2011

Fixed block configuration GDDs with block size 6 and (3, r)-regular graphs

Melanie R. Laffin
Michigan Technological University

FIXED BLOCK CONFIGURATION GDDs WITH BLOCK SIZE 6 AND $(3, r)$-REGULAR GRAPHS

By
MELANIE R. LAFFIN

A THESIS
Submitted in partial fulfillment of the requirements
for the degree of
MASTER OF SCIENCE
(Mathematical Sciences)

MICHIGAN TECHNOLOGICAL UNIVERSITY
2011

This thesis, "Fixed block configuration GDDs with block size six and $(3, r)$-regular graphs," is hereby approved in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE in the field of MATHEMATICAL SCIENCES.

Department of Mathematical Sciences

Signatures:

Thesis Advisor ______________________________
Melissa S. Keranen, PhD

Thesis Co-Advisor ______________________________
Sibel Özkan, PhD

Department Chair ______________________________
Mark S. Gockenbach, PhD

Date ______________________________

Contents

List of Figures

List of Tables

Preface

This thesis is divided into two topics, "Group divisible designs with block size six" (Chapter 2) and "$(3, r)$-regular graphs" (Chapter 3). The results presented in Chapter 2 were the result of a collaborative effort with Melissa S. Keranen, and the results in Chapter 3 were also a collaborative effort with Sibel Özkan. Both chapters have separately been submitted for publication.

Acknowledgements

I first deeply thank my advisors, Missy Keranen and Sibel Özkan. I am greatly indebted to them for not only my love of design theory and graph theory, but their infinite patience and kindness. I thank Dr. Nilufer Onder for agreeing to be my external committee member and reading my thesis. I would also like to thank Dinesh Sarvate, Spencer Hurd and Pete Johnson for their useful comments and help throughout working on the projects in my thesis.

Next, I thank my family. I owe the greatest thanks to my parents, Radomir and Liliana, who have always believed that I could do anything I set my mind to. I also need to thank my mother-in-law, Mariza Sardis, who has always treated me as one of her children. I need to thank my siblings, Ksenija, Novak, Marina and Sara, who have provided me with the strength to continue my studies more than they will ever know. Finally, I need to thank my relatives in Serbia: Jelena, Dragan, Žana, Stevan, Biljana, Milica, Jelisaveta, Nemanja and Lazar. They have always made space for me in their home (not a trivial problem) for the past few summers as well as helped me maintain my knowledge of the Serbian language and thus deserve special thanks.

I must also thank several of my friends for their support. First, I thank my fellow grad students, John Asplund, Andrew Healy, James Wright, Andrew Boettcher, Mike Mission, Richard Fears, David Kamin, Jack Kelly, and Erik Westlund. Next I thank my long-distance friends who helped me to escape Houghton winters. For this, I thank Alyssa Baron-Klask, Julie Emerick, Rob Chandler, Shahad Atiya, Christine Dascalu, Omaima Bokhari and Justin Marinelli.

Finally, I place special attention on my husband, Bill Laffin. I cannot even begin to enumerate what I owe him for, so I simply will just say I love you more than anything. ,#1

Abstract

Chapter 1 is used to introduce the basic tools and mechanics used within this thesis. Most of the definitions used in the thesis will be defined, and we provide a basic survey of topics in graph theory and design theory pertinent to the topics studied in this thesis.

In Chapter 2, we are concerned with the study of fixed block configuration group divisible designs, GDD$(n, m, k; \lambda_1, \lambda_2)$. We study those GDDs in which each block has configuration (s, t), that is, GDDs in which each block has exactly s points from one of the two groups and t points from the other. Chapter 2 begins with an overview of previous results and constructions for small group size and block sizes 3,4 and 5. Chapter 2 is largely devoted to presenting constructions and results about GDDs with two groups and block size 6. We show the necessary conditions are sufficient for the existence of $GDD(n, 2, 6; \lambda_1, \lambda_2)$ with fixed block configuration $(3, 3)$. For configuration $(1, 5)$, we give minimal or near-minimal index constructions for all group sizes $n \geq 5$ except $n = 10, 15, 160$, or 190. For configuration $(2, 4)$, we provide constructions for several families of $GDD(n, 2, 6; \lambda_1, \lambda_2)$s.

Chapter 3 addresses characterizing $(3, r)$-regular graphs. We begin with providing previous results on the well studied class of $(2, r)$-regular graphs and some results on the structure of large (t, r)-regular graphs. In Chapter 3, we completely characterize all $(3, 1)$-regular and $(3, 2)$-regular graphs, as well has sharpen existing bounds on the order of large $(3, r)$-regular graphs of a certain form for $r \geq 3$.

Finally, the appendix gives computational data resulting from Sage and C programs used to generate $(3, 3)$-regular graphs on less than 10 vertices.

Chapter 1

Introduction

We begin by providing the necessary background knowledge in the fields of graph theory and design theory. We borrow much of the notation that is standard in graph theory and design theory and obtain the definitions from [5, 7, 21, 23].

1.1 Graph Theory

A graph G is an ordered pair $(V(G), E(G))$ or (V, E), where $V(G)$ is a finite set called the vertex set (or points or nodes) and $E(G)$ is the edge set which is comprised of 2 element subsets of $V(G)$. If $\{x, y\} \in E(G)$ we write xy for short. If there are no repeated sets of edges or pairs of nondistinct vertices we call G simple, else G is called a multigraph. From now on, we assume all graphs are simple.

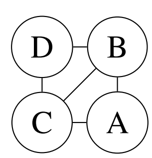


Figure 1.1: An example of a graph, $G = (V, E)$

Figure 1.1 is a graph G with $V = \{A, B, C, D\}$ and $E = \{AB, AC, BC, BD, CD\}$. The cardinality of the set of vertices is called the order of the graph, while the cardinality of the edge set is called the size of the graph. If two vertices v and u form an edge, we say that they are adjacent.

For $v \in V$, we say the (open) neighborhood of v, $N(v)$ are the set of vertices that form an edge with v. We will denote $N(v_1) \cup N(v_2) \cup \ldots N(v_n)$ as $N(v_1), N(v_2), \ldots, N(v_n)$.

The degree of a vertex $x \in G$, $d_G(v)$ or $d(v)$, is $|N(v)|$. A graph where every vertex has the same degree is called a regular graph, or k-regular indicating that every vertex is of degree k. The maximum degree of the graph is denoted $\Delta(G)$ while the minimum degree is denoted $\delta(G)$. A vertex with degree 1 is called a leaf, and a vertex with degree 0 is often said to be isolated.

We call $H = (V', E')$ a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. H is an induced subgraph of G (or subgraph of G induced by V') if $V' \subseteq V$ and H has all the edges whose end vertices are connected to V'. If a vertex is an endpoint of an edge the vertex is incident to the edge. Figure 1.2 is an induced subgraph of the graph in Figure 1.1 on the vertex set $\{A, B, C\}$.

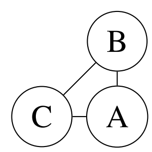


Figure 1.2: An induced subgraph of G in Figure 1.1

The complementary graph or complement of G, $\overline{G} = (V, \overline{E})$, has the same vertex set of G, and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

There are several important and commonly studied classes of graphs. A graph of order n where each pair of vertices forms an edge is called the complete graph and is denoted as K_n. The complete graph has $\binom{n}{2}$ edges and is $(n-1)$-regular. Figure 1.2 is the complete graph on 3 vertices.

If we can partition the vertex set V into two subsets X and Y such that each edge of G has one vertex in X and one vertex in Y we call G bipartite, and the pair X, Y a bipartition of G. A bipartite graph is denoted $K_{n,m}$ where n and m are the cardinalities of the partitions X and Y respectively. A complete bipartite graph is denoted $K_{n,n}$ and has the property that every vertex in X is adjacent to every vertex of Y. The bipartite graph $K_{1,n}$ is sometimes referred to as the star. If G has k-partitions we call the graph multipartite, or more precisely, k-partite.

An independent set of vertices is a subset of vertices so that no pair is adjacent. For example, in the graph found in Figure 1.1, an independent set is $\{A, D\}$. The largest number of vertices in an independent set is called the independence number of a graph G and is denoted by $\alpha(G)$. The determination of the independence number in general is difficult computational problem (NP-hard). A clique in a graph G is a subset U of vertices where each pair is adjacent, i.e. the subgraph induced by U is a complete graph. The largest number of vertices in a clique is called the clique number and is denoted $\omega(G)$. It follows that U is an independent set of G if and only if U is a clique of $\overline{G}$ and that U is a clique of G if and only if U is an independent set of $\overline{G}$. By definition, $\alpha(G) = \omega(\overline{G})$ and $\alpha(\overline{G}) = \omega(G)$. Thus finding a maximum clique is equivalent to finding a maximal independent set, and is also a NP-hard problem.

A walk in a graph G is an alternating sequence of vertices and edges which starts at a vertex and ends at a vertex. If we have the restriction that no two vertices can be repeated in a walk then we call the walk a path. A path on n edges is denoted P_n. A cycle is a closed walk where no two vertices are repeated except for the inital and the final vertex which are the same. We denote cycles with n vertices C_n.

