$  be defined by

$$\theta : a_i + (j-1)s_3 \mapsto (a_i, b_j) \text{ for } 1 \leq i \leq n \text{ and } 1 \leq j \leq m.$$

Clearly,  $H \cup F = A_n \square_r B_m$ , where  $A_n = a_{\pi(1)} a_{\pi(2)} \cdots a_{\pi(n)} a_{\pi(1)}$ . ■

We now recall an important class of graphs, established in [32], and discuss their relationship with

quotient graphs of Cayley graphs.

**Definition 3.1.4** (Liu [32]). For  $m, n \geq 3$ , let a  $D(3, m, n)$ -graph be a 6-regular graph  $G = (V, E)$  satisfying:

1.  $V(G) = \{(a_i, b_j) : 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}$ , and
2.  $E(G)$  can be partitioned into the three sets  $F$ ,  $H_1$ , and  $H_2$ , where

$$\begin{aligned} F &= \{(a_i, b_j), (a_i, b_{j+1})\}, \{(a_i, b_1), (a_i, b_m)\} : 1 \leq i \leq n, 1 \leq j < m\}, \\ H_1 &= \{(a_{\pi_1(i)}, b_j), (a_{\pi_1(i+1)}, b_j)\}, \{(a_{\pi_1(n)}, b_j), (a_{\pi_1(1)}, b_{j+r_1})\} : 1 \leq i < n, 1 \leq j \leq m\}, \\ H_2 &= \{(a_{\pi_2(i)}^2, b_j), (a_{\pi_2(i+1)}^2, b_j)\}, \{(a_{\pi_2(n)}^2, b_j), (a_{\pi_2(1)}^2, b_{j+r_2})\} : 1 \leq i < n, 1 \leq j \leq m\}, \end{aligned}$$

where  $r_1$  and  $r_2$  are integers, and for  $k = 1, 2$ ,  $0 \leq r_k < m$ ,  $\pi_k \in \text{SYM}(n)$ , and  $(a_i^2, b_j) = (a_t, b_{j+h_t})$  for some integer  $h_t$ , where  $0 \leq h_t < m$ .

*Remark 3.1.5* (Liu [32]).  $D(3, m, n)$ -graphs can be viewed as two pseudo-cartesian products that share a common vertical 2-factor. Indeed,  $F$ ,  $H_1$ , and  $H_2$  are each 2-factors,  $H_1 \cup F \simeq A_n^{(1)} \square_{r_1} B_m$ , and  $H_2 \cup F \simeq A_n^{(2)} \square_{r_2} B_m$ , with the edges of  $F$  vertical, the edges of  $H_j$  horizontal, and for  $k = 1, 2$ ,

$$A_n^{(k)} := a_{\pi_k(1)}^2 a_{\pi_k(2)}^2 \cdots a_{\pi_k(n)}^2 a_{\pi_k(1)}^2.$$

When the labeling is defined so that  $\pi_1 = (1)$ , we will write  $A_n$  in place of  $A_n^{(1)}$ . For the remainder of this thesis, whenever a  $D(3, m, n)$ -graph is discussed, the edges in  $F$  will be colored red, the edges in  $H_1$  colored blue, and the edges in  $H_2$  colored black (e.g., see Figure 3.1).

**Theorem 3.1.6** (Liu [32]). If  $\Gamma = \text{CAY}(A, \{s_1, s_2, s_3\})$  and if  $\Delta = \text{CAY}(A/\langle s_3 \rangle, \{\overline{s_1}, \overline{s_2}\})$  can be decomposed into two Hamilton cycles,

$$\overline{H_1} := \overline{a_{\pi_1(1)}}, \overline{a_{\pi_1(2)}}, \dots, \overline{a_{\pi_1(n)}}, \overline{a_{\pi_1(1)}} \quad \text{and} \quad \overline{H_2} := \overline{a_{\pi_2(1)}}, \overline{a_{\pi_2(2)}}, \dots, \overline{a_{\pi_2(n)}}, \overline{a_{\pi_2(1)}},$$

then  $\Gamma$  is a  $D(3, m, n)$ -graph, with  $m = |s_3|$ ,  $n = [A : \langle s_3 \rangle]$ , where  $H_i$  is the 2-factor that  $\overline{H_i}$  lifts to, and  $F$  the 2-factor generated by  $s_3$ .

*Proof.* By Lemma 3.1.3, we have  $H_1 \cup F \simeq A_n \square_{r_1} B_m$ , and without loss of generality  $\pi_i = (1)$ , so  $A_n := a_1 a_2 \cdots a_n a_1$ . By the same lemma,  $H_2 \cup F \simeq A_n^{(2)} \square_{r_2} B_m$  where

$$A_n^{(2)} := a_{\pi_2(1)}^2 a_{\pi_2(2)}^2 \cdots a_{\pi_2(n)}^2 a_{\pi_2(1)}^2,$$

where  $(a_i^2, b_j)$  is the vertex  $a_i^2 + (j-1)s_3$  in  $\Gamma$ . Now,  $\overline{a_i^2} = \overline{a_i}$ , and so  $a_i^2 = a_i + h_i s_3$ . Therefore,

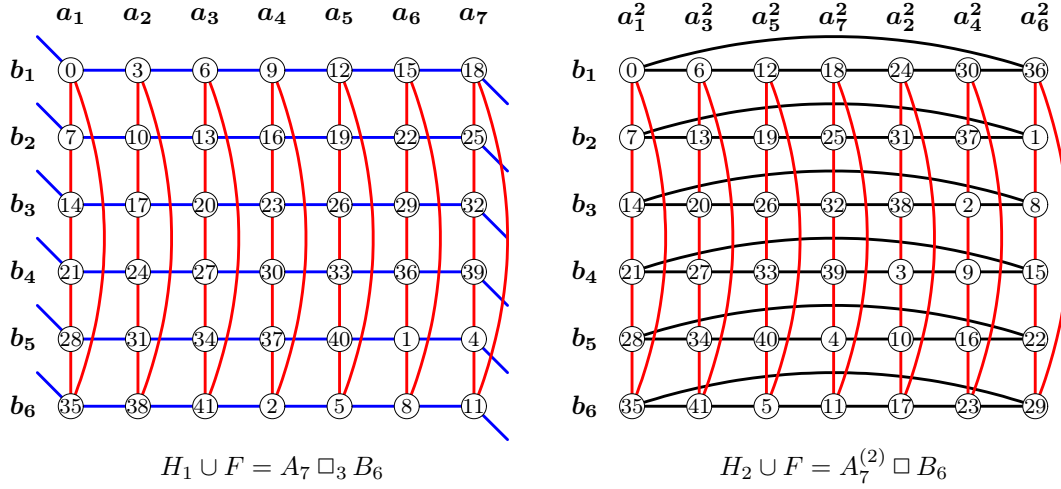
$$(a_i^2, b_j) \leftrightarrow a_i^2 + (j-1)s_3 = a_i + h_i s_3 + (j-1)s_3 = a_i + (h_i + j-1)s_3 \leftrightarrow (a_i, b_{j+h_i}).$$

Hence,  $\Gamma$  is a  $D(3, m, n)$ -graph, and the  $a_i^2$ -column of  $A_n^{(2)} \square_{r_2} B_m$  is just a cyclic shift of the elements in the  $a_i$ -column of  $A_n \square_{r_1} B_m$ . ■

**Example 3.1.7.** Consider the circulant graph  $\Gamma = \text{CAY}(\mathbb{Z}_{42}, \{s_1 = 3, s_2 = 6, s_3 = 7\})$  (see Figure 3.1). If  $J = \langle 7 \rangle$ , then the quotient graph is  $\Delta = \text{CAY}(\mathbb{Z}_{42}/\langle 7 \rangle, \{\overline{3}, \overline{6}\})$ .  $\Delta$  is Hamilton decomposable into  $\overline{H_1}$  and  $\overline{H_2}$ , where  $\pi_1 = (1)$  and  $\pi_2 = (235)(476)$ , so that

$$\overline{H_1} = \overline{a_1}, \overline{a_2}, \overline{a_3}, \overline{a_4}, \overline{a_5}, \overline{a_6}, \overline{a_7}, \overline{a_1} = \overline{0}, \overline{3}, \overline{6}, \overline{2}, \overline{5}, \overline{1}, \overline{4}, \overline{0},$$





**Figure 3.1:** The graph  $\text{CAY}(\mathbb{Z}_{42}, \{3, 6, 7\})$  from Example 3.1.7 is a  $D(3, 6, 7)$ -graph.

and

$$\overline{H_2} = \overline{a_1}, \overline{a_3}, \overline{a_5}, \overline{a_7}, \overline{a_2}, \overline{a_4}, \overline{a_6}, \overline{a_1} = \overline{0}, \overline{6}, \overline{5}, \overline{4}, \overline{3}, \overline{2}, \overline{1}, \overline{0}.$$

Thus, by Theorem 3.1.6,  $\Gamma$  is a  $D(3, 6, 7)$ -graph.  $F$ ,  $H_1$ , and  $H_2$  are the 2-factors generated by 7, 3 and 6, respectively, and  $H_1 \cup F = A_7 \square_3 B_6$  and  $H_2 \cup F = A_7^{(2)} \square B_6$ . Also,  $(a_i^2, b_t) = (a_i, b_{t+h_i})$ , where  $h_i = 0$  for  $i \in \{1, 3, 5, 7\}$  and  $h_i = 3$  for  $i \in \{2, 4, 6\}$ .

*Remark 3.1.8.* For brevity, we will write  $\{a_i^2, b_j, b_{j+d}\}$ -RAVS or LAVS to mean an alternating vertical switch between the  $a_{\pi(k-1)}^2$ ,  $a_{\pi(k)}^2$ , and  $a_{\pi(k+1)}^2$ -columns, where  $\pi(k) = i$ . When we say *apply an*  $\{a_i, a_{i+1}, b_j, b_{j+1}\}$ -CS, it is assumed the switch is applied to  $H_1 \cup F \simeq A_n \square_{r_1} B_m$ . When we say *apply an*  $\{a_i^2, a_{i+1}^2, b_j, b_{j+1}\}$ -CS, it is assumed the switch is applied to  $H_2 \cup F \simeq A_n^{(2)} \square_{r_2} B_m$ .

The following two results are fundamental in proving Theorem 3.2.1.

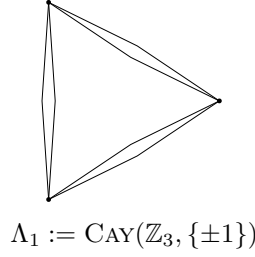
**Theorem 3.1.9** (Fan et al. [21]). *Let  $A$  be a finite abelian group, and  $S = \{s_1, s_2, s_3\}$  be a generating set for  $A$ . If  $S$  has an element  $s_i$  of odd order such that  $\langle s_i \rangle$  is a subgroup of index at least nine, and  $|\overline{s_j}| \geq 3$  for  $j \neq i$ , then  $\text{CAY}(A, S)$  has a Hamilton decomposition.*

**Theorem 3.1.10** (Fan et al. [21]). *Let  $A$  be a finite abelian group of odd order, with generating set  $S = \{s_1, s_2, s_3\}$ . If there exists an element of strictly smallest order, then  $\text{CAY}(A, S)$  has a Hamilton decomposition.*

## 3.2 Decompositions for Odd Order Groups

Note, if  $\Gamma = \text{CAY}(A, S)$  is connected, 6-regular, and  $|A|$  is odd, then  $S = \{s_1, s_2, s_3\}$ , and  $|s_i| \geq 3$  for  $i = 1, 2, 3$ .

**Theorem 3.2.1** ([53]). *Every connected 6-regular Cayley graph on an abelian group of odd order has a Hamilton decomposition.*



**Figure 3.2:** The quotient graph  $\Delta$  of Case 1.i. of Theorem 3.2.1.

*Proof.* Suppose  $\Gamma = \text{CAY}(A, S)$  is a connected 6-regular Cayley graph on a finite abelian group  $A$  of odd order. Then, without loss of generality, we may assume that the generating set is  $S = \{s_1, s_2, s_3\}$  where  $|s_1| \geq |s_2| \geq |s_3|$ . Furthermore by Lagrange's Theorem,  $|s_i| \geq 3$  is odd, for  $i = 1, 2, 3$ . The cases where  $|s_i| = |A|$  for some  $i = 1, 2, 3$ ,  $|s_2| > |s_3|$ , or  $S$  is a minimal generating set, are completely solved by Theorems 1.5.2, 3.1.10, or 1.5.3, respectively. Thus we may assume  $|s_2| = |s_3|$ . If  $s_1 \in \langle s_3 \rangle$  and  $s_2 \in \langle s_3 \rangle$ , then  $\langle s_3 \rangle = A$ , i.e.  $\Gamma$  is a circulant, which is resolved by Theorem 1.5.2. If  $s_1 \in \langle s_3 \rangle$ , but  $s_2 \notin \langle s_3 \rangle$ , then  $|s_1| = |s_2| = |s_3|$ , and so without loss of generality, there are two cases to consider. Either  $s_1 \notin \langle s_3 \rangle$  and  $s_2 \notin \langle s_3 \rangle$  or  $s_1 \notin \langle s_3 \rangle$  and  $s_2 \in \langle s_3 \rangle$ .

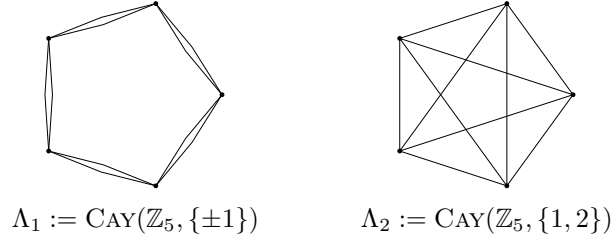
CASE 1:  $s_1 \notin \langle s_3 \rangle$  and  $s_2 \notin \langle s_3 \rangle$

Let  $J = \langle s_3 \rangle$ ,  $m = |J|$ , and  $n = |A/J|$ . Thus,  $|A| = nm$ , and by Lagrange's Theorem, both  $m$  and  $n$  are odd. Let  $\overline{S} := \{\overline{s_1}, \overline{s_2}\}$ . Then, as  $\overline{s_i} \neq \overline{0}$  and  $n$  is odd, we have that  $|\overline{s_i}| \geq 3$  is odd, for  $i = 1, 2$ . The quotient graph  $\Delta := \text{CAY}(A/J, \overline{S})$  is a 4-regular connected Cayley graph. As  $J \neq A$ , and the case  $n \geq 9$  is settled by Theorem 3.1.9, we may assume  $3 \leq n \leq 7$ . As  $n$  is prime,  $|\overline{s_1}| = |\overline{s_2}| = n$ , and  $A/J \cong \mathbb{Z}_n$ . Then, letting  $a_i := (i-1)s_1$ , where  $1 \leq i \leq n$ , we have:

$$\overline{H_1} := \overline{a_1}, \overline{a_2}, \dots, \overline{a_n}, \overline{a_1} \quad \text{and} \quad \overline{H_2} := \overline{a_{\pi(1)}}, \overline{a_{\pi(2)}}, \dots, \overline{a_{\pi(n)}}, \overline{a_{\pi(1)}}$$

is a Hamilton decomposition of  $\Delta$ , where  $\overline{H_i}$  is generated by  $\overline{s_i}$ , and  $\pi$  is a permutation of  $\{1, 2, \dots, n\}$ . The cycles  $\overline{H_1}$  and  $\overline{H_2}$  lift to two 2-factors,  $H_1$  and  $H_2$ , which are generated by  $s_1$  and  $s_2$ , respectively.  $H_1$  consists of  $t$  cycles of length  $mn/t \geq m$ , and  $H_2$  consists of  $n$  cycles of length  $m$ . If  $t = 1$ , then  $\Gamma$  is a circulant, and we may apply Theorem 1.5.2. Hence we may assume that  $3 \leq t \leq n$  and that  $t$  is odd. If  $3 \leq m \leq 7$ , then  $|A|$  is the product of two odd primes, and we can apply Theorem 1.5.5. Thus, we may further assume that  $nm > |s_1| \geq |s_2| = m \geq 9$ . By Theorem 3.1.6,  $\Gamma$  is a  $D(3, m, n)$ -graph. Color the edges of  $F$  red, the edges of  $H_1$  black, and the edges of  $H_2$  blue. Hence,  $H_1 \cup F \simeq A_n \square_{r_1} B_m$  and  $H_2 \cup F \simeq A_n^2 \square_{r_2} B_m$ . Also, by Remark 2.1.2,  $t = \gcd(r_1, m)$  and  $n = \gcd(r_2, m)$ , so that  $m = (2k+1)n$  for some  $k > 0$ . If  $r_i = 0$ , then  $|s_i| = n < m$ , a contradiction. Thus  $r_i \neq 0$  for  $i = 1, 2$ .

- i. If  $n = 3$ , then  $t = 3$ , and any three consecutive rows in  $A_3 \square_{r_1} B_m$ , respectively  $A_3^{(2)} \square_{r_2} B_m$ , lie on three different cycles. As  $\overline{H_1} = \overline{H_2}$ , up to a relabeling of the vertices,  $\Delta = \Delta_1 := \text{CAY}(\mathbb{Z}_3, \{\pm 1\})$ , a fat triangle, shown Figure 3.2. By Remark 2.2.2, applying an  $\{a_1, a_3, b_2\}$ -RAHS to  $C_3 \square_{r_1} B_m$ , will join the three red cycles of  $F$ , respectively the three black cycles of  $H_1$ , into Hamilton cycles. As  $m \geq 9$ , there exists an integer  $d$  such that all vertical edges between the  $b_j$ -rows for  $d \leq j \leq d+2$  in  $A_3^{(2)} \square_{r_2} B_m$  are red. Then, a Hamilton decomposition is obtained by applying an  $\{a_1^2, a_3^2, b_{d+1}\}$ -RAHS.



**Figure 3.3:** The quotient graphs of Case 1.ii. of Theorem 3.2.1 and Case 1 of Lemma 5.3.1.

- ii. If  $n = 5$ , then  $t \in \{3, 5\}$ . If  $t = 5$  (i.e.  $|s_i| = m$  for  $1 \leq i \leq 3$ ), then by Remark 2.2.2, the application of an  $\{a_2, b_1, b_5\}$ -LAVS to  $A_5 \square_{r_1} B_m$  yields a black Hamilton cycle, and joins together 3 of the 5 red cycles of  $F$ . Note, up to a relabeling of the vertices,  $\Delta = \Lambda_1 := \text{CAY}(\mathbb{Z}_5, \{\pm 1\})$ , a fat 5-cycle, or  $\Delta = \Lambda_2 := \text{CAY}(\mathbb{Z}_5, \{1, 2\})$ , a complete graph, shown in Figure 3.3.

If  $\Delta = \Lambda_1$ , then  $\overline{H_1} = \overline{H_2}$ , and without loss of generality,  $\pi = (1)$ . The red and black vertical edges in the  $a_3$ -column form a matching. Let  $d$  be any integer such that  $e = (a_3^2, b_{1+d})(a_3^2, b_{2+d})$  is a red edge. Apply an  $\{a_4^2, b_{1+d}, b_{5+d}\}$ -LAVS to  $A_5^{(2)} \square_{r_2} B_m$ , to obtain a set of three monochromatic Hamilton cycles in  $\Gamma$ .

If  $\Delta = \Lambda_2$ , without loss of generality,  $\pi = (2354)$ , so that

$$A_5^{(2)} = a_1^2 a_3^2 a_5^2 a_2^2 a_4^2 a_1^2.$$

Let  $h$  be the integer such that all vertical edges in the  $a_2^2$ -column that are between the  $b_{j+h}$ -rows, where  $1 \leq j \leq 5$ , are black. Now, as  $5 \mid m \Rightarrow m \geq 15$ , and so

$$(a_2^2, b_{5+h}), (a_2^2, b_{6+h}), (a_2^2, b_{7+h}), (a_2^2, b_{8+h})$$

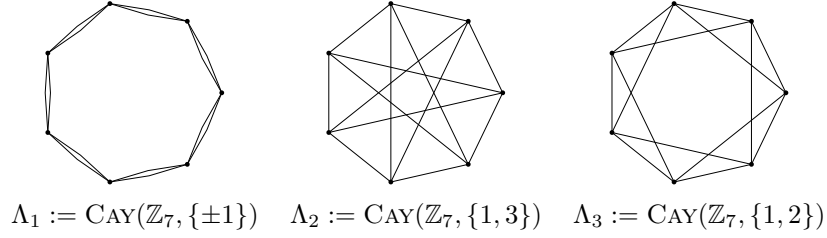
is a red 5-path. All vertical edges in the  $a_4^2$ - and  $a_5^2$ -columns are red, and so applying an  $\{a_2^2, b_{5+h}, b_{9+h}\}$ -LAVS or -RAVS to  $H_2 \cup F$  yields 3 monochromatic Hamilton cycles. In the case  $t = 3$ , apply an  $\{a_1, a_2, b_2\}$ -LAVS to  $H_1 \cup F$ , and use the aforementioned technique to obtain the result.

- iii. If  $n = 7$ , then  $t \in \{3, 5, 7\}$ . If  $t = 7$ , then  $|s_i| = m \geq 21$  for  $i = 1, 2, 3$ . By Remark 2.2.2, applying an  $\{a_2, b_1, b_7\}$ -LAVS to  $A_7 \square_{r_1} B_m$  produces a black Hamilton cycle, and joins all vertical red edges in the  $a_i$ -columns, where  $i = 1, 2, 3$ , to one cycle. Similar to the previous cases, up to a relabeling of the vertices, the quotient graph  $\Delta$  may be viewed as one of the following graphs shown in Figure 3.4.

If  $\Delta = \Lambda_1$ , then without loss of generality,  $\pi = (1)$ , and apply an  $\{a_3, a_5, b_2\}$ -RAHS to  $A_7 \square_{r_1} B_m$  to join two more of the red cycles of  $F$  into one red  $5m$ -cycle. By Lemma 2.2.15, this preserves the black cycle. Let  $d$  be any integer such that  $e = (a_5^2, b_{1+d})(e_5^2, b_{2+d})$  is a red edge and apply an  $\{a_6^2, b_{1+d}, b_{7+d}\}$ -LAVS to  $A_7^{(2)} \square_{r_2} B_m$  to obtain a Hamilton decomposition.

If  $\Delta = \Lambda_2$ , then without loss of generality,  $\pi = (243756)$ , so that

$$A_7^{(2)} = a_1^2 a_4^2 a_7^2 a_3^2 a_6^2 a_2^2 a_5^2 a_1^2.$$



**Figure 3.4:** The quotient graphs of Case 1.iii of Theorem 3.2.1 and Case 2 of Lemma 5.3.1.

Apply an  $\{a_3, a_5, b_2\}$ -RAHS to  $A_7 \square_{r_1} B_m$  to join all red edges in the  $a_i$ -columns into one monochromatic cycle, where  $1 \leq i \leq 5$ . Let  $h$  be any integer such that the vertical edges in the  $a_3^2$ -column and between the  $b_{j+h}$ -rows,  $1 \leq j \leq 7$  contain all four black edges. Clearly,  $\{(a_3^2, b_{i+h}) : 7 \leq i \leq 13\}$  is a red 6-path. Thus, apply an  $\{a_3^2, b_{7+h}, b_{13+h}\}$ -LAVS or -RAVS to obtain a Hamilton decomposition. The case where  $t = 3$  or  $t = 5$  follow similarly.

If  $\Delta = \Lambda_3$ , then without loss of generality,  $\pi = (235)(476)$ , so that

$$A_7^2 = a_1^2 a_3^2 a_5^2 a_7^2 a_2^2 a_4^2 a_6^2 a_1^2.$$

Let  $h$  be an integer such that all vertical edges in the  $a_2^2$ -column that are between the  $b_{j+h}$ -rows, where  $1 \leq j \leq 7$ , are black. Apply an  $\{a_5^2, b_{3+h}, b_{7+h}\}$ -LAVS or -RAVS, depending on if  $(a_3^2, b_{3+h})(a_3^2, b_{4+h})$  is red or not, to join 5 of the 7 blue cycles of  $H_2$  into one cycle and join the red edges in the  $a_i$ -columns into one cycle where  $i = 1, 2, 3, 5, 7$ . By Remark 2.2.2, the application of an  $\{a_2^2, a_6^2, b_{8+h}\}$ -RAHS or -LAHS to produce a Hamilton decomposition.

CASE 2:  $s_1 \notin \langle s_3 \rangle$  and  $s_2 \in \langle s_3 \rangle$ .

In this case,  $\langle s_2 \rangle = \langle s_3 \rangle$  and  $s_2, s_3 \notin \langle s_1 \rangle$ , for otherwise  $\Gamma$  is a circulant graph. Hence, we apply the technique of Case 1, by setting  $J := \langle s_1 \rangle$ . ■



## Chapter 4

# A Decomposition for Non-Minimal Connection Sets

### 4.1 Using a Subgroup of Index 2

In this chapter, we obtain results when the connection set  $S = \{s_1, s_2, s_3\}$  contains elements that have a linear dependency among themselves, i.e.,  $S$  is not a minimal set of generators, and one element generates a subgroup of index two. The following observation will be used in the proof of Theorem 4.1.5.

*Remark 4.1.1.* If  $\text{CAY}(A, \{s_1, s_2\}) \simeq A_n \square_r B_m$ , and  $x = (p-1)s_1 + (q-1)s_2 \leftrightarrow (a_p, b_q)$ , then  $-x$ , the inverse of  $x$ , corresponds to the vertex  $(a_{p^*}, b_{q^*})$  in  $A_n \square_r B_m$ , where  $p^* = n - p + 2$  and  $q^* = m - (q + r - 2)$ . This is because

$$\varphi : (a_{p^*}, b_{q^*}) \mapsto (n - p + 1)s_1 + (m - q - r + 1)s_2,$$

and as  $ns_1 = rs_2$  and  $ms_2 = 0$ , the above simplifies to  $(1 - p)s_1 + (1 - q)s_2 = -x$ .

**Definition 4.1.2.** If a CS-configuration has been applied to  $A_n \square_r C_m$  to create a Hamilton cycle of (red) vertical edges, and upon making this cycle a directed cycle, we find that all vertical edges in the  $a_i$ -column, for some  $1 \leq i \leq n$ , have the same direction (either  $\uparrow$  or  $\downarrow$ ), we say the  $a_i$ -column direction is  $\uparrow$  or  $\downarrow$ , respectively. More generally, we say  $A_n \square_r B_m$  has *column-direction pattern*  $\uparrow \downarrow \uparrow \downarrow \dots$  if neighboring columns have different directions (depending on the parity of  $n$ , the  $a_1$  and  $a_n$ -columns may have the same direction or not).

**Lemma 4.1.3.** *If  $\Gamma = \text{CAY}(A, \{s_1, s_2, s_3\})$  is a 6-regular, connected Cayley graph of even order, satisfying*

$$2h + 1 = |A : \langle s_3 \rangle| \geq |A : \langle s_2 \rangle| \geq |A : \langle s_1 \rangle| = 2,$$

*where  $s_1 \in \langle s_2, s_3 \rangle$ , and  $h \geq 1$ , then  $\Gamma$  has a Hamilton decomposition.*

*Proof.* Let  $J_i := \langle s_i \rangle$ , for  $1 \leq i \leq 3$ , so that  $|A|/2 = |s_1| \geq |s_2| \geq |s_3| = m \geq 3$ , where at least one of the inequalities is strict. It may further be assumed that  $s_3 \notin J_2$ , for otherwise  $\Gamma$  is a circulant graph, which is resolved by Theorem 1.5.2. Let  $t = |A : J_2|$  and  $n = |A : J_3|$ . Thus  $|A| = mn$ , and  $n = 2h_2 + 1 \geq t \geq 2$ . Let  $F_1 = \text{CAY}(A, \{s_1\})$ ,  $F_2 = \text{CAY}(A, \{s_2\})$ , and  $F_3 = \text{CAY}(A, \{s_3\})$ . Clearly,  $\{F_1, F_2, F_3\}$  is a 2-factorization of  $\Gamma$ . The subgraph  $F_2 \cup F_3 = \text{CAY}(A, \{s_2, s_3\}) \simeq A_n \square_r B_m$ , where  $n$  is the number of columns, and  $t$  is the number of distinct horizontal cycles. The vertex  $(i-1)s_2 + (j-1)s_3$  in  $F_2 \cup F_3$  is identified with the vertex  $(a_i, b_j)$ , for  $1 \leq i \leq n$ ,  $1 \leq j \leq m$ , where  $ns_2 = rs_3$ , and  $t = \gcd(r, m)$ . Hence,  $s_2 = (a_2, b_1)$  and  $s_3 = (a_1, b_2)$ . As  $s_1 \in \langle s_2, s_3 \rangle$ ,  $s_1 = (p-1)s_2 + (q-1)s_3 \mapsto (a_p, b_q)$ , for some  $1 \leq p \leq n$  and  $1 \leq q \leq m$ . Color the edges in  $F_1$  blue, edges in  $F_2$  black, and edges in  $F_3$  red.

**Case 1:  $n = 2h_2 + 1 > t = 2h_1 \geq 2$ .** Here,  $s_1 \notin J_3$  and  $s_3 \notin J_1$  for otherwise, either  $n \mid 2$  or  $2 \mid n$ , contradicting the choice of  $n$ . As  $|A|$  is even,  $m = 2s \geq 4$ .

1. If  $t = 2$  then apply an  $\{a_1, a_n, b_2\}$ -RAHS to  $A_n \square_r B_m$  to create a red Hamilton cycle,  $C_R$ . As  $n$  is odd, the column direction pattern is:  $\uparrow \downarrow \uparrow \cdots \downarrow \uparrow$ . Thus, by Remark 2.2.2 and Lemma 2.2.15, apply an  $\{a_n, a_1, b_1, b_2\}$ -CS to obtain a black Hamilton cycle, and preserve  $C_R$  (see Figure 4.2). Give  $C_R$  an orientation so that it becomes a directed cycle. Clearly, the column direction pattern is still  $\uparrow \downarrow \uparrow \cdots \downarrow \uparrow$  with the exception of the vertical  $\downarrow$ -edges in the  $a_1$ -column that are between the  $b_2$ - and  $b_{1+r}$ -rows. Note that  $|J_1| = |J_2|$  and  $A = J_1 \cup (J_1 + s_3) = J_2 \cup (J_2 + s_3)$ . Additionally,  $r \neq 0$ , for otherwise,  $ns_2 = 0 \Rightarrow |s_2| = \frac{nm}{2} \mid n \Rightarrow m \mid 2$ , a contradiction. Thus, consider the cases where  $J_1 = J_2$  and  $J_1 \neq J_2$ .

- (a)  **$J_1 = J_2$ :** For  $1 \leq j \leq m-1$  and  $1 \leq i \leq n$ , the vertices  $(a_i, b_j)$  and  $(a_i, b_{j+1})$  are on different blue cycles and all elements of  $J_1$  and  $J_2$  are on odd  $b_j$ -rows. In particular,  $s_1 = (a_p, b_q)$  for some  $1 < p \leq n$  and  $q = 2\hat{q} + 1$ , where  $1 \leq \hat{q} \leq m-1$ . We seek a vertical oblique color-switch to join the two blue cycles, and preserve  $C_R$ . Consider the following cases:

- i.  $(p, q, r, m) = (2\hat{p}, 1, 2, 4)$ , for some  $\hat{p} \geq 1$ . By Remark 4.1.1,  $-s_1 = (a_{p^*}, b_3)$ , where  $p^* = n - 2\hat{p} + 2$  is odd, and  $3 \leq p^* \leq n$ . By Table 1.1 of the Appendix, we may assume  $n \geq 5$ . The black edges  $(a_1, b_3)(a_1, b_4)$  and  $(a_{2i+1}, b_1)(a_{2i+1}, b_2)$ , for  $i \geq 0$  have opposite direction. In particular,  $(a_1, b_3)(a_1, b_4)$  and  $(a_{p^*}, b_1)(a_{p^*}, b_2)$  have opposite direction and  $(a_1, b_3)(a_{p^*}, b_1)$  and  $(a_1, b_4)(a_{p^*}, b_2)$  are blue edges that are on two different cycles. Apply an  $\{a_1, a_{p^*}, b_3, b_1\}$ -VOCS to create a blue Hamilton cycle and, by Remark 2.2.2, break the black Hamilton cycle into a 2-factor consisting of two cycles. In particular, the edges  $(a_{p^*-1}, b_2)(a_{p^*-1}, b_3)$  and  $(a_{p^*}, b_2)(a_{p^*}, b_3)$  are on two different black cycles and  $(a_{p^*-1}, b_2)(a_{p^*}, b_2)$  and  $(a_{p^*-1}, b_3)(a_{p^*}, b_3)$  are red edges that have the same direction. Apply an  $\{a_{p^*-1}, a_{p^*}, b_2, b_3\}$ -CS, which by Remark 2.2.2 will create a Hamilton decomposition.
- ii.  $(p, q, r, m) = (2\hat{p}, 1, 2, 2s)$ , where  $\hat{p} \geq 1$  and  $s \geq 3$ . Reflect all switches about the  $b_2$ -row. We still have a Hamilton decomposition of  $A_n \square_r B_m$  but now the column direction pattern is:  $\uparrow \uparrow \downarrow \cdots \uparrow \downarrow$ . Hence, as  $m \geq 6$ , the edge  $(a_1, b_4)(a_1, b_1)$  is red and has the same direction as  $(a_p, b_m)(a_p, b_1)$ . Apply an  $\{a_1, a_p, b_m, b_1\}$ -VOCS to obtain the result.

- iii.  $(p, q, r) = (2\hat{p}, 2\hat{q} + 1, 2u)$ , where  $\hat{q} \geq 0$  and  $u \geq 1$ . By Figure 4.2, the vertical oblique color-switches defined in (4.1) will produce a Hamilton decomposition of  $\Gamma$ .

$$\text{If } (p, q, r) = (2\hat{p}, 2\hat{q} + 1, 2u), \text{ apply an } \begin{cases} \{a_1, a_p, b_2, b_{q+1}\}\text{-VOCS} & \text{if } \hat{q} \geq 1 \\ \{a_1, a_p, b_3, b_3\}\text{-VOCS} & \text{if } \hat{q} = 0 \text{ and } u \neq 1 \end{cases} \quad (4.1)$$

- iv.  $p = 2\hat{p} + 1$  where  $\hat{p} \geq 1$ . Cases i-iii construct Hamilton decompositions when  $p = 2\hat{p}$ , for some  $\hat{p} \geq 1$ . By Remark 4.1.1,  $-s_1 = (a_{p^*}, b_{q^*})$ , where  $p^* = n - 2\hat{p} + 2$ , which is even, and  $q^* = m - (q + r - 2)$ . Hence, apply the technique of Cases i-iii by replacing  $p$  with  $p^*$  and  $q$  with  $q^*$ .

- (b)  $\mathbf{J_1 \neq J_2}$ : Here  $|J_1 \cap J_2| = mn/4$ , so that  $x \in J_1 \cap J_2 \Leftrightarrow x = 2ks_2$ , for some  $k \in \mathbb{Z}$ . As  $s_2, s_3 \notin J_1$ ,  $(a_i, b_j)$  is on one blue cycle and  $(a_{i+1}, b_j)$  and  $(a_i, b_{j+1})$  are on the other blue cycle, for all  $1 \leq i \leq n$  and  $1 \leq j \leq m - 1$ . All elements of  $J_2$  occur in odd  $b_j$ -rows, so  $s_1 = (a_p, b_{2\hat{q}})$  for some  $1 < p \leq n$ . If  $q = 2$ , then the edges  $(a_1, b_1)(a_2, b_1)$  and  $(a_p, b_2)(a_{p+1}, b_2)$  are red edges that have the same direction and  $(a_1, b_1)(a_p, b_2)$  and  $(a_2, b_1)(a_{p+1}, b_2)$  are blue edges that are not on the same cycle. By Remark 2.2.2, apply an  $\{a_1, a_p, b_1, b_2\}$ -HOCS to obtain a blue Hamilton cycle and preserve  $C_R$ , i.e., a Hamilton decomposition. If  $p = 2\hat{p} + 1$  and  $\hat{q} \neq 1$ , then  $(a_1, b_m)(a_1, b_1)$  and  $(a_p, b_{q-1})(a_p, b_q)$  are red edges that share the same direction, and we may apply an  $\{a_1, a_p, b_m, b_{q-1}\}$ -VOCS, which by Remark 2.2.2, preserves  $C_R$  and creates a blue Hamilton cycle. The result is a set of three monochromatic Hamilton cycles. If  $p = 2\hat{p}$  and  $q \neq m$ , apply an  $\{a_1, a_p, b_2, b_{q+1}\}$ -VOCS to obtain a Hamilton decomposition. If  $q = m$ , then  $-s_1$  lies in the odd  $a_{n-p+2}$ -column and even  $b_{m-(m+r-2)}$ -row. Apply a VOCS according to the case when  $p = 2\hat{p} + 1$  and using  $-s_1$  rather than  $s_1$  to obtain a Hamilton decomposition.

2. Suppose  $t \geq 4$ . Note,  $s_1 \notin J_2$  and so  $s_1 = (a_p, b_q)$  for some  $1 < p \leq n$  and  $1 < q \leq m$ . Furthermore, the elements of  $J_2$  consist of the vertices  $(a_i, b_{1+\ell t})$ , for  $1 \leq i \leq n$  and  $0 \leq \ell < k$ , where  $m = kt$ , and therefore  $q \neq 1 + \ell t$  for any  $\ell \geq 0$ . As before, the vertices  $(a_i, b_j)$  and  $(a_i, b_{j+1})$  lie on two different blue cycles. Apply an  $\{a_2, b_1, b_{t-1}\}$ -LAVS and an  $\{a_3, a_4, b_{t-1}, b_t\}$ -CS to  $F_2 \cup F_3 = A_n \square_r B_m$  to obtain a black Hamilton cycle and join the  $a_i$ -columns, where  $1 \leq i \leq 4$ , into one red cycle. Apply an  $\{a_4, a_{n-1}, b_t\}$ -LAHS and an  $\{a_{n-1}, a_n, b_t, b_{t+1}\}$ -CS to create a red Hamilton cycle,  $C_R$ , and break the black Hamilton cycle into two cycles. As  $n$  is odd, the column-direction pattern is:  $\uparrow \downarrow \uparrow \cdots \downarrow \uparrow$ . By Remark 2.2.2 and Lemma 2.2.15, apply an  $\{a_n, a_1, b_{t-1}, b_t\}$ -CS to reform a black Hamilton cycle and preserve  $C_R$ . Clearly, this switch is not  $r$ -incident with any  $b_j$ -row for  $1 \leq j \leq t - 2$ , for initially, the  $b_j$ -rows were on  $t$  different cycles, for  $1 \leq j \leq t$ . We now have a Hamilton decomposition of the subgraph  $F_2 \cup F_3$ . Make  $C_R$  a directed cycle as shown in Figure 4.3. Apply the following color-switch to obtain a Hamilton decomposition of  $\Gamma$ .

$$\begin{cases} \text{If } q \neq t, \text{ apply an } \begin{cases} \{a_1, a_p, b_m, b_{q-1}\}\text{-VOCS} & \text{if } p = 2\hat{p} + 1 > 3 \\ \{a_2, a_4, b_m, b_{q-1}\}\text{-VOCS} & \text{if } p = 3 \end{cases} \\ \text{If } q = t, \text{ apply an } \begin{cases} \{a_2, a_{2\hat{p}+2}, b_{t-1}, b_{2t-1}\}\text{-VOCS} & \text{if } p = 2\hat{p} + 1 < n \\ \{a_3, a_2, b_m, b_{r+t-1}\}\text{-VOCS} & \text{if } p = n \end{cases} \end{cases} \quad (4.2)$$

Each of the VOCS in (4.2) interchange the colors of two red vertical edges having the same direction, with the colors of two oblique blue edges that are on different cycles (refer to Figure 4.3). By Remark 2.2.2, the result is a Hamilton decomposition of  $\Gamma$ . If  $p = 2\hat{p}$ , then, by



Remark 4.1.1,  $-s_1$  is in the odd  $a_{n-2\hat{p}+2}$ -column. Apply the color-switches defined in (4.2) where  $p$  is replaced with  $p^* = n - p + 2$  and  $q$  is replaced with  $q^* = m - (q + r - 2)$ .