We call a graph connected if for any 2 vertices $u, v \in V$ there is a path from u to v. If G is not connected then we call the connected subgraphs of G components. A tree is a connected graph with no cycles, and a forest is a graph whose components are trees.

There are a few graph operations that we will use extensively in the later chapters. Let G_1 and G_2 be two graphs. Then the graph union $G_1 \cup G_2$ is a graph whose vertex set is the union of the set of vertices in G_1 and G_2 and whose edge set is the union of the set of edges in G_1 and G_2. The join of a graph, $G_1 \vee G_2$, is the disjoint union $G_1 \cup G_2$ plus the edges $\{uv : u \in V(G_1), v \in V(G_2)\}$.

1.2 Design Theory

Another important type of combinatorial configuration is a combinatorial design. A design is a pair (X, A) such that the following properties are satisfied:

1. X is a set of elements called points.
2. A is a multiset of nonempty subsets of X called blocks.

If two blocks in a design are identical, they are said to be repeated blocks. A design is said to be simple if there are no repeated blocks.

1.2.1 Balanced-incomplete block designs

Balanced-incomplete block designs are the most-studied type of design. A balanced-incomplete block design, BIBD(n, k, λ), is a design (X, A) such that the following properties are satisfied:

1. $|X| = n$,

2. each block contains exactly k points, and
3. every pair of distinct points is contained in exactly λ blocks.

A trivial BIBD(n, k, λ) has parameters BIBD$(n, k, \binom{n-t}{k-t})$ and is formed by taking λ copies of every pair of a n-set where $k < n$.

We will usually write the blocks in the form abc instead of $\{a, b, c\}$. We give two examples of BIBDs.The smallest non-trivial BIBD is BIBD$(7, 3, 1)$:

$$X = \{1, 2, 3, 4, 5, 6, 7\}, \text{ and}$$
$$A = \{123, 145, 167, 246, 257, 347, 356\}.$$

A BIBD$(7, 3, 1)$ is also known as the Fano plane. The following is a BIBD$(9, 3, 1)$.

$$X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}, \text{ and}$$
$$A = \{123, 456, 789, 147, 258, 369, 159, 267, 348, 168, 249, 357\}.$$

If $k = 3$ we may call the design a triple system, and abbreviate TS(n, λ).

A set of blocks in a design is called a parallel class if it partitions the point set. A partition of the blocks of a design into parallel classes is a resolution, and such a design is called resolvable. An α-parallel class in a design is a set of blocks which contain every point of the design exactly α times. A design that can be resolved into α-parallel classes is called α-resolvable. We may abbreviate an α-resolvable design as an α-RBIBD(n, k, λ). If $\alpha = 1$ then we abbreviate RBIBD(n, k, λ).

The necessary conditions for the existence of a α-RBIBD(n, k, λ) were given by Jungnickle, Mullin and Vanstone in [20].

Theorem 1.1 ([20])**.** *The necessary conditions for the existence of an α-resolvable BIBD(n, k, λ) are*

1. $\lambda(n-1) \equiv 0 \mod (k-1)\alpha$
2. $\lambda n(n-1) \equiv 0 \mod k(k-1)$
3. $\alpha n \equiv 0 \mod k$

In the same paper, they also showed that these conditions were sufficient when $k = 3$.

Lemma 1.2 ([20])**.** *The necessary conditions for the existence of an α-resolvable BIBD$(n, 3, \lambda)$ are sufficient, except for $n = 6, \alpha = 1$ and $\lambda \equiv 2 \mod 4$.*

Vasiga, Furino and Ling [22] showed that the necessary conditions are sufficient for $k = 4$.

Lemma 1.3. *The necessary conditions for the existence of an α-resolvable BIBD$(n, 4, \lambda)$ are sufficient, with the exception of $(\alpha, n, \lambda) = (2, 10, 2)$.*

By Theorem 1.2, there exists a RBIBD$(9, 3, 1)$. The design was presented above, and we now resolve the design into 4 parallel classes:

$\Pi_1 = \{123, 456, 789\}$,

$\Pi_2 = \{147, 258, 369\}$,

$\Pi_3 = \{159, 267, 348\}$, and

$\Pi_4 = \{168, 249, 357\}$.

A near parallel class is a partial parallel class missing a single point. A near resolvable design, NRB$(n, k, k-1)$, is a BIBD$(n, k, k-1)$ with the property that the blocks can be partitioned into near parallel classes. For such a design, every point is absent from exactly one class. The necessary condition for the existence of an NRB$(n, k, k-1)$ is $n \equiv 1 \mod k$. It is know that the necessary condition is sufficient for the existence of a NRB$(n, k, k-1)$ if $k \leq 7$ (see [7]).

1.2.2 Two-Associate-Class PBIBDs

The definitions and examples for this section were largely taken from [7]. Let X be a set of n points. A partially balanced incomplete block design with m associate classes, PBIBD(m) is a design on X with b blocks of cardinality k with each point appearing in r blocks. Sometimes points are refered to as treatments. Any two points that are ith associates appear together in λ_i blocks of the PBIBD(m). It is custom to abbreviate PBIBD(n, k, λ_i).

PBIBDs were initially used in plant breeding work, in survey sampling, and in group testing. They were introduced as generalizations of BIBDs. We will focus on PBIBD(2), which were classified by Bose and Shimamoto in 1952 in [3] The classified PBIBDs into 6 types based on the association scheme: group divisible, triangular, latin square, cyclic, partial geometry and miscellaneous. We will focus now on group divisible type.

We say a group divisible design GDD$(n, m, k; \lambda_1, \lambda_2)$ is a collection of k element subsets of a v-set **X** called blocks which satisfies the following properties: each point of **X** appears in r of the b blocks; the $v = nm$ elements of **X** are partitioned into m subsets (called groups) of size n each; pairs of points within the same group are called first associates of each other and appear in λ_1 blocks; pairs of points not in the same group are second associates and appear in λ_2 blocks together.

We make the remark that a very well studied class of group divisible designs has $\lambda_1 = 0$ [11], but it is not necessary for either λ values to be 0.

In 1952, Bose and Conner in[2] divided the GDD class of PBIBD(2) into three classes: singular, semiregular and regular. The differences between the three are that a singular GDD has that $r - \lambda_1 > 0$, a GDD is semiregular if $r - \lambda_1 > 0$ and $rk - n\lambda_2 = 0$ and it is regular if $r - \lambda_1 > 0$ and $k - n\lambda_2 > 0$. For all three classes, many GDDs have been studied and put into tables by Clatworthy (see [6]). For more information on PBIBDs, BIBDs and other designs we invite the reader to read the Handbook of Combinatorial Designs ([7]).

Chapter 2

Fixed Block Configuration Group Divisible Designs

In this chapter[1] we study a class of group divisible type PBIBDs with two associate classes. We say that fixed block configuration GDD has the property that every block contains s points from one group and $t = k - s$ points from the other. We focus our study on fixed block configuration GDDs with block size 6 and two groups with all possible configurations.

2.1 History

We begin by providing a recent survey of GDDs with first and second associates. In [9, 10], Fu, Rodger and Sarvate completely settle the existence question for group divisible designs with block size three and first and second associates for $n \geq 2$ and 2 or more groups.

Fixed block configuration GDDs began with Hurd and Sarvate in [16, 17]. In these papers, the necessary conditions for the existence of such designs with block size 4 and configurations $(2, 2)$ and $(1, 3)$ were established, and it was shown that the necessary conditions are sufficient for all $(n, 2, 4; \lambda_1, \lambda_2)$. Henson, Hurd and Sarvate studied the class of GDD$(n, 3, 4; \lambda_1, \lambda_2)$. Here they presented constructions showing that the necessary conditions are sufficient for all GDDs with 3 groups and group sizes 2,3,5 and group size 4 with two exceptions [13]. More recently, Hurd and Sarvate studied $GDD(n, 2, 5; \lambda_1, \lambda_2)$. Once again, they established necessary conditions and showed that they were sufficient in [14, 15].

[1]The results presented in this chapter are a collaborative effort between myself and Melissa Keranen, and has been submitted for publication.

2.2 Necessary Conditions

For GDDs with block size six and two groups there are two necessary conditions on the number of blocks b, and the number of blocks a point appears in r.

Theorem 2.1. *The following conditions are necessary for the existence of a GDD*$(n, 2, 6; \lambda_1, \lambda_2)$.

1. *The number of blocks is* $b = \dfrac{\lambda_1(n)(n-1) + \lambda_2 n^2}{15}$.

2. *The number of blocks a point appears in is* $r = \dfrac{\lambda_1(n-1) + \lambda_2 n}{5}$.

   Proof. For condition (1), we count the total number of blocks, b. Each block has $\binom{6}{2} = 15$ pairs. Thus the total number of blocks must be divisible by 15. Consider a point v. There are exactly $\lambda_1(n-1)$ pairs containing another point from the same group, and $\lambda_2 n$ pairs with a point from the other group. Thus the total number of pairs is $15b = \lambda_1(n)(n-1) + \lambda_2 n^2$ and the total number of blocks is $b = \dfrac{\lambda_1(n)(n-1) + \lambda_2 n^2}{15}$. For condition (2), consider a point v. In any block with v there are 5 pairs containing v and thus the total number of blocks containing v must be divisible by 5. Further v appears in a block λ_1 times with every other point in its same group, which is $n-1$ points, and it appears λ_2 times with every point in the other group (n points in the other group). Thus the total number of blocks that v appears in is $r = \dfrac{\lambda_1(n-1) + \lambda_2 n}{5}$.

   □

These two necessary conditions on b and r determine possibilities for the parameter n and the indices λ_1 and λ_2. Table 2.1 summarizes this relationship.

There are at least two other necessary conditions:

Theorem 2.2. *Suppose a GDD*$(n, 2, 6; \lambda_1, \lambda_2)$ *exists. Then:*

1. $b \geq \max(2r - \lambda_1, 2r - \lambda_2)$

2. $\lambda_2 \leq 2\lambda_1(n-1)/n$

Table 2.1:
Possible values of n with respect to λ_1, λ_2

(mod 15)	$\lambda_1 \equiv 0 \mod 5$	$\lambda_1 \equiv 1 \mod 5$	$\lambda_1 \equiv 2 \mod 5$	$\lambda_1 \equiv 3 \mod 5$	$\lambda_1 \equiv 4 \mod 5$
$\lambda_2 \equiv 0$	Any n	$n \equiv 1 \mod 5$	$n \equiv 1 \mod 5$	$n \equiv 1 \mod 5$	$n \equiv 1 \mod 5$
$\lambda_2 \equiv 1$	impossible	$n \equiv 3, 8 \mod 15$	$n \equiv 9, 14 \mod 15$	$n \equiv 2, 12 \mod 15$	impossible
$\lambda_2 \equiv 2$	impossible	$n \equiv 12 \mod 15$	$n \equiv 3, 8 \mod 15$	impossible	$n \equiv 9 \mod 15$
$\lambda_2 \equiv 3$	$n \equiv 0 \mod 5$	$n \equiv 4 \mod 5$	impossible	$n \equiv 3 \mod 5$	$n \equiv 2 \mod 5$
$\lambda_2 \equiv 4$	$n \equiv 0 \mod 15$	impossible	$n \equiv 2, 12 \mod 15$	$n \equiv 9, 14 \mod 15$	$n \equiv 3, 8 \mod 15$
$\lambda_2 \equiv 5$	$n \equiv 0 \mod 3$	$n \equiv 6 \mod 15$	$n \equiv 6, 11 \mod 15$	$n \equiv 6, 11 \mod 15$	$n \equiv 6, 11 \mod 15$
$\lambda_2 \equiv 6$	$n \equiv 0 \mod 5$	$n \equiv 3 \mod 5$	$n \equiv 4 \mod 5$	$n \equiv 2 \mod 5$	impossible
$\lambda_2 \equiv 7$	impossible	$n \equiv 2, 12 \mod 15$	$n \equiv 3, 8 \mod 15$	impossible	$n \equiv 9, 14 \mod 15$
$\lambda_2 \equiv 8$	impossible	$n \equiv 4, 9 \mod 15$	impossible	$n \equiv 3, 8 \mod 15$	$n \equiv 2, 12 \mod 15$
$\lambda_2 \equiv 9$	$n \equiv 0 \mod 5$	impossible	$n \equiv 2 \mod 5$	$n \equiv 4 \mod 5$	$n \equiv 3 \mod 5$
$\lambda_2 \equiv 10$	$n \equiv 0 \mod 3$	$n \equiv 6, 11 \mod 15$	$n \equiv 6, 11 \mod 15$	$n \equiv 6 \mod 15$	$n \equiv 6, 11 \mod 15$
$\lambda_2 \equiv 11$	impossible	$n \equiv 3 \mod 15$	$n \equiv 9, 14 \mod 15$	$n \equiv 2, 12 \mod 15$	impossible
$\lambda_2 \equiv 12$	$n \equiv 0 \mod 5$	$n \equiv 2 \mod 5$	$n \equiv 3, 13 \mod 15$	impossible	$n \equiv 4, 9 \mod 15$
$\lambda_2 \equiv 13$	impossible	$n \equiv 9, 14 \mod 15$	impossible	$n \equiv 3, 8 \mod 15$	$n \equiv 2, 12 \mod 15$
$\lambda_2 \equiv 14$	impossible	impossible	$n \equiv 2, 12 \mod 15$	$n \equiv 9, 14 \mod 15$	$n \equiv 3 \mod 5$

Proof. For part (1), consider the set of blocks containing the points x and y. There are r blocks containing x and $r - \lambda_i$ blocks which contain y but do not contain x. So there are at least $2r - \lambda_i$ blocks. For part (2) let b_6 be the number of blocks with all 6 points from one group, b_5 be the number of blocks with 5 points from 1 group, and the remaining point from the other group, b_4 be the number of blocks with 4 points from 1 group, and the remaining 2 points from the other group, and b_3 be the number of blocks with 3 points from each group. Counting the contribution of these blocks towards the number of pairs of points from the same group in the blocks together gives: $15b_6 + 10b_5 + 7b_4 + 6b_3 = 2\lambda_1\binom{n}{2} = n(n-1)\lambda_1$. Counting the pairs of points from different groups gives $5b_5 + 8b_4 + 9b_3 = n^2\lambda_2$. Thus we have:

$$-15b_6 - 5b_5 + b_4 + 3b_3 = n^2\lambda_2 - n^2\lambda_1 + n\lambda_1 \leq b_4 + 3b_3 \leq 5b =$$
$$n[\lambda_1(n-1) + \lambda_2 n]/3$$
$$\Rightarrow 3n^2\lambda_2 - 3n^2\lambda_1 + 3n\lambda_1 \leq n^2\lambda_2 + n^2\lambda_1 - n\lambda_1$$
$$\Rightarrow 2n^2\lambda_2 \leq 4n^2\lambda_1 - 4n\lambda_1$$
$$\Rightarrow \lambda_2 \leq \frac{2(n-1)\lambda_1}{n}$$
□

Condition (2) shows that while $\lambda_2 \geq \lambda_1$ is possible, we always have $\lambda_2 < 2\lambda_1$. We can apply the theorem to assert the following:

Corollary 2.3. *The family GDD$(n, 2, 6; s, 2st)$ does not exist for any integers $s, t > 0$.*

In [15], Hurd, Mishra and Sarvate proved the following two results for GDDs with fixed block configuration. We repeat their results here.

Theorem 2.4 ([15])**.** *Suppose a GDD$(n, 2, k; \lambda_1, \lambda_2)$ has configuration (s, t). Then the number of blocks with s points (respectively t) from the first group is equal to the number* *of blocks with s points (respectively t) from the second group. Consequently, for any s and t, the number of blocks b is necessarily even.*

Theorem 2.5 ([15]). *For any GDD$(n, 2, k; \lambda_1, \lambda_2)$ with configuration (s, t), the second index is given by* $\lambda_2 = \left(\frac{\lambda_1(n-1)}{n}\right)\left(\frac{k(k-1)-2\beta}{2\beta}\right)$ *where* $\beta = \binom{s}{2} + \binom{t}{2}$.

For the remainder of this paper, we refer to the results presented in this section as the "necessary conditions."

2.3 GDDs with Configuration (3,3)

In this section, we introduce a basic construction for configuration (3,3) GDDs with specific indices and present the minimal indices for any configuration (3,3) GDD$(n, 2, 6;\ \lambda_1, \lambda_2)$. We begin by providing an example of a configuration (3,3) GDD where $\lambda_1 = 4$ and $\lambda_2 = 5$.
Example 1: GDD$(6, 2, 6; 4, 5)$. Let $A = \{0, 1, 2, 3, 4, 5\}$ and $B = \{a, b, c, d, e, f\}$. Then the $b = 20$ blocks are:

$$\begin{gathered}
\{0,1,2,a,b,c\}, \{0,1,2,d,e,f\}, \{0,1,3,a,b,d\}, \{0,1,3,c,e,f\}, \{0,2,4,a,c,e\}, \\
\{0,2,4,b,d,f\}, \{0,3,5,a,d,f\}, \{0,3,5,b,c,e\}, \{0,4,5,a,e,f\}, \{0,4,5,b,c,d\}, \\
\{1,2,5,b,c,f\}, \{1,2,5,a,e,d\}, \{1,3,4,b,d,e\}, \{1,3,4,a,c,e\}, \{1,4,5,b,e,f\}, \\
\{1,4,5,a,c,d\}, \{2,3,4,c,d,e\}, \{2,3,4,a,b,f\}, \{2,3,5,c,d,f\}, \{2,3,5,a,b,e\}
\end{gathered}$$

By applying Theorem 2.5 to configuration (3,3) GDDs, we get the following result.