**Case 2:  $n = 2h_1 + 1 \geq 2h_2 + 1 = t \geq 3$ .** If  $s_1 = (a_{2\hat{p}+2}, b_{2\hat{q}+1})$ , for some  $\hat{p}, \hat{q} \geq 0$ , then

$$0 = \left(\frac{nm}{2}\right) s_1 = \frac{nm}{2} [(2\hat{p} + 1)s_2 + (2\hat{q})s_3] = nm(\hat{p}s_2 + \hat{q}s_3) + \left(\frac{nm}{2}\right) s_2 = \left(\frac{nm}{2}\right) s_2 \Rightarrow |s_2| \mid \frac{nm}{2},$$

and so  $\frac{nm}{t} \mid \frac{nm}{2} \Rightarrow t \mid 2$ , a contradiction. Similarly,  $s_1 = (a_{2\hat{p}+1}, b_{2\hat{q}+2})$  produces a contradiction. Hence,  $(p, q) = (2\hat{p}, 2\hat{q})$  or  $(p, q) = (2\hat{p} + 1, 2\hat{q} + 1)$ . As before,  $s_1 \notin J_2, J_3$ , and  $s_2, s_3 \notin J_1$ , so  $1 < p \leq n$ ,  $1 < q \leq m$ , and  $q \neq 1 + \ell t$  for all  $\ell \geq 0$ . Hence,  $(a_i, b_j)$  is on one blue cycle while  $(a_i, b_{j+1})$  and  $(a_{i+1}, b_j)$  are on the other blue cycle, for any  $1 \leq i \leq n$  and  $1 \leq j \leq m$ . Furthermore,  $r \neq 0$ , for  $m$  and  $t$  have opposite parity. If  $n \geq 5$ , apply an  $\{a_2, b_1, b_t\}$ -LAVS to obtain a black Hamilton cycle and join the red cycles in the  $a_j$ -columns, where  $1 \leq j \leq 3$ , into one red cycle. Apply an  $\{a_3, a_n, b_2\}$ -RAHS, which by Lemma 2.2.15, produces a Hamilton decomposition of  $A_n \square_r B_m$ . Give the red Hamilton cycle,  $C_R$ , an orientation so it becomes a directed cycle (refer to Figure 4.4). Similarly to Case 1(b), we apply a final switch as follows:

$$\begin{cases} \text{If } p = 2\hat{p} + 1 \neq 3, \text{ apply an} & \begin{cases} \{a_1, a_p, b_m, b_{q-1}\}\text{-VOCS} & \text{if } q \neq 3 \\ \{a_1, a_p, b_2, b_{q+1}\}\text{-VOCS} & \text{if } q = 3 \end{cases} \\ \text{If } p = 3, \text{ apply an} & \begin{cases} \{a_3, a_5, b_m, b_{q-1}\}\text{-VOCS} & \text{if } q \neq 3 \\ \{a_3, a_5, b_3, b_5\}\text{-VOCS} & \text{if } q = 3 \end{cases} \end{cases} \quad (4.3)$$

If  $p = 2\hat{p}$ , for some  $\hat{p} \geq 0$ , then by Remark 4.1.1,  $-s_1$  is in the odd  $a_{n-p+2}$ -column, and the odd  $b_{q^*}$ -row, where  $q^* = m - (q + r - 2)$ . Replace  $p$  with  $p^* = n - p + 2$  and  $q$  with  $q^*$  in (4.3) to obtain a Hamilton decomposition of  $\Gamma$ . The case  $n = t = 3$ , is similar, and is omitted. ■

**Lemma 4.1.4.** Let  $\Gamma = \text{CAY}(A, \{s_1, s_2, s_3\})$  be a 6-regular, connected Cayley graph of even order, where

$$2h = |A : \langle s_3 \rangle| \geq |A : \langle s_2 \rangle| \geq |A : \langle s_1 \rangle| = 2.$$

If  $s_1 \in \langle s_2, s_3 \rangle$  and  $h \geq 2$ , then  $\Gamma$  has a Hamilton decomposition.

*Proof.* The proof is similar that of Lemma 4.1.3, where now  $n = |A : J_3| = 2h \geq 4$  and  $t = |A : J_2| \geq 2$ . We again seek a Hamilton decomposition of  $F_2 \cup F_2 = A_n \square_r B_m$  and then define HOCS or VOCS to create a blue Hamilton cycle and obtain the result.

**Case 1:  $n = 2h_1 > 2h_2 + 1 = t \geq 3$ .** Clearly,  $s_2 \notin J_1$  and so  $(a_i, b_j)$  is on one blue cycle, while  $(a_{i+1}, b_j)$  is on the other blue cycle, for  $1 \leq i \leq n$ . As  $s_1 \notin J_2, J_3$ , we have  $1 < p \leq n$ ,  $1 < q \leq m$ , and  $q \neq \ell t + 1$  for any  $\ell \geq 0$ . Apply an  $\{a_2, b_1, b_t\}$ -LAVS to  $A_n \square_r B_m$  to obtain a black Hamilton cycle and join the vertical red edges in the  $a_i$ -columns, where  $1 \leq i \leq 3$ , into one red cycle,  $C$ . Let  $m = kt$ , where  $k \geq 1$ , with equality if and only if  $r = 0$ . Let  $d$  be the integer,  $1 \leq d \leq k$ , such that  $(d - 1)t + 1 < q < dt + 1$ . By Corollary 2.2.8, apply an  $\{a_3, a_4, b_{dt}, b_{dt+1}\}$ -CS to preserve the black Hamilton cycle and join the red edges in the  $a_4$ -column to  $C$ . Next, apply an  $\{a_4, a_n, b_{dt}\}$ -RAHS, which by Lemma 2.2.15, will preserve the black cycle and create a red Hamilton cycle. We now have a Hamilton decomposition of  $A_n \square_r B_m$ . Give the red Hamilton cycle an orientation, so it becomes a directed cycle (refer to Figure 4.5). We shall now define horizontal oblique color-switches to join the blue cycles and preserve the red and black Hamilton cycles. Let  $j = dt - q + 1$ , so that  $1 \leq j \leq t - 1$ . The choice of  $j$  ensures that  $(a_1, b_j)(a_p, b_{dt})$  is a blue edge. To see this, note

$$(p - 1)s_2 + (dt - 1)s_3 - (j - 1)s_3 = (p - 1)s_2 + (q - 1)s_3 = s_1.$$

If  $d \equiv q \pmod{2}$ , then  $j$  is odd, and if  $3 \leq p \leq n-1$ , the red edges  $(a_1, b_j)(a_2, b_j)$  and  $(a_p, b_{dt})(a_{p+1}, b_{dt})$  have the same direction. If  $d \not\equiv q \pmod{2}$ , then  $j$  is even, and so  $(a_1, b_j)(a_2, b_j)$  and  $(a_p, b_{dt})(a_{p+1}, b_{dt})$  have opposite direction. Switching on that 4-cycle will create a blue Hamilton cycle but break the red Hamilton cycle into a 2-factor with two cycles, say  $R_1$  and  $R_2$ . The vertical red edges in the  $a_1$ -column that are between the  $b_j$ -row and the  $b_m$ -row and the edge  $(a_1, b_m)(a_1, b_1)$  are on  $R_1$ , while the red edge  $(a_n, b_{dt-1})(a_n, b_{dt})$  is on  $R_2$ . Furthermore, the black edges  $(a_n, b_{dt-1})(a_1, b_{dt-1+r})$  and  $(a_n, b_{dt})(a_1, b_{dt+r})$  have the same direction, therefore by Remark 2.2.2, the color-switches in (4.4) will produce a Hamilton decomposition of  $\Gamma$ . Similarly, if  $p = 2$ , then  $q \equiv 0 \pmod{2}$ , and  $(a_2, b_j)(a_3, b_j)$  and  $(a_3, b_{dt})(a_4, b_{dt+1})$  are both red edges, and  $(a_2, b_j)(a_3, b_{dt})$  and  $(a_3, b_j)(a_4, b_{dt})$  are blue edges that are on different cycles, and the switches in (4.5) will yield the result.

$$\text{If } 3 \leq p \leq n-1 \quad \begin{cases} \{a_1, a_p, b_j, b_{dt}\}\text{-HOCS} & \text{if } d \equiv q \pmod{2} \\ \{a_1, a_p, b_j, b_{dt}\}\text{-HOCS}, \{a_n, a_1, b_{dt-1}, b_{dt}\}\text{-CS} & \text{if } d \not\equiv q \pmod{2} \end{cases} \quad (4.4)$$

$$\text{If } p = 2 \quad \begin{cases} \{a_2, a_3, b_2, b_{dt+1}\}\text{-HOCS} & \text{if } q = dt \\ \{a_2, a_3, b_j, b_{dt}\}\text{-HOCS} & \text{if } d \equiv q \pmod{2} \\ \{a_2, a_3, b_j, b_{dt}\}\text{-HOCS}, \{a_n, a_1, b_{dt-1}, b_{dt}\}\text{-CS} & \text{if } d \not\equiv q \pmod{2} \end{cases} \quad (4.5)$$

Finally, if  $p = n$ , then  $-s_1$  is in the  $a_2$ -column. Apply the color-switches of (4.4) or (4.5), by replacing  $q$  with  $q^* = m - (q + r - 2)$ .

**Case 2:  $n = 2h_1 \geq 4$  and  $t = 2h_2 \geq 2$ .**  $m = 2s \geq 4$  We may assume that if  $s_2 \in J_1$ , then  $s_3 \notin J_1$ , for otherwise,  $A$  is cyclic,  $\Gamma$  is a circulant graph, and we are done by Theorem 1.5.2.

1.  $t \geq 4$ . We have  $1 < p \leq n$  and  $1 < q \leq m$ .

(a) Suppose  $s_2 \in J_1$ , so that  $A = J_1 \cup (J_1 + s_3)$  and  $(2k)s_3 \in J_1$  for all  $k \geq 0$ . Thus,  $x$  and  $x + s_3$  lie on different blue cycles and as  $(q-1)s_3 = s_1 - (p-1)s_2 \in J_1$ , we have  $q = 2\hat{q} + 1 < m$ , for some  $\hat{q} \geq 1$  and  $q \notin \{\ell t, \ell t + 1\}$  for any  $\ell \geq 0$ .

i. If  $p = 2\hat{p} + 1$ , then apply an  $\{a_2, b_1, b_{t-1}\}$ -LAVS or -RAVS, an  $\{a_3, a_4, b_{t-1}, b_t\}$ -CS, and an  $\{a_4, a_n, b_{t-1}\}$ -RAHS to  $A_n \square_r B_m$ . By Theorem 2.2.7 and Lemma 2.2.15, the aforementioned CS-configuration produces a Hamilton decomposition of  $A_n \square_r B_m$ . We seek a VOCS to join the two blue cycles. Give the red Hamilton cycle an orientation so it becomes a directed cycle. The column direction pattern is:  $\uparrow \downarrow \uparrow \cdots \downarrow \uparrow \downarrow$ . Apply the color-switches defined in (4.6) to obtain a Hamilton decomposition of  $\Gamma$ .

$$\text{Apply an } \begin{cases} \{a_1, a_p, b_m, b_{q-1}\}\text{-VOCS} & \text{if } q \neq t-1 \\ \{a_1, a_p, b_2, b_{q+1}\}\text{-VOCS} & \text{if } q = t-1 \end{cases} \quad (4.6)$$

If  $p = 3$ , we have the freedom to choose a LAVS or RAVS in order to guarantee that  $\{(a_3, b_{q-1}), (a_3, b_q)\}$  is a red edge.

ii. If  $p = 2\hat{p}$ , where  $2 \leq p \leq n$ , then as  $t$  is even, apply the CS-configuration defined in Lemma 2.2.9 to  $A_n \square_r B_m$  to obtain a black Hamilton cycle and connect the vertical red edges in the  $a_1$  and  $a_n$ -columns into one cycle. It is easily seen that applying a  $\{a_2, a_n, b_3\}$ -RAHS will produce a Hamilton decomposition of  $A_n \square_r B_m$ . Furthermore, upon orienting the red Hamilton cycle, the column-direction pattern is:  $\uparrow \uparrow \downarrow \uparrow \cdots \downarrow \uparrow$ . Apply the color-switches defined in (4.7) to obtain a Hamilton decomposition of  $\Gamma$ .

$$\text{Apply an } \begin{cases} \{a_1, a_p, b_m, b_{q-1}\}\text{-VOCS} & \text{if } q \neq 3 \\ \{a_1, a_p, b_1, b_q\}\text{-VOCS} & \text{if } q = 3 \text{ and } p \neq 2 \\ \{a_1, a_n, b_1, b_3\}\text{-VOCS} & \text{if } (p, q) = (2, 3) \end{cases} \quad (4.7)$$

- (b) Suppose  $s_2 \notin J_1$ , so that  $A = J_1 \cup (J_1 + s_2)$  and  $(2k)s_2 \in J_1$  for all  $k \geq 0$ . Thus,  $x$  and  $x + s_2$  lie on different blue cycles. We may safely assume that  $p \leq n - 1$ . If  $p = n$ , then  $-s_1$  is in the  $a_2$ -column, by Remark 4.1.1, and we may apply the technique to  $-s_1$  instead of  $s_1$  to obtain the result.

If  $m \geq 6$ , then apply the CS-configuration of Lemma 2.2.9 and an  $\{a_2, a_n, b_3\}$ -RAHS to  $A_n \square_r B_m$  which produces a Hamilton decomposition of  $A_n \square_r B_m$  by Lemma 2.2.10. All vertical red edges in the  $a_1$  and  $a_2$ -columns have the same direction, and all horizontal black edges in the  $b_j$  and  $b_{j+1}$ -rows have opposite directions. If  $5 \leq q \leq m$ , apply an  $\{a_1, a_p, b_1, b_q\}$ -HOCS. If  $q = 2\hat{q} + 1$ , we are done, for the horizontal edges have the same direction. If  $q = 2\hat{q}$ , a blue Hamilton cycle is created, but the black Hamilton cycle has been broken into a 2-factor consisting of two cycles. Therefore, as the  $b_q$  and  $b_{q+1}$ -rows lie on different black cycles, and the edges  $(a_1, b_q)(a_1, b_{q+1})$  and  $(a_2, b_q)(a_2, b_{q+1})$  are red edges with the same direction, apply an  $\{a_1, a_2, b_q, b_{q+1}\}$ -CS. By Remark 2.2.2, a Hamilton decomposition is obtained. The case  $q \in \{2, 3, 4\}$  is similar, by using either  $(a_1, b_q)(a_2, b_q)$  or  $(a_1, b_{q+1})(a_2, b_{q+2})$  in place of  $(a_1, b_1)(a_2, b_1)$ . If  $m < 6$ , then  $m = t = 4$  and so  $r = 0$ . Thus,  $2 \leq q \leq 4$ .

$$\begin{cases} \begin{cases} \{a_2, b_1, b_3\}\text{-RAVS}, \\ \{a_3, a_4, b_3, b_4\}\text{-CS}, \\ \{a_4, a_n, b_3\}\text{-RAHS} \end{cases} & \text{if } q = 2 \text{ and } \begin{cases} \{a_1, a_p, b_2, b_3\}\text{-HOCS} & \text{if } p \neq 2 \\ \{a_2, a_3, b_2, b_3\}\text{-HOCS} & \text{if } p = 2 \end{cases} \\ \begin{cases} \{a_2, b_1, b_3\}\text{-LAVS}, \\ \{a_3, a_4, b_3, b_4\}\text{-CS}, \\ \{a_4, a_n, b_3\}\text{-RAHS}, \\ \{a_1, a_p, b_1, b_3\}\text{-HOCS} \end{cases} & \text{if } q = 3 \end{cases}$$

If  $q = 4$ , apply the CS-configuration of Lemma 2.2.9 and an  $\{a_2, a_n, b_3\}$ -RAHS to obtain a Hamilton decomposition of  $A_n \square B_m$ . Now,  $(a_1, b_2)(a_2, b_2)$  and  $(a_p, b_1)(a_{p+1}, b_1)$  are both black edges that have opposite direction. Furthermore,  $(a_1, b_2)(a_p, b_1)$  and  $(a_2, b_2)(a_{p+1}, b_1)$  are blue edges that lie on different blue cycles. Apply a  $\{a_1, a_p, b_2, b_1\}$ -HOCS to obtain a blue Hamilton cycle and break the black Hamilton cycle into a 2-factor where  $(a_1, b_1)(a_2, b_1)$  and  $(a_1, b_4)(a_2, b_4)$  are black edges on different cycles. Now,  $(a_1, b_1)(a_1, b_4)$  and  $(a_2, b_1)(a_2, b_4)$  are red edges with the same direction. This is because upon applying the color-switch of Lemma 2.2.10,  $A_n \square B_m$  has column-direction pattern:  $\uparrow \uparrow \downarrow \uparrow \cdots \downarrow \uparrow$ . Apply an  $\{a_1, a_2, b_4, b_1\}$ -CS, which by Remark 2.2.2, yields a Hamilton decomposition.

2. If  $t = 2$ , then  $1 < p \leq n$  and  $1 \leq q \leq m$  and  $(p, q) \neq (2, 1)$ , because  $s_1 \neq s_2$ . Apply the CS-configuration of Lemma 2.2.9 to  $A_n \square_r B_m$  to obtain a black Hamilton cycle, and join the vertical red edges in the  $a_1$  and  $a_n$ -columns into one cycle. Consider the following cases:

- (a)  $s_2 \in J_1$ . Here,  $s_3 \notin J_1$ , for otherwise,  $A = J_1$ , a contradiction. Hence,  $J_1 = J_2$ , and the vertices  $x$  and  $x + s_3$  lie on different blue cycles, and the elements of  $J_1$  consist of all vertices in the odd  $b_j$ -rows. In particular,  $q = 2\hat{q} + 1 \geq 1$ , and  $1 < p \leq n$ . Apply the

following switches:

$$\begin{cases} \{a_1, a_{n-1}, b_q\}\text{-RAHS}, \{a_1, a_p, b_1, b_q\}\text{-VOCS} & \text{if } q \neq 1, p = 2\hat{p} + 1 \\ \{a_2, a_n, b_q\}\text{-RAHS}, \{a_1, a_p, b_1, b_q\}\text{-VOCS} & \text{if } q \neq 1, p = 2\hat{p} \neq 2 \\ \{a_2, a_n, b_q\}\text{-RAHS}, \{a_1, a_3, b_2, b_{q+1}\}\text{-VOCS} & \text{if } p = 2 \end{cases} \quad (4.8)$$

In each of the CS-configurations of (4.8), the application of the RAHS creates a red Hamilton cycle and preserves the black Hamilton cycle by Lemma 2.2.10. By orienting the black Hamilton cycle, we see that each VOCS defined in (4.8) switches blue edges on different cycles, and black edges that have the same direction. By Remark 2.2.2, a Hamilton decomposition is obtained. If  $q = 1$ , then  $-s_1$  lies in the  $a_{p^*}$ -column and  $b_{q^*}$ -row, where  $q^* = m - r + 1$ . As  $t = 2 \Rightarrow 0 < r < m$ , we have  $q^* \neq 1$ . Thus, apply the CS-configuration of (4.8) by replacing  $p$  with  $p^*$  and  $q$  with  $q^*$ .

- (b)  $s_2 \notin J_1$ . In this case,  $x$  and  $x + s_2$  are on different blue cycles. Furthermore,  $2 \leq p \leq n$  and  $2 \leq q \leq m$ , where  $q = 2\hat{q}$ . This is because all vertices in the odd  $b_j$ -rows are elements of  $J_2$ , and by assumption,  $s_1 \notin J_2$ . The result now follows exactly by using the technique of Case2.1(b). ■

Lemmata 4.1.3 and 4.1.4 combine to yield the main result of this Chapter.

**Theorem 4.1.5.** *If  $A = \langle s_2, s_3 \rangle$  is an abelian group of even order,  $|s_3| \geq 3$ , and  $|A|/2 = |s_1| \geq |s_2| \geq |s_3|$ , where at least one inequality is strict, then  $\text{CAY}(A, \{s_1, s_2, s_3\})$  has a Hamilton decomposition.*

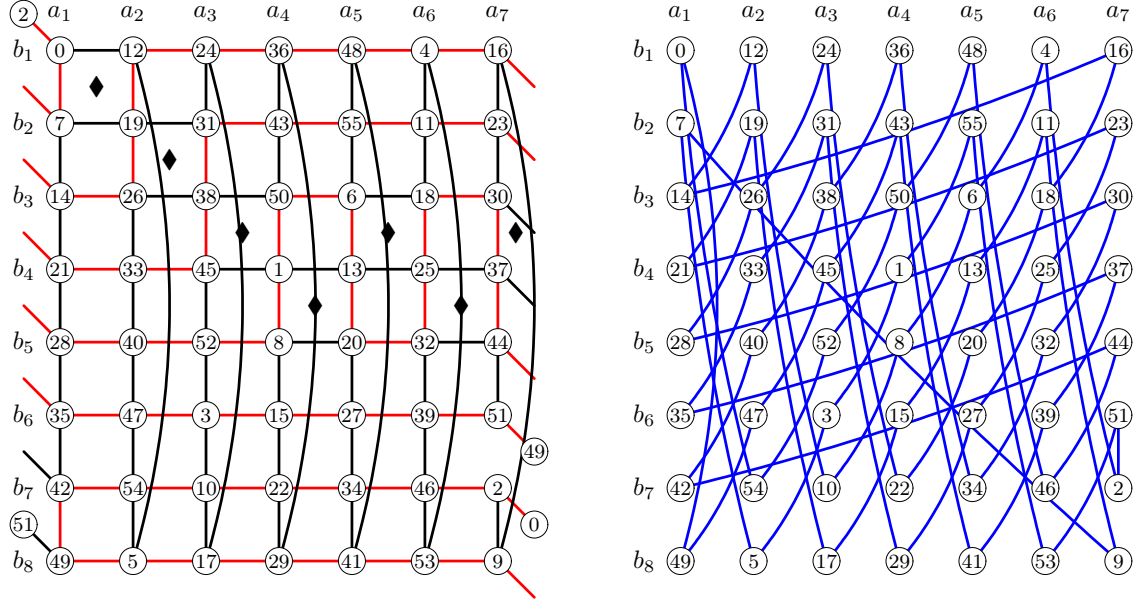
**Corollary 4.1.6.** *Let  $\Delta = \text{CAY}(A/\langle s_3 \rangle, \{\overline{s_1}, \overline{s_2}\})$  be a quotient of  $\Gamma = \text{CAY}(A, \{s_1, s_2, s_3\})$  of order at least three. If  $\overline{s_2}$  generates a Hamilton cycle in  $\Delta$ , and  $\langle s_1 \rangle$  has index 2 in  $A$ , then  $\Gamma$  has a Hamilton decomposition.*

*Proof.* As,  $\overline{s_2}$  generates a Hamilton cycle, we have  $\langle \overline{s_2} \rangle = A/\langle s_3 \rangle \Rightarrow A = \langle s_2, s_3 \rangle$ , hence,  $s_1 \in \langle s_2, s_3 \rangle$ . Furthermore,  $|A : \langle s_3 \rangle| \geq 3 > |A : \langle s_1 \rangle| = 2$ , and the result follows by Theorem 4.1.5. ■

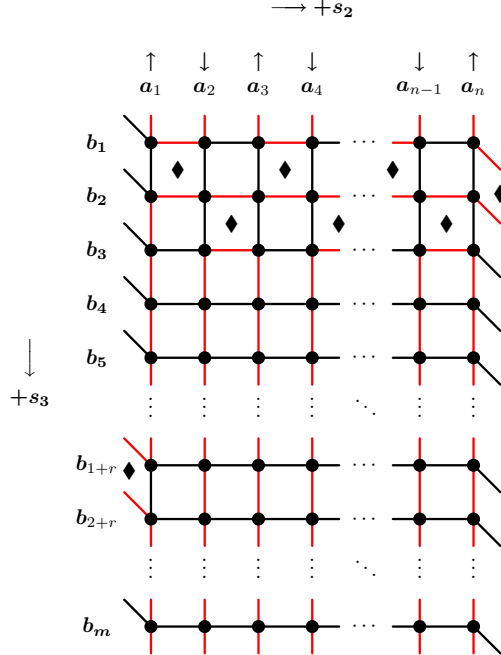
**Corollary 4.1.7.** *If  $\Gamma = \text{CAY}(\mathbb{Z}_{2m}, \{a, b, c\})$  is connected, 6-regular,  $|a| = m$ , and  $\gcd(2m, b, c) = 1$ , then  $\Gamma$  has a Hamilton decomposition.*

*Proof.* As  $\mathbb{Z}_{2m}$  has exactly one subgroup of order  $m$ , namely,  $A = \langle a \rangle$ , and  $\langle b, c \rangle = \mathbb{Z}_{2m}$ , it cannot be the case that  $|b| = |c| = |a|$ . The result follows from Theorem 4.1.5. ■

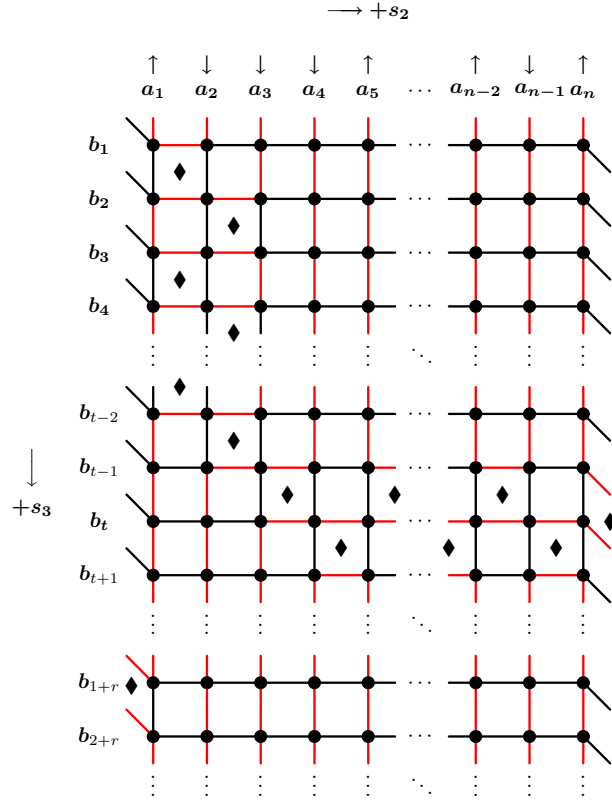
**Example 4.1.8.** Let  $\Gamma = \text{CAY}(\mathbb{Z}_{56}, \{7, 12, 2\})$ . Note  $\gcd(56, 7, 12) = 1$  and  $\langle 7, 12 \rangle = \mathbb{Z}_{56}$ , so  $\Gamma$  is 6-regular and connected. Let  $s_3 = 12$ ,  $s_2 = 7$ , and  $s_1 = 2$ . Note  $|s_3| = 14$ ,  $|s_2| = m = 8$ , and  $|s_1| = 28$ . Recall,  $\Gamma \simeq (C_7 \square_4 C_8) + F_3$ , where  $F_3$  is the 2-factor generated by 2. Here  $t = \gcd(4, 8) = 4$  and  $n = 7$ , so  $k = 2$ . Apply the switching configuration shown in Figure 4.3 to the subgraph  $C_7 \square_4 C_8$ . As  $s_3 = 2 = 6s_3 + 6s_2$ ,  $s_3$  is in the  $a_7$ -column and  $b_7$ -row, i.e.  $p = q = 7$ . By Case 1 of Lemma 4.1.3, the application of an  $\{a_2, b_1, b_3\}$ -LAVS, an  $\{a_3, a_4, b_3, b_4\}$ -CS, an  $\{a_4, a_6, b_4\}$ -LAHS, an  $\{a_6, a_7, b_4, b_5\}$ -CS, and  $\{a_7, a_1, b_3, b_4\}$ -CS creates red and black Hamilton cycles. Then applying an  $\{a_1, a_7, b_8, b_6\}$ -VOCS gives the required Hamilton decomposition, which is illustrated in Figure 4.1.



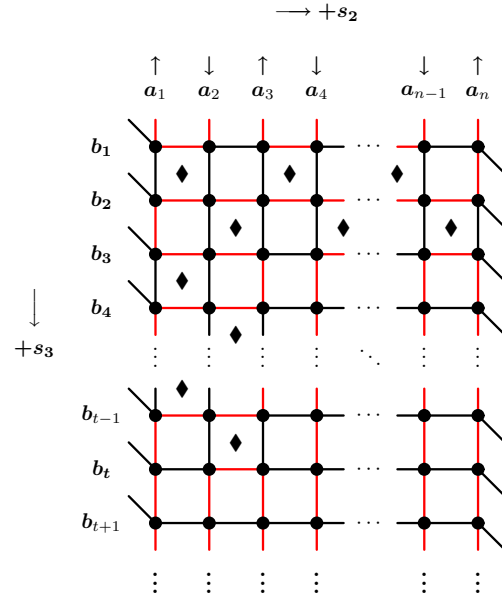
**Figure 4.1:** A Hamilton decomposition of  $\text{CAY}(\mathbb{Z}_{56}, \{7, 12, 2\}) \simeq (A_7 \square_4 B_8) \cup \text{CAY}(\mathbb{Z}_{56}, \{2\})$ , illustrating Case 2 of Lemma 4.1.3.



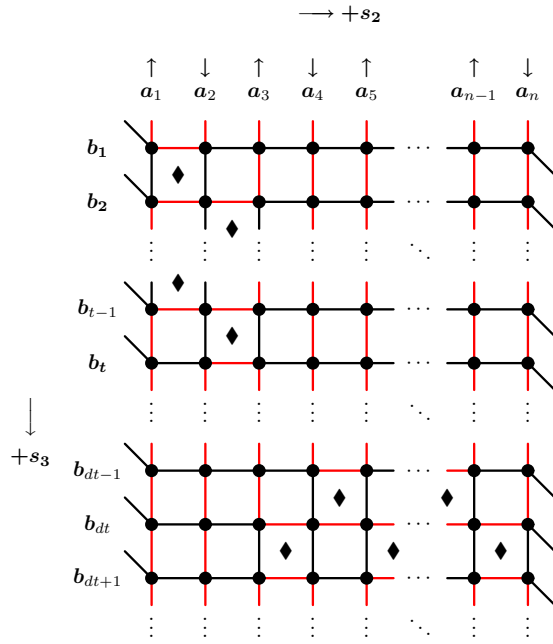
**Figure 4.2:** A CS-configuration of  $F_2 \cup F_3$  for Case 1.1(a) of Theorem 4.1.3.



**Figure 4.3:** A CS-configuration of  $F_2 \cup F_3$  for Case 1.2 of Theorem 4.1.3.



**Figure 4.4:** A CS-configuration of  $F_2 \cup F_3$  for Case 2 of Theorem 4.1.3.



**Figure 4.5:** A CS-configuration of  $F_2 \cup F_3$  for Case 1 of Theorem 4.1.4.

## Chapter 5

# Hamilton Decompositions Using Quotient Graphs

### 5.1 Preliminaries and Previous Work

In this chapter, the techniques used in Chapter 3 are adapted to obtain partial results for the even order, 6-regular, case of Alspach's conjecture. Recall from Remark 3.1.5, in a  $D(3, m, n)$ -graph, the permutations  $\pi_1$  and  $\pi_2$  determine the order of the columns in the two layers. We examine the graph union,  $A_n^{(1)} \cup A_n^{(2)}$ , whose vertices are  $\{a_1, a_2, \dots, a_n\}$ , and edges  $\{a_k, a_\ell\}$  if and only if either  $k = \pi_1(i)$  and  $\ell = \pi_1(i + 1)$  for some  $1 \leq i \leq n$ , or  $k = \pi_2(j)$  and  $\ell = \pi_2(j + 1)$  for some  $1 \leq j \leq n$ . Hence,  $A_n^{(1)} \cup A_n^{(2)}$  is a 4-regular multigraph that is, by construction, Hamilton decomposable. Throughout this chapter,  $t_1$  and  $t_2$  will denote the number of cycles in the 2-factors,  $H_1$  and  $H_2$ , respectively, in a  $D(3, m, n)$ -graph. The following lemmata were obtained in [34] and [21] by using the CS-configurations of Lemmata 2.2.9 and 2.2.11.

**Lemma 5.1.1** (Liu [34]). *If  $n \geq 8$  is even,  $m \geq 6$ ,  $t_1 = 2k_1 \geq 2$ , and  $\Pi_1 \cap \Pi_2 = \emptyset$ , where*

$$\Pi_1 = \{\pi_1(1), \pi_1(n)\} \text{ and } \Pi_2 = \{\pi_2(i) : 1 \leq i \leq 6\},$$

*for some permutations  $\pi_1$  and  $\pi_2$  of  $[n]$ , then a  $D(3, m, n)$ -graph has a Hamilton decomposition.*

**Lemma 5.1.2** (Fan et al. [21], Liu [34]). *Consider the sets*

$$\Pi_1 = \{\pi_1(1), \pi_1(2), \pi_1(3), \pi_1(4)\} \text{ and } \Pi_2 = \{\pi_2(1), \pi_2(2), \pi_2(3)\},$$

*where  $\pi_1$  and  $\pi_2$  are permutations of  $[n]$ , and  $n \geq 6$ . If  $\Pi_1 \cap \Pi_2 = \{\pi_1(4) = \pi_2(1)\}$ , then a*



$D(3, m, n)$ -graph has a Hamilton decomposition if one of the following holds:

1.  $t_i = 2k_i + 1$ , for  $i = 1, 2$ .
2.  $n$  is even, and  $t_1 = 2k_1$  and  $t_2 = 2k_2 + 1$ .

**Definition 5.1.3.** Suppose that  $G$  is a multigraph that can be decomposed into two Hamilton cycles  $C_1$  and  $C_2$ . We define the following properties:

**Property I :** there exists a path  $P = u_1u_2u_3u_4u_5$  in  $G$  such that  $P_1 = u_1u_2u_3$  is on  $C_1$  and  $P_2 = u_3u_4u_5$  is on  $C_2$ .

**Property II :** there exists a path  $P = u_1u_2u_3u_4u_5u_6$  in  $G$  such that  $P_1 = u_1u_2u_3u_4$  is on  $C_1$  and  $P_2 = u_4u_5u_6$  is on  $C_2$ .

**Property III :** there exists a path  $P = u_1u_2u_3u_4u_5u_6u_7u_8$  in  $G$  such that  $P_1 = u_1u_2$  is on  $C_1$  and  $P_2 = u_3u_4u_5u_6u_7u_8$  is on  $C_2$ .

The following result follows directly from the Pigeonhole principle.

**Lemma 5.1.4** (Fan et al. [21], Liu [32, 34]). *If  $G$  is a 4-regular multigraph of order  $n$  that can be decomposed into two Hamilton cycles,  $C_1$  and  $C_2$ , then*

$$G \text{ has } \begin{cases} \text{PROPERTY I} & \text{if } n \geq 7 \\ \text{PROPERTY II} & \text{if } n \geq 9 \\ \text{PROPERTY III} & \text{if } n \geq 13 \end{cases}$$

By combining Theorems 1.5.2, 1.5.5, 3.1.9, 3.2.1, and Lemmata 5.1.1, 5.1.2, 5.1.4, we have the following summary:

**Corollary 5.1.5** ([9, 15, 16, 21, 30, 32, 33, 34, 53]). *A  $D(3, m, n)$ -graph has a Hamilton decomposition if any one of the following are true:*

- (a)  $m \geq 3$  is odd,  $n = 3, 5, 7$ , or  $n \geq 9$ .
- (b)  $m \geq 4$  is even,  $n \geq 9$ ,  $t_1$  and  $t_2$  are odd.
- (c)  $m \geq 4$  is even,  $n \geq 10$  is even,  $t_1$  is even,  $t_2$  is odd.
- (d)  $m \geq 6$  is even,  $n \geq 14$  is even,  $t_1$  and  $t_2$  are even.

**Remark 5.1.6.** The following cases are not covered in Corollary 5.1.5:

- (e)  $m \geq 4$  is even,  $n \geq 9$  is odd,  $t_1$  and  $t_2$  are both even.
- (f)  $m \geq 4$  is even,  $n \geq 9$  is odd,  $t_1$  is even,  $t_2$  is odd.
- (g)  $m \geq 4$  is even,  $n = 10, 12$ ,  $t_1$  and  $t_2$  are even.
- (h)  $m \geq 3$ ,  $n = 3, 4, 5, 6, 7, 8$ .

Constructions for Cases (e)–(g) in Remark 5.1.6 are developed in Section 5.2, and constructions for Case (h) in Section 5.3.

## 5.2 Decomposing Layered Pseudo-Cartesian Products

**Lemma 5.2.1.** *If  $G$  is a multigraph with  $V(G) = \{a_1, a_2, \dots, a_n\}$ , that can be decomposed into two cycles,*

$$C_1 = a_{\pi_1(1)} a_{\pi_1(2)} \cdots a_{\pi_1(n)} \text{ and } C_2 = a_{\pi_2(1)} a_{\pi_2(2)} \cdots a_{\pi_2(n)},$$

*for some permutations  $\pi_1$  and  $\pi_2$  of  $[n]$ , then*

1. *if  $n = 12$ , and  $G$  does not have PROPERTY III, there exists an edge  $u_1 u_2$  in  $C_1$  dividing  $C_2$  into two paths, each on six vertices, so w.l.o.g.,  $\pi_1(1) = \pi_2(6) = 1$  and  $\pi_1(12) = \pi_2(12) = 12$ .*
2. *if  $n = 10$ , there exists a path  $P = u_1 u_2$  in  $C_1$  that either divides  $C_2$  into*
  - (a) *one path on six vertices and one path on four vertices, so w.l.o.g.,  $\pi_1(1) = \pi_2(6) = 1$ , and  $\pi_1(10) = \pi_2(10) = 10$ , or*
  - (b) *two paths on five vertices, so w.l.o.g.,  $\pi_1(1) = \pi_2(5)$  and  $\pi_1(10) = \pi_2(10)$ .*

**Lemma 5.2.2.** *If  $t_1 = 2k_1 \geq 2$ ,  $m > 2k_2 = t_2$ , and  $m \geq 6$ , then a  $D(3, m, 12)$ -graph has a Hamilton decomposition.*

*Proof.* We closely follow the technique of the proof of Lemma 5.1.1 (Lemma 3.18 in [34]). If the multigraph  $A_{12}^{(1)} \cup A_{12}^{(2)}$  corresponding to the  $D(3, m, 12)$ -graph has PIII, we are done by Lemma 5.1.1. In particular, we may assume  $\pi_1 \neq \pi_2$ . Thus, by Lemma 5.2.1, it is assumed that, up to a relabeling of the vertices,

$$A_{12}^{(2)} = a_{\pi_2(1)}^2 a_{\pi_2(2)}^2 a_{\pi_2(3)}^2 a_{\pi_2(4)}^2 a_{\pi_2(5)}^2 a_1^2 a_{\pi_2(7)}^2 a_{\pi_2(8)}^2 a_{\pi_2(9)}^2 a_{\pi_2(10)}^2 a_{\pi_2(11)}^2 a_{12}^2 a_{\pi_2(1)}^2.$$

Apply the CS-configuration of Lemma 2.2.9 to  $H_1 \cup F \simeq A_{12} \square_{r_1} B_m$  to obtain a blue Hamilton cycle and join the red edges in the  $a_1$  and  $a_{12}$ -columns into a single cycle. Note that the vertical red and blue edges in the  $a_1$ - and  $a_{12}$ -columns also form a matching. We now define color-switches in  $A_{12}^{(2)} \square_{r_2} B_m$ . Assume that  $t_2 \geq 6$ , because the proof for the cases  $t_2 = 2$  or  $4$  is very similar, and we have more freedom to define a CS-configuration. Apply the CS-configuration of Lemma 2.2.11 to  $H_2 \cup F \simeq A_{12}^{(2)} \square_{r_2} B_m$ , by starting with  $X_{1+\ell}$ , where  $\ell$  is any integer such that the edge  $e = (a_1^2, b_{5+\ell})(a_1^2, b_{6+\ell})$  is red. By Remark 2.2.2 and Lemma 2.2.11, there now exists a black Hamilton cycle, and a cycle consisting of vertical red edges in the  $a_1^2$ ,  $a_{12}^2$ , and  $a_{\pi_2(i)}^2$ -columns, where  $1 \leq i \leq 5$ . Call this red cycle the  $\star$ -cycle. Every column except the  $a_1^2$  and  $a_{12}^2$ -columns, has a red path of length at least four. For any integers  $2 \leq i \leq 11$  and  $1 \leq j \leq m$ , at least one of the two edges,  $e_j = (a_i, b_j)(a_i, b_{j+1})$  and  $e_{j+2} = (a_i, b_{j+2})(a_i, b_{j+3})$ , is red. We now define additional color-switches in  $H_1 \cup F \simeq A_{12} \square_{r_1} B_m$  to create three monochromatic Hamilton cycles. Let

$$\{x_1, x_2, x_3, x_4, x_5\} = \{\pi_2(i) : 7 \leq i \leq 11\}, \text{ where } 1 < x_1 < x_2 < x_3 < x_4 < x_5 < 12.$$

Define  $X_i = \{a_{x_i}, a_{x_i+1}, b_{w_i}, b_{w_i+1}\}$ -CS, and let  $\mathcal{X} = \{X_i : 1 \leq i \leq 5\}$ . By Remark 2.2.2, if  $\mathcal{X}$  is applied to  $A_{12} \square_{r_1} B_m$ , the six red cycles are joined with the  $\star$ -cycle into a red Hamilton cycle,  $C_R$ . If  $C_R$  is given a direction, all vertical edges in a fixed column have the same direction, the  $a_{x_i}$  and  $a_{x_i+1}$ -columns have opposite directions, and by Lemma 2.2.9, the  $a_1$  and  $a_{12}$ -columns have the same direction. Thus, there exists an integer  $x$ ,  $1 \leq x \leq 11$ , such that the  $a_x$  and  $a_{x+1}$ -columns have the same direction. Any color-switch between these two columns will preserve the red cycle. Thus, let

$$\{x, x_1, x_2, x_3, x_4, x_5\} = \{y_i : 1 \leq i \leq 6\}, \text{ where } 1 \leq y_1 < y_2 < y_3 < y_4 < y_5 < y_6 < 12,$$

and define  $Y_i = \{a_{y_i}, a_{y_i+1}, b_{z_i}, b_{z_i+1}\}$ -CS, and let  $\mathcal{Y} = \{Y_i : 1 \leq i \leq 6\}$ . Upon applying the color-switches in  $\mathcal{Y}$  to  $A_{12} \square_{r_1} B_m$ , a red Hamilton cycle will be obtained. We now define good pairs (see Definition 2.2.14)  $\{Y_1, Y_2\}$ ,  $\{Y_3, Y_4\}$ , and  $\{Y_5, Y_6\}$ , so that applying  $\mathcal{Y}$  will preserve the blue Hamilton cycle, and yield the result. There are two cases:

1.  $x \notin \{y_{2j-1}, y_{2j}\}$ . The  $a_{y_{2j-1}}$ -column has at most one non-red edge, the  $a_{y_{2j}}$ -column has all red edges, and  $1 < y_{2j-1} < 11$ . Let  $y \equiv 0 \pmod{2}$  be any integer such that there exists a red 3-path in the  $a_{y_{2j-1}}$ -column between the  $b_y$  and  $b_{y+3}$ -rows. Let

$$e_y = (a_{y_{2j-1}+1}, b_y)(a_{y_{2j-1}+1}, b_{y+1}) \quad \text{and} \quad f_y = (a_{y_{2j}+1}, b_y)(a_{y_{2j}+1}, b_{y+1}).$$

At least one of  $e_y$  and  $e_{y+2}$  is red. Define the good pair  $\{Y_{2j-1}, Y_{2j}\}$  as follows:

$$Y_{2j-1} = \begin{cases} \{a_{y_{2j-1}}, a_{y_{2j-1}+1}, b_y, b_{y+1}\}\text{-CS} & \text{if } e_y \text{ is red} \\ \{a_{y_{2j-1}}, a_{y_{2j-1}+1}, b_{y+2}, b_{y+3}\}\text{-CS} & \text{if } e_y \text{ is not red} \end{cases} \quad (5.1)$$

$$Y_{2j} = \begin{cases} \{a_{y_{2j}}, a_{y_{2j}+1}, b_{y+1}, b_{y+2}\}\text{-CS} & \text{if } f_{y+1} \text{ is red} \\ \{a_{y_{2j}}, a_{y_{2j}+1}, b_{y-1}, b_y\}\text{-CS} & \text{if } f_{y+1} \text{ is not red, } e_y \text{ is red} \\ \{a_{y_{2j}}, a_{y_{2j}+1}, b_{y+3}, b_{y+4}\}\text{-CS} & \text{if } f_{y+1} \text{ is not red, } e_y \text{ is not red} \end{cases} \quad (5.2)$$

2.  $x \in \{y_{2j-1}, y_{2j}\}$ . Clearly,  $x \in \{1, \pi_2(1), \dots, \pi_2(5)\}$ , the  $a_x$ -column has at most one blue edge, and the  $a_{x+1}$ -column has no blue edges. Furthermore, by Equation (5.2), any blue edge in the  $a_x$ -column lies between the  $b_z$  and  $b_{z+1}$ -rows, for some integer  $z \equiv 1 \pmod{2}$ .