Corollary 2.6. *For any configuration* $(3, 3)$ *GDD$(n, 2, 6; \lambda_1, \lambda_2)$, we have*

$$\lambda_2 = \frac{3\lambda_1(n-1)}{2n}$$

.

2.3.1 A Basic Construction for Configuration $(3, 3)$

In this section we use triple systems, TS(n, λ) or BIBD$(n, 3, \lambda)$, extensively to construct a family of configuration $(3, 3)$ GDDs.

Theorem 2.7. *If there exists a TS(n, λ) with b blocks and repetition number r, then there exists a configuration $(3, 3)$ GDD$(n, 2, 6; \lambda b, r^2)$. Further if such a GDD exists, then there exists a TS$(n, \lambda b)$.*

Proof. Suppose there exists a TS(n,λ). Consider two copies of this triple system, TS$_1(n,\lambda)$ and TS$_2(n,\lambda)$. Form the complete bipartite graph G with bipartitions G_1 and G_2 where $V(G_1)$ is the set of blocks of TS$_1(n,\lambda)$ and $V(G_2)$ is the set of blocks of TS$_2(n,\lambda)$. The blocks of the desired design are the edge set of G. Consider a pair of first associates. They will appear λ times in TS$_i(n,\lambda), i = 1, 2$. Therefore, in the given construction they will appear together exactly λb times, where b is the number of blocks in a TS(n,λ). Now consider a pair of second associates $\{v_1, v_2\}$ where $v_i \in TS_i(n,\lambda)$. Any point appears exactly r times in a TS(n,λ), thus the pair $\{v_1, v_2\}$ is contained in exactly r^2 blocks of this design.

Now suppose a GDD exists with groups G_1 and G_2. For each block, remove the points contained in G_1, and then remove G_1. What remains is a set of blocks of size 3 on G_2 which have the property that any pair of points occurs in exactly λb blocks. Thus it is a TS$(n,\lambda b)$. □

The construction given in Theorem 2.7 can easily be generalized to any configuration (k,k) GDD. Thus we have the following corollary.

Corollary 2.8. *If there exists a BIBD(n,k,λ) with b blocks and repetition number r then there exists a configuration (k,k) GDD$(n,2,2k;\lambda b,r^2)$.*

2.3.2 Minimal Indices

There exists a TS$(7,1)$, and thus by Theorem 2.7 there exists a GDD$(7,2,6;7,9)$. From Theorem 2.6, $\lambda_2 = \frac{3\lambda_1(6)}{14} = \frac{9\lambda_1}{7}$, so the construction given in Theorem 2.7 gives a design with the minimum possible indices. However, there also exists a TS(9,1) which means that there exists a GDD(9,2,6;12,16) by Theorem 2.7. In this case we have that $\lambda_2 = \frac{3\lambda_1(8)}{18} = \frac{4\lambda_1}{3}$. Here the minimum values for (λ_1,λ_2) are (3,4). So the construction given in Theorem 2.7 does not give a design with the minimum possible indices. In general, Theorem 2.6 says that for any configuration (3,3) GDD, if for some value of n, the minimum possible indices are (λ_1,λ_2), then any other GDD with that configuration will have the indices ($w\lambda_1$,$w\lambda_2$) for some positive integer w. We can find the minimal indices by using Theorem 2.6 and by the equations given in Theorem 2.1. Any configuration (3,3) GDD with indices ($w\lambda_1$,$w\lambda_2$) can be obtained by taking w copies of the blocks in the minimal design. Therefore, we focus on constructing configuration (3,3) GDDs with indices (λ_1, λ_2). We may then say that the necessary conditions are sufficient for the existence of any configuration (3,3) GDD with that n.

Theorem 2.9. *The minimal indices (λ_1,λ_2) for any configuration $(3,3)$ GDD(n,2, 6;λ_1,λ_2) are summarized in Table 2.2.*

Table 2.2:
Summary of Minimal Indices for Configuration $(3,3)$

n	λ_1	λ_2
$n \equiv 0 \mod 6$	$2n/3$	$(n-1)$
$n \equiv 1 \mod 6$	n	$3(n-1)/2$
$n \equiv 2 \mod 6$	$6n$	$9(n-1)$
$n \equiv 3 \mod 6$	$n/3$	$(n-1)/2$
$n \equiv 4 \mod 6$	$2n$	$3(n-1)$
$n \equiv 5 \mod 6$	$3n$	$9(n-1)/2$

Proof. We know that $\lambda_2 = \frac{3\lambda_1(n-1)}{2n}$ from Theorem 2.6. If $n \equiv 0 \mod 3$ and $n \equiv 1 \mod 2$, then $n \equiv 3 \mod 6$. Thus λ_1 is a multiple of $n/3$ and λ_2 is a multiple of $(n-1)/2$. If $n \equiv 0 \mod 3$ and $n \equiv 0 \mod 2$, then $n \equiv 0 \mod 6$, so λ_1 is a multiple of $2n/3$ and λ_2 is a multiple of $(n-1)$. If $n \equiv 1 \mod 3$ and $n \equiv 1 \mod 2$, $n \equiv 1 \mod 6$, implying λ_1 is a multiple of n and λ_2 is a multiple of $3(n-1)/2$. If $n \equiv 1 \mod 3$ and $n \equiv 0 \mod 2$, $n \equiv 4 \mod 6$, and λ_1 is a multiple of $2n$ and λ_2 is a multiple of $3(n-1)$. If $n \equiv 2 \mod 3$ and $n \equiv 1 \mod 2$, then $n \equiv 5 \mod 6$. This implies that λ_1 is a multiple of n and λ_2 is a multiple of $3(n-1)/2$. However, if we take these values to be the minimal indices, these number of blocks given by Theorem 2.1 would not be integer valued. The smallest values for (λ_1, λ_2) that give integer values for b are $(\lambda_1, \lambda_2) = (3n, \frac{9}{2}(n-1))$. Finally consider when $n \equiv 2 \mod 3$ and $n \equiv 0 \mod 2$. Then $n \equiv 2 \mod 6$, which means that λ_1 is a multiple of $2n$ and λ_2 is a multiple of $3(n-1)$. If we take these values to be the minimal indices, these number of blocks given by Theorem 2.1 would not be integer valued so the smallest values for (λ_1, λ_2) that give integer values for b are $(\lambda_1, \lambda_2) = (6n, 9(n-1))$.

$\square$

2.4 Constructing Configuration (3,3) GDDs

In this section, we give a similar construction to the one given in Theorem 2.7 based on α-resolvable triple systems. We then show that this construction produces designs with minimal indices for all configuration (3,3) GDDs with block size 6 and 2 groups.

We use α-resolvable designs to obtain the following result.

Lemma 2.10. *Suppose there exists an α-resolvable TS(n, λ) with s α-parallel classes, where each parallel class contains t blocks. Then there exists a configuration $(3,3)$ GDD$(n, 2, 6; \lambda t, \alpha^2 s)$.*

Proof. For $i = 1, 2$, let D_i be an α-resolvable TS(n, λ). Resolve the blocks of D_i into α-parallel classes $C_1^i, C_2^i, \ldots C_s^i$. Construct a graph G in the following manner. For $j = 1, 2, \ldots, s$, create the complete bipartite graph G_j with bipartitions G_j^1 and G_j^2 where $V(G_j^1)$ are the blocks of C_j^1 and $V(G_j^2)$ are the blocks of C_j^2. Let $G = \bigcup_{j=1}^{s} G_j$. The edge set of G will form the blocks of the desired design.

Consider a pair of first associates. It will appear in exactly λ blocks of D_i. Therefore, in the given construction, it will appear in λt blocks of size 6. Now consider a pair of second associates $\{v_1, v_2\}$ where $v_1 \in D_1$ and $v_2 \in D_2$. Here v_1 will be matched with v_2 exactly α times per α-parallel class, thus $\lambda_2 = \alpha^2 s$. $\square$

We now consider values of $n \mod 6$ and apply Lemma 2.10 in each case to obtain the desired configuration (3,3) GDD with minimal indices (λ_1, λ_2).

Theorem 2.11. *The necessary conditions are sufficient for the existence of a configuration* $(3, 3)$
GDD $(n, 2, 6; \frac{n}{3}, \frac{n-1}{2})$ *when* $n \equiv 3 \mod 6$.

Proof. Let $n \equiv 3 \mod 6$. Then by Lemma 1.2 there exists a 1-resolvable TS$(n, 1)$ with $\frac{n-1}{2}$ parallel classes, each containing $\frac{n}{3}$ blocks. By applying the construction in Lemma 2.10 we obtain a GDD with indices $(\lambda_1, \lambda_2) = (\frac{n}{3}, \frac{n-1}{2})$, which are the minimal indices given in Theorem 2.9. $\square$

Theorem 2.12. *The necessary conditions are sufficient for the existence of*
GDD $(n, 2, 6; n, \frac{3}{2}(n-1))$ *when* $n \equiv 1 \mod 6$ *with configuration* $(3, 3)$.