- (a)  $x = y_{2j-1}$ . If  $x = 1$ , select an integer  $y \equiv 0 \pmod{2}$  such that both

$$(a_1, b_y)(a_1, b_{y+1}) \quad \text{and} \quad (a_1, b_{y+2})(a_1, b_{y+3})$$

are red. Define  $Y_{2j-1}$  as in Equation (5.1). If  $x \neq 1$ , let  $P$  be a longest path of red edges in the  $a_{x+1}$ -column. Because  $x+1 \neq 1, 12$ , and  $m \geq 2t_2$ ,  $P$  has length at least four. Therefore, there exists an integer  $y \equiv 0 \pmod{2}$  such that both  $(a_{x+1}, b_y)(a_{x+1}, b_{y+1})$  and  $(a_{x+1}, b_{y+2})(a_{x+1}, b_{y+3})$  lie on  $P$ . If  $e_y = (a_x, b_y)(a_x, b_{y+1})$  is red, then define  $Y_{2j-1} = \{a_x, a_{x+1}, b_y, b_{y+1}\}$ -CS. If  $e_y$  is black, then define  $Y_{2j-1} = \{a_x, a_{x+1}, b_{y+2}, b_{y+3}\}$ -CS. Having defined  $Y_{2j-1}$ , it is clear that the  $a_{y_{2j}}$ -column contains at most one non-red edge. Define  $Y_{2j}$  similarly to Equation (5.2) to obtain a good pair  $\{Y_{2j-1}, Y_{2j}\}$ .

- (b)  $x = y_{2j}$ . There exists an integer  $y \equiv 1 \pmod{2}$ , such that both  $(a_{y_{2j}}, b_y)(a_{y_{2j}}, b_{y+1})$  and  $(a_{y_{2j}+1}, b_y)(a_{y_{2j}+1}, b_{y+1})$  are red. Define  $Y_{2j} = \{a_{y_{2j}}, a_{y_{2j}+1}, b_y, b_{y+1}\}$ -CS. Again, the  $a_{y_{2j-1}}$ -column contains all red edges, except possibly one blue edge

$$(a_{y_{2j-1}}, b_z)(a_{y_{2j-1}}, b_{z+1}),$$

for some  $z \equiv 1 \pmod{2}$ . Thus, either

$$Y_{2j-1} = \{a_{y_{2j-1}}, a_{y_{2j-1}+1}, b_{y-1}, b_y\}\text{-CS}$$

or

$$Y_{2j-1} = \{a_{y_{2j-1}}, a_{y_{2j-1}+1}, b_{y+1}, b_{y+2}\}\text{-CS}$$

is a good switch, making  $\{Y_{2j-1}, Y_{2j}\}$  a good pair.

Finally, the blue Hamilton cycle is preserved upon applying  $Y_{2j-1}$  and  $Y_{2j}$ . To see this, after applying

$Y_{2j-1}$ , the blue Hamilton cycle is broken into one cycle

$$(a_{y_{2j-1}+1}, b_{\hat{y}})(a_{y_{2j-1}+1}, b_{\hat{y}+1})(a_{y_{2j-1}+2}, b_{\hat{y}+1}) \cdots (a_{12}, b_{\hat{y}+1})(a_{12}, b_{\hat{y}})(a_{11}, b_{\hat{y}}) \cdots (a_{y_{2j-1}+1}, b_{\hat{y}}),$$

for some  $\hat{y} \equiv 0 \pmod{2}$ , and one cycle on the remaining vertices. By Remark 2.2.2,  $Y_{2j}$  restores the blue Hamilton cycle. ■

**Lemma 5.2.3.** *If  $t_1 = 2k_1 \geq 2$  and  $t_2 = m = 2k \geq 6$ , then a  $D(3, m, 12)$ -graph has a Hamilton decomposition.*

*Proof.* We proceed similarly to the proof of Lemma 5.2.2 with only slight modifications. Again, apply the CS-configuration of Lemma 2.2.9 to  $H_1 \cup F \simeq A_{12} \square_{r_1} B_m$ .

1.  $m \equiv 2 \pmod{4}$ . Apply the CS-configuration of Lemma 2.2.11 to  $H_2 \cup F \simeq A_{12}^{(2)} \square_{r_2} B_m$ , by starting with  $X_{1+\ell}$ , where  $\ell$  is any integer such that the edge

$$e = (a_1^2, b_{5+\ell})(a_1^2, b_{6+\ell})$$

is red. For any integers  $2 \leq i \leq 11$  and  $1 \leq j \leq m$ , at least one of the two edges,  $e_j = (a_i, b_j)(a_i, b_{j+1})$  and  $e_{j+2} = (a_i, b_{j+2})(a_i, b_{j+3})$ , is red. In this case, every column other than the  $a_1^2$  and  $a_{12}^2$ -columns contain either at least a 4-path of red edges, or contain an 8-path, consisting of three red edges, then two black edges, then three red edges again. We denote the former  $P_4^R$  and the latter  $P_3^R - P_2^B - P_3^R$ . We construct the CS-configuration

$$\mathcal{Y} = \{Y_i : 1 \leq i \leq 6\},$$

defined in Lemma 5.2.2 to again preserve the blue Hamilton cycle. If  $x \notin \{y_{2j-1}, y_{2j}\}$ , define the good pair  $\{Y_{2j-1}, Y_{2j}\}$  as in Equations (5.1) and (5.2). If  $x \in \{y_{2j-1}, y_{2j}\}$ , then the  $a_x$ -column initially had either a  $P_4$  or  $P_3^R - P_2^B - P_3^R$ . Again, any blue edge lies between the  $b_z$  and  $b_{z+1}$ -rows, for some integer  $z \equiv 1 \pmod{2}$ . It is easy to see that there exists an integer  $y \equiv 0 \pmod{2}$  such that both  $(a_{x+1}, b_y)(a_{x+1}, b_{y+1})$  and  $(a_{x+1}, b_{y+2})(a_{x+1}, b_{y+3})$  are red. Apply the technique of Case 2 of Lemma 5.2.2 to define a good switch.

2.  $m \equiv 0 \pmod{4}$ . Let  $m = 8 + 4d$ , where  $d \geq 0$ . Define

$$D_i = \{a_{\pi_2(i)}^2, a_{\pi_2(i+1)}^2, b_{i+\ell}, b_{i+1+\ell}\}\text{-CS},$$

and apply the CS-configuration  $\mathcal{D} = \{D_i : 1 \leq i \leq 5\}$ , where  $\ell$  is chosen so that  $e$ , defined as in Case 1, is red. Apply an  $\{a_1^2, a_{\pi_2(8)}^2, b_{7+\ell}\}$ -LAHS. If  $m > 8$ , define the CS-configuration,

$$\mathcal{E}_i = \begin{cases} \{a_{\pi_2(1)}^2, a_{\pi_2(2)}^2, b_{8+4i+\ell}, b_{8+4i+1+\ell}\}\text{-CS}, \\ \{a_{\pi_2(2)}^2, a_{\pi_2(3)}^2, b_{9+4i+\ell}, b_{9+4i+1+\ell}\}\text{-CS}, \\ \{a_{\pi_2(3)}^2, a_{\pi_2(4)}^2, b_{10+4i+\ell}, b_{10+4i+1+\ell}\}\text{-CS}, \\ \{a_{\pi_2(4)}^2, a_{\pi_2(5)}^2, b_{11+4i+\ell}, b_{11+4i+1+\ell}\}\text{-CS} \end{cases}$$

Apply  $\mathcal{E} = \{\mathcal{E}_i : 0 \leq i \leq d-1\}$ . We now have a black Hamilton cycle, and a red 2-factor consisting of one cycle on the  $a_{12}^2$  and  $a_{\pi_2(i)}^2$ -columns, where  $1 \leq i \leq 8$ , and three cycles, one on each of the  $a_{\pi_2(i)}^2$ -columns, where  $9 \leq i \leq 11$ . Furthermore, every column, except the  $a_1^2$  and  $a_{12}^2$ -columns, have a red 5-path, and for any integers  $2 \leq i \leq 11$  and  $1 \leq j \leq m$ , at least one

of the two edges,  $e_j = (a_i, b_j)(a_i, b_{j+1})$  and  $e_{j+2} = (a_i, b_{j+2})(a_i, b_{j+3})$ , is red. Use the method of Lemma 5.2.2 to obtain the result. ■

**Lemma 5.2.4.** *Let  $\Gamma = \text{CAY}(A, \{s_1, s_2, s_3\})$  be a connected, 6-regular abelian Cayley graph. If  $|s_3| \geq 6$ ,  $[A : \langle s_3 \rangle] = 10$ , and  $2s_1, 2s_2 \notin \langle s_3 \rangle$ , then  $\Gamma$  has a Hamilton decomposition.*

*Proof.*  $\Gamma$  is a  $D(3, m, 10)$ -graph, and by Corollary 5.1.5, we may assume  $t_1$  and  $t_2$  are even.  $A/\langle s_3 \rangle \cong \mathbb{Z}_{10}$ , and so

$$\Delta = \text{CAY}(A/\langle s_3 \rangle, \{\overline{s_1}, \overline{s_2}\}) \simeq \text{CAY}(\mathbb{Z}_{10}, \{x, y\}),$$

where  $\langle x, y \rangle = \mathbb{Z}_{10}$ , and  $|x|, |y| \geq 3$ . Without loss of generality, up to a relabeling of the vertices,

$$\{x, y\} \in \{\{\pm 1\}, \{\pm 3\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{3, 4\}\}.$$

If  $\{x, y\} \in \{\{\pm 1\}, \{\pm 3\}\}$ , then  $\Delta$  is a multigraph. Thus  $\overline{H_1} = \overline{H_2}$ , and both cycles are generated by a single element. Trivially, the path  $P = 0, 1, 2, 3, 4, 5, 6, 7$  has PROPERTY III of Lemma 5.1.4, so  $\Gamma$  has a Hamilton decomposition by Lemma 5.1.1. If  $\{x, y\} = \{1, 3\}$ , then

$$\overline{H_1} := 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0 \quad \text{and} \quad \overline{H_2} := 0, 3, 6, 9, 2, 5, 8, 1, 4, 7, 0$$

is a Hamilton decomposition, and  $P = 0, 3, 4, 5, 6, 7, 8, 9$  has PROPERTY III. Similarly, if  $\{x, y\} = \{1, 2\}$ , then

$$\overline{H_1} := 0, 2, 1, 3, 4, 5, 6, 7, 8, 9, 0 \quad \text{and} \quad \overline{H_2} := 7, 5, 3, 2, 4, 6, 8, 0, 1, 9, 7$$

is a Hamilton decomposition, and  $P = 0, 9, 7, 5, 3, 2, 4, 6$  has PROPERTY III. If  $\{x, y\} = \{2, 3\}$ , then

$$\overline{H_1} := 0, 2, 9, 1, 3, 5, 8, 6, 4, 7, 0 \quad \text{and} \quad \overline{H_2} := 5, 2, 4, 1, 8, 0, 3, 6, 9, 7, 5$$

is a Hamilton decomposition of  $\Delta$ . Now, by Lemma 5.2.1, we may assume  $\pi_1(1) = \pi_2(6) = 1$  and  $\pi_1(10) = \pi_2(10) = 10$ . Thus, we may use the technique of Lemma 5.2.2 to find a Hamilton decomposition of  $\Gamma$ . The result is obtained similarly for  $\{x, y\} = \{1, 4\}$  and  $\{x, y\} = \{3, 4\}$ . ■

**Lemma 5.2.5.** *If  $\Delta = \text{CAY}(B, \{s_1, s_2\})$  is a 4-regular, abelian Cayley graph on  $B = \{b_1, b_2, \dots, b_n\}$ , and  $E(\Delta) = H_1 \cup H_2$  is a Hamilton decomposition of  $\Delta$ , where*

$$H_k := b_{\pi_k(1)} b_{\pi_k(2)} b_{\pi_k(3)} \cdots b_{\pi_k(n)} b_{\pi_k(1)},$$

*for  $k = 1, 2$ , such that  $\pi_1(i) = \pi_2(j)$  for some  $1 \leq i, j \leq n$ , then*

$$|\{\pi_2(j-1), \pi_2(j+1)\} \cap \{\pi_1(i-1), \pi_1(i+1)\}| \geq 1 \Leftrightarrow s_1 = \pm s_2.$$

*Proof.* If  $s_1 = \pm s_2$ , then  $\Delta$  is a multigraph, and so  $H_1 = H_2$ , and the result follows. Conversely, if

$$\pi_1(i+1) \in \{\pi_2(j-1), \pi_2(j+1)\}$$

or

$$\pi_1(i-1) \in \{\pi_2(j-1), \pi_2(j+1)\},$$

then either the edge  $e = \{b_{\pi_1(i-1)}, b_{\pi_1(i)}\}$  or  $f = \{b_{\pi_1(i)}, b_{\pi_1(i+1)}\}$  appears on both  $H_1$  and  $H_2$ . Without loss of generality, suppose  $e$  is a multiedge, and  $b_{\pi_1(i-1)} - b_{\pi_1(i)} = s_1$ . If  $b_{\pi_1(i-1)} - b_{\pi_1(i)} = -s_1$ , then  $|s_1| = 2$ , a contradiction, as  $|s_i| \geq 3$ . Thus, we must have  $s_1 = \pm s_2$ , so that  $H_1 = H_2$ . ■

*Remark 5.2.6.* By Lemma 5.2.5, any  $D(3, m, n)$ -graph arising from a 6-regular Cayley graph  $\text{CAY}(A, \{s_1, s_2, s_3\})$  with a 4-regular quotient  $\text{CAY}(A/\langle s_3 \rangle, \{\overline{s_1}, \overline{s_2}\})$ , will have two columns adjacent in both  $H_1 \cup F = A_n^{(1)} \square_{r_1} B_m$  and  $H_2 \cup F = A_n^{(2)} \square_{r_2} B_m$  if and only if  $\overline{s_1} = \pm \overline{s_2}$ .

**Lemma 5.2.7.** *If  $\Gamma = \text{CAY}(A, \{s_1, s_2, s_3\})$  is a connected, 6-regular, Cayley graph on  $A$ , where  $|s_3| \geq 5$ ,  $[A : \langle s_3 \rangle] \geq 9$ , and  $2s_1, 2s_2 \notin \langle s_3 \rangle$ , then  $\Gamma$  has a Hamilton decomposition.*

*Proof.* We follow the technique and notation of ([32], Lemma 3.14). Let  $J = \langle s_3 \rangle$ ,  $|J| = m$ , and  $[A : J] = n$ . Clearly,  $\Delta = \text{CAY}(A/J, \{\overline{s_1}, \overline{s_2}\})$  is 4-regular and connected, where  $A/J = \{\overline{a_1}, \overline{a_2}, \dots, \overline{a_n}\}$ . By Theorem 1.5.1,  $\Delta$  has a Hamilton decomposition into two cycles:

$$\overline{H_1} = \overline{a_{\pi_1(1)}} \overline{a_{\pi_1(2)}} \cdots \overline{a_{\pi_1(n)}} \overline{a_{\pi_1(1)}} \text{ and } \overline{H_2} = \overline{a_{\pi_2(1)}} \overline{a_{\pi_2(2)}} \cdots \overline{a_{\pi_2(n)}} \overline{a_{\pi_2(1)}},$$

for some permutations  $\pi_1$  and  $\pi_2$  of  $[n]$ . By Theorem 3.1.6,  $\Gamma$  is a  $D(3, m, n)$ -graph and by Remark 3.1.5,  $H_1 \cup F \simeq A_n^{(1)} \square_{r_1} B_m$  and  $H_2 \cup F \simeq A_n^{(2)} \square_{r_2} B_m$ , where  $F$  is the 2-factor generated by  $s_3$ , and the 2-factors  $H_1$  and  $H_2$  each consist of  $t_1$  and  $t_2$  cycles, respectively. By Theorem 5.1.5, we may assume  $m = 2k \geq 6$  and  $n = 2x + 1 \geq 9$ . Hence, there are exactly two cases to consider: (1) both  $t_1$  and  $t_2$  are even, and (2) one of  $t_1$  and  $t_2$  are even, the other is odd.

**Case 1:** Let  $t_i = 2k_i \geq 2$  for  $i = 1, 2$ .

I.  $\overline{s_1} = \pm \overline{s_2}$ . Without loss of generality,  $\pi_1 = \pi_2 = (1)$ . Apply the CS-configuration of Lemma 2.2.11 to  $A_n \square_{r_1} B_m$  to obtain a blue Hamilton cycle and join the vertical red cycles in the  $a_i$ -columns into one red cycle for  $1 \leq i \leq d$ , where  $d = 2$  if  $t_1 = 2$ ,  $d = 4$  if  $t_1 = 4$ , and  $d = 6$  if  $t_1 \geq 6$ . For convenience of notation, we may assume  $t_1 \geq t_2$ .

(a) If  $t_1 \geq 6$ , let  $h$  be the integer such that  $e = (a_6^2, b_{1+h})(a_6^2, b_{2+h})$  is blue. If  $t_2 \geq 4$ , then apply an  $\{a_7^2, b_{1+h}, b_{t_2-1+h}\}$ -RAVS to  $H_2 \cup F = A_n^{(2)} \square_{r_2} B_m$ . By Theorem 2.2.7(b), this creates a black Hamilton cycle and joins the vertical red edges in the  $a_i$ -columns where  $1 \leq i \leq 9$  into one cycle. Let  $d = n - 9$ . As  $d \equiv 0 \pmod{2}$ , we may apply  $\{a_9^2, a_n^2, b_{t_2-1+h}\}$ -RAHS to  $H_2 \cup F$  to join the red edges in the  $a_i$ -columns with  $1 \leq i \leq n$  into a Hamilton cycle and preserve the black cycle, by Lemma 2.2.15. The result is a Hamilton decomposition of  $\Gamma$ . If  $t_2 = 2$ , apply an  $\{a_6^2, a_7^2, b_{2+h}, b_{3+h}\}$ -CS to  $H_2 \cup F$  to obtain a black Hamilton cycle and join all red edges in the  $a_i$ -columns with  $1 \leq i \leq 7$  into one red cycle. Then apply an  $\{a_7^2, a_n^2, b_{2+h}\}$ -RAHS to  $H_2 \cup F$  to join the red edges in the  $a_i$ -columns with  $1 \leq i \leq n$  into a Hamilton cycle, preserve the black cycle, and obtain a Hamilton decomposition of  $\Gamma$ .

(b) If  $t_1 \in \{2, 4\}$ , apply to  $H_1 \cup F$  an  $\{a_2, a_6, b_2\}$ -LAHS if  $t_1 = 2$  or an  $\{a_4, a_6, b_3\}$ -RAHS if  $t_1 = 4$ . This joins all vertical red edges in the  $a_i$ -columns with  $1 \leq i \leq 6$  into one cycle, and preserves the blue Hamilton cycle by Lemma 2.2.15. Apply the technique of (a) to  $H_2 \cup F$  to obtain the result.

II.  $\overline{s_1} \neq \pm \overline{s_2}$ . By Lemma 5.1.4,  $\Delta$  has PROPERTY II, so there exists a path  $P$  in  $\Delta$ , where

$$P = \overline{a_{\pi_1(1)}} \overline{a_{\pi_1(2)}} \overline{a_{\pi_1(3)}} \overline{a_{\pi_1(4)}} \overline{a_{\pi_2(2)}} \overline{a_{\pi_2(3)}}.$$

W.l.o.g., we may take  $\pi_1$  to be a cyclic permutation of  $[n]$ , so that  $\pi_1(4) = \pi_2(1) = p$ , and  $\{\pi_2(2), \pi_2(3)\} = \{q, n\}$ , where  $4 \leq p < q < n$ . Thus, relocate the  $r_1$ -jump to be between the  $a_1$  and  $a_n$ -columns rather than the  $a_{\pi_1(1)}$  and  $a_{\pi_1(n)}$ -columns. Thus,

$$A_n^{(1)} = a_1 a_2 a_3 \cdots a_p \cdots a_q \cdots a_{n-1} a_n a_1, \text{ and } A_n^{(2)} = a_p^2 a_{\pi_2(2)}^2 a_{\pi_2(3)}^2 \cdots a_{\pi_2(n)}^2 a_p^2,$$

where, by Remark 5.2.6,  $q \neq n - 1$ . This time, as opposed to Case 1.I, assume  $t_2 \geq t_1$ . Apply the CS-configuration to  $H_1 \cup F \simeq A_n^{(1)} \square_{r_1} B_m$ ,

$$\begin{cases} \{\{a_1, a_2, b_2, b_3\}\text{-CS}, \{a_2, a_3, b_1, b_2\}\text{-CS}, \{a_3, a_4, b_2, b_3\}\text{-CS}\} & \text{if } t_1 = 2 \\ \{\{a_2, b_1, b_{t_1-1}\}\text{-RAVS}, \{a_3, a_4, b_{t_1-1}, b_{t_1}\}\text{-CS}\} & \text{if } t_1 \geq 4 \end{cases} \quad (5.3)$$

Upon applying (5.3), by Remark 2.2.2, and either Lemma 2.2.15 or Theorem 2.2.7(b), a blue Hamilton cycle is created, and red edges in the  $a_i$ -columns are joined into a single cycle, where  $1 \leq i \leq 4$ . Next, as  $n$  is odd, apply the following CS-configuration to  $H_1 \cup F$ :

$$\begin{cases} \{a_4, a_{n-1}, b_1\}\text{-RAHS or an } \{a_4, a_{n-1}, b_2\}\text{-LAHS} & \text{if } t_1 = 2 \\ \{a_4, a_{n-1}, b_{t_1-1}\}\text{-RAHS or an } \{a_4, a_{n-1}, b_{t_1}\}\text{-LAHS} & \text{if } t_1 \geq 4 \end{cases} \quad (5.4)$$

By Lemma 2.2.15, the color-switches in (5.4) will preserve the blue Hamilton cycle. Remove the two color-switches that are incident to the  $a_q$ -column. The blue Hamilton cycle is still preserved, by Corollary 2.2.16, and there exists a red 2-factor consisting of four cycles, call them the  $\diamond$ ,  $\clubsuit$ ,  $\heartsuit$ , and  $\spadesuit$ -cycles\*, where

- (a) the  $\diamond$ -cycle consisting of the  $a_i$ -columns with  $1 \leq i \leq q - 1$ ,
- (b) the  $\clubsuit$ -cycle of red edges of the  $a_{\pi_2(2)}$ -column,
- (c) the  $\heartsuit$ -cycle of red edges of the  $a_{\pi_2(3)}$ -column, and
- (d) the  $\spadesuit$ -cycle consisting of the  $a_i$ -columns with  $q + 1 \leq i \leq n - 1$ .

Note, the  $a_p$ -column contains exactly one blue edge if  $p = q - 1$ , (which, by Lemma 5.2.5, is only possible if  $q = \pi_2(3)$ ) or two consecutive blue edges if  $p \neq q - 1$ . In the latter case, these edges are  $\{e_1, e_2\}$  if a LAHS was applied in (5.4) or  $\{e_1, e_3\}$  if a RAHS was applied in (5.4):

$$\begin{aligned} e_1 &= (a_p, b_y)(a_p, b_{y+1}), \\ e_2 &= (a_p, b_{y+1})(a_p, b_{y+2}), \\ e_3 &= (a_p, b_{y-1})(a_p, b_y), \end{aligned}$$

and  $y = 1$  if  $t_1 = 2$  or  $y = t_1 - 1$  if  $t_1 \geq 4$ . We now define a CS-configuration for  $H_2 \cup F$ . Let  $\ell$  be the integer such that  $(a_p^2, b_{m-1+\ell}) = (a_p, b_y)$ . In  $H_2 \cup F \simeq A_n^{(2)} \square_{r_2} B_m$ , the edges  $e_1$ ,  $e_2$ , and  $e_3$  are:

$$\begin{aligned} e_1 &= (a_p^2, b_{m-1+\ell})(a_p^2, b_{m+\ell}) \\ e_2 &= (a_p^2, b_{m+\ell})(a_p^2, b_{1+\ell}) \\ e_3 &= (a_p^2, b_{m-2+\ell})(a_p^2, b_{m-1+\ell}) \end{aligned}$$

Apply the following CS-configuration to  $H_2 \cup F \simeq A_n^{(2)} \square_{r_2} B_m$ :

$$\begin{cases} \{a_{\pi_2(2)}^2, b_{1+\ell}, b_{3+\ell}\}\text{-LAVS or -RAVS} & \text{if } t_2 = 2 \\ \{a_{\pi_2(2)}^2, b_{1+\ell}, b_{t_2-1+\ell}\}\text{-LAVS or -RAVS} & \text{if } t_2 \geq 4 \end{cases} \quad (5.5)$$

(Note, if  $t_2 = m$  and  $e_3$  is blue, we are forced to apply a LAVS in (5.5).) By Remark 2.2.2, the color-switches of (5.5) create a black 2-factor consisting of two cycles. Furthermore, as these

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\*named in honor of Brian Alspach, who is an avid poker player and poker author.

color-switches are between the  $a_p^2$ ,  $a_q^2$ , and  $a_n^2$ -columns (in some order), a red 2-factor is created that consists of the  $\spadesuit$ -cycle, and a new cycle formed by joining the  $\diamond$ ,  $\clubsuit$ , and  $\heartsuit$ -cycles together, which will be called the  $\star$ -cycle. Let  $3 \leq z \leq n-1$  be the smallest integer such that the red edges in the  $a_{\pi_2(z)}^2$ -column are in the  $\star$ -cycle, but the red edges in the  $a_{\pi_2(z+1)}^2$ -column are in the  $\spadesuit$ -cycle. By Remark 2.2.2, a color-switch,  $X$ , between the  $a_{\pi_2(z)}^2$  and  $a_{\pi_2(z+1)}^2$ -columns will create a red Hamilton cycle. The  $\spadesuit$ -cycle  $a_{\pi_2(i)}^2$ -columns have either two consecutive blue edges, or exactly one blue edge if  $\pi_2(i) \in \{q+1, n-1\}$ . On the other hand, the  $\star$ -cycle  $a_{\pi_2(i)}^2$ -columns, where  $4 \leq i \leq n-1$ , have the following possible forms:

- i.  $\pi_2(i) = 1$ , thus containing an alternating path of  $(t_1 - 2)/2$  red and blue edges when  $t_1 \geq 4$  or one blue edge when  $t_1 = 2$ .
- ii.  $\pi_2(i) = 2$ , thus containing a path of  $t_1 - 2$  blue edges when  $t_1 \geq 4$  or two consecutive blue edges when  $t_1 = 2$ .
- iii.  $\pi_2(i) = 3$ , thus containing an alternating path of  $t_1/2$  red and blue edges when  $t_1 \geq 4$  or two consecutive blue edges when  $t_1 = 2$ .
- iv. contain either one or two (consecutive) blue edges. Call these columns,  $\star_2$ -columns.

We will now define  $X$  according to which property (i)–(iv), the  $a_{\pi_2(z)}^2$ -column has.

- (a) The  $a_{\pi_2(z)}^2$ -column is a  $\star_2$ -column. First consider  $m > t_2$  and  $z \neq 3$ . If  $t_2 \geq 4$ , then consider the edges  $g_i$  and  $h_i$ :

$$g_i := (a_{\pi_2(z)}^2, b_{t_2-1+\ell+i})(a_{\pi_2(z)}^2, b_{t_2+\ell+i})$$

$$h_i := (a_{\pi_2(z+1)}^2, b_{t_2-1+\ell+i})(a_{\pi_2(z+1)}^2, b_{t_2+\ell+i})$$

If both  $g_0$  and  $h_0$  are red, define

$$X := \{a_{\pi_2(z)}^2, a_{\pi_2(z+1)}^2, b_{t_2-1+\ell}, b_{t_2+\ell}\}\text{-CS},$$

to obtain a Hamilton decomposition, by Remark 2.2.2. If both  $g_0$  and  $h_0$  are blue, then because  $m \geq 2t_2$ , both  $(a_{\pi_2(z)}^2, b_{m+\ell})(a_{\pi_2(z)}^2, b_{1+\ell})$  and  $(a_{\pi_2(z+1)}^2, b_{m+\ell})(a_{\pi_2(z+1)}^2, b_{1+\ell})$  are red. Furthermore, the black edges

$$(a_{\pi_2(z)}^2, b_{m+\ell})(a_{\pi_2(z+1)}^2, b_{m+\ell}) \text{ and } (a_{\pi_2(z)}^2, b_{1+\ell})(a_{\pi_2(z+1)}^2, b_{1+\ell})$$

lie on different cycles. Thus, define

$$X := \{a_{\pi_2(z)}^2, a_{\pi_2(z+1)}^2, b_{m+\ell}, b_{1+\ell}\}\text{-CS},$$

which by Remark 2.2.2 will produce a Hamilton decomposition. If  $g_0$  is blue and  $h_0$  is red, then clearly, both  $g_2$  and  $(a_{\pi_2(z)}^2, b_{m+\ell})(a_{\pi_2(z)}^2, b_{1+\ell})$  are red. Furthermore, either  $h_2$  is red or  $w = (a_{\pi_2(z+1)}^2, b_{m+\ell})(a_{\pi_2(z+1)}^2, b_{1+\ell})$  is red. If  $w$  is red, define

$$X := \{a_{\pi_2(z)}^2, a_{\pi_2(z+1)}^2, b_{m+\ell}, b_{1+\ell}\}\text{-CS},$$

or if  $h_2$  is red, define

$$X := \{a_{\pi_2(z)}^2, a_{\pi_2(z+1)}^2, b_{t_2+1+\ell}, b_{t_2+2+\ell}\}\text{-CS},$$



and replace the RAVS or LAVS of (5.5) with an  $\{a_{\pi_2(2)}^2, b_{3+\ell}, b_{t_2+1+\ell}\}$ -LAVS or RAVS. The result, by Remark 2.2.2, is a Hamilton decomposition. If  $g_0$  is red and  $h_0$  is blue, use the same technique to find a good switch. In the case  $t_2 = 2$ , and

$$(a_{\pi_2(z)}^2, b_{1+\ell})(a_{\pi_2(z)}^2, b_{2+\ell}) \text{ and } (a_{\pi_2(z+1)}^2, b_{1+\ell})(a_{\pi_2(z+1)}^2, b_{2+\ell})$$

are red edges, then define

$$X := \{a_{\pi_2(z)}^2, a_{\pi_2(z+1)}^2, b_{1+\ell}, b_{2+\ell}\}\text{-CS}$$

to obtain a Hamilton decomposition. If not, as  $m \geq 6$ , apply a similar technique as when  $t_2 \geq 4$  to obtain the result.

Now consider  $m = t_2$ , i.e.,  $H_2 \cup F \simeq A_n^{(2)} \square B_m$ . W.l.o.g., assume that  $e_1$  and  $e_2$  are blue edges, and define  $g_i$  and  $h_i$  as before.

$$X = \begin{cases} \{a_{\pi_2(z)}^2, a_{\pi_2(z+1)}^2, b_{m-1+\ell}, b_{m+\ell}\}\text{-CS} & \text{if } g_0 \text{ and } h_0 \text{ are both red} \\ \{a_{\pi_2(z)}^2, a_{\pi_2(z+1)}^2, b_{m+\ell}, b_{1+\ell}\}\text{-CS} & \text{if } g_1 \text{ and } h_1 \text{ are both red} \end{cases} \quad (5.6)$$

If  $g_{-1}$  and  $h_{-1}$  are both red, define  $X = \{a_{\pi_2(z)}^2, a_{\pi_2(z+1)}^2, b_{m-2+\ell}, b_{m-1+\ell}\}\text{-CS}$ , remove the LAVS or RAVS from (5.5) and apply an  $\{a_{\pi_2(2)}^2, b_{m+\ell}, b_{m-2+\ell}\}$ -RAVS. Similarly, if  $g_2$  and  $h_2$  are both red, define  $X = \{a_{\pi_2(z)}^2, a_{\pi_2(z+1)}^2, b_{1+\ell}, b_{2+\ell}\}\text{-CS}$ , remove the LAVS or RAVS from (5.5) and apply an  $\{a_{\pi_2(2)}^2, b_{2+\ell}, b_{m+\ell}\}$ -LAVS. Now, if there does not exist an integer  $i \in \{-1, 0, 1, 2\}$ , such that both  $g_i$  and  $h_i$  are red, then all vertical edges in both the  $a_{\pi_2(z)}^2$  and  $a_{\pi_2(z+1)}^2$ -columns that are between the  $b_{2+\ell}$  and  $b_{m-2+\ell}$ -rows are red. In this case, define  $X = \{a_{\pi_2(z)}^2, a_{\pi_2(z+1)}^2, b_{3+\ell}, b_{4+\ell}\}\text{-CS}$ , and remove the LAVS or RAVS from (5.5), and apply an  $\{a_{\pi_2(2)}^2, b_{1+\ell}, b_{3+\ell}\}$ -LAVS and an  $\{a_{\pi_2(2)}^2, b_{4+\ell}, b_{m+\ell}\}$ -LAVS. We now have a Hamilton decomposition of  $\Gamma$ . The case  $z = 3$  follows similarly and is omitted.

- (b)  $\pi_2(z) \in \{1, 3\}$ . If  $t_2 = 2$ , then  $t_1 = 2$ , and so the  $a_1^2$  and  $a_3^2$ -columns are  $\star_2$ -columns, which is resolved in Case 1.II(a). Hence, assume that  $t_2 \geq 4$  and w.l.o.g., we may assume  $e_1$  and  $e_2$  are blue. Again, define  $g_i$  and  $h_i$  as in Case 1.II(a). The  $a_{\pi_2(z)}^2$ -column contains an *alternating path* of red and blue edges and as the  $a_{\pi_2(z+1)}^2$ -column contains at most two consecutive blue edges, there exists at least one integer  $i \in \{-1, 0, 1 - t_2, 2 - t_2\}$  such that both  $g_i$  and  $h_i$  are red. Use the technique of Case 1.II(a), to define  $X$ .
- (c)  $\pi_2(z) = 2$ . If  $t_2 = 2$ , then  $t_1 = 2$ , so that the  $a_2^2$ -column is a  $\star_2$ -column, which is resolved in Case 1.II(a). Hence, assume that  $t_2 \geq 4$  and assume that  $e_1$  and  $e_2$  are blue edges. First consider  $m > t_2$  and let  $j$  be the smallest integer  $1 \leq j \leq t_2 + 1$  such that both  $(a_2^2, b_{j+\ell})(a_2^2, b_{j+1+\ell})$  and  $(a_{\pi_2(z+1)}^2, b_{j+\ell})(a_{\pi_2(z+1)}^2, b_{j+1+\ell})$  are red edges. As there are at most  $t_2 - 2$  blue edges in the  $a_2^2$ -column and at most two blue edges in the  $a_{\pi_2(z+1)}^2$ -column, such a  $j$  exists. Define

$$X := \{a_2^2, a_{\pi_2(z+1)}^2, b_{j+\ell}, b_{j+1+\ell}\}\text{-CS}$$

and remove the LAVS or RAVS of (5.5). Apply an

$$\{a_{\pi_2(2)}^2, b_{j+1+\ell}, b_{j+t_2-1+\ell}\}\text{-LAVS}.$$

As

$$j + t_2 - 1 + \ell \leq t_2 + 1 + t_2 - 1 + \ell = 2t_2 + \ell \leq m + \ell,$$

and  $e_3$  is red, the aforementioned LAVS always defines a proper red/black color-switching configuration. By Remark 2.2.2, the result is a Hamilton decomposition. If  $m = t_2$ , then the  $a_2^2$ -column contains at most  $m - 2$  blue edges, and so it is possible that there exists no pair of edges  $g = (a_2^2, b_{x+\ell})(a_2^2, b_{x+1+\ell})$  and  $h = (a_{\pi_2(z+1)}^2, b_{x+\ell})(a_{\pi_2(z+1)}^2, b_{x+1+\ell})$  that are both red. However, this is only possible if no pair  $g$  and  $h$  are both blue, as there are at most two blue edges in the  $a_{\pi_2(z+1)}^2$ -column. In this case, we may remove the LAHS from (5.4) and apply the RAHS of (5.4) so that  $e_1$  and  $e_3$  are now blue. Clearly, the switch reflect does not change the position of the blue edges in the  $a_2^2$ -column, but it does shift the blue edges in the  $a_{\pi_2(z+1)}^2$ -column up or down one row, thereby creating a pair  $g$  and  $h$  that are *both red*. The conclusion now follows.

**Case 2:** One of  $t_1$  and  $t_2$  is even, the other is odd. W.l.o.g., let  $t_2 \geq t_1$ . The case  $\overline{s_1} = \pm \overline{s_2}$  follows from the technique of Case 1.I. Hence, assume  $\overline{s_1} \neq \pm \overline{s_2}$ . If  $t_2 = 2k_2 + 1 \geq 3$  and  $t_1 = 2k_1 \geq 2$ , then clearly  $m \geq 2t_2 > t_2 > t_1$ . Apply the same CS-configuration of (5.3) and (5.4) to  $H_1 \cup F \simeq A_n^{(1)} \square_{r_1} B_m$ , where we may assume  $A_n^{(1)}$  and  $A_n^{(2)}$  are the same as in Case 1.II. Following this technique, to  $H_2 \cup F \simeq A^{(2)} \square_{r_2} B_m$ , apply an

$$\{a_{\pi_2(2)}^2, b_{1+\ell}, b_{t_2+\ell}\}\text{-RAVS or -LAVS}$$

in place of the CS-configuration of (5.5). Define the color-switch  $X$ , between the  $a_{\pi_2(z)}^2$  and  $a_{\pi_2(z+1)}^2$ -columns, as in Case 1.II. As  $m > t_2$ , we now have more freedom to define  $X$ . By Corollary 2.2.8, the application of  $X$  will produce a Hamilton decomposition of  $\Gamma$ .

Likewise, if  $t_2 = 2k_2 \geq 4$  and  $t_1 = 2k_1 + 1 \geq 3$ , we may apply follow the technique of Case 1.II by applying an  $\{a_2, b_1, b_{t_1}\}$ -RAVS and an  $\{a_3, a_4, b_{t_1}, b_{t_1+1}\}$ -CS in place of (5.3) and an  $\{a_4, a_{n-1}, b_{t_1}\}$ -RAHS or  $\{a_4, a_{n-1}, b_{t_1+1}\}$ -LAHS in place of (5.4). By Corollary 2.2.8, the cycle structure is the same as in Case 1.II. We now have more freedom to define  $X$ , as  $m > t_1$ , and the result follows. ■

We now extend Lemma 5.2.7 to allow for  $|s_3| = 4$ .