Proof. Let $n \equiv 1 \mod 6$. By Lemma 1.2 there exists a 3-resolvable TS$(n, 1)$ each containing $\frac{n-1}{6}$ 3-parallel classes with n blocks. If we apply the construction in Lemma 2.10, we obtain a GDD with minimal indices $(\lambda_1, \lambda_2) = (n, \frac{3(n-1)}{2})$. $\square$

Theorem 2.13. *The necessary conditions are sufficient for the existence of*
GDD $(n, 2, 6; 6n, 9(n-1))$ *when* $n \equiv 2 \mod 6$ *with configuration* $(3, 3)$.

Proof. Let $n \equiv 2 \mod 6$. Then by Lemma 1.2 there exists a 3-resolvable TS$(n, 6)$ with $(n-1)$ 3-parallel classes, each containing n blocks. Applying Lemma 2.10 yields a GDD with minimal indices $(\lambda_1, \lambda_2) = (6n, 9(n-1))$. $\square$

Theorem 2.14. *The necessary conditions are sufficient for the existence of*
GDD $(n, 2, 6; 2n, 3(n-1))$ *when* $n \equiv 4 \mod 6$ *with configuration* $(3, 3)$.

Proof. Let $n \equiv 4 \mod 6$. By Lemma 1.2, there exists a 3-resolvable TS$(n, 2)$ with $\frac{n-1}{3}$ 3-parallel classes each containing n blocks. We may apply Lemma 2.10 to obtain a GDD with minimal indices $(\lambda_1, \lambda_2) = (2n, 3(n-1))$. $\square$

Theorem 2.15. *The necessary conditions are sufficient for the existence of GDD* $(n, 2, 6; 3n, \frac{9}{2}(n-1))$ *when* $n \equiv 5 \mod 6$ *with configuration* $(3, 3)$.

Proof. Let $n \equiv 5 \mod 6$. Then by Lemma 1.2 there exists a 3-resolvable TS$(n, 3)$ with $\frac{n-1}{2}$ 3-parallel classes, each containing n blocks. We may apply Lemma 2.10 to obtain a GDD with minimal indices $(\lambda_1, \lambda_2) = (3n, \frac{9(n-1)}{2})$. □

Theorem 2.16. *The necessary conditions are sufficient for the existence of GDD*$(n, 2, 6; \frac{2}{3}n, n-1)$ *for* $n \equiv 0 \mod 6$ *with configuration* $(3, 3)$.

Proof. Let $n \equiv 0 \mod 6$ with $n \geq 12$. Then by Lemma 1.2 there exists a 1-resolvable TS$(n, 2)$ with $n-1$ parallel classes, each containing $\frac{n}{3}$ blocks. If we apply the construction given in Lemma 2.10 we obtain a GDD with minimal indices $(\lambda_1, \lambda_2) = (\frac{2n}{3}, n-1)$. If $n = 6$, we may not use the construction described in Lemma 1.2. However if $n = 6$, the minimal indices $(\lambda_1, \lambda_2) = (4, 5)$ and Example 1 gives a GDD(6,2,6;4,5). □

Since we have given a construction for all possible values of $n \mod 6$, we may give the following result.

Theorem 2.17. *The necessary conditions are sufficient for the existence of all configuration* $(3, 3)$ *GDD*$(n, 2, 6; \lambda_1, \lambda_2)$ *with minimal indices.*

2.5 GDDs with Configuration (2,4)

In this section we present the minimal indices for any configuration (2,4) GDD$(n, 2, 6; \lambda_1, \lambda_2)$. By Theorem 2.5 we have the following relation between λ_1 and λ_2 for any configuration $(2, 4)$ GDD.

Theorem 2.18. *For any configuration* $(2, 4)$ *GDD*$(n, 2, 6; \lambda_1, \lambda_2)$ *we have* $\lambda_2 = \frac{8\lambda_1(n-1)}{7n}$.

For any configuration (2,4) GDD if for some value of n, the minimum possible indices are (λ_1, λ_2), then any other GDD with that configuration will have the indices $(w\lambda_1, w\lambda_2)$ for some positive integer w. We may find the minimum indices by using the equation in Theorem 2.18, the equations in Theorem 2.1, and the condition in Theorem 2.4. As in the case with configuration $(3, 3)$, we focus on constructing GDDs with minimal indices since we may then say the necessary conditions are sufficient for the existence of any configuration (2,4) GDD with that n.

Table 2.3:
Summary of Minimal Indices for Configuration $(2, 4)$

n	λ_1	λ_2
$n \equiv 0, 16, 24, 32, 40, 48 \mod 56$	$7n/8$	$n-1$
$n \equiv 2, 6, 10, 14, 18, 26, 30, 34, 38, 42, 46, 54 \mod 56$	$7n/2$	$4(n-1)$
$n \equiv 4, 12, 20, 28, 44, 52 \mod 56$	$7n/4$	$2(n-1)$
$n \equiv 8 \mod 56$	$n/8$	$(n-1)/7$
$n \equiv 22, 50 \mod 56$	$n/2$	$4(n-1)/7$
$n \equiv 36 \mod 56$	$n/4$	$2(n-1)/7$
$n \equiv 3, 5, 7, 9, 11, 13, 17, 19, 21, 23, 25, 27, 31,$ $33, 35, 37, 39, 41, 45, 47, 49, 51, 53, 55 \mod 56$	$7n$	$8(n-1)$
$n \equiv 1, 15, 29, 43 \mod 56$	n	$8(n-1)/7$

Theorem 2.19. *The minimal indices* (λ_1, λ_2) *for any configuration* $(2, 4)$ *GDD*$(n, 2,6; \lambda_1, \lambda_2)$ *are summarized in Table 2.3.*

Proof. By Theorem 2.18, we know that $\lambda_2 = \frac{8\lambda_1(n-1)}{7n}$. If $n \not\equiv 1 \mod 7$ and n is odd, then this implies that $n \equiv 3, 5, 7, 9, 11, 13 \mod 14$. Thus λ_1 is a multiple of $7n$ and λ_2 is a multiple of $8(n-1)$. If $n \equiv 1 \mod 7$ and n is odd, then $n \equiv 1 \mod 14$. In this case, λ_1 must be a multiple of n and λ_2 a multiple of $(8/7)(n-1)$. If $n \not\equiv 1 \mod 7$ and $n \equiv 0 \mod 8$, we have that $n \equiv 0, 16, 24, 32, 40, 48 \mod 56$, so λ_1 is a multiple of $7n/8$ and λ_2 is a multiple of $n-1$. If $n \not\equiv 1 \mod 7$ and $n \equiv 2 \mod 8$ then $n \equiv 2, 10, 18, 26, 34, 42 \mod 56$ implying λ_1 is a multiple of $7n/2$ and λ_2 is a multiple of $4(n-1)$. If $n \not\equiv 1 \mod 7$ and $n \equiv 4 \mod 8$, $n \equiv 4, 12, 20, 28, 44, 52 \mod 56$. Then λ_1 is a multiple of $7n/4$ and λ_2 is a multiple of $2(n-1)$. If $n \not\equiv 1 \mod 7$ and $n \equiv 6 \mod 8$, $n \equiv 6, 14, 30, 38, 46, 54 \mod 56$, then λ_1 is a multiple of $7n/2$ and λ_2 is a multiple of $4(n-1)$. If $n \equiv 1 \mod 7$ and $n \equiv 0 \mod 8$, we have that $n \equiv 8 \mod 56$. Here, it follows that λ_1 is a multiple of $n/8$ and λ_2 is a multiple of $(n-1)/7$. If $n \equiv 1 \mod 7$ and $n \equiv 2 \mod 8$, we have that $n \equiv 50 \mod 56$. Here, it follows that λ_1 is a multiple of $n/2$ and λ_2 is a multiple of $4(n-1)/7$. If $n \equiv 1 \mod 7$ and $n \equiv 4 \mod 8$, we have that $n \equiv 36 \mod 56$. Here, it follows that λ_1 is a multiple of $n/4$ and λ_2 is a multiple of $2(n-1)/7$. If $n \equiv 1 \mod 7$ and $n \equiv 6 \mod 8$, $n \equiv 22 \mod 56$, and it follows λ_1 is a multiple of $n/2$ and λ_2 is a multiple of $4(n-1)/7$. □

2.6 Constructing $(2, 4)$ GDD$(n, 2, 6; \lambda_1, \lambda_2)$

We use the Theorem 2.19 and Lemma 1.3 to construct configuration (2,4) GDDs with minimal indices, when possible. We begin with a general construction.

Lemma 2.20. *If there exists an α-resolvable BIBD$(n, 4, \lambda)$ with n even and $\lambda = 3\alpha$, then there exists a configuration $(2, 4)$ GDD$(n, 2, 6; \frac{n}{2}(\lambda + \frac{\alpha}{2}), 2\alpha(n-1))$.*

Proof. Let the two groups be $A = \{1, 2, \ldots, n\}$, and $A' = \{1', 2', \ldots, n'\}$. Let D be an α-resolvable BIBD$(n, 4, \lambda)$ on the point set of A. Let F be a 1-factorization of K_n on the point set of A'. Resolve the blocks into α parallel classes. There will be $\lambda(n-1)/3\alpha = (n-1)$ classes with $(n\alpha)/4$ blocks in each class. Construct a graph G in the following manner. For $j = 1, 2, \ldots, (n-1)$, create the complete bipartite graph G_j with bipartitions G_j^1 and G_j^2 where $V(G_j^1)$ are the blocks of an α parallel class and $V(G_j^2)$ are a 1-factor of K_n. If we switch A with A' and repeat the construction, we obtain all desired blocks.

Consider a pair of first associates, $\{x, y\} \in A$. It will appear exactly λ times in D. Therefore in the given construction, it will appear $n\lambda/2$ times when in the first part of the construction. This pair will appear an additional $n\alpha/4$ times when the second part of the construction. Thus $\lambda_1 = \frac{n}{2}(\lambda + \frac{\alpha}{2})$. Now consider a pair of second associates $\{x, y'\}$, where $x \in A$ and $y' \in A'$. Here x will appear with y' exactly $\alpha(n-1)$ in both parts of the construction, so $\lambda_2 = 2\alpha(n-1)$. □

We use the above construction to obtain the following results:

Corollary 2.21. *Let $n \equiv 2, 6, 10, 14, 18, 26, 30, 34, 38, 42, 46, 54 \mod 56$. Then the necessary conditions are sufficient for the existence of a configuration $(2, 4)$ GDD$(n,\ 2, 6; \frac{7n}{2}, 4(n-1))$.*

Proof. Let n be assumed as above. By Lemma 1.3, there exists a 2-resolvable BIBD$(n, 4, 6)$. Apply Lemma 2.20 to obtain a GDD with minimal indices $(\lambda_1, \lambda_2) = (\frac{7n}{2}, 4(n-1))$. □

Corollary 2.22. *Let $n \equiv 4, 12, 20, 28, 44, 52 \mod 56$. Then the necessary conditions are sufficient for the existence of a configuration $(2, 4)$ GDD$(n, 2, 6; \frac{7n}{4}, 2(n-1))$.*

Proof. Let n be assumed as above. By Lemma 1.3, there exists a resolvable BIBD$(n, 4, 3)$. So we may apply Lemma 2.20 to obtain a GDD with minimal indices $(\lambda_1, \lambda_2) = (\frac{7n}{4}, 2(n-1))$. □

We define a near-minimal GDD as a GDD which has indices exactly twice the minimal size.

Corollary 2.23. *If $n \equiv 0, 8 \mod 24$ then there exists a near minimal configuration $(2, 4)$ GDD$(n, 2, 6; \frac{7n}{4}, 2(n-1))$.*

Proof. Let n be assumed as above. By Lemma 1.3, there exists a resolvable BIBD$(n, 4, 3)$. Apply Lemma 2.20 to obtain a near-minimal GDD with indices $(\frac{7n}{4}, 2(n-1))$. $\square$

The above construction gives near-minimal GDDs for $n \equiv 0, 8 \mod 24$. The next theorem shows that for $n = 8$, the minimal indices can not be obtained.

Theorem 2.24. *There does not a exist a configuration* $(2, 4)$ *GDD*$(8, 2, 6; 1, 1)$.

Proof. Assume such a design exists with groups A and B. Then it would have 8 blocks and every point would appear 3 times. Consider a point in the design, x and let its first associates be $\{1, 2, 3, 4, 5, 6, 7\}$. Suppose x appears in 3 blocks which intersect A in 4 points, and $x \in A$ in each of these blocks. Then because there are only 7 other points, there must be a repeated pair in one of these blocks. However, we assumed $\lambda_1 = 1$, so this is not possible. Now suppose x appears in 2 blocks which intersect A in 4 points and $x \in A$ in those blocks. Then x must also appear in a block which intersects B in 2 points and $x \in B$. Let the two partial blocks containing $x \in A$ be $\{x, 1, 2, 3\}$ and $\{x, 4, 5, 6\}$. Without loss, assume the last partial block containing x also contains 1, and $1 \in A$. The part of this block which intersects A may not contain $x, 2, 3$, and we cannot repeat pairs, so 1 must be in a partial block with $\{4, 7\}$. However, there is no additional first associate available to complete this block. Finally assume x appears in one block which intersects A in 4 points and $x \in A$. Without los, we may assume the partial block containing $x \in A$ be $\{x, 1, 2, 3\}$. Then x appears in 2 blocks which intersect B in 2 points, and $x \in B$. One of these blocks must contain the pair $\{x, 1\}$ where $1 \in A$ and the other block must contain the pair $\{x, 2\}$ where $2 \in A$. However, we have no way to cover the pair $\{x, 3\}$ where $x \in A$ and $3 \in B$ or $x \in B$ or $3 \in A$. Thus this design cannot exist. $\square$

We use a slightly different construction for $n \equiv 16 \mod 24$.

Theorem 2.25. *If* $n \equiv 16 \mod 24$ *then the necessary conditions are sufficient for the existence of a configuration* $(2, 4)$ *GDD*$(n, 2, 6; 7n/8, (n-1))$.

Proof. Let $n \equiv 16 \mod 24$, and let $A = \{1, 2, \ldots, n\}$ and $A' = \{1', 2', \ldots, n'\}$ be the point set for the two groups in the desired design. By Theorem 1.3, there exists a RBIBD$(n, 4, 1)$. Let D be such a design with point set A. Resolve the blocks of D into parallel classes, $C_1, \ldots, C_{(n-1)/3}$. There will be $n/4$ blocks in each parallel class. We construct a 1-factorization of K_n on the point set of A'. On each parallel class $C_j, j = 1, 2, \ldots, (n-1)/3$, decompose the blocks of C_j into three 1-factors as follows. For each block $\{a, b, c, d\} \in C_j$ we let $\{\{a', b'\}, \{c', d'\}\} \in F_{j,1}$, $\{\{a', c'\}, \{b', d'\}\} \in F_{j,2}$ and $\{\{a', d'\}, \{b', c'\}\} \in F_{j,3}$.

Now construct a graph G in the following manner. For $j = 1, 2, \ldots, (n-1)/3$, construct the complete bipartite graph $G_{j,1}$ with bipartitions $G^1_{j,1}$ and $G^2_{j,1}$ where $V(G^1_{j,1})$ are, without loss, the first $n/8$ blocks of parallel class C_j and $V(G^2_{j,1})$ are the 1-factor $F_{j,1}$. Also, create the complete bipartite graph $G_{j,2}$ with bipartitions $G^1_{j,2}$ and $G^2_{j,2}$ where $V(G^1_{j,2})$ are the last, without loss, $n/8$ blocks of parallel class C_j and $V(G^2_{j,2})$ are the 1-factor $F_{j,2}$. Construct the complete bipartite graph $G_{j,3}$ with bipartitions $G^1_{j,3}$ and $G^2_{j,3}$ where $V(G^1_{j,3})$ are the first $n/8$ blocks of parallel class C_j and $V(G^2_{j,3})$ are the edges of $F_{j,3}$ which were obtained from the first $n/8$ blocks of C_j. Finally construct the complete bipartite graph $G_{j,4}$ with bipartitions $G^1_{j,4}$ and $G^2_{j,4}$ where $V(G^1_{j,4})$ are the last $n/8$ blocks of parallel class C_j and $V(G^2_{j,4})$ are the edges of $F_{j,3}$ which are obtained from the last $n/8$ blocks of C_j. If we take the union of all these bipartite graphs, then we obtain half of the blocks of size 6 in the GDD.

To obtain the other half, we switch the roles of A and A' in the design and the 1-factorization. We construct a graph H on the vertex set A, A' in a similar manner to G. For $j = 1, 2, \ldots, (n-1)/3$, construct the complete bipartite graph $H_{j,1}$ with bipartitions $H^1_{j,1}$ and $H^2_{j,1}$ where $V(H^1_{j,1})$ are the last $n/8$ blocks of parallel class C_j and $V(H^2_{j,1})$ are the 1-factor $F_{j,1}$. Also, construct the complete bipartite graph $H_{j,2}$ with bipartitions $H^1_{j,2}$ and $H^2_{j,2}$ where $V(H^1_{j,2})$ are the first $n/8$ blocks of parallel class C_j and $V(H^2_{j,2})$ are the 1-factor $F_{j,2}$. Construct the complete bipartite graph $H_{j,3}$ with bipartitions $H^1_{j,3}$ and $H^2_{j,3}$ where $V(H^1_{j,3})$ are the first $n/8$ blocks of parallel class C_j and $V(H^2_{j,3})$ are the edges of $F_{j,3}$ which were obtained from the last $n/8$ blocks of C_j. Finally construct the complete bipartite graph $H_{j,4}$ with bipartitions $H^1_{j,4}$ and $H^2_{j,4}$ where $V(H^1_{j,4})$ are the last $n/8$ blocks of parallel class C_j and $V(H^2_{j,4})$ are the edges of $F_{j,3}$ which are obtained from the first $n/8$ blocks of C_j. If we take the union of all these bipartite graphs, then we obtain the other half of the blocks of size 6 in the GDD.