**Lemma 5.2.8.** *If  $\Gamma = \text{CAY}(A, \{s_1, s_2, s_3\})$  is a connected, 6-regular, Cayley graph on  $A$ , where  $|A : \langle s_3 \rangle| \geq 9$ , and  $2s_1, 2s_2 \notin \langle s_3 \rangle$ , then  $\Gamma$  has a Hamilton decomposition.*

*Proof.* Note,  $|s_3| \geq 3$ , and by Lemmata 5.1.5(a) and 5.2.7, it suffices to prove the result when  $|s_3| = 4$ . W.l.o.g.,  $t_1 \geq t_2$ . Similarly to Lemma 5.2.7,  $\Gamma$  is a  $D(3, 4, n)$ -graph with  $H_1 \cup F \simeq A_n^{(1)} \square_{r_1} B_4$  and  $H_1 \cup F \simeq A_n^{(2)} \square_{r_2} B_4$ . W.l.o.g.,

$$A_n^{(1)} = a_1 a_2 a_3 \cdots a_p \cdots a_q \cdots a_n a_1,$$

$$A_n^{(2)} = a_{\pi_2(2)}^2 a_{\pi_2(3)}^2 a_{\pi_2(4)}^2 \cdots a_{\pi_2(n)}^2 a_p^2 a_{\pi_2(2)}^2 = c_1 c_2 c_3 \cdots c_n c_1,$$

where  $\{\pi_2(2), \pi_2(3)\} = \{q, n\}$ , and for simplicity of notation, we relabel the elements as  $c_1, c_2, \dots, c_n$ . Apply the CS-configuration of Lemma 2.2.9 to  $H_2 \cup F$ , to obtain a black Hamilton cycle, and join the red edges in the  $c_1$ - and  $c_n$ -columns into one cycle. Apply the following CS-configuration to  $A_n^{(1)} \square_{r_1} B_4$ :

$$\begin{cases} \{a_{p-3}, a_{p-1}, b_2\}\text{-RAHS}, \{a_{p-1}, a_p, b_1, b_2\}\text{-CS} & \text{if } t_1 = 2 \\ \{a_{p-3}, a_{p-1}, b_2\}\text{-RAHS}, \{a_{p-1}, a_p, b_3, b_4\}\text{-CS} & \text{if } t_1 = 4 \end{cases} \quad (5.7)$$

In either case, a blue Hamilton cycle is created by Remark 2.2.2, and the  $a_i$ -columns,  $p-3 \leq i \leq p-1$ , are joined to the red cycle containing the  $a_{\pi_2(2)}^2$  and  $a_p^2$ -columns.

**Case 1.** Let  $n \geq 9$  odd. A red 2-factor now exists, consisting of  $n - 4$  cycles. We now closely follow the technique of the proof of Lemma 5.1.1 (Lemma 3.18 in [34]). Let  $\{z_i : 1 \leq i \leq n - 5\}$  be a sequence of integers satisfying  $z_1 < z_2 < z_3 < \dots < z_{n-5}$ , such that, for  $1 \leq i \leq n - 5$ , all vertical edges in the  $c_{z_i+1}$ -column are red. Clearly,  $z_1 = 1$  and  $z_{n-5} < n - 1$ . Consider color-switches,  $X_i$ , that are c-incident to the  $c_{z_i}$ - and  $c_{z_i+1}$ -columns. By Remark 2.2.2, applying  $X_1, X_2, \dots, X_{n-5}$  to  $A_n^{(2)} \square_{r_2} B_4$ , will create a red Hamilton cycle. Having already defined  $X_1, X_2, \dots, X_{2i-3}, X_{2i-2}$ , define  $X_{2i-1}$  and  $X_{2i}$  as follows:

$$X_{2i-1} = \begin{cases} \{c_{z_{2i-1}}, c_{z_{2i-1}+1}, b_2, b_3\}\text{-CS} & \text{if } (c_{z_{2i-1}}, b_2)(c_{z_{2i-1}}, b_3) \text{ is red} \\ \{c_{z_{2i-1}}, c_{z_{2i-1}+1}, b_4, b_1\}\text{-CS} & \text{if } (c_{z_{2i-1}}, b_4)(c_{z_{2i-1}}, b_1) \text{ is red} \end{cases} \quad (5.8)$$

$$X_{2i} = \begin{cases} \{c_{z_{2i}}, c_{z_{2i}+1}, b_1, b_2\}\text{-CS} & \text{if } (c_{z_{2i}}, b_1)(c_{z_{2i}+1}, b_2) \text{ is red} \\ \{c_{z_{2i}}, c_{z_{2i}}, b_3, b_4\}\text{-CS} & \text{if } (c_{z_{2i}}, b_3)(c_{z_{2i}}, b_4) \text{ is red} \end{cases} \quad (5.9)$$

Clearly, by Remark 2.2.2, applying  $X_{2i-1}$  will break the black Hamilton cycle into two cycles, and applying  $X_{2i}$  will rejoin the black cycles again. Thus,  $\Gamma$  has a Hamilton decomposition.

**Case 2.** Let  $n \geq 10$  even. We consider  $\overline{s_1} \neq \pm \overline{s_2}$ , for the case of equality is very similar and is omitted. By Remark 5.2.6, if two columns in  $H_1 \cup F = A_n^{(1)} \square_{r_1} B_4$  are adjacent, they are not adjacent in  $H_2 \cup F = A_n^{(2)} \square_{r_2} B_4$ . Relocate the  $r_2$ -jump to be between the  $a_{\pi_2(3)}^2$  and  $a_{\pi_2(2)}^2$ -columns, and relabel the columns of  $A_n^{(2)} \square_{r_2} B_4$  by  $c_1, c_2, \dots, c_n$ , with the  $c_1, c_{n-1}$ , and  $c_n$ -columns representing the  $a_{\pi_2(3)}^2, a_p^2$ , and  $a_{\pi_2(2)}^2$ -columns, respectively. To  $H_1 \cup F$ , apply an  $\{a_{p-1}, a_p, b_1, b_2\}$ -CS if  $t_1 = 2$ , or if  $t_1 = 4$ , apply the switching configuration

$$\{\{a_{p-3}, a_{p-2}, b_{1+\ell}, b_{2+\ell}\}\text{-CS}, \{a_{p-2}, a_{p-1}, b_{2+\ell}, b_{3+\ell}\}\text{-CS}, \{a_{p-1}, a_p, b_{3+\ell}, b_{4+\ell}\}\text{-CS}\},$$

to obtain a blue Hamilton cycle, and join the red edges in the involved columns into one cycle, where  $\ell$  may vary depending on the context. To  $H_2 \cup F$ , apply the CS-configuration of Lemma 2.2.9, to create a black Hamilton cycle. Let  $c_{p_i}$  denote the  $a_{p-i}^2$ -column, where  $i = 1, 2, 3$ . There now exists a red 2-factor consisting of one cycle on the  $c_1$  and  $c_n$ -columns, one cycle containing the  $c_{n-1}, c_{p_1}, c_{p_2}$ , and  $c_{p_3}$ -columns (or the  $c_{n-1}$  and  $c_{p_1}$ -columns if  $t_1 = 2$ ), and  $n - y$  cycles of length 4, each one a single column. Call these  $n - y$  columns, *free columns*, where  $y = 6$  if  $t_1 = 4$ , and  $y = 4$  if  $t_1 = 2$ . Let the total number of red cycles be  $2d$ . Consider the integer sequence  $\{z_i\}_{i=1}^{2d-1}$ , where  $1 = z_1 < z_2 < \dots < z_{2d-1}$ , such that the  $c_{z_i+1}$ -column is a free column or the left-most  $c_{p_i}$ -column. Let  $\mathcal{X} = \{X_1, X_2, \dots, X_{2d-1}\}$  be a set of edge-disjoint color-switches where  $X_i$  is c-incident to the  $c_{z_i}$ - and  $c_{z_i+1}$ -columns. After applying  $\mathcal{X}$  to  $H_2 \cup F$ , we have a red Hamilton cycle. However, the black Hamilton cycle is now split into two cycles. If the red Hamilton cycle is given a direction, all edges in a fixed column have the same direction, edges in the  $c_{z_i}$ - and  $c_{z_i+1}$ -columns have opposite direction, and edges in the  $c_1$ - and  $c_n$ -columns have the same direction. As  $n$  is even, there exists an integer  $z$ , such that the edges in the  $c_z$  and  $c_{z+1}$ -columns have the same direction. Applying any color-switch,  $X'$ , between those two columns, will preserve the red Hamilton cycle. Insert  $z$  into  $\{z_i\}_{i=1}^{2d-1}$ , and  $X'$  into  $\mathcal{X}$ , and relabel, so that we have  $1 \leq z'_1 < z'_2 < \dots < z'_{2d} < n$  and  $\mathcal{X} = \{X_1, X_2, \dots, X_{2d}\}$ . Clearly,  $z \neq z_i$ , and so the  $c_{z+1}$ -column is not a free column. We now construct the set  $\mathcal{X}$ . Having already defined  $X_1, X_2, \dots, X_{2i-3}, X_{2i-2}$ , we now define  $X_{2i-1}$  and  $X_{2i}$ .

1.  $z \notin \{z'_{2i-1}, z'_{2i}, 1, n-1\}$ . Let

$$X_{2i-1} := \begin{cases} \{c_{z'_{2i-1}}, c_{z'_{2i-1}+1}, b_2, b_3\}\text{-CS} & \text{if } (c_{z'_{2i-1}}, b_2)(c_{z'_{2i-1}}, b_3) \text{ is red,} \\ \{c_{z'_{2i-1}}, c_{z'_{2i-1}+1}, b_4, b_1\}\text{-CS} & \text{if } (c_{z'_{2i-1}}, b_4)(c_{z'_{2i-1}}, b_1) \text{ is red.} \end{cases}$$

and

$$X_{2i} := \begin{cases} \{c_{z'_{2i}}, c_{z'_{2i}+1}, b_3, b_4\}\text{-CS} & \text{if } (c_{z'_{2i}}, b_3)(c_{z'_{2i}}, b_4) \text{ is red,} \\ \{c_{z'_{2i}}, c_{z'_{2i}+1}, b_1, b_2\}\text{-CS} & \text{if } (c_{z'_{2i}}, b_1)(c_{z'_{2i}}, b_2) \text{ is red.} \end{cases}$$

2.  $z \in \{z'_{2i-1}, z'_{2i}\}$ , but  $z \notin \{1, n-1\}$ . We consider two subcases: (1) where  $c_z$  is a free column and (2) where  $c_z$  is one of the  $c_{p_i}$ -columns. First, let the  $c_z$ -column be a free column. Now the  $c_{z+1}$ -column is one of the three  $c_{p_i}$ -columns that is not the left-most one, and has at most two consecutive non-red edges. If  $z = z'_{2i-1}$ , then as both  $(c_z, b_2)(c_z, b_3)$  and  $(c_z, b_4)(c_z, b_1)$  are red, define  $X_{2i-1} := \{c_z, c_{z+1}, b_2, b_3\}\text{-CS}$  or  $X_{2i-1} := \{c_z, c_{z+1}, b_4, b_1\}\text{-CS}$ , depending on which of  $(c_{z+1}, b_2)(c_{z+1}, b_3)$  or  $(c_{z+1}, b_4)(c_{z+1}, b_1)$  is red. Similarly, if  $z = z'_{2i}$ , then as both  $(c_z, b_1)(c_z, b_2)$  and  $(c_z, b_3)(c_z, b_4)$  are red, define  $X_{2i} := \{c_z, c_{z+1}, b_1, b_2\}\text{-CS}$  or  $X_{2i} := \{c_z, c_{z+1}, b_3, b_4\}\text{-CS}$ , depending on which of  $(c_{z+1}, b_1)(c_{z+1}, b_2)$  or  $(c_{z+1}, b_3)(c_{z+1}, b_4)$  is red.

Now suppose the  $c_z$ -column is the  $c_{p_3}$ -column. By Lemma 5.2.5, the  $c_{z+1}$ -column can be either the  $c_{n-1}$  or  $c_{p_1}$ -column. Consider

$$e_{f+i} := (c_z, b_{1+i})(c_z, b_{2+i}) \text{ and } g_{f+i} := (c_{z+1}, b_{1+i})(c_{z+1}, b_{2+i}).$$

If  $z = z'_{2i-1}$ , choose  $\ell$  in the switching configuration of  $H_1 \cup F$ , so that either  $e_{f+1}$  and  $g_{f+1}$  are red, or  $e_{f+3}$  and  $g_{f+3}$  are red, depending on if the color switch,  $X_{2i-2}$  was r-incident to the  $b_1$  and  $b_2$ -rows or the  $b_3$  and  $b_4$ -rows. Define  $X_{2i-1}$  to be the corresponding switch r-incident to either the  $b_1$  and  $b_2$ -rows, or the  $b_4$  and  $b_1$ -rows. The result is obtained similarly for  $z = z'_{2i}$  by choosing  $\ell$  so that either  $e_f$  and  $g_f$  are red or  $e_{f+2}$  and  $g_{f+2}$  are red. Suppose the  $c_z$ -column is the  $c_{p_2}$ -column. Hence, by Lemma 5.2.5, the  $c_{z+1}$ -column must be the  $c_{n-1}$ -column. As the  $c_z$ -column is not the left-most column in its cycle, it has exactly two non-red edges. Therefore, if  $z = z'_{2i-1}$ , we may choose  $\ell$  so that either  $(c_z, b_2)(c_z, b_3)$  and  $(c_z, b_4)(c_z, b_1)$  are red or  $(c_{n-1}, b_2)(c_{n-1}, b_3)$  and  $(c_{n-1}, b_4)(c_{n-1}, b_1)$  are red, and let  $X_{2i-1}$  be the corresponding switch. The result is obtained similarly for  $z = z'_{2i}$ . If the  $c_z$ -column is the  $c_{p_1}$ -column, then by Lemma 5.2.5, the  $c_{z+1}$ -column must be the  $c_{p_3}$ -column, and result follows similarly.

3.  $z \in \{1, n-1\}$ . If  $z = 1$ , define the color switch  $X_1 := \{c_1, c_2, b_2, b_3\}\text{-CS}$  or  $X_1 := \{c_1, c_2, b_4, b_1\}\text{-CS}$  depending which forms a good switch. Likewise, if  $z = n-1$ , then define the color switch  $X_{2d} := \{c_{n-1}, c_n, b_1, b_2\}\text{-CS}$  or  $X_{2d} := \{c_{n-1}, c_n, b_3, b_4\}\text{-CS}$  depending on which forms a good switch. ■

Lemmata 5.2.7 and 5.2.8 combine to yield the following.

**Theorem 5.2.9.** *If  $A$  is an abelian group, and for some  $1 \leq i \leq 3$ ,  $\text{CAY}(A/\langle s_i \rangle, \{\overline{s_{j_1}}, \overline{s_{j_2}}\})$  is 4-regular and of order at least nine, then  $\text{CAY}(A, \{s_1, s_2, s_3\})$  has a Hamilton decomposition.*

## 5.3 Decompositions for Low-Order Quotient Graphs

In this section, we consider Case (h) of Remark 5.1.6. The technique used in the proofs of Lemmata 5.3.1 and 5.3.3 is similar to that used in the proof of Theorem 3.2.1.

### 5.3.1 Odd-order quotients

**Lemma 5.3.1.** *If  $A$  is an abelian group, and for some  $1 \leq i \leq 3$ ,  $\text{CAY}(A/\langle s_i \rangle, \{\overline{s_{j_1}}, \overline{s_{j_2}}\})$  is 4-regular and of order  $n \in \{1, 5, 7\}$ , then  $\text{CAY}(A, \{s_1, s_2, s_3\})$  has a Hamilton decomposition.*

*Proof.* Let  $[A : \langle s_i \rangle] = n$ . If  $n = 1$ , then  $s_i$  generates  $A$ , a case resolved by Theorem 1.5.2. By assumption,  $\overline{s_j} \neq \overline{0}$ , so  $|\overline{s_j}| \geq 3$ . Let  $\langle \overline{s_{j_1}}, \overline{s_{j_2}} \rangle = A/\langle s_i \rangle$ , where  $|s_{j_1}| \geq |s_{j_2}|$  and  $|\langle s_i \rangle| = m$ . If  $m = 2j + 1$ , then  $|A|$  is odd, a case resolved by Theorem 3.2.1. Hence, without loss of generality,  $m = 2d \geq 4$ . As  $n$  is prime,  $|\overline{s_{j_i}}| = n$  for  $i = 1, 2$ . The quotient graph,  $\Delta = \text{CAY}(A/J, \{\overline{s_{j_1}}, \overline{s_{j_2}}\})$  is 4-regular and connected. Thus, by Theorem 1.5.1,  $\Delta$  has a Hamilton decomposition into two cycles  $\overline{H_1}$  and  $\overline{H_2}$ , where  $\overline{H_i}$  is just the cycle generated by  $\overline{s_{j_i}}$ . The lift of  $\overline{H_i}$ , denoted  $H_i$ , is the 2-factor generated by  $s_{j_i}$  in  $\Gamma$ . By Theorem 3.1.6,  $\Gamma$  is a  $D(3, m, n)$ -graph, where  $F$  is the red 2-factor generated by  $s_i$ . Thus  $H_i \cup F = A_n^{(i)} \square_{r_i} B_m$ , for  $i = 1, 2$ , where

$$A_n^{(i)} := a_{\pi_i(1)}^i a_{\pi_i(2)}^i \cdots a_{\pi_i(n)}^i a_{\pi_i(1)}^i.$$

Note, for  $i = 1, 2$ ,  $H_i$  consists of  $t_i = \gcd(r_i, m) = |A : \langle s_{j_i} \rangle|$  cycles of length  $|s_{j_i}| = nm/t_i$ . If  $t_i = 1$  for some  $i = 1, 2$ , then  $s_{j_i}$  generates  $A$ , a case resolved by Theorem 1.5.2. Furthermore, if  $t_1 = 2$ , then we are done by Corollary 4.1.6. Thus, assume  $m \geq t_2 \geq t_1 \geq 3$ .

**Case 1:  $n = 5$ .** Up to relabeling,  $\Delta$  is either,  $\Lambda_1 := \text{CAY}(\mathbb{Z}_5, \{\pm 1\})$ , a multigraph on five vertices, or  $\Lambda_2 := \text{CAY}(\mathbb{Z}_5, \{1, 2\}) = K_5$ . If  $\Delta = \Lambda_1$ , without loss of generality,  $\pi_1 = \pi_2 = (1)$ . If  $\Delta = \Lambda_2$ , then without loss of generality,  $\pi_1 = (1)$  and  $\pi_2 = (2354)$  so that  $A_5^{(2)} := a_1^2 a_3^2 a_5^2 a_2^2 a_4^2 a_1^2$ . The corresponding quotient graphs are shown in Figure 3.3 of Chapter 3.

- 1.i.  $t_2 = 2k_2 \geq 2k_1 = t_1 \geq 4$ . Apply the CS-configuration of Lemma 2.2.9 to  $H_1 \cup F$  to obtain a blue Hamilton cycle and join the vertical red edges in the  $a_1$  and  $a_5$ -columns into one cycle  $C$ .

If  $\Delta = \Lambda_1$ , let  $\ell$  be any integer such that  $(a_1^2, b_{1+\ell})(a_2^2, b_{2+\ell})$  is blue, and apply a switching configuration of Theorem 2.2.7(b), using a RAVS with  $i = 2$  to obtain a set of three monochromatic Hamilton cycles in  $\Gamma$ .

If  $\Delta = \Lambda_2$  then let  $(a_1^2, b_{1+\ell})(a_2^2, b_{2+\ell})$  be blue as before, and apply an  $\{a_3^2, b_{1+\ell}, b_{t_2-1+\ell}\}$ -RAVS and either an  $\{a_2^2, a_4^2, b_{t_2-1+\ell}, b_{t_2+\ell}\}$ -CS or an  $\{a_2^2, a_4^2, b_{m+\ell}, b_{1+\ell}\}$ -CS, depending on which of  $(a_2^2, b_{t_2-1+\ell})(a_2^2, b_{t_2+\ell})$  or  $(a_2^2, b_{1+\ell})(a_2^2, b_{m+\ell})$  is red. The result is a Hamilton decomposition of  $\Gamma$ .

- 1.ii.  $t_2 = 2k_2 > 2k_1 + 1 = t_1 \geq 3$ . As  $t_2$  is even, apply the CS-configuration of Lemma 2.2.9 to  $H_2 \cup F$  to obtain a black Hamilton cycle.

If  $\Delta = \Lambda_1$ , this joins all vertical red edges in the  $a_1^2$  and  $a_5^2$ -columns into one cycle. Let  $\ell$  be the integer such that  $(a_1, b_{1+\ell})(a_1, b_{2+\ell})$  is a black edge. As the red and black edges in the  $a_1$ -column form a matching, apply an  $\{a_2, b_{1+\ell}, b_{t_1+\ell}\}$ -RAVS to obtain a blue Hamilton cycle and join all vertical red edges in the  $a_i$ -columns, where  $i \in \{1, 2, 3, 5\}$ , into one cycle. Finally, by Theorem 2.2.7, applying an  $\{a_3, a_4, b_{t_1+\ell}, b_{t_1+1+\ell}\}$ -CS preserves the blue cycle and produces a red Hamilton cycle. The result is a Hamilton decomposition of  $\Gamma$ .

If  $\Delta = \Lambda_2$ , the color switching configuration of Lemma 2.2.9 joins all vertical red edges in the  $a_1^2$  and  $a_4^2$ -columns into one cycle. Let  $\ell$  be the integer such that  $(a_1, b_{1+\ell})(a_1, b_{2+\ell})$  is black. Apply an  $\{a_2, b_{1+\ell}, b_{t_1+\ell}\}$ -RAVS, which by Theorem 2.2.7, generates a blue Hamilton cycle and joins all vertical red edges in the  $a_i$ -columns, where  $1 \leq i \leq 4$ , into one cycle. The vertical red edges in the  $a_4$ -column, which also form a matching with black edges, have the

property that either  $(a_4, b_{t_1+\ell})(a_4, b_{t_1+1+\ell})$  is red, or  $(a_4, b_{m+\ell})(a_4, b_{1+\ell})$  is red. Apply either an  $\{a_4, a_5, b_{t_1+\ell}, b_{t_1+1+\ell}\}$ -CS or an  $\{a_4, a_5, b_{m+\ell}, b_{1+\ell}\}$ -CS, accordingly. By Corollary 2.2.8, this creates three monochromatic Hamilton cycles in  $\Gamma$ .

1.iii.  $t_2 = 2k_2 + 1 > 2k_1 = t_1 \geq 4$ . This case follows identically to 2.ii. by applying the CS-configuration of Lemma 2.2.9 to  $H_1 \cup F$  rather than  $H_2 \cup F$ .

1.iv.  $t_2 = 2k_2 + 1 \geq 2k_1 + 1 = t_1 \geq 3$ .

If  $\Delta = \Lambda_1$ , then apply an  $\{a_2, b_1, b_{t_1}\}$ -LAVS to obtain a blue Hamilton cycle and join the vertical red edges in the  $a_i$ -columns, where  $1 \leq i \leq 3$ , into one cycle  $C$ . As the vertical red and blue edges in the  $a_3$ -column form a matching, let  $\ell$  be the integer such that  $(a_3^2, b_{1+\ell})(a_3^2, b_{2+\ell})$  is red. Apply an  $\{a_4^2, b_{1+\ell}, b_{t_2+\ell}\}$ -LAVS to obtain a black Hamilton cycle and join the vertical red edges in the  $a_4^2$  and  $a_5^2$ -columns to  $C$ . The result is a Hamilton decomposition of  $\Gamma$ .

If  $\Delta = \Lambda_2$ , then  $m \geq 2t_2$ . Apply an  $\{a_2^2, b_1, b_{t_2}\}$ -LAVS to obtain a black Hamilton cycle and join all vertical red edges in the  $a_{\pi_2(i)}^2$ -columns, where  $3 \leq i \leq 5$ , into one cycle  $C$ . Let  $\ell$  be the integer such that all vertical edges in the  $a_2$ -column that are between the  $b_{1+\ell}$  and  $b_{t_2+\ell}$ -rows are black. By our assumption on  $m$ , all vertical edges in the  $a_2$ -column between the  $b_{t_2+\ell}$  and  $b_{t_2+t_1-1+\ell}$ -rows are red. Thus, apply an  $\{a_2, b_{t_2+\ell}, b_{t_2+t_1-1+\ell}\}$ -RAVS to obtain a blue Hamilton cycle, and join the vertical red edges in the  $a_1$  and  $a_2$ -columns to  $C$ . The result is a Hamilton decomposition of  $\Gamma$ .

**Case 2:  $n = 7$ .** Again, up to relabeling,  $\Delta$  is either  $\Lambda_1 := \text{CAY}(\mathbb{Z}_7, \{\pm 1\})$ ,  $\Lambda_2 := \text{CAY}(\mathbb{Z}_7, \{1, 3\})$ , or  $\Lambda_3 := \text{CAY}(\mathbb{Z}_7, \{1, 2\})$ . If  $\Delta = \Lambda_1$ , then without loss of generality,  $\pi_1 = \pi_2 = (1)$ , if  $\Delta = \Lambda_2$ , then without loss of generality,  $\pi_1 = (1)$ ,  $\pi_2 = (243756)$  and so

$$A_7^{(2)} := a_1^2 a_4^2 a_7^2 a_3^2 a_6^2 a_2^2 a_5^2 a_1^2.$$

If  $\Delta = \Lambda_3$ , then without loss of generality,  $\pi_1 = (1)$ ,  $\pi_2 = (235)(476)$  and

$$A_7^{(2)} := a_1^2 a_3^2 a_5^2 a_7^2 a_2^2 a_4^2 a_6^2 a_1^2.$$

The corresponding quotient graphs are shown in Figure 3.4 of Chapter 3.

2.i.  $t_2 = 2k_2 \geq 2k_1 = t_1 \geq 4$ .

If  $\Delta = \Lambda_1$ , then apply the CS-configuration of Lemma 2.2.9 to  $H_1 \cup F$  to obtain a blue Hamilton cycle and join the  $a_1$  and  $a_7$ -columns into one red cycle  $C$ . Apply an  $\{a_2^2, b_{1+\ell}, b_{t_2-1+\ell}\}$ -RAVS, where  $\ell$  is chosen as before. By Theorem 2.2.7(b), applying an  $\{a_2^2, a_3^2, b_{t_2-1+\ell}, b_{t_2+\ell}\}$ -CS will produce a black Hamilton cycle and join the vertical red edges in the  $a_i^2$ -columns, where  $i = 2, 3, 4$  to the cycle  $C$ . Finally, by Lemma 2.2.15, applying an  $\{a_4^2, a_6^2, b_{t_2-1+\ell}\}$ -RAHS will preserve the black cycle, and create three monochromatic Hamilton cycles in  $\Gamma$ .

If  $\Delta = \Lambda_2$ , then apply a  $\{a_4^2, b_1, b_{t_2-1}\}$ -LAVS and a  $\{a_7^2, a_3^2, b_{t_2-1}, b_{t_2}\}$ -CS. The result is a black Hamilton cycle and a red 2-factor consisting of one cycle, call it  $C$ , on the vertical red edges in the  $a_i^2$ -columns, where  $i = 1, 4, 7, 3$ , and three remaining cycles on the  $a_6^2$ ,  $a_2^2$ , and  $a_5^2$ -columns, respectively. Let  $x$  be the integer such that the edge  $(a_7, b_{1+x})(a_7, b_{2+x})$  is non-red. As the red and non-red edges in the  $a_7$ -column form a matching, apply a  $\{a_6, b_{1+x}, b_{t_1-1+x}\}$ -LAVS to join all vertical red edges in the  $a_5$  and  $a_6$ -columns to  $C$ . We now have a red 2-factor consisting of the red vertical edges in the  $a_i$ -columns, where  $i \neq 2$ , and a red  $m$ -cycle on the  $a_2$ -column. If  $t_1 < m$ , then one of the edges  $e := (a_3, b_{m+x})(a_3, b_{1+x})$  or  $e' := (a_3, b_{t_1-1+x})(a_3, b_{t_1+x})$  are red.

Apply either a  $\{a_2, a_3, b_{m+x}, b_{1+x}\}$ -CS or a  $\{a_2, a_3, b_{t_1-1+x}, b_{t_1+x}\}$ -CS, accordingly to obtain a Hamilton decomposition. If  $t_1 = m$ , then  $e = e'$ . If  $e$  is red, we are done. Otherwise,  $e$  is the only non-red edge in the  $a_3$ -column, and we may replace  $x$  with  $x + 2$ , to obtain the result.

If  $\Delta = \Lambda_3$ , then apply a  $\{a_2, b_1, b_{t_1-1}\}$ -RAVS and a  $\{a_3, a_4, b_{t_1-1}, b_{t_1}\}$ -CS to obtain a blue Hamilton cycle and join the vertical red edges in the  $a_i$ -columns, where  $1 \leq i \leq 4$  into one cycle,  $C$ . The red and non-red edges in the  $a_3$ -column form a matching. Thus, let  $\ell$  be the integer such that the edge  $(a_3^2, b_{1+\ell})(a_3^2, b_{2+\ell})$  is non-red. As  $(a_3^2, b_{2+\ell})(a_3^2, b_{3+\ell})$  must be red, apply a  $\{a_5^2, b_{1+\ell}, b_{t_2-1+\ell}\}$ -RAVS to connect all vertical red edges in the  $a_i$ -columns, where  $i = 1, 2, 3, 4, 5, 7$  into one cycle. The red edges in the  $a_6^2$ -column lie in their own  $m$ -cycle. Let  $e := (a_4^2, b_{m+\ell})(a_4^2, b_{1+\ell})$  and  $f := (a_4^2, b_{t_2-1+\ell})(a_4^2, b_{t_2+\ell})$ . If  $t_2 < m$ , then  $e \neq f$ . Thus, apply a  $\{a_4^2, a_6^2, b_{m+\ell}, b_{1+\ell}\}$ -CS if  $e$  is red, or a  $\{a_4^2, a_6^2, b_{t_2-1+\ell}, b_{t_2+\ell}\}$ -CS otherwise. As the black horizontal edges in the  $b_{m+\ell}$  and  $b_{t_2+\ell}$ -rows lie on the same cycle, and the black edges in the  $b_{1+\ell}$  and  $b_{t_2-1+\ell}$ -rows lie together on a different cycle, the appropriately chosen color switch will create Hamilton decomposition of  $\Gamma$ . If  $t_2 = m$ , then  $e = f$ . If  $e$  is red, proceed as before to obtain the result. If  $e$  is non-red, then clearly  $(a_4^2, b_{2+\ell})(a_4^2, b_{3+\ell})$  is red. Apply a  $\{a_5^2, b_{3+\ell}, b_{1+\ell}\}$ -RAVS rather than the  $\{a_5^2, b_{1+\ell}, b_{t_2-1+\ell}\}$ -RAVS (i.e. replace  $\ell$  with  $\ell + 2$ ) to obtain the result.

2.ii.  $t_2 = 2k_2 > 2k_1 + 1 = t_1 \geq 3$ .

If  $\Delta = \Lambda_1$ , then apply an  $\{a_2, b_1, b_{t_1}\}$ -RAVS and an  $\{a_3, a_4, b_{t_1}, b_{t_1+1}\}$ -CS. By Corollary 2.2.15(a), the result is a blue Hamilton cycle and a red cycle on the  $a_i$ -columns, where  $1 \leq i \leq 4$ . Choose  $\ell$  so that  $(a_4^2, b_{1+\ell})(a_4^2, b_{2+\ell})$  is red, and apply an  $\{a_5^2, b_{1+\ell}, b_{t_2-1+\ell}\}$ -LAVS and an  $\{a_6^2, a_7^2, b_{t_2-1+\ell}, b_{t_1+\ell}\}$ -CS to obtain three monochromatic Hamilton cycles.

If  $\Delta = \Lambda_2$ , apply a  $\{a_2, b_1, b_{t_1}\}$ -RAVS to obtain a blue Hamilton cycle and join the  $a_i$ -columns into one red cycle, where  $1 \leq i \leq 3$ . By Corollary 2.2.8, applying an  $\{a_5, a_6, b_{t_1}, b_{t_1+1}\}$ -CS will preserve the blue cycle, and join the  $a_5$  and  $a_6$ -columns into one red cycle. As  $m \geq 6$ , we can find an integer  $\ell$  such that the edge  $(a_3^2, b_{1+\ell})(a_3^2, b_{2+\ell})$  is non-red and the edge  $(a_6^2, b_{m+\ell})(a_6^2, b_{1+\ell})$  is red. Note the red and non-red edges in the  $a_3^2$ -column form a matching. Apply a  $\{a_7^2, b_{1+\ell}, b_{t_2-1+\ell}\}$ -LAVS to join the  $a_{\pi_2(i)}^2$ -columns into one red cycle, where  $i \in \{1, 2, 3, 4, 6\}$ . Clearly, the black edges in the  $b_{m+\ell}$ -row are on a different cycle than the black edges in the  $b_{j+\ell}$ -rows, where  $1 \leq j \leq t_2 - 1$ . Furthermore, the  $a_3^2$  and  $a_6^2$ -columns are on two different red cycles. Thus, by Corollary 2.2.8(b), apply an  $\{a_3^2, a_6^2, b_{m+\ell}, b_{1+\ell}\}$ -CS to obtain a Hamilton decomposition.

If  $\Delta = \Lambda_3$ , then apply the CS-configuration of Theorem 2.2.7(a) using a RAVS and  $\ell = 0$  to obtain a blue Hamilton cycle. Choose  $x$  so that  $(a_3^2, b_{1+x})(a_3^2, b_{2+x})$  is blue, and apply an  $\{a_5^2, b_{1+x}, b_{t_2-1+x}\}$ -RAVS. Now, there exists only one non-red edge in the  $a_4^2$ -column and so one of  $(a_4^2, b_{t_2-1+x})(a_4^2, b_{t_2+x})$  or  $(a_4^2, b_{m+x})(a_4^2, b_{1+x})$  is red. Apply an appropriate color-switch c-incident to the  $a_4^2$  and  $a_6^2$ -columns, which by Corollary 2.2.8(b), yields a set of three monochromatic Hamilton cycles.

2.iii.  $t_2 = 2k_2 + 1 > 2k_1 = t_1 \geq 4$ .

If  $\Delta = \Lambda_1$ , apply an  $\{a_2, b_1, b_{t_1-1}\}$ -RAVS and an  $\{a_3, a_4, b_{t_1-1}, b_{t_1}\}$ -CS. The result is a blue Hamilton cycle and a red cycle consisting of the vertical red edges in the  $a_i$ -columns, where  $1 \leq i \leq 4$ . Choose  $\ell$  so that  $(a_4^2, b_{1+\ell})(a_4^2, b_{2+\ell})$  is blue. Apply the CS-configuration of Theorem 2.2.7(a) using an  $\{a_5^2, b_{1+\ell}, b_{t_2+\ell}\}$ -RAVS to obtain three monochromatic Hamilton cycles.

If  $\Delta = \Lambda_2$ , then apply an  $\{a_2, b_1, b_{t_1-1}\}$ -RAVS and an  $\{a_3, a_4, b_{t_1-1}, b_{t_1}\}$ -CS to create a blue Hamilton cycle and a matching of red and blue edges in the  $a_3$ -column and a path of  $t_1 - 2$  blue vertical edges in the  $a_2$ -column. Choose  $x$  so that all vertical edges in the  $a_3^2$ -column between

the  $b_{1+x}$  and  $b_{t_2+x}$ -rows are red. As  $m \geq 2t_2$ , such a path exists. Apply an  $\{a_3^2, b_{1+x}, b_{t_2+x}\}$ -RAVS to obtain a black Hamilton cycle. At this point, we have a red 2-factor consisting of all vertical red edges in the  $a_i^2$ -columns, where  $i = 1, 2, 3, 4, 5, 6$ . Now, as  $t_2 > t_1$ , at least one of the edges  $(a_2^2, b_{m+x})(a_2^2, b_{1+x})$  and  $(a_2^2, b_{t_2+x})(a_2^2, b_{t_2+1+x})$  is red. Call this red edge  $e$ . By Corollary 2.2.8, we may apply one color switch that is  $c$ -incident to  $e$  and the  $a_5^2$ -column. The result is a Hamilton decomposition of  $\Gamma$ . The case  $\Delta = \Lambda_3$  follows similarly.

2.iv.  $t_2 = 2k_2 + 1 \geq 2k_1 + 1 = t_1 \geq 3$ .

If  $\Delta = \Lambda_1$ , then apply an  $\{a_2, b_1, b_{t_1}\}$ -RAVS and an  $\{a_3, a_4, b_{t_1}, b_{t_1+1}\}$ -CS to obtain a blue Hamilton cycle by Theorem 2.2.7(a), and join the vertical red edges in the  $a_i$ -columns, where  $1 \leq i \leq 4$ , into one red cycle. Choose  $\ell$  so that  $(a_4^2, b_{2+\ell})(a_4^2, b_{3+\ell})$  is red, and apply an  $\{a_5^2, b_{1+\ell}, b_{t_2+\ell}\}$ -RAVS and an

$$\{a_6^2, a_7^2, b_{t_2+\ell}, b_{t_2+1+\ell}\}\text{-CS},$$

which by Theorem 2.2.7(a), produces three monochromatic Hamilton cycles.

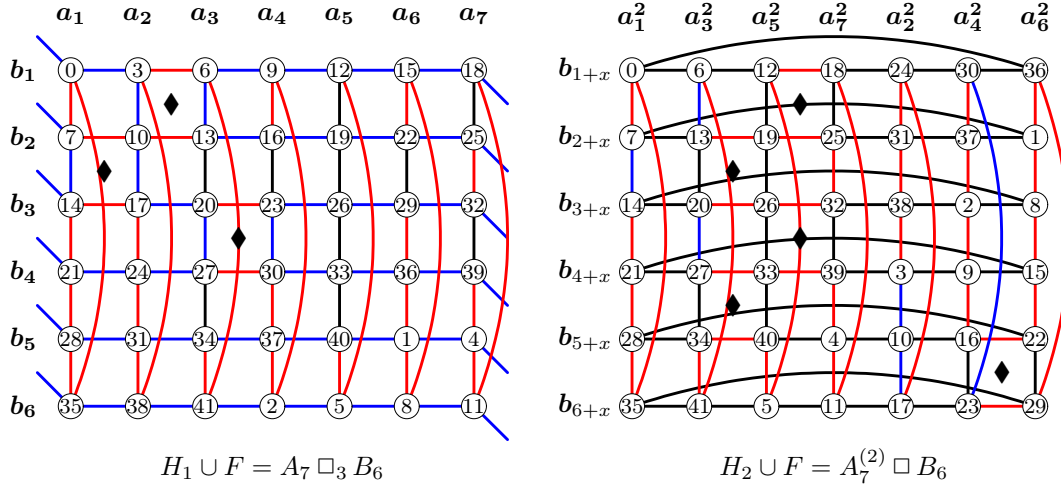
If  $\Delta = \Lambda_2$ , then apply an  $\{a_2, b_1, b_{t_1}\}$ -LAVS to obtain a blue Hamilton cycle. By Lemma 2.2.15, applying an  $\{a_3, a_5, b_2\}$ -RAHS will preserve the blue cycle and join the vertical red edges in the  $a_i$ -columns, where  $1 \leq i \leq 5$  into one red cycle,  $C$ . As  $m \geq 2t_2 \geq 2t_1$ , we can choose  $x$  so that all vertical edges between the  $b_{1+x}$  and  $b_{t_2+x}$ -rows are red. Then, apply an  $\{a_3^2, b_{1+x}, b_{t_2+x}\}$ -LAVS or -RAVS to produce a black Hamilton cycle and join the vertical red edges in the  $a_6^2$  and  $a_7^2$ -columns to  $C$ . The result is a Hamilton decomposition of  $\Gamma$ .