Consider a pair of first associates. In the first part of the construction, when $\{x, y\} \in A$ appears in the BIBD, it will appear exactly once. Thus in the construction, it will be in a block of size 6 exactly $n/2 + n/4 = 3n/4$ times. In the second part of the construction when $\{x, y\}$ is in the role of a 1-factor, it will appear $n/8$ times. Thus $\lambda_1 = 7n/8$. Now consider a pair of second associates, $\{x, y'\}$ where $x \in A$ and $y' \in A'$. Without loss, we may assume $\{x, y'\} \in C_j$ for some j. In part one of the construction, there are 4 cases to consider. Each point is either in the first $n/8$ blocks of C_j or in the last $n/8$ blocks of C_j. Let $C_{j,1}$ denote the first $n/8$ blocks of C_j and $C_{j,2}$ denote the last $n/8$ blocks of C_j. Suppose $x \in C_{j,1}$ and $y' \in C_{j,1}$. Then in the construction, $\{x, y'\}$ appears twice. If $x \in C_{j,1}$ and $y' \in C_{j,2}$, then $\{x, y'\}$ appears once. If $x \in C_{j,2}$ and $y' \in C_{j,1}$, then $\{x, y'\}$ appears once and if $x \in C_{j,2}$ and $y' \in C_{j,2}$, then $\{x, y'\}$ appears twice. In the second part of the construction when we reverse the roles, if $x \in C_{j,1}$ and $y' \in C_{j,1}$, then $\{x, y'\}$ appears once. If $x \in C_{j,1}$ and $y' \in C_{j,2}$, then $\{x, y'\}$ appears twice. If $x \in C_{j,2}$ and $y' \in C_{j,1}$, then $\{x, y'\}$ appears twice, and if if $x \in C_{j,2}$ and $y' \in C_{j,2}$, then $\{x, y'\}$ appears once. Thus for each parallel class, each pair $\{x, y'\}$ appears a total of 3 times. Thus each pair of second associates will appear a total of $3(n - 1/3) = n - 1$ times in the construction. $\square$

Theorem 2.26. *Let* $n \equiv 3, 5, 7, 9, 11, 13 \mod 14$*. Then the necessary conditions are sufficient for the existence of a configuration* $(2, 4)$ *GDD*$(n, 2, 6; 7n, 8(n-1))$.

Proof. Let the two groups be $A = \{1, 2, \ldots, n\}$ and $A' = \{1', 2', \ldots, n'\}$. By Lemma 1.3, there exists a 4-resolvable BIBD$(n, 4, 6)$. Let D be such a design with point set A. Resolve the blocks of D into 4-parallel classes. There will be $(n-1)/2$ classes with n blocks in each class. Construct a graph G in the following manner. For $j = 1, 2, \ldots, (n-1)/2$ create the complete bipartite graph G_j with bipartitions G_j^1 and G_j^2 where $V(G_j^1)$ are the blocks of a 4-parallel class and $V(G_j^2)$ are the pairs obtained by developing $\{0', j'\} \mod n$. If we switch A with A' and repeat the same construction, we obtain all desired blocks.

Consider a pair of first associates, $\{x, y\} \in A$. It will appear exactly 6 times in D. Therefore, in the given construction, it will appear $6n$ times in the first part of the construction. This pair will appear an additional n times when in the second part. Thus $\lambda_1 = 7n$. Now consider a pair of second associates $\{x, y'\}$ where $x \in A$ and $y' \in A'$. Here x will be matched with y' exactly $4(n-1)$ times, in each part of the construction, and thus $\lambda_2 = 8(n-1)$.

□

If $n \equiv 1, 15, 29, 43 \mod 56$, then the above construction gives a GDD with 7 times the minimal indices. However, there is a construction for a configuration $(2, 4)$ GDD$(15, 2, 6; 15, 16)$.

Theorem 2.27. *The necessary conditions are sufficient for the existence of a configuration* $(2, 4)$ *GDD*$(15, 2, 6; 15, 16)$.

Proof. By Lemma 1.3, there exists a RBIBD$(16, 4, 1)$. It has 5 parallel classes with 4 blocks in each class, for a total of 20 blocks. Let $X = \{\infty, 0, 1, 2, ..., 14\}$ be the points in the RBIBD$(16, 4, 1)$. Because ∞ appears with every other point exactly once, the blocks of the form $\{\infty, x, y, z\}$ form a partition the set $X \backslash \{\infty\}$. Each block is in one of the 5 parallel classes. For each block $\{\infty, x, y, z\}$, form the pairs $\{x, y\}, \{x, z\}, \{y, z\}$. Let the two groups be $A = \{0, 1, ..., 14\}$ and $A' = \{0', 1', ..., 14'\}$. For $j = 1, 2, 3, 4, 5$, create the complete bipartite graph G$_j$ with bipartitions G_{j_1} and G_{j_2} where $V(G_{j_1})$ are the blocks of parallel class j except the block containing ∞, and $V(G_{j_2})$ are the 15 pairs obtained from the blocks containing ∞. This gives us half of the desired blocks. To get the rest of the blocks repeat the construction with $V(G_{j_1})$ as the 15 pairs and $V(G_{j_2})$ as the blocks of PC_j.

Consider a pair of first associates, $\{x, y\}$. If $\{x, y\}$ was in a block with ∞ in the RBIBD, then it appears exactly 0 times in the first part of the construction and 15 times in the second part. If $\{x, y\}$ was not in a block with ∞ in the RBIBD, then it appears exactly 15 times in the first part and 0 times in the second part. Therefore, each pair of first associates appears

$\lambda_1 = 15$ times. Now consider a pair of second associates $\{x, y'\}$. In the first part, x is in 4 of the blocks and y' is in 2 of the blocks, so $\{x, y'\}$ is in 8 blocks. In the second part, x is in 2 blocks and y' is in 4 blocks, so $\{x, y'\}$ is again in 8 blocks. Thus, $\lambda_2 = 16$. □

2.6.1 Summary of Minimality

Table 2.4:
Summary of Constructions and Minimality for Configuration $(2, 4)$

n	λ_1	λ_2	
$n \equiv 0, 16, 24, 32, 40, 48 \mod 56$ and $n \equiv 16 \mod 24$	$7n/8$	$n-1$	minimal
$n \equiv 0, 16, 24, 32, 40, 48 \mod 56$ and $n \equiv 0, 8 \mod 24$	$7n/8$	$n-1$	near minimal
$n \equiv 2, 10, 18, 26, 34, 42, 6, 14, 30, 38, 46, 54 \mod 56$	$7n/2$	$4(n-1)$	minimal
$n \equiv 4, 12, 20, 28, 44, 52 \mod 56$	$7n/4$	$2(n-1)$	minimal
$n \equiv 8 \mod 56$ and $n \equiv 16 \mod 24$	$n/8$	$(n-1)/7$	7 times the minimal
$n \equiv 8 \mod 56$ and $n \equiv 0, 8 \mod 24$	$n/8$	$(n-1)/7$	14 times the minimal
$n \equiv 22, 50 \mod 56$	$n/2$	$4(n-1)/7$	7 times the minimal
$n \equiv 36 \mod 56$	$n/4$	$2(n-1)/7$	7 times the minimal
$n \equiv 3, 5, 7, 9, 11, 13 \mod 14$	$7n$	$8(n-1)$	minimal
$n \equiv 1 \mod 14, n \neq 15$	n	$8(n-1)/7$	7 times the minimal
$n = 15$	15	16	minimal

Table 2.4 summarizes when the previous results show the necessary conditions are sufficient for (2,4) GDDs with minimal indices. Further, the table indicates where the previous results show the necessary conditions are sufficient for configuration $(2, 4)$ GDDs with near minimal, seven times the minimal possible or fourteen times the minimal possible indices.

2.7 GDDs with Configuration (1,5)

In this section we focus on the minimal indices for configuration $(1, 5)$ GDD(n,2, 6;λ_1, λ_2). Hurd and Sarvate gave a construction for configuration $(1, k)$ GDD$(n, 2, k+1; \lambda_1, \lambda_2)$ using a BIBD(n, k, Λ)s [15]. We repeat their result here:

Theorem 2.28. *The existence of a BIBD(n, k, Λ) implies the existence of a configuration $(1, k)$ GDD$(n, 2, k + 1; \lambda_1, \lambda_2)$ with $\lambda_1 = \Lambda n$ and $\lambda_2 = 2\Lambda(n-1)/(k-1)$.*

Further, in [11] Hanani showed the existence of some classses of BIBD$(n, 5, \lambda)$. Using his result and Theorem 2.28 we obtain the following $(1, 5)$ configuration GDD(n, 2, 6;λ_1,λ_2)s summarized in Table 2.5.

Table 2.5:
Existence of BIBD$(n, 5, \lambda)$ and Resulting Configuration $(1, 5)$ GDDs.