If  $\Delta = \Lambda_3$ , then apply an  $\{a_2, b_1, b_{t_1}\}$ -LAVS to obtain a blue Hamilton cycle. As  $m \geq 2t_2 \geq 2t_1$ , we can choose  $x$  so that all vertical edges between the  $b_{1+x}$  and  $b_{t_1+x}$ -rows are blue. Hence, the vertical edges between the  $b_{t_1+x}$  and  $b_{t_1+t_2-1+x}$ -rows forms a red path of length  $t_2 - 1$ . Apply an  $\{a_7^2, b_{t_1+x}, b_{t_1+t_2-3+x}\}$ -LAVS and either an  $\{a_4^2, b_{t_1+t_2-3+x}, b_{t_1+t_2-1+x}\}$ -RAVS or -LAVS to create a black Hamilton cycle and join all vertical red edges into a single cycle, for a Hamilton decomposition of  $\Gamma$ . ■

**Example 5.3.2.** Consider the graph  $\text{CAY}(\mathbb{Z}_{42}, \{s_1 = 3, s_2 = 6, s_3 = 7\})$  from Example 3.1.7. The labeling that was applied allows us to view the quotient graph  $\Delta = \text{CAY}(\mathbb{Z}_{42}/\langle 7 \rangle, \{\bar{3}, \bar{6}\})$  as  $\Lambda_3 = \text{CAY}(\mathbb{Z}_7, \{1, 2\})$ . Furthermore,  $t_2 = 6 > t_1 = 3$ , and by Case 2.ii with  $n = 7$  in Lemma 5.3.1, we may apply an  $\{a_2, b_1, b_3\}$ -RAVS and an  $\{a_3, a_4, b_3, b_4\}$ -CS to  $A_7 \square_3 B_6$  to obtain a blue Hamilton cycle and join the red cycles in the  $a_i$ -columns, where  $1 \leq i \leq 4$  together. Next, we choose  $x = 0$ , so that  $(a_3^2, b_1)(a_3^2, b_2)$  is a blue edge and apply an  $\{a_5^2, b_1, b_5\}$ -RAVS and an  $\{a_4^2, a_6^2, b_5, b_6\}$ -CS to create three monochromatic Hamilton cycles, i.e., a Hamilton decomposition,  $\{D_1, D_2, D_3\}$ , of  $\text{CAY}(\mathbb{Z}_{42}, \{3, 6, 7\})$  (see Figure 5.1).

$$\begin{aligned} D_1 &= 0, 3, 10, 17, 20, 27, 24, 21, 18, 15, 12, 9, 6, 13, 16, 19, 22, 25, 28, 31, 34, 37, 40, 1, 4, 7, 14, 11, 8, 5, \\ &\quad 2, 41, 38, 35, 32, 29, 26, 23, 30, 33, 36, 39 \\ D_2 &= 0, 7, 10, 13, 19, 25, 32, 26, 20, 23, 29, 36, 1, 8, 15, 22, 16, 9, 2, 37, 30, 27, 33, 39, 4, 11, 18, 12, 5, 40, \\ &\quad 34, 41, 6, 3, 38, 31, 24, 17, 14, 21, 28, 35 \\ D_3 &= 0, 6, 12, 19, 26, 33, 40, 4, 10, 16, 23, 17, 11, 5, 41, 35, 29, 22, 28, 34, 27, 21, 15, 9, 3, 39, 32, 38, 2, 8, \\ &\quad 14, 20, 13, 7, 1, 37, 31, 25, 18, 24, 30, 36 \end{aligned}$$





**Figure 5.1:** A Hamilton decomposition of  $\text{CAY}(\mathbb{Z}_{42}, \{3, 6, 7\}) = H_1 \cup H_2 \cup F$ , from Example 5.3.2 with  $x = 0$ . (A “♦” represents a color switch on the 4-cycle surrounding it.)

### 5.3.2 Even-order quotients

**Lemma 5.3.3.** *If  $A$  is an abelian group, and for some  $1 \leq i \leq 3$ ,  $\text{CAY}(A/\langle s_i \rangle, \{\overline{s_{j_1}}, \overline{s_{j_2}}\})$  is 4-regular and of order  $n \in \{4, 6, 8\}$ , then  $\text{CAY}(A, \{s_1, s_2, s_3\})$  has a Hamilton decomposition.*

*Proof.* As in Lemma 5.3.1, we have  $|\overline{s_{j_1}}|, |\overline{s_{j_2}}| \geq 3$ , where, without loss of generality,  $|s_{j_1}| \geq |s_{j_2}| \geq 3$  and  $|\langle s_i \rangle| = m \geq 3$ . By Theorem 3.1.6,  $\Gamma$  is a  $D(3, m, n)$ -graph, where  $F$  is the red 2-factor generated by  $s_i$ , and  $H_i \cup F = A_n^{(i)} \square_{r_i} B_m$ , for  $i = 1, 2$ . If  $t_i = 1$  for some  $i = 1, 2$ , then  $H_i$  is itself a Hamilton cycle, and upon deletion, leaves a pseudo-cartesian product of cycles, which is Hamilton decomposable by Theorem 1.5.1. Thus, assume  $t_2 \geq t_1 \geq 2$ .

**Case 1:  $n = 4$ .** Clearly,  $|\overline{s_{j_1}}| = |\overline{s_{j_2}}| = 4$ , thus  $A/J \cong \mathbb{Z}_4$ , i.e.,  $\Delta = \text{CAY}(\mathbb{Z}_4, \{\pm 1\})$ , as unlabeled graphs. Without loss of generality,  $\pi_1 = \pi_2 = (1)$  and  $H_i$  is just the 2-factor generated by  $s_{j_i}$  for  $i = 1, 2$ . If  $m = 3$ , then  $|A| = 12$ , and so  $A \cong \mathbb{Z}_{12}$  or  $A \cong \mathbb{Z}_2 \times \mathbb{Z}_6$ . Hamilton decompositions for all non-isomorphic, 6-regular, connected Cayley graphs on these groups are given in the Appendix in Table 1.1. Thus, assume  $m \geq 4$  and, by Corollary 4.1.6, assume  $t_1 \geq 3$ .

- 1.i.  $t_2 = 2k_2 \geq 2k_1 = t_1 \geq 4$ . Apply the CS-configuration of Lemma 2.2.9 to  $H_1 \cup F$  to obtain a blue Hamilton cycle and join the red edges in the  $a_1^2$  and  $a_4^2$ -columns. Choose  $x$  so that  $(a_1^2, b_{1+x})(a_1^2, b_{2+x})$  is a red edge. Apply an  $\{a_2^2, b_{1+x}, b_{t_2-1+x}\}$ -LAVS to create a red Hamilton cycle. Make the red cycle a directed cycle. By Lemma 2.2.9, all vertical red edges in the  $a_1^2$  and  $a_4^2$ -columns share the same orientation, call them  $\uparrow$ -edges. The application of the LAVS forces the red edges in the  $a_2^2$ -column to be  $\downarrow$ -edges, and the red edges in the  $a_3^2$ -column to be  $\uparrow$ -edges. Thus, we may apply a color-switch, call it  $\mathcal{X}$ , that is c-incident to the  $a_3^2$  and  $a_4^2$ -columns, and preserve the red Hamilton cycle. As the vertical edges in the  $a_4^2$ -column form a matching of red and blue edges, exactly one of  $e := (a_4^2, b_{m+x})(a_4^2, b_{1+x})$  or  $f := (a_4^2, b_{t_2-1+x})(a_4^2, b_{t_2+x})$  is

red. Therefore, define  $\mathcal{X}$  as follows:

$$\mathcal{X} := \begin{cases} \{a_3^2, a_4^2, b_{t_2-1+x}, b_{t_2+x}\}\text{-CS} & \text{if } f \text{ is red.} \\ \{a_3^2, a_4^2, b_{m+x}, b_{1+x}\}\text{-CS} & \text{if } f \text{ is non-red.} \end{cases}$$

The result is a set of three monochromatic Hamilton cycles.

- 1.ii.  $t_2 = 2k_2 > 2k_1 + 1 = t_1 \geq 3$ . Apply the CS-configuration of Lemma 2.2.9 to  $H_2 \cup F$  to obtain a black Hamilton cycle and join the red edges in the  $a_1^2$  and  $a_4^2$ -columns. Choose  $x$  so that  $(a_1, b_{1+x})(a_1, b_{2+x})$  is a red edge. Then apply an  $\{a_2, b_{1+x}, b_{t_1+x}\}$ -LAVS to obtain three monochromatic Hamilton cycles.
- 1.iii.  $t_2 = 2k_2 + 1 > 2k_1 = t_1 \geq 4$ . Apply the CS-configuration of Lemma 2.2.9 to  $H_1 \cup F$  to obtain a blue Hamilton cycle and join the red edges in the  $a_1$  and  $a_4$ -columns. Choose  $x$  so that  $(a_1^2, b_{1+x})(a_1^2, b_{2+x})$  is a red edge. Then apply an  $\{a_2^2, b_{1+x}, b_{t_2+x}\}$ -LAVS to obtain three monochromatic Hamilton cycles.
- 1.vi.  $t_2 = 2k_2 + 1 \geq 2k_1 + 1 = t_1 \geq 3$ . First suppose that  $m = t_2$ . Hence,  $|s_{j_2}| = 4$  and so  $\langle s_{j_2} \rangle = \{0, \pm s_{j_2}, 2s_{j_2}\}$ . By definition of  $\Gamma$ ,  $s_{j_1} \notin \langle s_{j_2} \rangle$ , and as  $m \geq 4$ , and  $m = 2k_2 + 1$ , we have  $m \geq 5$ , so that  $s_i \notin \langle s_{j_2} \rangle$ . Therefore, as  $\langle s_{j_2} \rangle$  is a subgroup of odd index, we are done by Lemma 5.3.1. Now suppose  $m > t_2$ . Hence,  $m \geq 6$ . Apply an  $\{a_2, b_1, b_{t_1}\}$ -LAVS to obtain a blue Hamilton cycle. By the justification of Theorem 2.2.7(a), all horizontal edges in the  $b_{t_1}$  and  $b_{t_1+1}$ -rows have the same direction upon making the blue cycle a directed cycle. Now, apply an  $\{a_1, a_2, b_{t_1}, b_{t_1+1}\}$ -CS to preserve the blue cycle. There now exists a red 2-factor consisting of three cycles, call them the  $\spadesuit$ -cycle, the  $\clubsuit$ -cycle, and the  $\diamond$ -cycle. The  $\spadesuit$ -cycle consists of the following:

$$(a_1, b_1)(a_2, b_1)(a_2, b_m)(a_2, b_{m-1}) \cdots (a_2, b_{t_1+1})(a_1, b_{t_1+2}) \cdots (a_1, b_m)(a_1, b_1)$$

The  $\clubsuit$ -cycle consists of all vertical red edges in the  $a_3$ -column and all vertical red edges in the  $a_1$ -column that are between the  $b_1$  and  $b_{t_1}$ -rows. The latter set forms a matching with the blue edges in the  $a_1$ -column. Finally, the  $\diamond$ -cycle consists of all  $m$  vertical red edges in the  $a_4$ -column. Cyclically shift the columns in  $H_2 \cup F$  so as to relocate the  $r_2$ -jump to be between the  $a_1^2$  and  $a_2^2$ -columns. In this manner, we view  $H_2 \cup F$  as  $C_4^{(2)} \square_{r_2} C_m$  where  $C_4^{(2)} := a_2^2 a_3^2 a_4^2 a_1^2 a_2^2$ . Let  $x$  be the integer such that the set of vertical edges in the  $a_1^2$ -column between the  $b_{1+x}$  and  $b_{t_1+1+x}$ -rows is precisely the red-blue partial matching in that column, i.e., the edges  $(a_1^2, b_{1+x})(a_1^2, b_{2+x})$  and  $(a_1^2, b_{t_1+x})(a_1^2, b_{t_1+1+x})$  are both blue. As  $m > t_2$ , all vertical edges in the  $a_4^2$  and  $a_1^2$ -columns between the  $b_{t_1+1+x}$  and  $b_{m+x}$ -rows are red. This forms a total of

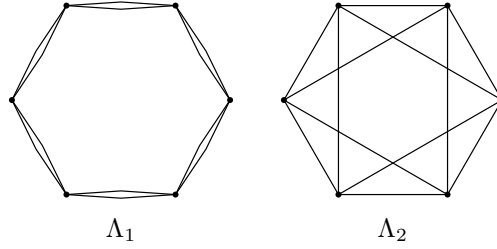
$$(m+x) - (t_1+1+x) + 1 = m - t_1 \geq m - t_2 \geq t_2$$

pairs of red edges. If  $e := (a_3^2, b_{t_1+1+x})(a_3^2, b_{t_1+2+x})$  is red, apply an

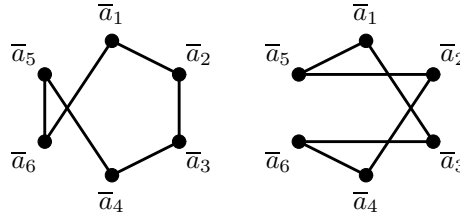
$$\{a_4^2, b_{t_1+1+x}, b_{t_2+t_1+1+x}\}\text{-LAVS}$$

and if  $e$  is not red, apply the corresponding RAVS. The result joins the  $\spadesuit$ ,  $\clubsuit$ , and  $\diamond$ -cycles and produces three monochromatic Hamilton cycles.

**Case 2:  $n = 6$ .** If  $m = 3$ , then  $A \cong \mathbb{Z}_{18}$  or  $A \cong \mathbb{Z}_3 \times \mathbb{Z}_6$ . Thus, by the computational results of Table 1.2 in Appendix A, we may assume  $m \geq 4$ .



**Figure 5.2:** The quotient graphs of Case 2 of Lemma 5.3.3.



**Figure 5.3:** A Hamilton decomposition of  $\Lambda_2 := \text{CAY}(\mathbb{Z}_6, \{1, 2\})$  from Case 2 of Lemma 5.3.3.

Clearly,  $A/J \cong \mathbb{Z}_6$ , and up to relabeling,  $\Delta = \Lambda_1 := \text{CAY}(\mathbb{Z}_6, \{\pm 1\})$  or  $\Delta = \Lambda_2 := \text{CAY}(\mathbb{Z}_6, \{1, 2\})$ , both of which are depicted in Figure 5.2. If  $\Delta = \Lambda_1$ , then without loss of generality,  $\pi_1 = \pi_2 = (1)$ , and by Corollary 4.1.6, we may assume  $t_1 \geq 3$ . If  $\Delta = \Lambda_2$ , then using the Hamilton decomposition shown in Figure 5.3, we may assume  $\pi_1 = (1)$  and  $A_6^{(2)} := a_1^2 a_3^2 a_6^2 a_4^2 a_2^2 a_5^2$ .

2.i.  $t_2 = 2k_2 \geq 2k_1 = t_1 \geq 2$ . Here  $m \geq 4$ . If  $m = 4$ , then  $|A| = 24$ , hence,

$$A \cong \mathbb{Z}_{24}, \mathbb{Z}_2 \times \mathbb{Z}_{12}, \text{ or } \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6.$$

By Theorem 1.5.2 or Tables 1.3, 1.4, and 1.5 of Appendix A, we may assume  $m \geq 6$ .

If  $\Delta = \Lambda_1$ , apply the CS-configuration of Lemma 2.2.9 to  $H_1 \cup F$  to obtain a blue Hamilton cycle. Suppose a RAVS or LAVS were applied between the  $a_1^2$  and  $a_3^2$ -columns, followed by the color-switches  $\{\mathcal{X}_1, \mathcal{X}_2\}$ , where  $\mathcal{X}_1$  is c-incident to the  $a_3^2$  and  $a_4^2$ -columns, and  $\mathcal{X}_2$  is c-incident to the  $a_4^2$  and  $a_5^2$ -columns. Clearly, a red Hamilton cycle would be formed. Making the red cycle a directed cycle, we see that all vertical red edges in the  $a_i^2$ -columns are  $\uparrow$ -edges, where  $i = 1, 3, 5$ , and all vertical red edges in the  $a_i^2$ -columns are  $\downarrow$ -edges, where  $i = 2, 4$ . By Lemma 2.2.9, the vertical red edges in the  $a_6^2$ -column are also  $\uparrow$ -edges. Hence, by Remark 2.2.2, applying the color-switch  $\mathcal{X}_3$  that is c-incident to the  $a_5^2$  and  $a_6^2$ -columns, will preserve the red Hamilton cycle. We shall now define  $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3\}$ . Choose  $x$  so that  $(a_1^2, b_{1+x})(a_1^2, b_{2+x})$  is red, and let  $e := (a_6^2, b_{t_2-1+x})(a_6^2, b_{t_2+x})$ . If  $e$  is red, then apply an  $\{a_2^2, b_{1+x}, b_{t_2-1+x}\}$ -LAVS and define

$$\mathcal{X}_1 := \{a_3^2, a_4^2, b_{t_2-1+x}, b_{t_2+x}\}\text{-CS}.$$

We now have a black Hamilton cycle. By Lemma 2.2.15, define

$$\mathcal{X}_2 := \{a_4^2, a_5^2, b_{t_2-2+x}, b_{t_2-1+x}\}\text{-CS and } \mathcal{X}_3 := \{a_5^2, a_6^2, b_{t_2-1+x}, b_{t_2+x}\}\text{-CS}$$

to preserve the black and red Hamilton cycles. The result is a Hamilton decomposition of  $\Gamma$ . If  $e$  is non-red, i.e.,  $e$  is blue, then clearly,  $(a_6^2, b_{t_2-2+x})(a_6^2, b_{t_2-1+x})$  is red. Hence, apply an  $\{a_2^2, b_{m+x}, b_{t_2-2+x}\}$ -RAVS and define

$$\mathcal{X}_1 := \{a_3^2, a_4^2, b_{t_2-2+x}, b_{t_2-1+x}\}\text{-CS}$$

to obtain a black Hamilton cycle. By Lemma 2.2.15, define

$$\mathcal{X}_2 := \{a_4^2, a_5^2, b_{t_2-1+x}, b_{t_2+x}\}\text{-CS and } \mathcal{X}_3 := \{a_5^2, a_6^2, b_{t_2-2+x}, b_{t_2-1+x}\}\text{-CS}$$

to preserve the black and red Hamilton cycles. The result is a Hamilton decomposition of  $\Gamma$ .

If  $\Delta = \Lambda_2$ , and  $t_2 = 2$ , then apply an  $\{a_1, a_2, b_1, b_2\}$ -CS to obtain a blue Hamilton cycle. By Lemma 2.2.15, the application of a  $\{a_2, a_6, b_2\}$ -LAHS will preserve the blue cycle, and create a red Hamilton cycle. Clearly, the direction pattern of the red cycle is:  $\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow$ , in particular, all vertical red edges in the  $a_1$  and  $a_3$ -columns have the same direction. As  $m \geq 4$ , there exists some integer  $x$  such that both  $e = (a_1^2, b_{1+x})(a_1^2, b_{2+x})$  and  $f = (a_3^2, b_{1+x})(a_3^2, b_{2+x})$  are red edges. Furthermore,  $e$  and  $f$  have the same direction, and the application of an  $\{a_1^2, a_3^2, b_{1+x}, b_{2+x}\}$ -CS yields the result. If  $t_2 \geq 4$ , then relocate the  $r_1$ -jump in  $H_1 \cup F$  to between the  $a_5$  and  $a_6$ -columns. In this manner, we view  $H_1 \cup F$  as  $A_6 \square_{r_1} B_m$ , where  $A_6 := a_6 a_1 a_2 a_3 a_4 a_5 a_6$ . Apply the CS-configuration of Lemma 2.2.9 to  $H_1 \cup F$  to obtain a blue Hamilton cycle and connect the vertical red edges in the  $a_5$  and  $a_6$ -columns into one cycle. Choose  $x$  so that  $(a_6^2, b_{1+x})(a_6^2, b_{2+x})$  is red, and let  $e := (a_5^2, b_{t_2-1+x})(a_5^2, b_{t_2+x})$ . We subdivide into two cases on the color of  $e$ .

- (a) If  $e$  is red, then apply an  $\{a_3^2, b_{1+x}, b_{t_2-1+x}\}$ -RAVS and an

$$\{a_6^2, a_4^2, b_{t_2-1+x}, b_{t_2+x}\}\text{-CS}.$$

At this point, there exists a black Hamilton cycle. If we apply a color-switch c-incident to the  $a_4^2$  and  $a_5^2$ -columns, we will obtain a red Hamilton cycle and break the black cycle. Making the red cycle a directed cycle, we see that, by Lemma 2.2.9, the vertical red edges in the  $a_6^2$  and  $a_5^2$ -columns are  $\uparrow$ -edges. It follows that the vertical red edges in the  $a_4^2$ -column are  $\downarrow$ -edges, and thus the vertical red edges in the  $a_2^2$ -column are forced to be  $\uparrow$ -edges. By Remark 2.2.2, an additional color-switch c-incident to the  $a_2^2$  and  $a_5^2$ -columns will preserve the red Hamilton cycle. Hence, apply an

$$\{a_4^2, a_2^2, b_{t_2-2+x}, b_{t_2-1+x}\}\text{-CS and an } \{a_2^2, a_5^2, b_{t_2-1+x}, b_{t_2+x}\}\text{-CS}.$$

By Lemma 2.2.15, the result is three monochromatic Hamilton cycles.

- (b) If  $e$  is non-red, i.e.,  $e$  is blue, then relocate the  $r_2$ -jump to be between the  $a_6^2$  and  $a_3^2$ -columns. In this manner, we view  $H_2 \cup F = A_6^{(2)} \square_{r_2} B_m$  where

$$A_6^{(2)} := a_6^2 a_4^2 a_2^2 a_5^2 a_1^2 a_3^2 a_6^2.$$

Apply an  $\{a_6^2, a_4^2, b_{1+x}, b_{2+x}\}$ -CS and an  $\{a_4^2, a_2^2, b_{2+x}, b_{3+x}\}$ -CS to connect all vertical red edges in the  $a_i^2$ -columns, where  $i \in \{6, 4, 2, 5\}$ , into one cycle. Apply an  $\{a_1^2, b_{3+x}, b_{t_2-1+x}\}$ -RAVS to create a red Hamilton cycle. As before, making the red cycle a directed cycle, it is clear the the vertical red edges in the  $a_i^2$ -columns take the form:  $\uparrow\downarrow\uparrow\downarrow\uparrow$ . Hence, we may apply an additional switch,  $\mathcal{X}$ , c-incident to the  $a_3^2$  and  $a_6^2$ -columns, and preserve

the red cycle, by Remark 2.2.2. Consider the following 4-cycle:

$$(a_3^2, b_{t_2-1+x}) \xrightarrow{e_1} (a_6^2, b_{t_2-1+r_2+x}) \xrightarrow{e_2} (a_6^2, b_{t_2+r_2+x}) \xrightarrow{e_3} (a_3^2, b_{t_2+x}) \xrightarrow{e_4} (a_3^2, b_{t_2-1+x}).$$

Clearly,  $e_4$  is red, and  $e_1$  and  $e_3$  are black. By our choice of  $x$ , and the fact that  $r_2$  is even, it can never be the case that  $e_2$  is blue. If  $e_2$  is red, then let

$$\mathcal{X} := \{a_3^2, a_6^2, b_{t_2-1+x}, b_{t_2+x}\}\text{-CS}$$

to obtain a black Hamilton cycle. If  $e_2$  is non-red, then

$$(a_6^2, b_{t_2-1+r_2+x}) = (a_6^2, b_{1+x}),$$

and hence, it is the only black edge in the  $a_6^2$ -column. In this case, remove the

$$\{a_1^2, b_{3+x}, b_{t_2-1+x}\}\text{-RAVS},$$

and apply an  $\{a_1^2, b_{4+x}, b_{t_2+x}\}$ -LAVS. Define  $\mathcal{X} := \{a_3^2, a_6^2, b_{3+x}, b_{4+x}\}$ -CS to obtain a Hamilton decomposition of  $\Gamma$ .

2.ii.  $t_2 = 2k_2 > 2k_1 + 1 = t_1 \geq 3$ . Note  $m \geq 6$ .

If  $\Delta = \Lambda_1$ , then apply the CS-configuration of Lemma 2.2.15 to  $H_2 \cup F$  and let  $x$  be the integer such that  $(a_1, b_{1+x})(a_2, b_{2+x})$  is red. Apply an  $\{a_2, b_{1+x}, b_{t_1+x}\}$ -LAVS and an  $\{a_3, a_5, b_{t_1+x}\}$ -LAHS to obtain the result.

If  $\Delta = \Lambda_2$ , then apply an  $\{a_2, b_1, b_{t_1}\}$ -LAVS to obtain a blue Hamilton cycle. There exists a path of  $t_1 - 1 < m$  vertical blue edges in the  $a_2$ -column. Let  $x$  be any integer such that  $(a_2^2, b_{t_2-1+x})(a_2^2, b_{t_2+x})$  is red and apply an  $\{a_2^2, a_5^2, b_{t_2-1+x}, b_{t_2+x}\}$ -CS. Finally, apply an

$$\begin{cases} \{a_6^2, b_{1+x}, b_{t_2-1+x}\}\text{-LAVS} & \text{if } (a_3^2, b_{1+x})(a_3^2, b_{2+x}) \text{ is red} \\ \{a_6^2, b_{1+x}, b_{t_2-1+x}\}\text{-RAVS} & \text{if } (a_3^2, b_{1+x})(a_3^2, b_{2+x}) \text{ is not red.} \end{cases}$$

2.iii.  $t_2 = 2k_2 + 1 > 2k_1 = t_1 \geq 2$ . Here  $m \geq 2t_2 \geq 6$ .

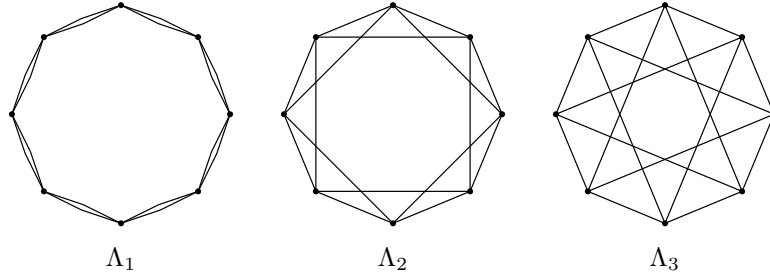
If  $\Delta = \Lambda_1$ , then use the method of Case 2.ii by applying the CS-configuration of Lemma 2.2.15 to  $H_1 \cup F$ .

If  $\Delta = \Lambda_2$ , and  $t_1 = 2$ , apply an  $\{a_4^2, b_{1+x}, b_{t_2+x}\}$ -LAVS to  $H_2 \cup F$  to obtain a black Hamilton cycle, where  $x$  is chosen so that  $(a_4, b_1)(a_4, b_2)(a_4, b_3)$  is a red 3-path. Apply the following three switches to  $H_1 \cup F$  to obtain a Hamilton decomposition:

$$\begin{cases} \{a_1, a_3, b_2\}\text{-RAHS and an } \{a_4, a_5, b_1, b_2\}\text{-CS} & \text{if } (a_3, b_2)(a_3, b_3) \text{ is red} \\ \{a_1, a_3, b_2\}\text{-LAHS and an } \{a_4, a_5, b_2, b_3\}\text{-CS} & \text{if } (a_3, b_2)(a_3, b_3) \text{ is not red.} \end{cases}$$

If  $t_1 > 2$ , then apply an  $\{a_4^2, b_{1+x}, b_{t_2+x}\}$ -LAVS to  $H_2 \cup F$ , where this time,  $x$  is chosen so that  $(a_4, b_{t_1-1})(a_4, b_{t_1})$  is red and apply an  $\{a_4, a_5, b_{t_1-1}, b_{t_1}\}$ -CS. Finally, apply an

$$\begin{cases} \{a_2, b_1, b_{t_1-1}\}\text{-RAVS} & \text{if } (a_3, b_1)(a_3, b_2) \text{ is red} \\ \{a_2, b_1, b_{t_1-1}\}\text{-LAVS} & \text{if } (a_3, b_1)(a_3, b_2) \text{ is not red.} \end{cases}$$



**Figure 5.4:** The quotient graphs of Case 3 of Lemma 5.3.3.

2.vi.  $t_2 = 2k_2 + 1 \geq 2k_1 + 1 = t_1 \geq 3$ . Here  $m \geq 3$ .

If  $\Delta = \Lambda_1$ , then apply an  $\{a_2, b_1, b_{t_1}\}$ -LAVS and choose  $x$  so that  $(a_3^2, b_{1+x})(a_3^2, b_{2+x})$  is red. Apply an  $\{a_4^2, b_{1+x}, b_{t_2+x}\}$ -LAVS and an  $\{a_5^2, a_6^2, b_{t_2+x}, b_{t_2+1+x}\}$ -CS to obtain the result.

If  $\Delta = \Lambda_2$ , then apply an  $\{a_2, b_1, b_{t_1}\}$ -LAVS to obtain a blue Hamilton cycle. There exists a path of  $t_1 - 1 < m$  vertical blue edges in the  $a_2$ -column. Let  $x$  be any integer such that  $(a_2^2, b_{t_2+x})(a_2^2, b_{t_2+1+x})$  is red and apply an  $\{a_2^2, a_5^2, b_{t_2+x}, b_{t_2+1+x}\}$ -CS. Finally, apply an

$$\begin{cases} \{a_6^2, b_{1+x}, b_{t_2+x}\}\text{-LAVS} & \text{if } (a_3^2, b_{1+x})(a_3^2, b_{2+x}) \text{ is red} \\ \{a_6^2, b_{1+x}, b_{t_2+x}\}\text{-RAVS} & \text{if } (a_3^2, b_{1+x})(a_3^2, b_{2+x}) \text{ is not red.} \end{cases}$$

By Theorem 2.2.7(a), the result is a Hamilton decomposition.

**Case 3:  $n = 8$ .** As before, we must have  $|\overline{s_{j_1}}| \in \{4, 8\}$ . If  $|\overline{s_{j_1}}| = 8$ , then  $A/J \cong \mathbb{Z}_8$ , and, up to a relabeling of the vertices,  $\Delta$  is either the graph  $\Lambda_1 := \text{CAY}(\mathbb{Z}_8, \{\pm 1\})$ , so without loss of generality,  $\pi_1 = \pi_2 = (1)$ , or  $\Lambda_2 := \text{CAY}(\mathbb{Z}_8, \{1, 2\})$ , so without loss of generality,  $\pi_1 = (1)$  and  $\pi_2 = (2356487)$ . In this case,

$$A_8^{(2)} := a_1^2 a_3^2 a_5^2 a_8^2 a_6^2 a_4^2 a_2^2 a_7^2 a_1^2.$$

If  $|\overline{s_{j_1}}| = |\overline{s_{j_2}}| = 4$ , then  $A/J \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ , and, up to a relabeling of the vertices,  $\Delta$  is the graph  $\Lambda_3 := \text{CAY}(\mathbb{Z}_4 \times \mathbb{Z}_2, \{(1, 0), (1, 1)\})$ , so without loss of generality,  $\pi_1 = (1)$  and  $\pi_2 = (24)(37)(68)$ . In this case,

$$A_8^{(2)} := a_1^2 a_4^2 a_7^2 a_2^2 a_5^2 a_8^2 a_3^2 a_6^2 a_1^2.$$

In what follows, we shall write  $\Delta = \Lambda_i$  to mean,  $\Delta$  and  $\Lambda_i$  are equal as unlabeled graphs, and these are shown in Figure 5.4. As before, if  $\Delta \in \{\Lambda_1, \Lambda_3\}$ , then by Corollary 4.1.6, we may assume  $t_1 \geq 3$ . Cases 3.ii-3.iv for  $\Delta = \Lambda_1$ , follow from the technique of Cases 2.ii-2.iv., respectively. If  $m = 3$ , then  $|A| = 24$ , which is resolved in Tables 1.3, 1.4, and 1.5 of Appendix A. If  $m = 4$ , then  $|A| = 32$ , and so  $A$  is isomorphic to one of four possible abelian groups. Note,  $A \not\cong \mathbb{Z}_2^5$ , for we are requiring that  $|s_i| \geq 3$  for all  $1 \leq i \leq 3$ , and  $A \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ , because  $|S| = 3$ . Thus,

$$A \cong \mathbb{Z}_{32}, \mathbb{Z}_4 \times \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_{16}, \text{ or } \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8.$$

These small cases are resolved in Tables 1.6, 1.7, and 1.8 of Appendix A. Thus, we assume  $m \geq 5$ .

3.i.  $t_2 = 2k_2 \geq 2k_1 = t_1 \geq 4$ .

If  $\Delta = \Lambda_1$ , then apply the CS-configuration of Lemma 2.2.9 to  $H_1 \cup F$  and choose  $x$  so that  $(a_1^2, b_{1+x})(a_2^2, b_{2+x})$  is red. Let  $e := (a_8^2, b_{t_2-1+x})(a_8^2, b_{t_2+x})$ . If  $e$  is not red, then apply an  $\{a_1^2, a_2^2, b_{1+x}, b_{2+x}\}$ -CS, and an  $\{a_2^2, a_4^2, b_{t_2-1+x}\}$ -LAHS. We now have obtained a blue Hamilton cycle, a black Hamilton cycle, and a red 2-factor consisting of one cycle on the vertical red edges in the  $a_i^2$ -columns, where  $i \in \{1, 2, 3, 4, 8\}$ . As  $(a_8^2, b_{t_2-2+x})(a_8^2, b_{t_2-1+x})$  is red, we may apply an  $\{a_4^2, a_8^2, b_{t_2-1+x}\}$ -LAHS to preserve the black cycle, by Lemma 2.2.15. To see that this produces a red Hamilton cycle, we note that prior to applying the  $\{a_7^2, a_8^2, b_{t_2-1+x}, b_{t_2+x}\}$ -CS, a red Hamilton cycle exists. As the red edges in the  $a_1^2$  and  $a_8^2$ -columns are  $\uparrow$ -edges, the direction pattern is:

$$\uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow \uparrow.$$

Hence, the  $\{a_7^2, a_8^2, b_{t_2-1+x}, b_{t_2+x}\}$ -CS preserves the red Hamilton cycle. If  $e$  is red, then apply an  $\{a_2^2, b_{1+x}, b_{t_2-1+x}\}$ -LAVS, and an  $\{a_3^2, a_4^2, b_{t_2-1+x}, b_{t_2+x}\}$ -CS. The application of an  $\{a_4^2, a_8^2, b_{t_2-1+x}\}$ -RAHS creates a set of three monochromatic Hamilton cycles.

If  $\Delta = \Lambda_2$ , and  $t_2 = 2$ , follow the technique of Case 2.i. Relocate the  $r_1$ -jump in  $H_1 \cup F$  to between the  $a_2$  and  $a_3$ -columns. In this manner, we view  $H_1 \cup F$  as  $A_8 \square_{r_1} B_m$ , where

$$A_8 := a_3 a_4 a_5 a_6 a_7 a_8 a_1 a_2 a_3.$$

Apply the CS-configuration of Lemma 2.2.9 to  $H_1 \cup F$  to obtain a blue Hamilton cycle and connect the vertical red edges in the  $a_2$  and  $a_3$ -columns into one cycle. Choose  $x$  so that  $(a_3^2, b_{1+x})(a_2^2, b_{2+x})$  is red, and let  $e := (a_2^2, b_{t_2-1+x})(a_2^2, b_{t_2+x})$ . We subdivide into two cases on the color of  $e$ .

- (a) If  $e$  is red, then apply an  $\{a_1^2, b_{1+x}, b_{t_2-1+x}\}$ -RAVS and an  $\{a_3^2, a_5^2, b_{t_2-1+x}, b_{t_2+x}\}$ -CS to obtain a black Hamilton cycle. Next, apply an  $\{a_5^2, a_6^2, b_{t_2-1+x}\}$ -RAHS, which by Lemma 2.2.15, preserves the black cycle. As before, if an

$$\{a_6^2, a_4^2, b_{t_2-2+x}, b_{t_2-1+x}\}$$
-CS

is applied, a red Hamilton cycle is created. Upon making this a directed cycle, the column direction pattern of

$$A_8^{(2)} := a_7^2 a_1^2 a_3^2 a_5^2 a_8^2 a_6^2 a_4^2 a_2^2$$

is

$$\uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow \uparrow.$$

Thus, applying an  $\{a_4^2, a_2^2, b_{t_2-1+x}, b_{t_2+x}\}$ -CS, will preserve the red cycle, and by Lemma 2.2.15, preserve the black cycle. The result is now obtained.

- (b) If  $e$  is non-red, i.e.,  $e$  is blue, then relocate the  $r_2$ -jump to be between the  $a_3^2$  and  $a_1^2$ -columns. In this manner, we view  $H_2 \cup F = A_6^{(2)} \square_{r_2} B_m$  where

$$A_6^{(2)} := a_3^2 a_5^2 a_8^2 a_6^2 a_4^2 a_2^2 a_7^2 a_1^2 a_3^2.$$

Apply an  $\{a_3^2, a_4^2, b_{2+x}\}$ -RAHS to connect all horizontal black edges in the  $b_{j+x}$ -rows, where  $j \in \{1, 2, 3\}$ , to one cycle, and connect all vertical red edges in the  $a_i^2$ -columns, where  $i \in \{3, 5, 8, 6, 4, 2\}$ , into one cycle. The result now follows by finding three additional switches,  $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3\}$ , between the  $a_2^2$ ,  $a_7^2$ ,  $a_1^2$ , and  $a_3^2$ -columns, by using the technique of Case 2.i.(b).

If  $\Delta = \Lambda_3$ , then apply the CS-configuration of Lemma 2.2.9 to  $H_2 \cup F = A_8^{(2)} \square_{r_2} B_m$  to obtain a black Hamilton cycle and join the vertical red edges in the  $a_1^2$  and  $a_6^2$ -columns into one cycle. We can assume the switch was applied so that  $(a_1, b_1)(a_1, b_2)$  is a red edge in  $A_8^{(1)} \square_{r_2} B_m$ . Let  $e := (a_6, b_{t_1-1})(a_6, b_{t_1})$ , and let  $v$  be the vertex  $(a_1, b_{t_1-1+r_1})$ . Note,  $v \neq (a_1, b_1)$ , for otherwise, the blue edges in the  $b_1$  and  $b_{t_1-1}$ -rows must have been on the same cycle, a contradiction.

(a) If  $e$  is red, then apply an

$$\{a_2, b_1, b_{t_1-1}\}\text{-LAVS and an } \{a_3, a_4, b_{t_1-1}, b_{t_1}\}\text{-CS.}$$

This produces a blue Hamilton cycle, and connects the  $a_i$ -columns,  $i \in \{1, 2, 3, 4, 6\}$  into a single red cycle. As  $e$  is red, apply an  $\{a_4, a_6, b_{t_1-1}\}$ -RAHS, an  $\{a_7, a_8, b_{t_1-2}, b_{t_1-1}\}$ -CS, and an  $\{a_8, a_1, b_{t_1-1}, b_{t_1}\}$ -CS (because  $r_1$  is even, the last switch is well-defined). We claim this provides a Hamilton decomposition of  $\Gamma$ . To see this, note that if the  $\{a_5, a_6, b_{t_1-1}, b_{t_1}\}$ -CS is removed, we have a red Hamilton cycle with column-direction pattern is:

$$\uparrow \downarrow \uparrow \downarrow \uparrow \uparrow \uparrow \downarrow.$$

Hence, applying an  $\{a_5, a_6, b_{t_1-1}, b_{t_1}\}$ -CS will preserve the red cycle by Remark 2.2.2, and by Lemma 2.2.15, the blue Hamilton cycle is preserved.

(b) If  $e$  is not red, and  $t_1 \geq 6$ , then apply an  $\{a_1, a_5, b_2\}$ -RAHS, an  $\{a_7, b_3, b_{t_1-1}\}$ -RAVS, and an  $\{a_8, a_1, b_{t_1-1}, b_{t_1}\}$ -CS to obtain the result. Use a similar technique for the case  $t_1 = 4$ .

3.ii.  $t_2 = 2k_2 > 2k_1 + 1 = t_1 \geq 3$ .

If  $\Delta = \Lambda_2$ , then apply an  $\{a_2, b_1, b_{t_1}\}$ -LAVS to obtain a blue Hamilton cycle. Note that as  $m \geq t_2 > t_1$ , there exists a red 3-path  $P$  in the  $a_2$ -column. Let  $x$  be the integer such that  $P$  lies in between the  $b_{t_2-2+x}$  and  $b_{t_2+x}$ -rows. Apply an  $\{a_8^2, b_{1+x}, b_{t_2-1+x}\}$ -LAVS and an

$$\{a_6^2, a_4^2, b_{t_2-1+x}, b_{t_2+x}\}\text{-CS}$$

to create a black Hamilton cycle. By Lemma 2.2.15, applying an  $\{a_4^2, a_7^2, b_{t_2-1+x}\}$ -RAHS will preserve the black cycle, and create a Hamilton decomposition of  $\Gamma$ .

If  $\Delta = \Lambda_3$ , let  $x$  be the integer such that  $e := (a_6, b_m)(a_6, b_1)$  in  $H_1 \cup F$  corresponds to  $(a_6^2, b_{t_2-1+x})(a_6^2, b_{t_2+x})$  in  $H_2 \cup F$ . Apply an

$$\{a_4^2, b_{1+x}, b_{t_2-1+x}\}\text{-RAVS and an } \{a_7^2, a_2^2, b_{t_2-1+x}, b_{t_2+x}\}\text{-CS}$$

to obtain a black Hamilton cycle and join all vertical red edges in the  $a_i^2$ -columns, where  $i \in \{1, 4, 7, 2\}$ , into one cycle  $C$ . Apply an  $\{a_8^2, a_6^2, b_{t_2-1+x}\}$ -RAHS to preserve the black cycle, by Lemma 2.2.15, and create a red cycle,  $C'$ , on the  $a_i^2$ -columns, where  $i \in \{8, 3, 6\}$ . Note, that  $e$  is now the only non-red edge in the  $a_6$ -column, and the  $a_7$ -column contains a  $t_2$ -path of alternating red and black edges. Therefore, apply an  $\{a_6, b_1, b_{t_1}\}$ -LAVS or -RAVS, depending on which defines a valid switching configuration, to obtain a Hamilton decomposition of  $\Gamma$ .

3.iii.  $t_2 = 2k_2 + 1 > 2k_1 = t_1 \geq 4$ .

If  $\Delta = \Lambda_2$ , then apply an  $\{a_2, b_1, b_{t_1-1}\}$ -LAVS and an  $\{a_6, a_7, b_{t_1-1}, b_{t_1}\}$ -CS. We now have a blue Hamilton cycle and a red 2-factor consisting, among other things, of a red cycle,  $C$ , on the  $a_i$ -columns, where  $i \in \{1, 2, 3\}$ , and a red cycle,  $C'$ , on the  $a_6$  and  $a_7$ -columns. Let  $x$  be an integer such that the  $(a_6^2, b_{1+x})(a_6^2, b_{2+x})$  is a blue edge (note this is the only non-red edge in the  $a_6^2$ -column). Apply an  $\{a_8^2, b_{1+x}, b_{t_2+x}\}$ -LAVS to create a black Hamilton cycle and join



the vertical red edges in the  $a_5^2$  and  $a_8^2$ -columns to  $C'$ . Call this new cycle,  $C''$ . The application of an  $\{a_6^2, a_2^2, b_{t_2+x}\}$ -LAHS will join  $C$  and  $C''$  into a red Hamilton cycle. By Lemma 2.2.15, this also will preserve the black Hamilton cycle. The result now follows.

If  $\Delta = \Lambda_3$ , then  $t_1 \geq 4$ , and  $t_2 \geq 5$ . Apply the color switching configuration of Lemma 2.2.9 to  $H_1 \cup F$ . Let  $x$  be the integer such that  $(a_1^2, b_{1+x})(a_1^2, b_{2+x})$  is red. Apply an  $\{a_4^2, b_{1+x}, b_{t_2+x}\}$ -LAVS to obtain a black Hamilton cycle. If  $h := (a_8^2, b_{t_2+x})(a_8^2, b_{t_2+1+x})$  is red, then apply an  $\{a_7^2, a_5^2, b_{t_2+x}\}$ -LAHS and an  $\{a_8^2, a_6^2, b_{t_2+x}\}$ -LAHS to obtain the result. If  $h$  is not red, then clearly  $(a_8^2, b_{2j+x})(a_8^2, b_{2j+1+x})$  is red, for all  $j$ . Apply an  $\{a_4^2, a_8^2, b_{2+x}\}$ -RAHS to connect all horizontal black edges in the  $b_{1+x}$ ,  $b_{2+x}$ , and  $b_{3+x}$ -rows into a single cycle. Next, apply an  $\{a_3^2, b_{3+x}, b_{t_2+x}\}$ -RAVS to create a black Hamilton cycle, and obtain the result.

3.vi.  $t_2 = 2k_2 + 1 \geq 2k_1 + 1 = t_1 \geq 3$ .

If  $\Delta = \Lambda_2$ , then apply an  $\{a_2, b_1, b_{t_1}\}$ -LAVS and an  $\{a_3, a_5, b_{t_1}\}$ -LAHS, which by Lemma 2.2.15 produces a blue Hamilton cycle, and joins the vertical red edges of the  $a_i$ -columns, where  $1 \leq i \leq 5$ , into one cycle,  $C$ . There exists at least one red edge in the  $a_2^2$ -column. Let  $x$  be the integer such that  $(a_2^2, b_{t_2+x})(a_2^2, b_{t_2+1+x})$  is red. Furthermore, note that

$$h := (a_5, b_{t_1-1})(a_5, b_{t_1}) = (a_5^2, b_y)(a_5^2, b_{y+1})$$

is the only non-red edge in the  $a_5^2$ -column. If  $y$  is odd, then apply an  $\{a_8^2, b_{1+x}, b_{t_2+x}\}$ -RAVS, and if  $y$  is even, then apply an  $\{a_8^2, b_{1+x}, b_{t_2+x}\}$ -LAVS to obtain a black Hamilton cycle. Finally, by Corollary 2.2.8(a), the application of an  $\{a_2^2, a_7^2, b_{t_2+x}, b_{t_2+1+x}\}$ -CS preserves the black cycle, and creates a red Hamilton cycle, to obtain a Hamilton decomposition of  $\Gamma$ .

If  $\Delta = \Lambda_3$ , then apply an  $\{a_4^2, b_{1+x}, b_{t_2+x}\}$ -LAVS, where  $x$  is chosen so that

$$e := (a_4, b_{t_1})(a_4, b_{t_1+1}) = (a_4^2, b_{t_2+x})(a_4^2, b_{t_2+1+x}).$$

The choice of  $x$  guarantees that  $e$  is red. Let  $f := (a_7, b_{t_1})(a_7, b_{t_1+1})$ . If  $f$  is red, then apply an  $\{a_4, a_6, b_{t_1}\}$ -LAHS and an  $\{a_7, a_8, b_{t_1}, b_{t_1+1}\}$ -CS. Apply either an  $\{a_2, b_1, b_{t_1}\}$ -LAVS or -RAVS, depending on which defines a good red-blue CS-configuration. In either case, by Corollary 2.2.8(a) and Lemma 2.2.15, the result is a Hamilton decomposition. If  $f$  is not red, then switch reflect the  $\{a_4^2, b_{1+x}, b_{t_2+x}\}$ -LAVS so it becomes a -RAVS. Now, clearly,  $f$  is red, and  $e$  is still red. Use the previous technique to obtain a Hamilton decomposition of  $\Gamma$ . ■

## 5.4 Main Results

If  $\Delta_i = \text{CAY}(A/\langle s_i \rangle, \{\overline{s_{j_1}}, \overline{s_{j_2}}\})$  is a connected, 4-regular, quotient graph of  $\text{CAY}(A, \{s_1, s_2, s_3\})$ , it cannot have order  $n = 2$ , for each connection set element generates a 1-factor. Furthermore, if  $|V(\Delta_i)| = 3$ , then  $\Delta_i \simeq \text{CAY}(\mathbb{Z}_3, \{1, 1\})$ , which is a multigraph. Thus, Theorem 5.2.9 and Lemmata 5.3.1 and 5.3.3 combine to yield the following result.

**Theorem 5.4.1.** *If  $\Gamma = \text{CAY}(A, \{s_1, s_2, s_3\})$  is a connected, 6-regular, abelian Cayley graph of even order, and for some  $1 \leq i \leq 3$ ,  $\Delta_i = \text{CAY}(A/\langle s_i \rangle, \{\overline{s_{j_1}}, \overline{s_{j_2}}\})$  is 4-regular, and  $\Delta_i \not\simeq \text{CAY}(\mathbb{Z}_3, \{1, 1\})$ , then  $\Gamma$  has a Hamilton decomposition.*

Alternatively, Theorem 5.4.1 may be stated as follows.

**Theorem 5.4.2.** *If  $\Gamma = \text{CAY}(A, S)$  is a connected, 6-regular, abelian Cayley graph of even order, then  $\Gamma$  has a Hamilton decomposition if  $S$  has no involutions, and for some  $s \in S$ ,  $\text{CAY}(A/\langle s \rangle, \bar{S})$  is 4-regular, and of order at least 4.*

**Corollary 5.4.3.** *If  $\Gamma = \text{CAY}(A, \{s_1, s_2, s_3\})$  is a connected, 6-regular, abelian Cayley graph of even order, then  $\Gamma$  has a Hamilton decomposition if one of the following holds:*

- (a)  $s_1 \in \langle s_2, s_3 \rangle$ ,  $s_2 \in \langle s_1, s_3 \rangle$ , and  $[A : \langle s_3 \rangle] \geq 4$ , or
- (b)  $|s_1| \geq |s_2| > 2|s_3|$ , or
- (c)  $|s_1| \geq |s_2| > |s_3|$ , and either
  - i.  $|A| = (2k + 1)|s_3|$ , with  $k \geq 2$ , or
  - ii.  $|A| \geq 4|s_3|$  and  $|s_1|$  and  $|s_2|$  are odd.

*Proof.* Clearly,  $|s_i| \geq 3$  for  $1 \leq i \leq 3$ . Suppose (a),  $\langle \bar{s}_1 \rangle = \langle \bar{s}_2 \rangle = A/\langle s_3 \rangle$ , and the result follows from Theorem 5.4.1. Suppose (b) holds, and note that if  $|s_i|$  is even, then  $|2s_i| = |s_i|/2$ , and if  $|s_i|$  is odd, then  $|2s_i| = |s_i|$ . Therefore, by Lagrange's Theorem,  $2s_i \notin \langle s_3 \rangle$ , for  $i = 1, 2$  and by Theorem 1.5.2, it is assumed  $|A| \geq 2|s_1| > 4|s_3|$ , so that  $[A : \langle s_3 \rangle] > 4$ . Hence, the quotient graph  $\text{CAY}(A/\langle s_3 \rangle, \{\bar{s}_1, \bar{s}_2\})$  satisfies Theorem 5.4.1. Now suppose (c) holds. If  $s_3$  generates a subgroup of odd index, at least five, clearly  $\text{CAY}(A/\langle s_3 \rangle, \{\bar{s}_1, \bar{s}_2\})$  is 4-regular, and we are done by Theorem 5.4.1. A similar result is obtained when  $|s_1|$  and  $|s_2|$  are odd, and  $s_3$  generates a subgroup of index at least four. ■



## Chapter 6

# Conclusions and Further Research Problems

### 6.1 Open cases

One glaring omission to Theorem 5.4.1 is the following:

**Open Problem 1.** *If  $\Gamma = \text{CAY}(A, \{s_1, s_2, s_3\})$  is a connected, 6-regular, abelian Cayley graph of even order, and for some  $1 \leq i \leq 3$ , the graph  $\text{CAY}(A/\langle s_i \rangle, \{\overline{s_{j_1}}, \overline{s_{j_2}}\}) \simeq \text{CAY}(\mathbb{Z}_3, \{1, 1\})$ , prove that  $\Gamma$  has a Hamilton decomposition.*

We give a partial solution to the above problem, in the case when  $s_{j_1}$  and  $s_{j_2}$  generate subgroups of even index in  $A$ .

**Lemma 6.1.1.** *If  $\Gamma = \text{CAY}(A, \{s_1, s_2, s_3\})$  is a connected, 6-regular, abelian Cayley graph of even order, and for some  $1 \leq i \leq 3$ , the graph  $\text{CAY}(A/\langle s_i \rangle, \{\overline{s_{j_1}}, \overline{s_{j_2}}\}) \simeq \text{CAY}(\mathbb{Z}_3, \{1, 1\})$ , where  $s_{j_1}$  and  $s_{j_2}$  generate subgroups of even index in  $A$ , then  $\Gamma$  has a Hamilton decomposition.*

*Proof.* Let  $m = |s_i|$ , and without loss of generality,  $|s_{j_1}| \geq |s_{j_2}|$ . By Theorem 3.1.6,  $\Gamma$  is a  $D(3, m, 3)$ -graph, where  $H_1 \cup F = A_3^{(1)} \square_{r_1} B_m$  consists of  $t_1 = \gcd(r_1, m) = [A : \langle s_{j_1} \rangle] = 2k_1$  horizontal blue cycles. Similarly,  $H_2 \cup F = A_3^{(2)} \square_{r_2} B_m$  consists of  $t_2 = \gcd(r_2, m) = [A : \langle s_{j_2} \rangle] = 2k_2$  horizontal black cycles. We may assume that

$$A_3^{(1)} = a_1 a_2 a_3 a_1 \text{ and } A_3^{(2)} = a_1^2 a_2^2 a_3^2 a_1^2.$$

The case  $m \in \{4, 6, 8\}$  is resolved in Appendix A. Hence, assume  $m \geq 10$  is even, and by Corollary

4.1.6, assume  $t_2 \geq t_1 \geq 4$ . Apply the CS-configuration of Lemma 2.2.9 to  $H_1 \cup F$  to obtain a blue Hamilton cycle and join the  $a_1$  and  $a_3$ -columns into one red cycle. Relocate the  $r_2$ -jump in  $H_2 \cup F$  so that it is between the  $a_2^2$  and  $a_3^2$ -columns. In this manner, we view  $H_2 \cup F$  as  $A_3^{(2)} \square_{r_2} B_m$  with  $A_3^{(2)} := a_3^2 a_1^2 a_2^2 a_3^2$ . Let  $h$  be any integer such that  $(a_3^2, b_{1+h})(a_3^2, b_{2+h})$  is red. Now, apply the CS-configuration of Lemma 2.2.9 to  $H_2 \cup F$  to create a black Hamilton cycle, and obtain a Hamilton decomposition of  $\Gamma$ . ■

### 6.1.1 Quotient connection sets with involutions

We briefly examine the case where, for all  $i \in \{1, 2, 3\}$ , the quotient graph  $\text{CAY}(A/\langle s_i \rangle, \{\overline{s_{j_1}}, \overline{s_{j_2}}\})$  of  $\text{CAY}(A, \{s_1, s_2, s_3\})$ , is  $k$ -regular, for some  $k \leq 3$ . Note,  $k = 0$  if and only if  $\{s_{j_1}, s_{j_2}\} \subseteq \langle s_i \rangle$ , i.e.  $A$  is a cyclic group generated by  $s_i$ . By Theorem 1.5.2,  $\Gamma$  is Hamilton decomposable. Thus, we assume  $k \in \{1, 2, 3\}$ . Equivalently, for all  $i \in \{1, 2, 3\}$ , the subgroup  $\langle s_i \rangle$  contains  $2s_j$ , for some  $j \neq i$ , i.e.,  $\overline{S}$  contains at least one involution.

Suppose that  $\Delta = \text{CAY}(A/\langle s_i \rangle, \{\overline{s_{j_1}}, \overline{s_{j_2}}\})$  is 3-regular, of order  $n$ . It is clear that, without loss of generality,  $|\overline{s_{j_2}}| = 2$ , and thus,  $n$  is even, and

$$|A/\langle s_i \rangle| = n = \frac{|\langle \overline{s_{j_1}} \rangle| |\langle \overline{s_{j_2}} \rangle|}{|\langle \overline{s_{j_1}} \rangle \cap \langle \overline{s_{j_2}} \rangle|}.$$

Hence,  $|\overline{s_{j_1}}| \in \{n/2, n\}$ , where  $|\overline{s_{j_1}}| = n$  if  $\overline{s_{j_2}} \in \langle \overline{s_{j_1}} \rangle$ , and  $|\overline{s_{j_1}}| = n/2$  if the intersection is trivial. If  $|\overline{s_{j_1}}| = n \Rightarrow A/\langle s_i \rangle = \langle \overline{s_{j_1}} \rangle$ , and so  $A = \langle s_i, s_{j_1} \rangle$ .

If  $\{\overline{x}, \overline{y}\} \in E(\Delta)$  is generated by  $\overline{s_{j_2}}$ , then as  $\overline{x} - \overline{y} = \overline{s_{j_2}} = -\overline{s_{j_2}} = \overline{-s_{j_2}}$ , we have

$$(x + J) - (y + J) = s_{j_2} + J = -s_{j_2} + J,$$

so that

$$x + h_1 s_i - (y + k_1 s_i) = s_{j_2} \text{ and } x + h_2 s_i - (y + k_2 s_i) = -s_{j_2}.$$

Thus, as  $s_{j_2} \neq -s_{j_2}$ ,  $\{x, y + (k_1 - h_1)s_i\} = \{x, x - s_{j_2}\}$  and  $\{x, y + (k_2 - h_2)s_i\} = \{x, x + s_{j_2}\}$  are distinct edges in  $\Gamma$ . Thus, the multi-edge  $\{\overline{x}, \overline{y}\}$  lifts to:

$$L_\Delta\{\overline{x}, \overline{y}\} = \{\{x + ts_i - s_{j_2}, x + ts_i\} : t = 0, \dots, m-1\} \cup \{\{x + ts_i, x + ts_i + s_{j_2}\} : t = 0, \dots, m-1\}$$

As  $2s_{j_2} \in \langle s_i \rangle$ , we have  $2s_{j_2} = ds_i$  for some  $d > 0$ . Thus, the union of the edges corresponding to values of  $t \in \{0, d, 2d, \dots\}$  is just the cycle containing  $x$ , that is generated by  $s_{j_2}$  in  $\Gamma$ .

**Open Problem 2:** If  $\Gamma = \text{CAY}(A, \{s_1, s_2, s_3\})$  is a connected, 6-regular, abelian Cayley graph on  $A$ , such that for all  $1 \leq i \leq 3$ , the quotient graph  $\text{CAY}(A/\langle s_i \rangle, \{\overline{s_{j_1}}, \overline{s_{j_2}}\})$  has a connection set containing at least one involution, then show  $\Gamma$  has a Hamilton decomposition.

### 6.1.2 General connection sets with involutions

Any connected, 6-regular abelian Cayley graph of  $A$  with connection set  $S$  must take the form:

1.  $\text{CAY}(A, \{s_1, s_2, s_3\})$  with  $|s_1| \geq |s_2| \geq |s_3| \geq 3$ .
2.  $\text{CAY}(A, \{s_1, s_2, s_3, s_4\})$  with  $|s_3| = |s_4| = 2$  and  $|s_1| \geq |s_2| \geq 3$ .
3.  $\text{CAY}(A, \{s_1, s_2, s_3, s_4, s_5\})$  with  $|s_2| = |s_3| = |s_4| = |s_5| = 2$  and  $|s_1| \geq 3$ .
4.  $\text{CAY}(A, \{s_1, s_2, s_3, s_4, s_5, s_6\})$  with  $|s_i| = 2$  for all  $1 \leq i \leq 6$ .

A solution to Open Problems 1 and 2, together with Theorem 5.4.2, would prove Alspach's conjecture for Cayley graphs of type (1) above.

**Open Problem 3:** *If  $\Gamma = \text{CAY}(A, S)$  is a connected, 6-regular, abelian Cayley graph on  $A$ , find a Hamilton decomposition of  $\Gamma$  if  $S$  contains involutions.*

**Lemma 6.1.2.** (Liu [32]) *Let  $S = \{s_1, s_2, \dots, s_k\}$  be a generating set for a finite abelian group  $A$ . Let  $S' = \{s_1, s_2, \dots, s_{k-1}\}$ ,  $\langle S' \rangle = A' \leq A$ , and  $J = \langle s_k \rangle$ . If  $A' \cap J = \{0\}$ , then*

$$\text{CAY}(A, S) \cong \text{CAY}(A', S') \square \text{CAY}(J, \langle s_k \rangle).$$

**Lemma 6.1.3.** *If  $\Gamma = \text{CAY}(A, S)$  is a connected, 6-regular, abelian Cayley graph with a minimal connection set of involutions, then  $\Gamma$  has a Hamilton decomposition.*

*Proof.* As each element generates a 1-factor of  $\Gamma$ , we have  $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ , where  $|s_i| = 2$  for  $1 \leq i \leq 6$ . We will assume that  $s_i \neq s_j$ , for all  $i \neq j$ . Now, every element in  $A$  must have order 2, and so  $A$  is isomorphic to an elementary abelian 2-group. By repeatedly applying Lemma 6.1.2, we see

$$\Gamma \cong K_2 \square K_2 \square K_2 \square K_2 \square K_2 \square K_2 = C_4 \square C_4 \square C_4.$$

By Theorem 1.3.7,  $\Gamma$  is decomposable into three Hamilton cycles. ■

## 6.2 Fundamental Questions

We close by offering the following is a list of fundamental questions about Cayley graphs.

1. What are necessary and sufficient conditions for two Cayley graphs on the same group to be isomorphic?
2. For which Cayley graphs is  $G_R$  a normal subgroup of automorphisms?
3. Do almost all Cayley graphs have automorphism group as small as possible?
4. Are all nontrivial, connected, Cayley graphs hamiltonian? (Not even known for dihedral Cayley graphs, see [6].)
5. Which circulant digraphs are hamiltonian? ( $\overrightarrow{\text{CAY}}(\mathbb{Z}_{12}, \{3, 4\})$  is not.)
6. Is every hamiltonian Cayley graph also edge-hamiltonian?
7. What Cayley graphs are Hamilton decomposable? (see Li-Yao [31].) In particular, are all Cayley graphs of  $p$ -groups Hamilton decomposable?



# Appendix A

## Data

This section contains Hamilton decompositions for certain 6-regular Cayley graphs (up to isomorphism) on abelian groups of orders 12, 18, 24, and 32, with a corresponding 3-element connection set that are not covered by previous theorems. The data was obtained via MAGMA programs and a random backtracking algorithm that finds Hamilton cycles (see Chapter B for source code).

### 1.1 Abelian Groups of Order 12

Table 1.1: Hamilton decompositions for Cayley graphs of order 12.

$\text{CAY}(\mathbb{Z}_{12}, \{2, 3, 4\})$
$H_1: 9, 6, 2, 0, 4, 1, 5, 3, 11, 8, 10, 7$
$H_2: 4, 7, 3, 0, 8, 5, 2, 11, 9, 1, 10, 6$
$H_3: 1, 3, 6, 8, 4, 2, 10, 0, 9, 5, 7, 11$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_6, \{(0, 1), (0, 2), (1, 1)\})$
$H_0: (1, 1), (1, 0), (0, 1), (0, 0), (0, 4), (0, 2), (0, 3), (0, 5), (1, 4), (1, 2), (1, 3), (1, 5)$
$H_1: (1, 2), (0, 3), (0, 1), (0, 2), (0, 0), (1, 5), (1, 4), (1, 0), (0, 5), (0, 4), (1, 3), (1, 1)$
$H_2: (0, 1), (1, 2), (1, 0), (1, 5), (0, 4), (0, 3), (1, 4), (1, 3), (0, 2), (1, 1), (0, 0), (0, 5)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_6, \{(0, 1), (1, 1), (1, 2)\})$
$H_0: (0, 2), (1, 4), (1, 5), (0, 0), (1, 2), (0, 3), (0, 4), (1, 0), (1, 1), (0, 5), (1, 3), (0, 1)$
$H_1: (1, 0), (0, 2), (0, 3), (1, 4), (1, 3), (1, 2), (1, 1), (0, 0), (0, 5), (0, 4), (1, 5), (0, 1)$
$H_2: (1, 0), (0, 5), (1, 4), (0, 0), (0, 1), (1, 2), (0, 4), (1, 3), (0, 2), (1, 1), (0, 3), (1, 5)$

### 1.2 Abelian Groups of Order 18

Table 1.2: Hamilton decompositions for Cayley graphs of order 18.

$\text{CAY}(\mathbb{Z}_{18}, \{2, 3, 4\})$
$H_1: 0, 14, 12, 9, 7, 3, 6, 4, 1, 17, 15, 11, 13, 10, 8, 5, 2, 16$
$H_2: 5, 3, 17, 14, 10, 12, 16, 1, 15, 13, 9, 6, 2, 0, 4, 8, 11, 7$
$H_3: 4, 2, 17, 13, 16, 14, 11, 9, 5, 1, 3, 0, 15, 12, 8, 6, 10, 7$

Continued on next page



Table 1.2 – continued from previous page

$\text{CAY}(\mathbb{Z}_{18}, \{3, 4, 6\})$
$H_1: 11, 5, 9, 3, 0, 4, 10, 6, 2, 8, 14, 17, 13, 16, 12, 15, 1, 7$
$H_2: 4, 8, 11, 17, 2, 5, 1, 13, 7, 3, 6, 12, 9, 15, 0, 14, 10, 16$
$H_3: 9, 6, 0, 12, 8, 5, 17, 3, 15, 11, 14, 2, 16, 1, 4, 7, 10, 13$
$\text{CAY}(\mathbb{Z}_3 \times \mathbb{Z}_6, \{(1, 3), (0, 1), (1, 2)\})$
$H_0: (1, 5), (1, 0), (2, 2), (2, 1), (2, 0), (1, 3), (0, 0), (0, 1), (2, 5), (2, 4), (2, 3), (1, 1), (1, 2), (0, 5), (0, 4), (0, 3), (0, 2), (1, 4)$
$H_1: (2, 1), (0, 3), (1, 0), (0, 4), (1, 1), (2, 4), (0, 0), (2, 3), (0, 5), (2, 2), (1, 5), (0, 2), (2, 0), (2, 5), (1, 2), (1, 3), (0, 1), (1, 4)$
$H_2: (2, 0), (0, 3), (1, 5), (2, 1), (0, 4), (2, 2), (2, 3), (1, 0), (1, 1), (0, 5), (0, 0), (1, 2), (2, 4), (0, 1), (0, 2), (2, 5), (1, 3), (1, 4)$
$\text{CAY}(\mathbb{Z}_3 \times \mathbb{Z}_6, \{(1, 3), (2, 1), (1, 1)\})$
$H_0: (0, 0), (2, 1), (0, 4), (1, 1), (2, 0), (0, 1), (2, 2), (0, 5), (1, 2), (0, 3), (1, 0), (2, 3), (1, 4), (2, 5), (0, 2), (1, 5), (2, 4), (1, 3)$
$H_1: (0, 3), (2, 2), (1, 5), (0, 0), (2, 3), (0, 2), (1, 1), (2, 4), (0, 1), (1, 2), (2, 1), (1, 4), (0, 5), (1, 0), (2, 5), (0, 4), (1, 3), (2, 0)$
$H_2: (0, 4), (1, 5), (2, 0), (0, 5), (2, 4), (0, 3), (1, 4), (0, 1), (1, 0), (2, 1), (0, 2), (1, 3), (2, 2), (1, 1), (0, 0), (2, 5), (1, 2), (2, 3)$
$\text{CAY}(\mathbb{Z}_3 \times \mathbb{Z}_6, \{(0, 1), (2, 2), (1, 0)\})$
$H_0: (0, 0), (1, 0), (2, 0), (0, 4), (0, 5), (2, 1), (0, 1), (1, 5), (1, 4), (1, 3), (1, 2), (0, 2), (0, 3), (1, 1), (2, 5), (2, 4), (2, 3), (2, 2)$
$H_1: (1, 3), (2, 3), (0, 1), (0, 0), (2, 0), (1, 2), (1, 1), (1, 0), (1, 5), (0, 5), (2, 5), (0, 3), (0, 4), (1, 4), (2, 4), (0, 2), (2, 2), (2, 1)$
$H_2: (0, 4), (1, 2), (2, 2), (1, 4), (0, 0), (0, 5), (1, 3), (0, 3), (2, 3), (1, 5), (2, 5), (2, 0), (2, 1), (1, 1), (0, 1), (0, 2), (1, 0), (2, 4)$
$\text{CAY}(\mathbb{Z}_3 \times \mathbb{Z}_6, \{(2, 1), (1, 1), (1, 2)\})$
$H_0: (0, 1), (2, 0), (1, 4), (0, 5), (1, 0), (2, 2), (1, 1), (0, 2), (2, 1), (0, 0), (1, 2), (2, 3), (0, 4), (1, 5), (0, 3), (2, 4), (1, 3), (2, 5)$
$H_1: (0, 1), (1, 3), (0, 2), (1, 4), (0, 3), (2, 1), (1, 2), (2, 4), (0, 5), (2, 3), (1, 1), (2, 0), (1, 5), (0, 0), (2, 5), (1, 0), (0, 4), (2, 2)$
$H_2: (0, 4), (2, 5), (1, 4), (2, 3), (0, 2), (2, 0), (0, 5), (1, 1), (0, 0), (2, 4), (1, 5), (2, 1), (1, 0), (0, 1), (1, 2), (0, 3), (2, 2), (1, 3)$
$\text{CAY}(\mathbb{Z}_3 \times \mathbb{Z}_6, \{(2, 1), (1, 0), (1, 2)\})$
$H_0: (0, 5), (1, 4), (0, 2), (2, 0), (1, 0), (0, 0), (2, 4), (0, 3), (2, 1), (1, 2), (2, 2), (0, 4), (1, 3), (2, 3), (1, 1), (0, 1), (2, 5), (1, 5)$
$H_1: (0, 0), (1, 2), (2, 4), (0, 4), (1, 0), (2, 2), (0, 1), (2, 1), (1, 1), (0, 2), (2, 3), (1, 4), (2, 0), (0, 5), (2, 5), (1, 3), (0, 3), (1, 5)$
$H_2: (2, 5), (1, 0), (0, 1), (1, 3), (2, 2), (0, 2), (1, 2), (0, 3), (2, 3), (0, 5), (1, 1), (2, 0), (0, 0), (2, 1), (1, 5), (2, 4), (1, 4), (0, 4)$

### 1.3 Abelian Groups of Order 24

Table 1.3: Hamilton decompositions for circulant graphs of order 24.

$\text{CAY}(\mathbb{Z}_{24}, \{3, 9, 10\})$
$H_0: 6, 9, 18, 4, 7, 16, 2, 5, 8, 22, 19, 10, 1, 11, 14, 17, 20, 23, 13, 3, 0, 15, 12, 21$
$H_1: 15, 1, 16, 19, 5, 14, 4, 13, 10, 0, 21, 7, 22, 12, 9, 23, 2, 11, 20, 6, 3, 17, 8, 18$
$H_2: 2, 12, 3, 18, 21, 11, 8, 23, 14, 0, 9, 19, 4, 1, 22, 13, 16, 6, 15, 5, 20, 10, 7, 17$
$\text{CAY}(\mathbb{Z}_{24}, \{2, 6, 9\})$
$H_0: 15, 0, 6, 4, 10, 1, 3, 9, 7, 22, 20, 14, 16, 18, 12, 21, 19, 17, 2, 8, 23, 5, 11, 13$
$H_1: 15, 6, 12, 10, 16, 7, 1, 19, 4, 13, 22, 0, 18, 9, 11, 2, 20, 5, 3, 21, 23, 14, 8, 17$
$H_2: 12, 3, 18, 20, 11, 17, 23, 1, 16, 22, 4, 2, 0, 9, 15, 21, 6, 8, 10, 19, 13, 7, 5, 14$
$\text{CAY}(\mathbb{Z}_{24}, \{3, 4, 10\})$
$H_0: 19, 5, 9, 13, 17, 20, 6, 10, 14, 0, 4, 18, 15, 11, 1, 21, 7, 3, 23, 2, 16, 12, 8, 22$
$H_1: 8, 5, 1, 15, 12, 2, 22, 18, 14, 17, 3, 6, 9, 19, 16, 13, 23, 20, 10, 0, 21, 11, 7, 4$
$H_2: 10, 13, 3, 0, 20, 16, 6, 2, 5, 15, 19, 23, 9, 12, 22, 1, 4, 14, 11, 8, 18, 21, 17, 7$
$\text{CAY}(\mathbb{Z}_{24}, \{2, 9, 10\})$
$H_0: 7, 21, 6, 20, 10, 8, 17, 3, 13, 11, 1, 15, 0, 9, 23, 14, 5, 19, 4, 18, 16, 2, 12, 22$
$H_1: 13, 15, 6, 8, 22, 0, 10, 1, 16, 7, 17, 19, 9, 18, 3, 5, 20, 11, 2, 4, 14, 12, 21, 23$
$H_2: 0, 2, 17, 15, 5, 7, 9, 11, 21, 19, 10, 12, 3, 1, 23, 8, 18, 20, 22, 13, 4, 6, 16, 14$
$\text{CAY}(\mathbb{Z}_{24}, \{6, 8, 9\})$
$H_0: 0, 9, 1, 10, 4, 20, 5, 23, 14, 8, 16, 7, 13, 22, 6, 15, 21, 12, 3, 19, 11, 17, 2, 18$
$H_1: 3, 18, 10, 2, 8, 0, 6, 12, 4, 19, 13, 21, 5, 11, 20, 14, 22, 16, 1, 7, 15, 23, 17, 9$
$H_2: 3, 11, 2, 20, 12, 18, 9, 15, 0, 16, 10, 19, 1, 17, 8, 23, 7, 22, 4, 13, 5, 14, 6, 21$

Continued on next page

Table 1.3 – continued from previous page

$\text{CAY}(\mathbb{Z}_{24}, \{3, 4, 9\})$
$H_0: 5, 20, 0, 3, 12, 15, 6, 2, 23, 19, 10, 7, 4, 8, 11, 14, 17, 21, 1, 16, 13, 22, 18, 9$
$H_1: 3, 6, 21, 18, 14, 23, 8, 5, 1, 10, 13, 17, 20, 16, 12, 9, 0, 4, 19, 15, 11, 2, 22, 7$
$H_2: 22, 1, 4, 13, 9, 6, 10, 14, 5, 2, 17, 8, 12, 21, 0, 15, 18, 3, 23, 20, 11, 7, 16, 19$
$\text{CAY}(\mathbb{Z}_{24}, \{2, 3, 8\})$
$H_0: 19, 22, 6, 3, 1, 4, 2, 18, 16, 14, 17, 9, 11, 13, 15, 23, 20, 12, 10, 7, 5, 8, 0, 21$
$H_1: 16, 0, 2, 5, 3, 11, 14, 6, 8, 10, 13, 21, 23, 7, 4, 12, 9, 1, 22, 20, 18, 15, 17, 19$
$H_2: 18, 10, 2, 23, 1, 17, 20, 4, 6, 9, 7, 15, 12, 14, 22, 0, 3, 19, 11, 8, 16, 13, 5, 21$
$\text{CAY}(\mathbb{Z}_{24}, \{3, 8, 9\})$
$H_0: 2, 18, 15, 0, 3, 11, 20, 12, 21, 6, 9, 17, 14, 5, 13, 10, 1, 22, 7, 4, 19, 16, 8, 23$
$H_1: 5, 2, 17, 8, 0, 9, 1, 4, 12, 3, 6, 22, 14, 11, 19, 10, 18, 21, 13, 16, 7, 15, 23, 20$
$H_2: 0, 16, 1, 17, 20, 4, 13, 22, 19, 3, 18, 9, 12, 15, 6, 14, 23, 7, 10, 2, 11, 8, 5, 21$
$\text{CAY}(\mathbb{Z}_{24}, \{4, 8, 9\})$
$H_0: 6, 21, 13, 4, 8, 12, 3, 7, 22, 2, 10, 18, 9, 17, 1, 16, 0, 15, 23, 19, 11, 20, 5, 14$
$H_1: 10, 19, 15, 7, 11, 2, 17, 13, 5, 21, 1, 9, 0, 4, 12, 20, 16, 8, 23, 3, 18, 14, 22, 6$
$H_2: 7, 16, 12, 21, 17, 8, 0, 20, 4, 19, 3, 11, 15, 6, 2, 18, 22, 13, 9, 5, 1, 10, 14, 23$
$\text{CAY}(\mathbb{Z}_{24}, \{4, 6, 9\})$
$H_0: 0, 18, 3, 23, 8, 12, 16, 1, 19, 10, 4, 22, 2, 6, 15, 21, 17, 11, 7, 13, 9, 5, 14, 20$
$H_1: 16, 7, 22, 18, 12, 3, 9, 15, 11, 5, 23, 14, 10, 1, 21, 6, 0, 4, 19, 13, 17, 8, 2, 20$
$H_2: 23, 17, 2, 11, 20, 5, 1, 7, 3, 21, 12, 6, 10, 16, 22, 13, 4, 8, 14, 18, 9, 0, 15, 19$

Table 1.4: Hamilton decompositions for Cayley graphs on  $\mathbb{Z}_2 \times \mathbb{Z}_{12}$ .

$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{12}, \{(1, 4), (1, 5), (0, 3)\})$
$H_0: (0, 2), (1, 6), (0, 11), (1, 4), (1, 1), (0, 8), (1, 3), (1, 0), (1, 9), (0, 1), (0, 4), (1, 8), (0, 3), (1, 7), (0, 0), (0, 9), (1, 5), (0, 10), (1, 2), (0, 7),$ $(1, 11), (0, 6), (1, 10), (0, 5)$
$H_1: (0, 3), (1, 11), (0, 4), (0, 7), (1, 3), (1, 6), (0, 10), (0, 1), (1, 5), (1, 8), (0, 0), (1, 4), (0, 8), (1, 0), (0, 5), (1, 9), (0, 2), (0, 11), (1, 7), (1, 10),$ $(1, 1), (0, 9), (1, 2), (0, 6)$
$H_2: (0, 1), (1, 8), (1, 11), (1, 2), (1, 5), (0, 0), (0, 3), (1, 10), (0, 2), (1, 7), (1, 4), (0, 9), (0, 6), (1, 1), (0, 5), (0, 8), (0, 11), (1, 3), (0, 10), (0, 7),$ $(1, 0), (0, 4), (1, 9), (1, 6)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{12}, \{(1, 3), (1, 4), (0, 3)\})$
$H_0: (0, 8), (0, 5), (1, 2), (0, 6), (0, 3), (1, 7), (0, 11), (1, 3), (1, 0), (1, 9), (0, 0), (1, 4), (0, 1), (0, 4), (1, 1), (0, 10), (0, 7), (1, 10), (0, 2), (1, 6),$ $(0, 9), (1, 5), (1, 8), (1, 11)$
$H_1: (1, 8), (0, 11), (0, 2), (0, 5), (1, 1), (0, 9), (0, 0), (0, 3), (1, 11), (1, 2), (1, 5), (0, 1), (0, 10), (1, 7), (1, 10), (0, 6), (1, 9), (1, 6), (1, 3), (0, 7),$ $(1, 4), (0, 8), (1, 0), (0, 4)$
$H_2: (0, 1), (1, 9), (0, 5), (1, 8), (0, 0), (1, 3), (0, 6), (0, 9), (1, 0), (0, 3), (1, 6), (0, 10), (1, 2), (0, 11), (0, 8), (1, 5), (0, 2), (1, 11), (0, 7), (0, 4),$ $(1, 7), (1, 4), (1, 1), (1, 10)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{12}, \{(0, 1), (0, 2), (1, 1)\})$
$H_0: (0, 6), (0, 7), (1, 8), (1, 10), (1, 0), (1, 1), (0, 2), (0, 4), (0, 3), (0, 1), (1, 2), (1, 3), (1, 4), (0, 5), (1, 6), (1, 5), (1, 7), (1, 9), (0, 10), (1, 11),$ $(0, 0), (0, 11), (0, 9), (0, 8)$
$H_1: (1, 3), (0, 2), (0, 0), (0, 1), (1, 0), (0, 11), (1, 10), (0, 9), (0, 10), (0, 8), (0, 7), (1, 6), (1, 7), (1, 8), (1, 9), (1, 11), (1, 1), (1, 2), (1, 4), (0, 3),$ $(0, 5), (0, 4), (0, 6), (1, 5)$
$H_2: (1, 9), (0, 8), (1, 7), (0, 6), (0, 5), (0, 7), (0, 9), (1, 8), (1, 6), (1, 4), (1, 5), (0, 4), (1, 3), (1, 1), (0, 0), (0, 10), (0, 11), (0, 1), (0, 2), (0, 3),$ $(1, 2), (1, 0), (1, 11), (1, 10)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{12}, \{(1, 3), (0, 4), (1, 2)\})$
$H_0: (0, 1), (1, 4), (0, 6), (1, 8), (1, 0), (0, 2), (1, 11), (0, 9), (1, 6), (0, 4), (0, 0), (1, 3), (1, 7), (0, 10), (1, 1), (1, 9), (0, 7), (0, 11), (0, 3), (1, 5),$ $(0, 8), (1, 10), (1, 2), (0, 5)$
$H_1: (0, 1), (1, 3), (0, 5), (1, 8), (1, 4), (0, 7), (0, 3), (1, 1), (0, 4), (1, 7), (0, 9), (1, 0), (0, 10), (0, 6), (0, 2), (1, 5), (1, 9), (0, 11), (1, 2), (1, 6),$ $(1, 10), (0, 0), (0, 8), (1, 11)$
$H_2: (0, 5), (1, 7), (1, 11), (1, 3), (0, 6), (1, 9), (0, 0), (1, 2), (0, 4), (0, 8), (1, 6), (0, 3), (1, 0), (1, 4), (0, 2), (0, 10), (1, 8), (0, 11), (1, 1), (1, 5),$

Continued on next page

Table 1.4 – continued from previous page

$(0,7),(1,10),(0,1),(0,9)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{12}, \{(1,3),(1,5),(0,3)\})$
$H_0: (0,4),(0,1),(1,4),(1,7),(0,2),(1,5),(0,8),(1,11),(1,2),(0,5),(1,8),(0,11),(1,6),(1,3),(0,6),(0,9),(1,0),(0,7),(0,10),(1,1),$ $(1,10),(0,3),(0,0),(1,9)$
$H_1: (0,7),(1,2),(0,9),(1,6),(0,3),(0,6),(1,1),(0,4),(1,7),(1,10),(0,1),(0,10),(1,3),(0,0),(1,5),(1,8),(1,11),(0,2),(1,9),(1,0),$ $(0,5),(0,8),(0,11),(1,4)$
$H_2: (0,8),(1,3),(1,0),(0,3),(1,8),(0,1),(1,6),(1,9),(0,6),(1,11),(0,4),(0,7),(1,10),(0,5),(0,2),(0,11),(1,2),(1,5),(0,10),(1,7),$ $(0,0),(0,9),(1,4),(1,1)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{12}, \{(0,4),(0,5),(1,2)\})$
$H_0: (0,3),(0,8),(0,0),(0,4),(0,11),(0,6),(1,8),(0,10),(1,0),(0,2),(0,9),(1,7),(1,2),(1,9),(1,4),(1,11),(0,1),(0,5),(1,3),(1,10),$ $(1,6),(1,1),(1,5),(0,7)$
$H_1: (0,1),(1,3),(1,7),(1,0),(1,4),(1,8),(1,1),(0,3),(1,5),(1,9),(0,11),(0,7),(0,2),(0,6),(0,10),(0,5),(0,0),(1,2),(1,10),(0,8),$ $(0,4),(1,6),(1,11),(0,9)$
$H_2: (1,8),(1,0),(1,5),(1,10),(0,0),(0,7),(1,9),(1,1),(0,11),(0,3),(0,10),(0,2),(1,4),(0,6),(0,1),(0,8),(1,6),(1,2),(0,4),(0,9),$ $(0,5),(1,7),(1,11),(1,3)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{12}, \{(0,2),(1,5),(1,2)\})$
$H_0: (1,2),(0,7),(1,0),(0,5),(0,3),(1,8),(1,6),(0,4),(0,2),(0,0),(1,10),(0,8),(0,6),(1,1),(1,3),(0,10),(1,5),(1,7),(1,9),(0,11),$ $(0,1),(1,11),(0,9),(1,4)$
$H_1: (1,11),(0,6),(1,4),(1,6),(0,8),(1,3),(0,5),(1,7),(0,2),(1,0),(1,10),(0,3),(0,1),(1,8),(0,10),(0,0),(1,5),(0,7),(1,9),(0,4),$ $(1,2),(0,9),(0,11),(1,1)$
$H_2: (1,1),(0,3),(1,5),(1,3),(0,1),(1,6),(0,11),(1,4),(0,2),(1,9),(1,11),(0,4),(0,6),(1,8),(1,10),(0,5),(0,7),(0,9),(1,7),(0,0),$ $(1,2),(1,0),(0,10),(0,8)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{12}, \{(1,3),(0,1),(1,5)\})$
$H_0: (0,3),(1,6),(0,1),(0,2),(1,11),(1,10),(1,9),(0,6),(0,5),(0,4),(1,7),(0,0),(1,3),(1,4),(0,7),(1,0),(1,1),(0,8),(1,5),(0,10),$ $(0,9),(1,2),(0,11),(1,8)$
$H_1: (1,0),(0,3),(0,2),(1,9),(0,4),(1,1),(1,2),(0,7),(0,6),(1,3),(0,10),(0,11),(0,0),(0,1),(1,10),(0,5),(1,8),(1,7),(1,6),(1,5),$ $(1,4),(0,9),(0,8),(1,11)$
$H_2: (0,5),(1,2),(1,3),(0,8),(0,7),(1,10),(0,3),(0,4),(1,11),(0,6),(1,1),(0,10),(1,7),(0,2),(1,5),(0,0),(1,9),(1,8),(0,1),(1,4),$ $(0,11),(1,6),(0,9),(1,0)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{12}, \{(0,1),(0,3),(1,2)\})$
$H_0: (1,9),(0,7),(1,5),(1,2),(1,11),(1,0),(0,10),(1,8),(1,7),(1,4),(0,2),(0,11),(1,1),(0,3),(0,0),(1,10),(0,8),(0,9),(0,6),(0,5),$ $(1,3),(0,1),(0,4),(1,6)$
$H_1: (1,5),(1,8),(0,6),(0,7),(0,4),(0,3),(0,2),(0,5),(1,7),(1,6),(0,8),(0,11),(0,0),(0,1),(0,10),(0,9),(1,11),(1,10),(1,9),(1,0),$ $(1,1),(1,2),(1,3),(1,4)$
$H_2: (1,6),(1,3),(1,0),(0,2),(0,1),(1,11),(1,8),(1,9),(0,11),(0,10),(0,7),(0,8),(0,5),(0,4),(1,2),(0,0),(0,9),(1,7),(1,10),(1,1),$ $(1,4),(0,6),(0,3),(1,5)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{12}, \{(1,3),(0,1),(0,4)\})$
$H_0: (1,1),(1,2),(0,5),(0,1),(1,4),(1,5),(0,8),(0,7),(0,6),(1,9),(0,0),(1,3),(1,11),(1,10),(1,6),(0,9),(1,0),(0,3),(0,2),(0,10),$ $(0,11),(1,8),(1,7),(0,4)$
$H_1: (0,10),(1,1),(1,5),(0,2),(0,1),(0,9),(0,5),(0,4),(0,3),(1,6),(1,2),(1,10),(1,9),(1,8),(1,0),(1,4),(0,7),(0,11),(0,0),(0,8),$ $(1,11),(1,7),(1,3),(0,6)$
$H_2: (1,5),(1,6),(1,7),(0,10),(0,9),(0,8),(0,4),(0,0),(0,1),(1,10),(0,7),(0,3),(0,11),(1,2),(1,3),(1,4),(1,8),(0,5),(0,6),(0,2),$ $(1,11),(1,0),(1,1),(1,9)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{12}, \{(0,2),(0,3),(1,2)\})$
$H_0: (0,0),(0,3),(0,1),(1,3),(1,5),(1,2),(0,4),(0,2),(1,4),(1,1),(0,11),(0,9),(0,6),(0,8),(1,6),(1,8),(1,11),(1,9),(1,7),(0,5),$ $(0,7),(0,10),(1,0),(1,10)$
$H_1: (1,5),(0,3),(0,6),(0,4),(0,7),(0,9),(0,0),(0,2),(1,0),(1,9),(1,6),(1,4),(1,2),(1,11),(1,1),(1,3),(0,5),(0,8),(0,11),(0,1),$ $(0,10),(1,8),(1,10),(1,7)$
$H_2: (1,8),(1,5),(0,7),(1,9),(0,11),(0,2),(0,5),(0,3),(1,1),(1,10),(0,8),(0,10),(0,0),(1,2),(1,0),(1,3),(1,6),(0,4),(0,1),(1,11),$ $(0,9),(1,7),(1,4),(0,6)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{12}, \{(1,4),(1,1),(1,2)\})$

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Table 1.4 – continued from previous page

$H_0$ :	(1,6),(0,4),(1,2),(0,6),(1,7),(0,5),(1,3),(0,2),(1,4),(0,3),(1,1),(0,9),(1,5),(0,7),(1,8),(0,10),(1,9),(0,1),(1,11),(0,0), (1,10),(0,11),(1,0),(0,8)
$H_1$ :	(0,3),(1,11),(0,7),(1,3),(0,4),(1,8),(0,6),(1,5),(0,1),(1,0),(0,10),(1,6),(0,5),(1,9),(0,11),(1,1),(0,2),(1,10),(0,9),(1,7), (0,8),(1,4),(0,0),(1,2)
$H_2$ :	(0,0),(1,1),(0,5),(1,4),(0,6),(1,10),(0,8),(1,9),(0,7),(1,6),(0,2),(1,0),(0,4),(1,5),(0,3),(1,7),(0,11),(1,3),(0,1),(1,2), (0,10),(1,11),(0,9),(1,8)
<hr/>	
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{12}, \{(1,4),(1,5),(0,5)\})$	
$H_0$ :	(1,9),(1,2),(0,7),(0,2),(1,10),(0,3),(1,8),(0,0),(0,5),(1,0),(1,7),(0,11),(0,4),(1,11),(0,6),(1,1),(0,8),(1,3),(0,10),(1,6), (0,1),(1,5),(0,9),(1,4)
$H_1$ :	(1,4),(0,8),(1,0),(0,7),(0,0),(1,5),(0,10),(1,2),(0,6),(1,10),(1,3),(0,11),(1,6),(1,1),(0,5),(1,9),(0,1),(1,8),(0,4),(0,9), (0,2),(1,7),(0,3),(1,11)
$H_2$ :	(1,9),(0,2),(1,6),(1,11),(0,7),(1,3),(1,8),(1,1),(0,9),(1,2),(1,7),(0,0),(1,4),(0,11),(0,6),(0,1),(0,8),(0,3),(0,10),(0,5), (1,10),(1,5),(1,0),(0,4)
<hr/>	
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{12}, \{(1,3),(0,1),(0,5)\})$	
$H_0$ :	(1,4),(0,1),(0,2),(0,9),(1,6),(1,5),(1,0),(1,1),(1,2),(1,3),(0,0),(0,5),(0,6),(0,11),(0,4),(0,3),(0,10),(1,7),(1,8),(1,9), (1,10),(0,7),(0,8),(1,11)
$H_1$ :	(1,0),(1,7),(1,2),(0,11),(0,0),(0,1),(0,6),(1,9),(1,4),(0,7),(0,2),(0,3),(0,8),(1,5),(1,10),(1,3),(1,8),(0,5),(0,4),(0,9), (0,10),(1,1),(1,6),(1,11)
$H_2$ :	(1,10),(0,1),(0,8),(0,9),(1,0),(0,3),(1,6),(1,7),(0,4),(1,1),(1,8),(0,11),(0,10),(0,5),(1,2),(1,9),(0,0),(0,7),(0,6),(1,3), (1,4),(1,5),(0,2),(1,11)
<hr/>	
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{12}, \{(0,2),(0,3),(1,1)\})$	
$H_0$ :	(0,7),(0,4),(1,3),(0,2),(0,5),(0,3),(1,2),(1,5),(0,6),(1,7),(1,10),(1,0),(0,11),(0,1),(0,10),(0,8),(1,9),(1,11),(0,0),(1,1), (1,4),(1,6),(1,8),(0,9)
$H_1$ :	(1,2),(1,4),(0,3),(0,6),(0,8),(1,7),(1,9),(1,6),(0,5),(0,7),(1,8),(1,5),(1,3),(1,0),(0,1),(0,4),(0,2),(1,1),(1,10),(0,11), (0,9),(0,0),(0,10),(1,11)
$H_2$ :	(0,9),(0,6),(0,4),(1,5),(1,7),(1,4),(0,5),(0,8),(0,11),(0,2),(0,0),(0,3),(0,1),(1,2),(1,0),(1,9),(0,10),(0,7),(1,6),(1,3), (1,1),(1,11),(1,8),(1,10)
<hr/>	
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{12}, \{(0,1),(0,4),(1,1)\})$	
$H_0$ :	(1,2),(0,3),(1,4),(1,3),(0,4),(0,5),(0,1),(1,0),(1,1),(0,0),(0,8),(1,7),(1,11),(1,10),(0,9),(1,8),(1,9),(1,5),(0,6),(0,2), (0,10),(0,11),(0,7),(1,6)
$H_1$ :	(1,1),(0,2),(0,3),(0,7),(0,6),(1,7),(1,6),(0,5),(0,9),(0,1),(0,0),(0,4),(0,8),(1,9),(0,10),(1,11),(1,3),(1,2),(1,10),(0,11), (1,0),(1,8),(1,4),(1,5)
$H_2$ :	(1,11),(0,0),(0,11),(0,3),(0,4),(1,5),(1,6),(1,10),(1,9),(1,1),(1,2),(0,1),(0,2),(1,3),(1,7),(1,8),(0,7),(0,8),(0,9),(0,10), (0,6),(0,5),(1,4),(1,0)
<hr/>	
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{12}, \{(0,1),(1,4),(0,5)\})$	
$H_0$ :	(0,6),(0,1),(0,8),(0,3),(1,11),(0,7),(0,2),(1,6),(1,5),(1,10),(1,3),(0,11),(0,0),(1,4),(1,9),(0,5),(1,1),(1,8),(1,7),(1,0), (0,4),(0,9),(0,10),(1,2)
$H_1$ :	(0,4),(0,3),(0,2),(0,9),(1,5),(0,1),(0,0),(0,5),(0,6),(0,7),(1,3),(1,2),(1,7),(0,11),(0,10),(1,6),(1,1),(1,0),(0,8),(1,4), (1,11),(1,10),(1,9),(1,8)
$H_2$ :	(0,2),(0,1),(1,9),(1,2),(1,1),(0,9),(0,8),(0,7),(0,0),(1,8),(1,3),(1,4),(1,5),(1,0),(1,11),(1,6),(1,7),(0,3),(0,10),(0,5), (0,4),(0,11),(0,6),(1,10)
<hr/>	
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{12}, \{(1,3),(0,3),(0,4)\})$	
$H_0$ :	(1,6),(1,2),(0,5),(0,1),(0,10),(1,1),(1,4),(1,8),(1,0),(0,3),(0,7),(0,11),(0,2),(1,11),(1,3),(0,6),(0,9),(0,0),(1,9),(1,5), (0,8),(0,4),(1,7),(1,10)
$H_1$ :	(0,5),(0,8),(0,11),(1,2),(1,5),(1,8),(1,11),(1,7),(1,3),(1,0),(1,4),(0,1),(0,4),(0,0),(0,3),(0,6),(0,2),(0,10),(0,7),(1,10), (1,1),(1,9),(1,6),(0,9)
$H_2$ :	(1,3),(1,6),(0,3),(0,11),(1,8),(0,5),(0,2),(1,5),(1,1),(0,4),(0,7),(1,4),(1,7),(0,10),(0,6),(1,9),(1,0),(0,9),(0,1),(1,10), (1,2),(1,11),(0,8),(0,0)
<hr/>	

Table 1.5: Hamilton decompositions for Cayley graphs on  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ .

$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6, \{(0,0,1), (1,1,1), (1,0,1)\})$
$H_0: (0,0,4), (0,0,5), (0,0,0), (0,0,1), (1,1,2), (0,1,3), (0,1,4), (1,0,3), (0,0,2), (1,0,1), (0,1,0), (0,1,1), (1,0,2), (0,0,3), (1,0,4),$ $(1,0,5), (1,0,0), (0,1,5), (1,1,0), (1,1,1), (0,1,2), (1,1,3), (1,1,4), (1,1,5)$ $H_1: (1,0,3), (0,0,4), (1,1,3), (0,1,4), (1,1,5), (0,0,0), (1,0,5), (0,1,0), (0,1,5), (1,0,4), (0,0,5), (1,0,0), (0,0,1), (1,1,0), (0,1,1),$ $(1,1,2), (1,1,1), (0,0,2), (0,0,3), (1,1,4), (0,1,3), (1,0,2), (1,0,1), (0,1,2)$ $H_2: (0,1,1), (0,1,2), (0,1,3), (1,0,4), (1,0,3), (1,0,2), (0,0,1), (0,0,2), (1,1,3), (1,1,2), (0,0,3), (0,0,4), (1,0,5), (0,1,4), (0,1,5),$ $(1,1,4), (0,0,5), (1,1,0), (1,1,5), (0,1,0), (1,1,1), (0,0,0), (1,0,1), (1,0,0)$

## 1.4 Abelian Groups of Order 32

Table 1.6: Hamilton decompositions for Cayley graphs on  $\mathbb{Z}_2 \times \mathbb{Z}_{16}$ .

$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{16}, \{(0,2), (1,5), (0,3)\})$
$H_0: (1,7), (1,4), (1,6), (1,3), (1,5), (0,10), (0,8), (0,6), (0,3), (0,1), (0,4), (0,7), (0,9), (0,11), (0,13), (0,0), (0,14), (0,12), (0,15),$ $(0,2), (0,5), (1,10), (1,8), (1,11), (1,13), (1,0), (1,2), (1,15), (1,1), (1,14), (1,12), (1,9)$ $H_1: (1,6), (0,1), (1,12), (1,10), (1,13), (0,8), (0,5), (1,0), (0,11), (0,14), (1,3), (1,1), (1,4), (0,15), (0,13), (1,2), (0,7), (0,10), (1,15),$ $(0,4), (0,6), (0,9), (0,12), (1,7), (0,2), (0,0), (1,5), (1,8), (0,3), (1,14), (1,11), (1,9)$ $H_2: (0,3), (0,0), (1,11), (0,6), (1,1), (0,12), (0,10), (0,13), (1,8), (1,6), (0,11), (0,8), (1,3), (1,0), (1,14), (0,9), (1,4), (1,2), (1,5),$ $(1,7), (1,10), (0,15), (0,1), (0,14), (1,9), (0,4), (0,2), (1,13), (1,15), (1,12), (0,7), (0,5)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{16}, \{(1,5), (1,6), (1,1)\})$
$H_0: (0,2), (1,8), (0,9), (1,14), (0,3), (1,9), (0,8), (1,2), (0,13), (1,7), (0,1), (1,11), (0,6), (1,1), (0,0), (1,10), (0,4), (1,5), (0,10),$ $(1,4), (0,15), (1,0), (0,11), (1,12), (0,7), (1,13), (0,12), (1,6), (0,5), (1,15), (0,14), (1,3)$ $H_1: (0,6), (1,5), (0,11), (1,10), (0,5), (1,0), (0,1), (1,6), (0,7), (1,1), (0,12), (1,2), (0,3), (1,8), (0,13), (1,3), (0,4), (1,14), (0,15),$ $(1,9), (0,10), (1,11), (0,0), (1,15), (0,9), (1,4), (0,14), (1,13), (0,8), (1,7), (0,2), (1,12)$ $H_2: (1,15), (0,4), (1,9), (0,14), (1,8), (0,7), (1,2), (0,1), (1,12), (0,13), (1,14), (0,8), (1,3), (0,9), (1,10), (0,15), (1,5), (0,0), (1,6),$ $(0,11), (1,1), (0,2), (1,13), (0,3), (1,4), (0,5), (1,11), (0,12), (1,7), (0,6), (1,0), (0,10)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{16}, \{(1,3), (0,5), (1,2)\})$
$H_0: (1,14), (0,1), (1,3), (0,5), (1,2), (0,0), (0,11), (1,8), (0,6), (1,9), (0,7), (0,2), (1,4), (1,15), (0,13), (0,8), (1,11), (0,9), (1,7),$ $(0,4), (1,6), (0,3), (1,1), (0,15), (1,13), (0,10), (1,12), (0,14), (1,0), (1,5), (1,10), (0,12)$ $H_1: (0,1), (1,4), (0,6), (1,3), (0,0), (0,5), (1,7), (0,10), (1,8), (1,13), (0,11), (1,14), (1,9), (0,12), (0,7), (1,5), (0,8), (0,3), (0,14),$ $(1,1), (1,12), (0,15), (1,2), (0,4), (0,9), (1,6), (1,11), (1,0), (0,2), (0,13), (1,10), (1,15)$ $H_2: (0,8), (1,6), (1,1), (0,4), (0,15), (0,10), (0,5), (1,8), (1,3), (1,14), (0,0), (1,13), (1,2), (1,7), (1,12), (0,9), (0,14), (1,11), (0,13),$ $(1,0), (0,3), (1,5), (0,2), (1,15), (0,12), (0,1), (0,6), (0,11), (1,9), (1,4), (0,7), (1,10)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{16}, \{(1,3), (0,7), (1,1)\})$
$H_0: (1,6), (0,5), (1,2), (0,15), (1,12), (1,3), (0,4), (1,7), (1,0), (0,1), (1,4), (0,7), (0,14), (1,1), (0,2), (1,5), (0,8), (1,9), (0,6),$ $(0,13), (1,14), (0,11), (1,10), (0,9), (1,8), (1,15), (0,0), (1,13), (0,10), (1,11), (0,12), (0,3)$ $H_1: (1,13), (0,12), (1,9), (1,0), (0,3), (1,2), (0,1), (0,10), (1,7), (0,8), (0,15), (1,14), (1,5), (0,6), (1,3), (0,2), (0,9), (0,0), (1,1),$ $(0,4), (0,11), (1,12), (0,13), (1,10), (0,7), (1,8), (0,5), (1,4), (1,11), (0,14), (1,15), (1,6)$ $H_2: (0,6), (1,7), (1,14), (0,1), (0,8), (1,11), (1,2), (1,9), (0,10), (0,3), (1,4), (1,13), (0,14), (0,5), (0,12), (1,15), (0,2), (0,11), (1,8),$ $(1,1), (1,10), (1,3), (0,0), (0,7), (1,6), (0,9), (1,12), (1,5), (0,4), (0,13), (1,0), (0,15)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{16}, \{(1,3), (0,1), (1,5)\})$
$H_0: (0,0), (0,15), (0,14), (1,1), (0,12), (1,15), (1,0), (0,11), (0,10), (1,7), (0,4), (0,3), (1,6), (0,9), (1,14), (1,13), (0,8), (0,7), (0,6),$ $(0,5), (1,8), (1,9), (1,10), (0,13), (1,2), (1,3), (1,4), (1,5), (0,2), (0,1), (1,12), (1,11)$ $H_1: (0,5), (1,2), (0,7), (1,4), (0,1), (1,6), (1,7), (0,2), (0,3), (1,14), (0,11), (1,8), (0,13), (0,12), (1,9), (0,6), (1,3), (0,14), (1,11),$ $(1,10), (0,15), (1,12), (0,9), (0,8), (1,5), (0,0), (1,13), (0,10), (1,15), (0,4), (1,1), (1,0)$ $H_2: (0,10), (1,5), (1,6), (0,11), (0,12), (1,7), (1,8), (0,3), (1,0), (0,13), (0,14), (1,9), (0,4), (0,5), (1,10), (0,7), (1,12), (1,13), (0,2),$ $(1,15), (1,14), (0,1), (0,0), (1,3), (0,8), (1,11), (0,6), (1,1), (1,2), (0,15), (1,4), (0,9)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{16}, \{(1,4), (0,3), (1,7)\})$
$H_0: (1,10), (0,1), (1,5), (1,8), (0,4), (1,11), (0,7), (1,3), (0,12), (0,9), (1,0), (1,13), (0,6), (1,2), (0,11), (0,14), (1,7), (0,0), (1,4),$

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Table 1.6 – continued from previous page

$(0,13),(1,9),(0,5),(0,2),(0,15),(1,6),(0,10),(1,14),(1,1),(0,8),(1,12),(1,15),(0,3)$
$H_1: (0,2),(1,6),(1,9),(1,12),(0,0),(0,3),(0,6),(1,10),(1,7),(1,4),(1,1),(0,13),(0,10),(1,3),(0,15),(1,8),(0,12),(1,0),(0,7),$ $(0,4),(0,1),(1,13),(0,9),(1,5),(0,14),(1,2),(1,15),(0,11),(0,8),(0,5),(1,14),(1,11)$
$H_2: (1,3),(1,0),(0,4),(1,13),(1,10),(0,14),(0,1),(1,8),(1,11),(0,15),(0,12),(1,5),(1,2),(0,9),(0,6),(1,15),(0,8),(1,4),(0,11),$ $(1,7),(0,3),(1,12),(0,5),(1,1),(0,10),(0,7),(1,14),(0,2),(1,9),(0,0),(0,13),(1,6)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{16}, \{(1,4),(0,2),(0,5)\})$
$H_0: (0,5),(0,0),(0,2),(0,4),(0,6),(0,1),(0,3),(1,15),(1,1),(0,13),(1,9),(1,4),(1,6),(0,10),(0,8),(1,12),(1,7),(0,11),(0,9),$ $(1,5),(1,0),(1,14),(1,3),(0,15),(1,11),(1,13),(1,2),(0,14),(1,10),(1,8),(0,12),(0,7)$
$H_1: (1,0),(1,2),(1,7),(0,3),(0,14),(0,12),(0,1),(0,15),(0,10),(0,5),(1,1),(1,3),(1,5),(1,10),(1,15),(1,13),(0,9),(0,7),(0,2),$ $(0,13),(0,11),(0,6),(0,8),(1,4),(0,0),(1,12),(1,14),(1,9),(1,11),(1,6),(1,8),(0,4)$
$H_2: (1,14),(0,2),(1,6),(1,1),(1,12),(1,10),(0,6),(1,2),(1,4),(1,15),(0,11),(0,0),(0,14),(0,9),(0,4),(0,15),(0,13),(0,8),(0,3),$ $(0,5),(1,9),(1,7),(1,5),(0,1),(1,13),(1,8),(1,3),(0,7),(1,11),(1,0),(0,12),(0,10)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{16}, \{(0,1),(1,6),(0,7)\})$
$H_0: (1,6),(0,0),(0,1),(0,2),(0,9),(1,3),(1,10),(1,1),(0,11),(0,4),(0,3),(1,9),(1,2),(0,8),(0,7),(0,6),(0,15),(1,5),(1,4),$ $(0,10),(1,0),(1,7),(0,13),(0,14),(1,8),(1,15),(1,14),(1,13),(1,12),(1,11),(0,5),(0,12)$
$H_1: (1,4),(1,11),(0,1),(0,8),(1,14),(1,7),(1,6),(1,15),(1,0),(0,6),(0,5),(0,4),(0,13),(1,3),(1,2),(1,1),(1,8),(0,2),(1,12),$ $(1,5),(0,11),(0,12),(0,3),(0,10),(0,9),(0,0),(1,10),(1,9),(0,15),(0,14),(0,7),(1,13)$
$H_2: (0,13),(0,6),(1,12),(1,3),(1,4),(0,14),(0,5),(1,15),(0,9),(0,8),(0,15),(0,0),(0,7),(1,1),(1,0),(1,9),(1,8),(1,7),(0,1),$ $(0,10),(0,11),(0,2),(0,3),(1,13),(1,6),(1,5),(1,14),(0,4),(1,10),(1,11),(1,2),(0,12)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{16}, \{(1,5),(0,4),(1,2)\})$
$H_0: (0,14),(1,3),(0,1),(0,5),(1,10),(0,12),(0,0),(1,14),(0,3),(0,15),(1,1),(0,6),(1,8),(0,13),(1,2),(0,4),(1,15),(1,11),(1,7),$ $(0,9),(1,4),(1,0),(0,2),(1,13),(0,8),(1,6),(0,11),(1,9),(1,5),(0,7),(1,12),(0,10)$
$H_1: (0,11),(0,15),(1,4),(0,6),(0,10),(1,8),(1,12),(0,14),(0,2),(1,7),(1,3),(0,8),(1,10),(1,14),(0,12),(1,1),(1,13),(1,9),(0,4),$ $(0,0),(1,5),(0,3),(0,7),(1,2),(1,6),(0,1),(1,15),(0,13),(1,11),(0,9),(0,5),(1,0)$
$H_2: (1,9),(0,7),(0,11),(1,13),(0,15),(1,10),(1,6),(0,4),(0,8),(0,12),(1,7),(0,5),(1,3),(1,15),(0,10),(1,5),(1,1),(0,3),(1,8),$ $(1,4),(0,2),(0,6),(1,11),(0,0),(1,2),(1,14),(0,9),(0,13),(0,1),(1,12),(1,0),(0,14)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{16}, \{(0,1),(1,4),(1,2)\})$
$H_0: (0,5),(1,1),(1,2),(0,4),(1,0),(1,15),(0,1),(0,2),(1,14),(0,0),(1,12),(1,13),(0,9),(1,5),(0,3),(1,7),(0,11),(1,9),(1,8),$ $(0,10),(1,6),(0,8),(1,4),(1,3),(0,15),(0,14),(0,13),(0,12),(1,10),(1,11),(0,7),(0,6)$
$H_1: (1,1),(0,15),(0,0),(0,1),(1,5),(1,6),(0,2),(1,4),(0,6),(1,10),(1,9),(0,5),(0,4),(0,3),(1,15),(0,13),(1,11),(0,9),(1,7),$ $(1,8),(0,12),(0,11),(1,13),(1,14),(0,10),(1,12),(0,8),(0,7),(1,3),(1,2),(0,14),(1,0)$
$H_2: (1,0),(0,2),(0,3),(1,1),(0,13),(1,9),(0,7),(1,5),(1,4),(0,0),(1,2),(0,6),(1,8),(0,4),(1,6),(1,7),(0,5),(1,3),(0,1),$ $(1,13),(0,15),(1,11),(1,12),(0,14),(1,10),(0,8),(0,9),(0,10),(0,11),(1,15),(1,14),(0,12)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{16}, \{(1,5),(1,6),(1,2)\})$
$H_0: (1,3),(0,1),(1,15),(0,10),(1,8),(0,2),(1,12),(0,14),(1,0),(0,11),(1,13),(0,3),(1,1),(0,6),(1,4),(0,9),(1,14),(0,4),(1,2),$ $(0,7),(1,9),(0,15),(1,5),(0,0),(1,6),(0,12),(1,7),(0,13),(1,11),(0,5),(1,10),(0,8)$
$H_1: (0,3),(1,5),(0,10),(1,4),(0,14),(1,3),(0,5),(1,7),(0,1),(1,11),(0,9),(1,15),(0,4),(1,9),(0,11),(1,6),(0,8),(1,14),(0,0),$ $(1,10),(0,15),(1,13),(0,2),(1,0),(0,6),(1,12),(0,7),(1,1),(0,12),(1,2),(0,13),(1,8)$
$H_2: (0,6),(1,11),(0,0),(1,2),(0,8),(1,13),(0,7),(1,5),(0,11),(1,1),(0,15),(1,4),(0,2),(1,7),(0,9),(1,3),(0,13),(1,15),(0,5),$ $(1,0),(0,10),(1,12),(0,1),(1,6),(0,4),(1,10),(0,12),(1,14),(0,3),(1,9),(0,14),(1,8)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{16}, \{(1,3),(0,7),(1,2)\})$
$H_0: (0,9),(0,2),(1,4),(0,1),(1,3),(0,0),(1,13),(0,11),(0,4),(0,13),(1,0),(1,7),(1,14),(0,12),(1,10),(0,8),(1,6),(1,15),(1,8),$ $(1,1),(0,3),(0,10),(1,12),(1,5),(0,7),(1,9),(0,6),(0,15),(1,2),(0,5),(0,14),(1,11)$
$H_1: (0,1),(0,10),(1,7),(0,5),(1,3),(0,6),(1,8),(0,11),(1,14),(0,0),(1,2),(0,4),(1,6),(0,9),(1,12),(0,14),(0,7),(1,4),(1,13),$ $(0,15),(1,1),(1,10),(0,13),(1,11),(0,8),(1,5),(0,3),(0,12),(1,9),(1,0),(0,2),(1,15)$
$H_2: (1,5),(0,2),(0,11),(1,9),(1,2),(1,11),(1,4),(0,6),(0,13),(1,15),(0,12),(0,5),(1,8),(0,10),(1,13),(1,6),(0,3),(1,0),(0,14),$ $(1,1),(0,4),(1,7),(0,9),(0,0),(0,7),(1,10),(1,3),(1,12),(0,15),(0,8),(0,1),(1,14)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{16}, \{(1,3),(1,4),(0,4)\})$
$H_0: (0,0),(1,12),(1,0),(0,12),(0,8),(1,5),(0,2),(1,6),(0,3),(1,15),(1,3),(0,15),(1,2),(1,14),(0,11),(1,8),(1,4),(0,7),(1,10),$ $(0,6),(1,9),(0,5),(0,1),(0,13),(0,9),(1,13),(0,10),(1,7),(1,11),(0,14),(1,1),(0,4)$
$H_1: (1,6),(1,2),(0,6),(1,3),(1,7),(0,3),(1,0),(0,4),(0,8),(1,4),(0,1),(1,5),(1,1),(0,13),(1,9),(1,13),(0,0),(0,12),(1,8),$

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Table 1.6 – continued from previous page

$(0,5),(0,9),(1,12),(0,15),(1,11),(0,7),(0,11),(1,15),(0,2),(1,14),(0,10),(0,14),(1,10)$
$H_2:(1,5),(0,9),(1,6),(0,10),(0,6),(0,2),(0,14),(1,2),(0,5),(1,1),(1,13),(0,1),(1,14),(1,10),(0,13),(1,0),(1,4),(0,0),(1,3),$ $(0,7),(0,3),(0,15),(0,11),(1,7),(0,4),(1,8),(1,12),(0,8),(1,11),(1,15),(0,12),(1,9)$
$CAY(\mathbb{Z}_2 \times \mathbb{Z}_{16}, \{(1,3),(0,5),(1,1)\})$
$H_0:(1,6),(0,3),(0,8),(1,11),(0,12),(0,1),(0,6),(1,3),(0,0),(0,11),(1,12),(0,9),(0,4),(1,5),(1,0),(0,15),(0,10),(1,7),(1,2),$ $(0,5),(1,4),(1,9),(1,14),(0,13),(0,2),(1,15),(1,10),(0,7),(1,8),(1,13),(0,14),(1,1)$
$H_1:(0,10),(0,5),(1,6),(0,7),(0,2),(1,5),(0,8),(1,7),(0,4),(1,3),(1,8),(0,9),(1,10),(0,11),(0,6),(1,9),(0,12),(1,13),(0,0),$ $(1,1),(1,12),(0,13),(1,0),(0,3),(1,2),(0,15),(1,14),(0,1),(1,4),(1,15),(0,14),(1,11)$
$H_2:(1,15),(0,0),(0,5),(1,8),(0,11),(1,14),(1,3),(0,2),(1,1),(0,4),(0,15),(1,12),(1,7),(0,6),(1,5),(1,10),(0,13),(0,8),(1,9),$ $(0,10),(1,13),(1,2),(0,1),(1,0),(1,11),(1,6),(0,9),(0,14),(0,3),(1,4),(0,7),(0,12)$
$CAY(\mathbb{Z}_2 \times \mathbb{Z}_{16}, \{(0,2),(1,6),(0,7)\})$
$H_0:(0,6),(1,0),(1,2),(1,9),(1,11),(1,13),(0,3),(0,1),(0,10),(1,4),(1,6),(1,8),(0,2),(0,0),(1,10),(0,4),(1,14),(1,12),(1,3),$ $(0,9),(1,15),(0,5),(0,12),(0,14),(0,7),(1,1),(0,11),(0,13),(1,7),(1,5),(0,15),(0,8)$
$H_1:(1,9),(1,7),(1,0),(0,10),(0,8),(1,14),(1,5),(0,11),(0,2),(0,4),(0,6),(1,12),(1,10),(1,1),(1,3),(0,13),(0,15),(0,1),(1,11),$ $(0,5),(0,14),(1,8),(1,15),(1,6),(0,0),(0,9),(0,7),(1,13),(1,4),(1,2),(0,12),(0,3)$
$H_2:(1,12),(0,2),(0,9),(0,11),(0,4),(0,13),(0,6),(0,15),(1,9),(1,0),(1,14),(1,7),(0,1),(0,8),(1,2),(1,11),(1,4),(0,14),(0,0),$ $(0,7),(0,5),(0,3),(0,10),(0,12),(1,6),(1,13),(1,15),(1,1),(1,8),(1,10),(1,3),(1,5)$
$CAY(\mathbb{Z}_2 \times \mathbb{Z}_{16}, \{(0,1),(1,5),(0,6)\})$
$H_0:(1,2),(1,1),(1,7),(0,12),(0,11),(1,0),(1,6),(1,5),(0,10),(0,4),(0,5),(0,6),(0,0),(0,1),(0,2),(0,8),(0,9),(0,3),(0,13),$ $(1,8),(1,9),(1,15),(1,14),(1,13),(1,3),(0,14),(0,15),(1,4),(1,10),(1,11),(1,12),(0,7)$
$H_1:(0,3),(1,8),(1,2),(1,12),(0,1),(0,7),(0,8),(0,14),(0,13),(0,12),(1,1),(0,6),(1,11),(1,5),(0,0),(0,10),(1,15),(0,4),(1,9),$ $(1,3),(1,4),(1,14),(0,9),(0,15),(1,10),(1,0),(0,5),(0,11),(1,6),(1,7),(1,13),(0,2)$
$H_2:(1,11),(0,0),(0,15),(0,5),(1,10),(1,9),(0,14),(0,4),(0,3),(1,14),(1,8),(1,7),(0,2),(0,12),(0,6),(0,7),(0,13),(1,2),(1,3),$ $(0,8),(1,13),(1,12),(1,6),(0,1),(0,11),(0,10),(0,9),(1,4),(1,5),(1,15),(1,0),(1,1)$
$CAY(\mathbb{Z}_2 \times \mathbb{Z}_{16}, \{(1,3),(1,4),(0,5)\})$
$H_0:(0,12),(1,0),(0,3),(0,14),(0,9),(1,5),(1,10),(0,6),(1,2),(0,5),(0,0),(1,13),(0,1),(1,4),(1,9),(0,13),(1,1),(1,6),(1,11),$ $(0,8),(1,12),(0,15),(1,3),(1,8),(0,4),(1,7),(0,10),(1,14),(0,11),(1,15),(0,2),(0,7)$
$H_1:(0,0),(1,3),(1,14),(0,1),(1,5),(0,2),(0,13),(1,0),(1,11),(0,14),(1,10),(0,7),(1,4),(0,8),(0,3),(1,15),(0,12),(1,8),(0,5),$ $(1,9),(0,6),(0,11),(1,7),(1,2),(1,13),(0,9),(1,6),(0,10),(0,15),(0,4),(1,1),(1,12)$
$H_2:(0,7),(1,3),(0,6),(0,1),(0,12),(1,9),(1,14),(0,2),(1,6),(0,3),(1,7),(1,12),(0,9),(0,4),(1,0),(1,5),(0,8),(0,13),(1,10),$ $(1,15),(1,4),(0,0),(0,11),(1,8),(1,13),(0,10),(0,5),(1,1),(0,14),(1,2),(0,15),(1,11)$
$CAY(\mathbb{Z}_2 \times \mathbb{Z}_{16}, \{(0,1),(1,5),(1,6)\})$
$H_0:(0,7),(0,8),(1,3),(0,13),(0,12),(1,2),(1,1),(0,11),(1,0),(0,6),(0,5),(0,4),(1,10),(1,11),(0,1),(1,6),(1,5),(0,0),(0,15),$ $(1,4),(0,9),(0,10),(1,15),(1,14),(0,3),(1,9),(0,14),(1,8),(1,7),(0,2),(1,12),(1,13)$
$H_1:(0,10),(1,4),(0,14),(0,13),(1,2),(1,3),(0,9),(1,14),(0,8),(1,13),(0,3),(0,4),(1,9),(1,8),(0,2),(0,1),(0,0),(1,10),(0,15),$ $(1,5),(0,11),(1,6),(1,7),(0,12),(1,1),(0,6),(0,7),(1,12),(1,11),(0,5),(1,15),(1,0)$
$H_2:(0,8),(1,2),(0,7),(1,1),(1,0),(0,5),(1,10),(1,9),(0,15),(0,14),(1,3),(1,4),(1,5),(0,10),(0,11),(0,12),(1,6),(0,0),(1,11),$ $(0,6),(1,12),(0,1),(1,7),(0,13),(1,8),(0,3),(0,2),(1,13),(1,14),(0,4),(1,15),(0,9)$
$CAY(\mathbb{Z}_2 \times \mathbb{Z}_{16}, \{(1,4),(1,5),(1,7)\})$
$H_0:(1,1),(0,8),(1,12),(0,3),(1,7),(0,11),(1,0),(0,4),(1,11),(0,0),(1,4),(0,13),(1,2),(0,14),(1,5),(0,1),(1,6),(0,2),(1,9),$ $(0,5),(1,10),(0,15),(1,8),(0,12),(1,3),(0,7),(1,14),(0,9),(1,13),(0,6),(1,15),(0,10)$
$H_1:(1,1),(0,12),(1,0),(0,7),(1,2),(0,6),(1,10),(0,14),(1,3),(0,10),(1,5),(0,9),(1,4),(0,15),(1,11),(0,2),(1,7),(0,0),(1,9),$ $(0,4),(1,8),(0,3),(1,14),(0,5),(1,12),(0,1),(1,13),(0,8),(1,15),(0,11),(1,6),(0,13)$
$H_2:(1,0),(0,9),(1,2),(0,11),(1,4),(0,8),(1,3),(0,15),(1,6),(0,10),(1,14),(0,2),(1,13),(0,4),(1,15),(0,3),(1,10),(0,1),(1,8),$ $(0,13),(1,9),(0,14),(1,7),(0,12),(1,5),(0,0),(1,12),(0,7),(1,11),(0,6),(1,1),(0,5)$
$CAY(\mathbb{Z}_2 \times \mathbb{Z}_{16}, \{(1,3),(1,4),(1,5)\})$
$H_0:(1,6),(0,1),(1,4),(0,8),(1,11),(0,0),(1,3),(0,6),(1,9),(0,4),(1,1),(0,14),(1,2),(0,15),(1,12),(0,7),(1,10),(0,5),(1,0),$ $(0,13),(1,8),(0,11),(1,7),(0,12),(1,15),(0,10),(1,5),(0,2),(1,13),(0,9),(1,14),(0,3)$
$H_1:(0,9),(1,5),(0,1),(1,12),(0,0),(1,4),(0,15),(1,3),(0,8),(1,13),(0,10),(1,7),(0,4),(1,0),(0,12),(1,9),(0,14),(1,10),(0,6),$ $(1,11),(0,7),(1,2),(0,13),(1,1),(0,5),(1,8),(0,3),(1,15),(0,2),(1,14),(0,11),(1,6)$
$H_2:(1,0),(0,3),(1,7),(0,2),(1,6),(0,10),(1,14),(0,1),(1,13),(0,0),(1,5),(0,8),(1,12),(0,9),(1,4),(0,7),(1,3),(0,14),(1,11),$

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Table 1.6 – continued from previous page

$(0,15),(1,10),(0,13),(1,9),(0,5),(1,2),(0,6),(1,1),(0,12),(1,8),(0,4),(1,15),(0,11)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{16}, \{(1,6),(0,6),(1,1)\})$
$H_0: (1,9),(0,3),(1,2),(0,12),(0,2),(1,3),(0,4),(0,14),(1,13),(0,7),(1,1),(1,7),(0,13),(1,12),(1,6),(1,0),(0,6),(0,0),(0,10),$ $(1,4),(1,10),(0,11),(0,1),(1,11),(0,5),(0,15),(1,5),(1,15),(0,9),(1,8),(1,14),(0,8)$
$H_1: (0,7),(1,6),(0,0),(1,1),(0,11),(0,5),(1,4),(0,3),(0,9),(0,15),(1,14),(0,4),(1,10),(1,0),(0,1),(1,7),(0,8),(0,2),(1,12),$ $(1,2),(1,8),(0,14),(1,15),(1,9),(0,10),(1,11),(1,5),(0,6),(0,12),(1,13),(1,3),(0,13)$
$H_2: (1,1),(0,2),(1,8),(0,7),(0,1),(1,2),(0,8),(0,14),(1,4),(1,14),(0,13),(0,3),(1,13),(1,7),(0,6),(1,12),(0,11),(1,5),(0,4),$ $(0,10),(1,0),(0,15),(1,9),(1,3),(0,9),(1,10),(0,0),(1,15),(0,5),(1,6),(0,12),(1,11)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{16}, \{(1,3),(0,1),(0,4)\})$
$H_0: (1,5),(0,8),(0,7),(1,4),(0,1),(0,5),(1,8),(1,7),(1,3),(0,0),(0,4),(0,3),(0,15),(1,12),(1,0),(1,15),(0,2),(0,6),(1,9),$ $(1,10),(1,11),(0,14),(1,1),(1,2),(1,14),(1,13),(0,10),(0,11),(0,12),(0,13),(0,9),(1,6)$
$H_1: (0,2),(1,5),(1,4),(1,0),(1,1),(0,4),(0,8),(0,9),(0,5),(1,2),(1,6),(1,10),(0,13),(0,1),(1,14),(0,11),(0,15),(0,14),(0,10),$ $(1,7),(1,11),(1,12),(1,8),(1,9),(1,13),(0,0),(0,12),(1,15),(1,3),(0,6),(0,7),(0,3)$
$H_2: (1,7),(0,4),(0,5),(0,6),(0,10),(0,9),(1,12),(1,13),(1,1),(1,5),(1,9),(0,12),(0,8),(1,11),(1,15),(1,14),(1,10),(0,7),(0,11),$ $(1,8),(1,4),(1,3),(1,2),(0,15),(0,0),(0,1),(0,2),(0,14),(0,13),(1,0),(0,3),(1,6)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_{16}, \{(0,1),(1,4),(0,2)\})$
$H_0: (1,1),(1,15),(1,0),(1,2),(1,3),(0,7),(0,6),(0,4),(0,2),(0,0),(1,12),(1,13),(1,14),(0,10),(0,9),(0,8),(1,4),(1,5),(1,7),$ $(1,6),(1,8),(0,12),(0,11),(0,13),(0,14),(1,10),(1,9),(1,11),(0,15),(0,1),(0,3),(0,5)$
$H_1: (0,1),(1,5),(1,3),(1,1),(1,2),(1,4),(0,0),(0,14),(0,15),(0,13),(1,9),(0,5),(0,4),(1,8),(1,7),(0,11),(0,9),(0,7),(0,8),$ $(0,6),(1,10),(1,11),(1,12),(1,14),(1,0),(0,12),(0,10),(1,6),(0,2),(0,3),(1,15),(1,13)$
$H_2: (0,4),(1,0),(1,1),(0,13),(0,12),(0,14),(1,2),(0,6),(0,5),(0,7),(1,11),(1,13),(0,9),(1,5),(1,6),(1,4),(1,3),(0,15),(0,0),$ $(0,1),(0,2),(1,14),(1,15),(0,11),(0,10),(0,8),(1,12),(1,10),(1,8),(1,9),(1,7),(0,3)$

Table 1.7: Hamilton decompositions for Cayley graphs on  $\mathbb{Z}_4 \times \mathbb{Z}_8$ .

$\text{CAY}(\mathbb{Z}_4 \times \mathbb{Z}_8, \{(3,1),(3,2),(2,3)\})$
$H_0: (1,1),(0,3),(2,0),(3,7),(1,4),(3,1),(1,6),(0,0),(3,2),(2,4),(3,3),(0,1),(1,7),(3,4),(2,5),(0,2),(2,7),(1,0),(2,6),(3,5),(1,2),$ $(2,1),(3,0),(0,6),(2,3),(1,5),(0,7),(2,2),(0,5),(1,3),(0,4),(3,6)$
$H_1: (0,3),(3,4),(1,1),(0,2),(1,0),(0,1),(3,2),(2,3),(0,0),(3,1),(0,7),(3,0),(2,2),(1,4),(0,5),(2,0),(1,2),(0,4),(3,5),(2,7),(3,6),$ $(1,3),(2,1),(3,7),(0,6),(1,5),(2,4),(1,6),(3,3),(2,5),(1,7),(2,6)$
$H_2: (3,3),(1,0),(3,5),(0,3),(1,2),(3,7),(0,5),(3,6),(2,0),(1,1),(2,7),(0,4),(2,1),(0,6),(1,4),(2,3),(3,1),(2,2),(1,3),(3,0),(1,5),$ $(3,2),(1,7),(0,0),(2,5),(1,6),(0,7),(2,4),(0,1),(2,6),(3,4),(0,2)$
$\text{CAY}(\mathbb{Z}_4 \times \mathbb{Z}_8, \{(3,1),(2,1),(2,2)\})$
$H_0: (1,1),(0,3),(2,0),(3,7),(1,4),(3,1),(1,6),(0,0),(3,2),(2,4),(3,3),(0,1),(1,7),(3,4),(2,5),(0,2),(2,7),(1,0),(2,6),(3,5),(1,2),$ $(2,1),(3,0),(0,6),(2,3),(1,5),(0,7),(2,2),(0,5),(1,3),(0,4),(3,6)$
$H_1: (0,3),(3,4),(1,1),(0,2),(1,0),(0,1),(3,2),(2,3),(0,0),(3,1),(0,7),(3,0),(2,2),(1,4),(0,5),(2,0),(1,2),(0,4),(3,5),(2,7),(3,6),$ $(1,3),(2,1),(3,7),(0,6),(1,5),(2,4),(1,6),(3,3),(2,5),(1,7),(2,6)$
$H_2: (3,3),(1,0),(3,5),(0,3),(1,2),(3,7),(0,5),(3,6),(2,0),(1,1),(2,7),(0,4),(2,1),(0,6),(1,4),(2,3),(3,1),(2,2),(1,3),(3,0),(1,5),$ $(3,2),(1,7),(0,0),(2,5),(1,6),(0,7),(2,4),(0,1),(2,6),(3,4),(0,2)$
$\text{CAY}(\mathbb{Z}_4 \times \mathbb{Z}_8, \{(3,1),(1,1),(1,2)\})$
$H_0: (1,1),(0,3),(2,0),(3,7),(1,4),(3,1),(1,6),(0,0),(3,2),(2,4),(3,3),(0,1),(1,7),(3,4),(2,5),(0,2),(2,7),(1,0),(2,6),(3,5),(1,2),$ $(2,1),(3,0),(0,6),(2,3),(1,5),(0,7),(2,2),(0,5),(1,3),(0,4),(3,6)$
$H_1: (0,3),(3,4),(1,1),(0,2),(1,0),(0,1),(3,2),(2,3),(0,0),(3,1),(0,7),(3,0),(2,2),(1,4),(0,5),(2,0),(1,2),(0,4),(3,5),(2,7),(3,6),$ $(1,3),(2,1),(3,7),(0,6),(1,5),(2,4),(1,6),(3,3),(2,5),(1,7),(2,6)$
$H_2: (3,3),(1,0),(3,5),(0,3),(1,2),(3,7),(0,5),(3,6),(2,0),(1,1),(2,7),(0,4),(2,1),(0,6),(1,4),(2,3),(3,1),(2,2),(1,3),(3,0),(1,5),$ $(3,2),(1,7),(0,0),(2,5),(1,6),(0,7),(2,4),(0,1),(2,6),(3,4),(0,2)$
$\text{CAY}(\mathbb{Z}_4 \times \mathbb{Z}_8, \{(1,4),(0,2),(2,3)\})$
$H_0: (1,1),(0,3),(2,0),(3,7),(1,4),(3,1),(1,6),(0,0),(3,2),(2,4),(3,3),(0,1),(1,7),(3,4),(2,5),(0,2),(2,7),(1,0),(2,6),(3,5),(1,2),$ $(2,1),(3,0),(0,6),(2,3),(1,5),(0,7),(2,2),(0,5),(1,3),(0,4),(3,6)$
$H_1: (0,3),(3,4),(1,1),(0,2),(1,0),(0,1),(3,2),(2,3),(0,0),(3,1),(0,7),(3,0),(2,2),(1,4),(0,5),(2,0),(1,2),(0,4),(3,5),(2,7),(3,6),$ $(1,3),(2,1),(3,7),(0,6),(1,5),(2,4),(1,6),(3,3),(2,5),(1,7),(2,6)$

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Table 1.7 – continued from previous page

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Table 1.7 – continued from previous page

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$H_2: (3,3), (1,0), (3,5), (0,3), (1,2), (3,7), (0,5), (3,6), (2,0), (1,1), (2,7), (0,4), (2,1), (0,6), (1,4), (2,3), (3,1), (2,2), (1,3), (3,0), (1,5),$ $(3,2), (1,7), (0,0), (2,5), (1,6), (0,7), (2,4), (0,1), (2,6), (3,4), (0,2)$
--

---

Table 1.8: Hamilton decompositions for Cayley graphs on  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8$ .

---

$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8, \{(1,0,3), (0,0,3), (0,1,2)\})$ $H_0: (0,1,0), (0,0,6), (0,0,3), (0,0,0), (0,0,5), (0,0,2), (0,0,7), (0,1,1), (0,1,6), (0,1,3), (1,1,6), (1,0,4), (1,0,7), (0,0,4), (0,0,1),$ $(0,1,7), (1,1,2), (0,1,5), (1,1,0), (1,1,3), (1,0,1), (1,0,6), (1,1,4), (1,0,2), (1,0,5), (1,0,0), (1,0,3), (1,1,1), (0,1,4),$ $(1,1,7), (0,1,2), (1,1,5)$ $H_1: (0,1,6), (1,1,3), (0,1,0), (0,0,2), (0,1,4), (0,0,6), (0,0,1), (0,1,3), (0,0,5), (1,0,0), (0,0,3), (0,1,5), (0,0,7), (1,0,4), (1,0,1),$ $(0,0,4), (0,1,2), (0,1,7), (1,1,4), (0,1,1), (1,1,6), (1,1,1), (1,0,7), (1,0,2), (1,1,0), (1,0,6), (1,0,3), (1,1,5), (1,1,2),$ $(1,1,7), (1,0,5), (0,0,0)$ $H_2: (1,1,2), (1,0,0), (1,1,6), (1,1,3), (1,0,5), (0,0,2), (1,0,7), (1,1,5), (1,1,0), (0,1,3), (0,1,0), (0,1,5), (0,1,2), (0,0,0), (1,0,3),$ $(0,0,6), (1,0,1), (1,1,7), (1,1,4), (1,1,1), (0,1,6), (0,0,4), (0,0,7), (1,0,2), (0,0,5), (0,1,7), (0,1,4), (0,1,1), (0,0,3),$ $(1,0,6), (0,0,1), (1,0,4)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8, \{(0,0,1), (1,0,3), (1,1,1)\})$ $H_0: (1,1,7), (0,0,6), (0,0,7), (0,0,0), (1,0,3), (0,1,2), (0,1,3), (1,0,2), (1,0,1), (0,1,0), (0,1,7), (1,0,0), (0,1,1), (1,1,4), (0,0,5),$ $(1,1,6), (1,1,5), (0,0,4), (0,0,3), (0,0,2), (0,0,1), (1,1,0), (0,1,5), (1,0,4), (1,0,5), (1,0,6), (1,0,7), (0,1,6), (1,1,3),$ $(1,1,2), (1,1,1), (0,1,4)$ $H_1: (0,0,2), (1,0,7), (0,1,0), (1,1,3), (0,0,4), (0,0,5), (0,0,6), (1,0,3), (1,0,2), (0,1,1), (1,1,6), (1,1,7), (1,1,0), (0,0,7), (1,0,4),$ $(0,1,3), (0,1,4), (0,1,5), (1,0,6), (0,0,1), (1,1,2), (0,0,3), (1,0,0), (1,0,1), (0,1,2), (1,1,5), (1,1,4), (0,1,7), (0,1,6),$ $(1,0,5), (0,0,0), (1,1,1)$ $H_2: (1,0,3), (0,1,4), (1,0,5), (0,0,2), (1,1,3), (1,1,4), (0,0,3), (1,0,6), (0,1,7), (1,1,2), (0,1,5), (0,1,6), (1,1,1), (1,1,0), (0,1,3),$ $(1,1,6), (0,0,7), (1,0,2), (0,0,5), (1,0,0), (1,0,7), (0,0,4), (1,0,1), (0,0,6), (1,1,5), (0,1,0), (0,1,1), (0,1,2), (1,1,7),$ $(0,0,0), (0,0,1), (1,0,4)$
$\text{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8, \{(1,0,2), (0,1,3), (1,1,2)\})$ $H_0: (1,0,2), (0,0,0), (0,1,3), (1,0,1), (0,0,3), (0,1,6), (1,0,0), (0,0,6), (1,0,4), (0,0,2), (0,1,7), (0,0,4), (1,1,2), (1,0,5), (1,1,0),$ $(1,0,3), (0,1,5), (1,0,7), (1,1,4), (0,1,2), (0,0,5), (1,1,7), (0,1,1), (1,1,3), (0,0,1), (0,1,4), (1,1,6), (0,1,0), (1,0,6),$ $(1,1,1), (0,0,7), (1,1,5)$ $H_1: (0,1,0), (0,0,3), (1,1,1), (0,1,3), (1,1,5), (0,1,7), (1,0,5), (0,0,7), (0,1,2), (1,0,4), (0,1,6), (1,1,0), (0,0,6), (0,1,1), (1,0,3),$ $(0,0,1), (1,0,7), (0,0,5), (1,1,3), (1,0,0), (0,0,2), (1,1,4), (1,0,1), (1,1,6), (0,0,4), (1,0,6), (0,1,4), (1,0,2), (1,1,7),$ $(0,1,5), (0,0,0), (1,1,2)$ $H_2: (1,0,3), (0,0,5), (0,1,0), (1,0,2), (0,0,4), (0,1,1), (1,0,7), (1,1,2), (0,1,4), (0,0,7), (1,0,1), (0,1,7), (1,1,1), (1,0,4), (1,1,7),$ $(0,0,1), (0,1,6), (1,1,4), (0,0,6), (0,1,3), (1,0,5), (0,0,3), (1,1,5), (1,0,0), (0,1,2), (1,1,0), (0,0,2), (0,1,5), (1,1,3),$ $(1,0,6), (0,0,0), (1,1,6)$

---



# Appendix B

## Source code

### 2.1 MAGMA code

The MAGMA function, `CayBuild3(A,s1,s2,s3)`, constructs a Cayley graph of  $A$  relative to connection set  $S = \{s_1, s_2, s_3\}$ . The function, `CGraphs3(A,Fout)`, outputs to file all generating sets  $S = \{s_1, s_2, s_3\}$  for an abelian group  $A$ , where  $2 < |s_i| < |A|$ , for  $i = 1, 2, 3$ , such that the corresponding set of 6-regular, connected, Cayley graphs are pairwise non-isomorphic.

```
>intrinsic CayBuild3(A::GrpAb, s1::GrpAbElt, s2::GrpAbElt, s3::GrpAbElt) ->
> GrphUnd,GrphVertSet,GrphEdgeSet
>{Constructs the Cayley Graph of A generated by s1, s2, and s3.}
> V:={a: a in A};
> E:={};
> for x in A do
>   e := {x, x*s1 };
>   f := {x, x*s2 };
>   g := {x, x*s3 };
>   Include(~E,e);
>   Include(~E,f);
>   Include(~E,g);
> end for;
> G,V,E := Graph< V | E >;
> return G,V,E;
>end intrinsic

>intrinsic CGraphs3(A::GrpAb, F::File)
>{Prints all Three-Element generating sets of A.}
> V := {a: a in A | Order(a) ne 2 and Order(a) ne 1 and Order(a) ne Order(A)};
> for a in A do
>   if a in V then
>     Exclude(~V, -a);
>   end if;
> end for;
> S := Subsets(V,3);
> SI := {};
> for s in S do
```

```

>   H := sub<A|s>;
>   if H eq A then
>       t := SetToIndexedSet(s);
>       Include(~SI, t);
>   end if;
> end for;
> SJ := SetToIndexedSet(SI);
> TJ := IndexedSetToSet(SJ);
> n := #SJ;
> for i := 1 to n-1 do
>     G1,E1,V1 := CayBuild3(A,SJ[i][1], SJ[i][2], SJ[i][3]);
>     for j := i+1 to n do
>         G2,E2,V2 := CayBuild3(A,SJ[j][1], SJ[j][2], SJ[j][3]);
>         if IsIsomorphic(G1,G2) then
>             TJj := IndexedSetToSet(SJ[j]);
>             Exclude(~TJ, TJj);
>         end if;
>     end for;
> end for;
> SK := SetToIndexedSet(TJ);
> m := #SK;
> fprintf F, "%o %o %o\n", m , Order(A), 3;
> for i := 1 to m do
>     fprintf F, "%o %o %o\n", SK[i][1],SK[i][2],SK[i][3];
> end for;
>end intrinsic;

```

## 2.2 C code

The following C-programs were used to find random Hamilton cycle decompositions for the graphs obtained using MAGMA in Section 2.1. They were written with Donald L. Kreher.

### 2.2.1 Constructing the Cayley graphs

```

/*-----GetGraph.c-----*/
#include<stdlib.h>
#include<stdio.h>

FILE *f;
int n,m,s;
int *S;

main( int ac, char * av[])
{
    int i,j;
    if(ac!=2)
    {
        fprintf(stderr,"Usage: %s fname\n",av[0]);
        exit(1);
    }
}

```

```

    if(!(f=fopen(av[1],"r") ))
    {
        fprintf(stderr,"%s cannot open %s\n",av[1]);
        exit(1);
    }
    fscanf(f," %d %d ",&n,&s);
    S = (int *)calloc(s,sizeof(int));
    for(i=0;i<s;i++) fscanf(f,"%d",&S[i]);
    m=n*s;
    printf(" %d %d\n",n,m);
    for(i=0;i<n;i++)
    {
        for(j=0;j<s;j++) printf(" %d %d\n",i , (n+i+S[j])%n);
    }
}

/*-----GetGraph2.c-----*/
#include<stdlib.h>
#include<stdio.h>

FILE *f;
int n,na,nb,m,s;
int S[10][2];

main( int ac, char * av[])
{
    int i,j;
    int i0,a0,b0;
    int i1,a1,b1;
    if(ac!=2)
    {
        fprintf(stderr,"Usage: %s fname\n",av[0]);
        exit(1);
    }
    if(!(f=fopen(av[1],"r") ))
    {
        fprintf(stderr,"%s cannot open %s\n",av[1]);
        exit(1);
    }
    fscanf(f," %d %d %d ",&na,&nb,&s);
    for(i=0;i<s;i++) fscanf(f,"%d %d",&S[i][0],&S[i][1]);
    n=na*nb;
    m=n*s;
    printf(" %d %d\n",n,m);
    for(i0=0;i0<n;i0++)
    {
        a0=i0%na;
        b0=(i0-a0)/na;
        for(j=0;j<s;j++)
        {
            a1=(na+a0+S[j][0])%na;
            b1=(nb+b0+S[j][1])%nb;
            i1= a1+na*b1;
            printf(" %d %d\n",i0 , i1);

```

```

    }
    }
}

/*-----GetGraph3.c-----*/
#include<stdlib.h>
#include<stdio.h>

FILE *f;
int n,na,nb,nc,m,g,nt;
int S[10][10][10];
int T[10][3];

main( int ac, char * av[] )
{
    int i,j,k,counter;
    int i0,a0,b0,c0,j0,k0,j1,k1,l;
    int i1,a1,b1,c1;
    if(ac!=2)
    {
        fprintf(stderr,"Usage: %s fname\n",av[0]);
        exit(1);
    }
    if(!(f=fopen(av[1],"r") ) )
    {
        fprintf(stderr,"%s cannot open %s\n",av[1]);
        exit(1);
    }
    fscanf(f," %d %d %d %d ",&na,&nb,&nc,&g);
    for(i=0;i<g;i++)
        fscanf(f,"%d %d %d",&T[i][0],&T[i][1],&T[i][2]);
    n = na*nb*nc;
    nt = na*nb;
    m = n*g;
    printf(" %d %d\n",n,m);
    counter = 0;
    for(i0=0;i0<na;i0++)
        for(j0=0;j0<nb;j0++)
            for(k0=0;k0<nc;k0++)
            {
                S[i0][j0][k0] = counter;
                counter++;
            }
    for(i0=0;i0<na;i0++)
        for(j0=0;j0<nb;j0++)
            for(k0=0;k0<nc;k0++)
                for(l=0;l<3;l++)
                {
                    i1 = (na+i0 + T[l][0])%na;
                    j1 = (nb+j0 + T[l][1])%nb;
                    k1 = (nc+k0 + T[l][2])%nc;
                    printf(" %d %d\n",S[i0][j0][k0],S[i1][j1][k1]);
                }
}

```

### 2.2.2 Hamilton cycles via a randomized greedy algorithm

```

/*-----RHC.c-----*/
#include<stdlib.h>
#include<stdio.h>

FILE *f;
int n,m;
int **A;
int **C;
int *X;
int *N;
int L;
int seed;
int *done;
int *R;

/* -----genalea -----*/

double genalea (x0)
    int *x0;
{
    int m = 2147483647;
    int a = 16807 ;
    int b = 127773 ;
    int c = 2836 ;
    int x1, k;

    k = (int) ((*x0)/b) ;
    x1 = a*(*x0 - k*b) - k*c ;
    if(x1 < 0) x1 = x1 + m;
    *x0 = x1;

    if(((double)x1/(double)m > 0.0001) &&
        ((double)x1/(double)m < 0.99999))
        return((double)x1/(double)m);
    else return(genalea(x0));
}

/* -----Randomize -----*/

void Randomize(int h)
{
    int i,j,x,y;
    for(i=0;i<N[h];i++)
        R[i]=N[h]*genalea(&seed);
    for(i=1;i<N[h];i++)
    {
        x=R[i];
        y=C[h][i];
        j=i-1;
        while(j >=0 && R[j]> x )
        {
            R[j+1]=R[j];

```



```

        C[h][j+1]=C[h][j];
        j=j-1;
    }
    R[j+1]=x;
    C[h][j+1]=y;
}
}

/* -----BT-----*/

void BT(int ell)
{
    int i;
    if(ell==n)
    {
        for(i=0;i<n;i++) printf(" %d",X[i]);
        printf("\n");
        f=fopen("seed","w");
        fprintf(f," %d\n",seed);
        fclose(f);
        exit(1);
    }
    N[ell]=0;
    if(ell==0)
    {
        for(i=0;i<n;i++) C[ell][N[ell]++]=i;
    }
    else if(ell == (n-1) )
    {
        for(i=0;i<n;i++)
            if(A[X[0]][i] && A[X[ell-1]][i] && !done[i])
                C[ell][N[ell]++]=i;
    }
    else
    {
        for(i=0;i<n;i++)
            if(A[X[ell-1]][i] && !done[i])
                C[ell][N[ell]++]=i;
    }
    Randomize(ell);

    for(i=0;i<N[ell];i++)
    {
        X[ell]=C[ell][i];
        done[X[ell]]=1;
        BT(ell+1);
        done[X[ell]]=0;
    }
}

/* -----Main-----*/

main( int ac, char * av[])
{

```

```

int i,j,x,y;
setbuf(stdout,0);
if(ac!=2)
{
    fprintf(stderr,"Usage: %s fname\n",av[0]);
    exit(1);
}
if(!(f=fopen(av[1],"r") ) )
{
    fprintf(stderr,"%s cannot open %s\n",av[1]);
    exit(1);
}
fscanf(f," %d %d ",&n,&m); //Scans #vertices #edges
A=(int **)calloc(n,sizeof(int*));
C=(int **)calloc(n,sizeof(int*));
N=(int *)calloc(n,sizeof(int));
X=(int *)calloc(n,sizeof(int));
done=(int *)calloc(n,sizeof(int));
R=(int *)calloc(n,sizeof(int));
for(i=0;i<n;i++) done[i]=0;
for(i=0;i<n;i++)
{
    A[i]=(int *)calloc(n,sizeof(int));
    C[i]=(int *)calloc(n,sizeof(int));
    for(j=0;j<n;j++) A[i][j]=0;
}
for(i=0;i<m;i++) //Constructs the adjacency matrix, A.
{
    fscanf(f," %d %d ",&x,&y);
    A[x][y]=1;
    A[y][x]=1;
}
fclose(f);

if((f=fopen("seed","r"))==NULL)
{
    f=fopen("seed","w");
    printf("Please enter the first seed \n");
    printf("for random number generator:");
    scanf("%d",&seed);
    fprintf(f," %d\n",seed );
    fclose(f);
}
f=fopen("seed","r");
fscanf(f," %d",&seed);
fclose(f);

BT(0);
}

```

### 2.2.3 Obtaining Hamilton decompositions

```
/*-----DelHCyc.c-----*/
```

```

#include<stdlib.h>
#include<stdio.h>

FILE *f;
int n,m;
int **A;

/*  Deletes the Hamiltonian cycle H from the graph with adjacency matrix A. */

main( int ac, char * av[])
{
    int i,j,x,y,z;
    setbuf(stdout,0);
    if(ac!=3)
    {
        fprintf(stderr,"Usage: %s graph H-cycle\n",av[0]);
        exit(1);
    }
    if(!(f=fopen(av[1],"r") ) )
    {
        fprintf(stderr,"%s cannot open %s\n",av[1]);
        exit(1);
    }
    fscanf(f," %d %d ",&n,&m);
    A=(int **) calloc(n,sizeof(int*));
    for(i=0;i<n;i++)
    {
        A[i]=(int *) calloc(n,sizeof(int));
        for(j=0;j<n;j++) A[i][j]=0;
    }
    for(i=0;i<m;i++)
    {
        fscanf(f," %d %d ",&x,&y);
        A[x][y]=1;
        A[y][x]=1;
    }
    fclose(f);

    if(!(f=fopen(av[2],"r") ) )
    {
        fprintf(stderr,"%s cannot open %s\n",av[2]);
        exit(1);
    }
    fscanf(f," %d ",&x);
    z=x;
    for(i=1;i<n;i++)
    {
        fscanf(f," %d ",&y);
        A[x][y]=0;
        A[y][x]=0;
        x=y;
    }
    y=z;
    A[x][y]=0;
}

```

```

    A[y][x]=0;
    printf(" %d %d\n",n,m-n);
    for(x=0;x<(n-1);x++)
        for(y=x+1;y<n;y++)
            if(A[x][y]) printf(" %d %d\n",x,y);
}

/*-----GraphLister.c-----*/
#include<stdlib.h>
#include<stdio.h>
int G[20];
int G2[10][2];
int G3[10][3];

main(int ac,char *av[])
{
    int i,j,m,n,g,ng,na,nb,nc ;
    char datafname[20];
    char command[20];
    FILE *FI;
    FILE *FO, *Fout;
    setbuf(stdout,0);
    if(ac!=3)
    {
        fprintf(stderr,"Usage %s fgens numGroups \n",av[0]);
        exit(1);
    }
    if( !(FI=fopen(av[1],"r")) )
    {
        fprintf(stderr,"%s cannot open %s\n", av[0],av[1]);
        exit(1);
    }
    ng = atoi(av[2]);
    if(ng != 1 && ng != 2 && ng!= 3)
    {
        fprintf(stderr,"%s only works with Z_a, Z_a x Z_b, \n");
        fprintf(stderr,"or Z_a x Z_b x Z_c\n", av[0]);
        exit(1);
    }
    if( ng == 1 )           //For Groups Z_n
    {
        fscanf(FI," %d %d %d", &m, &n, &g);
        for(i=0;i<m;i++)
        {
            for(j=0;j<g;j++) fscanf(FI," %d ", &G[j]);
            sprintf(datafname,"D%d",i);
            if( !(FO=fopen(datafname,"w")) )
            {
                fprintf(stderr,"%s cannot open %s\n", av[0],datafname);
                exit(1);
            }
            fprintf(FO," %d %d \n",n,g);
            for(j=0;j<g;j++)
                fprintf(FO,"%d ", G[j]);

```

```

        fprintf(F0, "\n");
        fclose(F0);
        sprintf(command, ". /RUN %s %d", datafname, ng);
        system(command);
    }
} else if( ng == 2 )    // For Groups Z_{na} \times Z_{nb}
{
    fscanf(FI, "%d %d %d %d", &m, &na, &nb, &g);
    for(i=0; i<m; i++)
    {
        for(j=0; j<g; j++)
            fscanf(FI, " %d %d", &G2[j][0], &G2[j][1]);
        sprintf(datafname, "D%d", i);
        if( !(F0=fopen(datafname, "w")) )
        {
            fprintf(stderr, "%s cannot open %s\n", av[0], datafname);
            exit(1);
        }
        fprintf(F0, " %d %d %d \n", na, nb, g);
        for(j=0; j<g; j++)
            fprintf(F0, "%d %d ", G2[j][0], G2[j][1]);
        fprintf(F0, "\n");
        fclose(F0);
        sprintf(command, ". /RUN %s %d", datafname, ng);
        system(command);
    }
} else // For Groups Z_{na} \times Z_{nb} \times Z_{nc}
{
    fscanf(FI, "%d %d %d %d %d", &m, &na, &nb, &nc, &g);
    for(i=0; i<m; i++)
    {
        for(j=0; j<g; j++)
            fscanf(FI, " %d %d %d", &G3[j][0], &G3[j][1], &G3[j][2]);
        sprintf(datafname, "D%d", i);
        if( !(F0=fopen(datafname, "w")) )
        {
            fprintf(stderr, "%s cannot open %s\n", av[0], datafname);
            exit(1);
        }
        fprintf(F0, " %d %d %d %d \n", na, nb, nc, g);
        for(j=0; j<g; j++)
            fprintf(F0, "%d %d %d ", G3[j][0], G3[j][1], G3[j][2]);
        fprintf(F0, "\n");
        fclose(F0);
        sprintf(command, ". /RUN %s %d", datafname, ng);
        system(command);
    }
}
}
}

```

## 2.2.4 Outputting to L<sup>A</sup>T<sub>E</sub>X

```
/*----- Convert1.c -----*/
```

```

#include<stdlib.h>
#include<stdio.h>

FILE * Fin, * Fout, * Fgens ;
int * H, ** K;

/* Takes a list, HCYC, of H-decompositions, a list, fgens,
of Cayley Graphs and prints the LaTeX to fout */

main(int ac, char* av[])
{
    int n,i,v,k,j,g;
    if( ac != 4 )
    {
        fprintf(stderr,"Usage %s HCYC fgens fout\n",av[0]);
        exit(1);
    }
    if( !(Fin=fopen(av[1],"r")) )
    {
        fprintf(stderr,"%s cannot open %s\n", av[0],av[1]);
        exit(1);
    }
    if( !(Fgens=fopen(av[2],"r")) )
    {
        fprintf(stderr,"%s cannot open %s\n", av[0],av[2]);
        exit(1);
    }
    if( !(Fout=fopen(av[3],"w")) )
    {
        fprintf(stderr,"%s cannot open %s\n", av[0],av[3]);
        exit(1);
    }
    fscanf(Fgens, "%d ", &n);
    fscanf(Fgens, "%d ", &v);
    fscanf(Fgens, "%d \n", &g);

    H = (int *)calloc(v,sizeof(int));
    K = (int **)calloc(n,sizeof(int*));
    for(i=0;i<n;i++)
    {
        K[i] = (int*)calloc(g,sizeof(int));
        for(j=0;j<g;j++)
            fscanf(Fgens, "%d ", &K[i][j]);
    }
    for(k=0;k<n;k++)
    {
        fprintf(Fout, "$\\cay(\\zed_{%d},\\{\\%d",v,K[k][0]);
        for(j=1;j<g-1;j++)
            fprintf(Fout, "%d,",K[k][j]);
        fprintf(Fout, "%d\\}) $ \\\\ \\hline\\n",K[k][g-1]);
        for(j=0;j<g;j++)
        {
            fprintf(Fout, "$H_{%d: ",j);
            for(i=0;i<v-1;i++)

```

```

    {
        fscanf(Fin, "%d ", &H[i]);
        fprintf(Fout, "%d,", H[i]);
    }
    fscanf(Fin, "%d ", &H[v-1]);
    fprintf(Fout, "%d ", H[v-1]);
    fprintf(Fout, "$ \\\n");
}
fprintf(Fout, "\\hline \n");
}
free(K);
free(H);
close(Fout);
close(Fin);
close(Fgens);
}

/*----- Convert2.c -----*/
#include<stdlib.h>
#include<stdio.h>

FILE *Fin, *Fgens, *Fout;
int * H, **K;
int n,na,nb,m,s,v;
int S[10][2];

main( int ac, char * av[])
{
    int i,j,g,k;
    int i0,a0,b0;
    int i1,a1,b1;
    char L[4];
    if(ac!=4)
    {
        fprintf(stderr,"Usage: %s HCYC fgens fout\n",av[0]);
        exit(1);
    }
    if( !(Fin=fopen(av[1],"r")) )
    {
        fprintf(stderr,"%s cannot open %s\n",av[0],av[1]);
        exit(1);
    }
    if( !(Fgens=fopen(av[2],"r")) )
    {
        fprintf(stderr,"%s cannot open %s\n",av[0],av[2]);
        exit(1);
    }
    if( !(Fout=fopen(av[3],"w")) )
    {
        fprintf(stderr,"%s cannot open %s\n",av[0],av[3]);
        exit(1);
    }
    fscanf(Fgens,"%d %d %d %d\n",&n,&na,&nb,&g);
    v=na*nb;

```

```

H = (int *)calloc(v,sizeof(int));
K = (int **)calloc(n,sizeof(int*));

for(i=0;i<n;i++)
{
    K[i] = (int*)calloc(2*g,sizeof(int));
    for(j=0;j<2*g;j++)
        fscanf(Fgens, "%d ", &K[i][j]);
}
for(k=0;k<n;k++)
{
    fprintf(Fout, "$\\cay(\\zed_{%d}\\times\\zed_{%d},\\{(%d,%d)",
na,nb,K[k][0],K[k][1]);
    for(j=2;j<2*g-1;j=j+2)
        fprintf(Fout, ",(%d,%d)",K[k][j],K[k][j+1]);
    fprintf(Fout, "\\}) $ \\ \\ \\ \\ \\hline\\n");
    for(j=0;j<g;j++)
    {
        fprintf(Fout, "$H_{%d}: ", j);
        for(i=0;i<v-1;i++)
        {
            fscanf(Fin, " %d ", &i0);
            a0=i0%na;
            b0=(i0-a0)/na;
            fprintf(Fout, "(%d,%d)", a0, b0);
            if( i == 18 || i == 36)
                fprintf(Fout, "$ \\ \\ \\ \\ \\hspace{.16in}$ ");
        }
        fscanf(Fin, " %d ", &i0);
        a0=i0%na;
        b0=(i0-a0)/na;
        fprintf(Fout, "(%d,%d)$ \\ \\ \\ \\ \\n", a0, b0);
    }
    fprintf(Fout, "\\hline \\n");
}
free(K);
free(H);
close(Fout);
close(Fin);
close(Fgens);
}

/*----- Convert3.c-----*/
#include<stdlib.h>
#include<stdio.h>

FILE *Fin, *Fgens, *Fout;
int * H, **K;
int n,na,nb,nc,m,s,v;
int S[10][10][10];

main( int ac, char * av[])
{
    int i,j,g,k,counter;

```



```

int i0,j0,k0;
int i1,a1,b1;
if(ac!=4)
{
    fprintf(stderr,"Usage: %s HCYC fgens fout\n",av[0]);
    exit(1);
}
if( !(Fin=fopen(av[1],"r")) )
{
    fprintf(stderr,"%s cannot open %s\n", av[0],av[1]);
    exit(1);
}
if( !(Fgens=fopen(av[2],"r")) )
{
    fprintf(stderr,"%s cannot open %s\n", av[0],av[2]);
    exit(1);
}
if( !(Fout=fopen(av[3],"w")) )
{
    fprintf(stderr,"%s cannot open %s\n", av[0],av[3]);
    exit(1);
}
fscanf(Fgens, "%d %d %d %d %d\n",&n,&na,&nb,&nc,&g);
v=na*nb*nc;

counter = 0;
for(i0=0;i0<na;i0++)
    for(j0=0;j0<nb;j0++)
        for(k0=0;k0<nc;k0++)
        {
            S[i0][j0][k0] = counter;
            counter++;
        }

H = (int *)calloc(v,sizeof(int));
K = (int **)calloc(n,sizeof(int*));

for(i=0;i<n;i++)
{
    K[i] = (int*)calloc(3*g,sizeof(int));
    for(j=0;j<3*g;j++)
        fscanf(Fgens, "%d ", &K[i][j]);
}
for(k=0;k<n;k++)
{
    fprintf(Fout,
"$\\cay(\\zed_{%d}\\times\\zed_{%d}\\times\\zed_{%d},"
"\\{(%d,%d,%d),(%d,%d,%d),(%d,%d,%d)\\})$\\\\\\\\\\\\hline\n"
,na,nb,nc,
K[k][0],K[k][1],K[k][2],
K[k][3],K[k][4],K[k][5],
K[k][6],K[k][7],K[k][8]);
    for(j=0;j<g;j++)
    {

```

```

fprintf(Fout, "$H_{%d}: ", j);
for(i=0;i<v;i++)
{
    fscanf(Fin," %d ", &i1 );
    for(i0=0;i0<na;i0++)
        for(j0=0;j0<nb;j0++)
            for(k0=0;k0<nc;k0++)
                if( S[i0][j0][k0] == i1 )
                {
                    if( i != v-1)
                    {
                        fprintf(Fout, "(%d,%d,%d)",i0,j0,k0);
                        break;
                    }else
                        fprintf(Fout, "(%d,%d,%d)$\\\\\\ \n",i0,j0,k0);
                }
                if( i == 14 || i == 28)
                    fprintf(Fout, "\n$\\\\\\\\\\hspace{.16in}$ ");
            }
        }
    fprintf(Fout, "\\hline \n");
}
free(K);
free(H);
close(Fout);
close(Fin);
close(Fgens);
}

```

## 2.3 Shell Scripts

The following shell scripts were used to find random Hamilton cycle decompositions for the graphs obtained using MAGMA in Section 2.1. They were written with Donald L. Kreher.

```

#!/bin/csh
if ( $#argv != 2 ) then
    echo "usage:  RUN GeneratorFile NumberOfGroups "
endif
if ( $2 == 1 ) then
    ./GetGraph $1 > G1
else if ( $2 == 2 ) then
    ./GetGraph2 $1 > G1
else if ( $2 == 3 ) then
    ./GetGraph3 $1 > G1
endif
set h3=0
while ( $h3 != 1 )
set h2=0
while ( $h2 != 1 )
./RHC G1 > H1;
./DelHCyc G1 H1 > G2
set h1='wc -l H1'

```

```

set h1='echo $h1 | sed -e 's/ .*/''
if ( $h1 != 1 ) then
echo "Not connected" >> CycleList
exit
endif
./RHC G2 > H2;
./DelHCyc G2 H2 > G3
set h2='wc -l H2'
set h2='echo $h2 | sed -e 's/ .*/''
end
./RHC G3 > H3 ;
set h3='wc -l H3'
set h3='echo $h3 | sed -e 's/ .*/''
end
cat H1 >> $1
cat H2 >> $1
cat H3 >> $1
cat H1 >> CycleList
cat H2 >> CycleList
cat H3 >> CycleList

echo " " >> CycleList

#!/bin/csh
if ( $#argv != 3 ) then
echo "usage: RUN CycleList SetofGenerators NumberOfGroups"
endif
if ( $3 == 1 ) then
./Convert1 $1 $2 LaTeXOut
else if ( $3 == 2 ) then
./Convert2 $1 $2 LaTeXOut
else if ( $3 == 3 ) then
./Convert3 $1 $2 LaTeXOut
endif

```

## 2.4 Mathematica Code

The following is Mathematica code written (with Matthew Miller and Raymond Molzon) to produce the graphics in Figure 2.1.

```

b:= $\frac{2\pi(n-1)}{n}$ ;
baseT:=RevolutionPlot3D[{2 + Cos[t], Sin[t]}, {t, 0, 2 $\pi$ }, { $\theta$ , 0, 2 $\pi$ },
ColorFunction -> "ArmyColors",
Mesh -> None];
Edges:=RevolutionPlot3D[{2 + Cos[t], Sin[t]}, {t, 0, 2 $\pi$ }, { $\theta$ , 0, b}, PlotStyle -> None,
Mesh -> {Range[0, 2 $\pi$ , 2 $\pi$ /m], Range[0, b, b/(n - 1)]},
MeshStyle -> {Directive[Blue, Thick], Directive[Black, Thick]};
kludge:=RevolutionPlot3D[{3, 0}, {3.01, 0}, {t, 0, 2 $\pi$ }, { $\theta$ , 0, b}, PlotStyle -> None,
MeshStyle -> Directive[Blue, Thick];
Pt[a_]:= {2 + Cos[a], 0, Sin[a]};

```

```

bpts:=Table[Pt[i*2π/m],{i,0,m}];
rotr[X_,i_]:=RotationMatrix[i*b/(n-1),{0,0,1}].X;
pts:=Table[rotr[bpts[[j]],i],{i,0,n-1},{j,1,m}];
Sphr[X_]:=Sphere[X,0.075];
Vertices:=Graphics3D[{Black,Map[Sphr,Flatten[pts,1]]]};
θ[φ_,i_]:= (nr)/m(φ-2π)+(2π(i-1+r))/m
x[φ_,i_]:=Cos[φ](2+Cos[θ[φ,i]])
y[φ_,i_]:=Sin[φ](2+Cos[θ[φ,i]])
z[φ_,i_]:= -Sin[θ[φ,i]]
JumpEdges:=
Table[ParametricPlot3D[{x[p,j],y[p,j],z[p,j]},{p,b,2π},
PlotStyle→Directive[Blue,Thick]],
{j,1,m}]

n=10;

(*Numberofcolumnsinred2-factor*)
m=8; (*Numberofrowsinblue2-factor*)
r=2; (* The Jump Number *)
GCD[r,m]; (*Numberofhorizontalcycles.*)
torus=Show[baseT,Edges,JumpEdges,kludge,Vertices,Axes→None,Boxed→False,
PlotRange→{-1.05,1.05}]

```



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