BIBD	Existence	Resulting GDD
$(n, 5, 1)$	$n \equiv 1, 5 \mod 20$	$GDD(n, 2, 6; n, (n-1)/2)$
$(n, 5, 2)$	$n \equiv 1, 5 \mod 10, n \neq 15$	$GDD(n, 2, 6; 2n, n-1)$
$(n, 5, 4)$	$n \equiv 0, 1 \mod 10, n \neq 10, 160, 190$	$GDD(n, 2, 6; 4n, 2(n-1))$
$(n, 5, 5)$	$n \equiv 1 \mod 4$	$GDD(n, 2, 6; 5n, 5/(2(n-1)))$
$(n, 5, 10)$	$n \equiv 1 \mod 2$	$GDD(n, 2, 6; 10n, 5(n-1))$
$(n, 5, 20)$	All n	$GDD(n, 2, 6; 20n, 10(n-1))$

However, this construction does not always give optimal values of λ_1 and λ_2. By Theorem 2.5, we have the following relation between λ_1 and λ_2.

Corollary 2.29. *For any configuration* $(1, 5)$ *GDD*$(n, 2, 6;\lambda_1,\lambda_2)$ *we have* $\lambda_2 = \frac{\lambda_1(n-1)}{2n}$.

From Theorem 2.29 we see that for some value of n the minimum possible indices are (λ_1, λ_2). As in the other two configurations, we may find the minimal indices by Theorem 2.29 and Theorem 2.1. Further, any other GDD with configuration $(1, 5)$ will have indices $(w\lambda_1, w\lambda_2)$ for some positive integer w. The minimal indices are summarized in the next theorem.

Theorem 2.30. *The minimal indices* (λ_1, λ_2) *for any configuration* $(1, 5)$ *GDD*$(n, 2, 6;\lambda_1,\lambda_2)$ *summarized in Table 2.6.*

Table 2.6:
Summary of Minimal Indices for Configuration $(1, 5)$

n	λ_1	λ_2
$n \equiv 0, 6, 10, 11, 15, 16 \mod 20$	$2n$	$(n-1)$
$n \equiv 1, 5 \mod 20$	n	$(n-1)/2$
$n \equiv 2, 4, 8, 12, 14, 18 \mod 20$	$10n$	$5(n-1)$
$n \equiv 3, 7, 9, 13, 17, 19 \mod 20$	$5n$	$5(n-1)/2$

Proof. By Theorem 2.29, we have that $\lambda_2 = \dfrac{\lambda_1(n-1)}{2n}$. This implies that if $n \equiv 1 \mod 2$ then λ_1 must be a multiple of n and λ_2 must be a a multiple of $(n-1)/2$. However, if

$n \equiv 11, 15 \mod 20$ then the indices given do not give an even number of blocks which is required by Theorem 2.4. So for $n \equiv 11, 15 \mod 20$, if we take two times the minimum possible indices, the number of blocks will be integer valued implying $(\lambda_1, \lambda_2) = (2n, (n-1))$. Also, using the given indices for $n \equiv 3, 7, 9 \mod 10$ results in a non-integer value for the number of blocks given by Theorem 2.1. Thus we must take 5 times these, so the minimal indices are $(\lambda_1, \lambda_2) = (5n, 5(n-1)/2)$. Finally, if $n \equiv 1, 5 \mod 20$, the necessary conditions in Theorem 2.1 are met.

If $n \equiv 0 \mod 2$, Theorem 2.29 tells us that λ_1 must be a multiple of $2n$ and λ_2 must be a multiple of $n-1$. However if $n \equiv 2, 4, 8 \mod 10$, then these values give a non-integer value for the number of blocks. If we take 5 times these indices then the necessary condition in Theorem 2.1 is satisfied, and so the minimal indices are $(\lambda_1, \lambda_2) = (10n, 5(n-1))$. Notice that for $n \equiv 0, 6 \mod 10$, the given indices are $(\lambda_1, \lambda_2) = (2n, n-1)$ which are the minimum possible. $\square$

2.8 Constructing Configuration (1,5) GDDs

In this section we focus on constructing $(1, 5)$ GDDs with minimal indices. Theorem 2.28 gives us the following results.

Corollary 2.31. *The necessary conditions are sufficient for the existence of a configuration* $(1, 5)$ *GDD*$(n, 2, 6; n, (n-1)/2)$ *for* $n \equiv 1, 5 \mod 20$.

Corollary 2.32. *The necessary conditions are sufficient for the existence of a configuration* $(1, 5)$ *GDD*$(n, 2, 6; 2n, n-1)$ *for* $n \equiv 11, 15 \mod 20, n \neq 15$.

Notice that in the previous two constructions, the design is minimal. We use a resolvable BIBD$(n, 5, 4)$ in the following construction. In [1], it is given that a resolvable BIBD$(n, 5, 4)$ exists for $n \equiv 0 \mod 10$ except for $n = 10, 160, 190$.

Theorem 2.33. *Let* $n \equiv 0 \mod 10, n \neq 10, 160, 190$. *Then the necessary conditions are sufficient for the existence of a configuration* $(1, 5)$ *GDD*$(n, 2, 6; 2n, n-1)$.

Proof. Let $n \equiv 0 \mod 10, n \neq 10, 160, 190$. Assume the two groups are $A = \{1, 2, \ldots, n\}$ and $A' = \{1', 2', \ldots, n'\}$. There exists a RBIBD$(n, 5, 4)$ with $b = n(n-1)/5$ blocks, and each point appearing $r = (n-1)$ times. Let D be such a design on A with parallel classes $C_1, C_2, \ldots, C_{n-1}$. Construct a graph G in the following manner. For $j = 1, 2, \ldots, n-1$, create the bipartite graph G_j with bipartitions G_j^1 and G_j^2 where $V(G_j^1)$ are the blocks of C_j and $V(G_j^2)$ are the points in A'. Each of the first $n/10$ vertices in G_j^1 are adjacent to the vertices in G_j^2 that correspond to the first $n/10$ blocks of C_j. Each of the last $n/10$ vertices in G_j^2 are adjacent to the vertices in G_j^2 that correspond to the last

$n/10$ blocks of C_j. Thus each vertex in G_j^1 has degree $n/2$ and each vertex in G_j^2 has degree $n/10$. This creates half of the desired blocks in the GDD. To obtain the other half, let D be an RBIBD$(n, 5, 4)$ on A' and repeat the construction. This time, each of the first $n/10$ vertices in G_j' will be adjacent to the vertices in G_j^2 that correspond to the last $n/10$ blocks of C_j, and each of the last $n/10$ vertices of G_j^1 will be adjacent to the vertices in G_j^2 that correspond to the 1st $n/10$ blocks of C_j.

In the design, each pair appears four times and will be matched $n/2$ times. Thus $\lambda_1 = 2n$. For λ_2, consider the pair xy'. It appears exactly once per parallel class, either in the first part or the second part. Thus λ_2 is the number of parallel classes or $n - 1$. □

Recall a near parallel class is a partial parallel class missing a single point. A near - resolvable design, NRB$(n, k, k-1)$, is a BIBD$(n, k, k-1)$ with the property that the blocks can be partitioned into near parallel classes. For such a design, every point is absent from exactly one class. We use near resolvable designs in the following construction.

Theorem 2.34. *Let* $n \equiv 6 \mod 10$. *Then the necessary conditions are sufficient for the existence of a configuration* $(1, 5)$ *GDD*$(n, 2, 6; 2n, n-1)$.

Proof. Let $n \equiv 6 \mod 10$, and the two groups have point sets $A = \{1, 2, \ldots, n\}$ and $A' = \{1', 2', \ldots, n'\}$. Since $n \equiv 6 \mod 10$, there exists a NRB$(n, 5, 4)$. It has n near parallel classes with $(n-1)/5$ blocks in them each. Let D be such a design on the point set of A, and resolve the blocks of D into near parallel classes $C_1, C_2, \ldots C_n$ where C_i misses point i. Construct a graph G in the following manner. For $j = 1, 2, \ldots n/2$, create the complete bipartite graph G_j with bipartitions G_j^1 and G_j^2 where $V(G_j^1)$ are the blocks of C_j and $V(G_j^2)$ are the points $\{1', 2', \ldots, n/2'\}$. For $j = n/2+1, \ldots, n$, create the complete bipartite graph G_j with bipartitions G_j^1 and G_j^2 where $V(G_j^2)$ are the points $\{(n/2+1)', (n/2+2)', \ldots, n'\}$. This creates half of the desired blocks. To get the other half, let D be the NRB$(n, 5, 4)$ on A' and repeat the construction with $V(G_j^2)$ being the points $\{1, 2, \ldots, n/2\}$ for $j = n/2+1, \ldots, n$.

Consider a pair of first associates. It will appear $4(n/2) = 2n$ times in a block of size 6. Now consider a pair of second associates where $x \in A$ and $y' \in A'$. If $x \in \{1, 2, \ldots, n/2\}$ and $y \in \{1', 2', \ldots, (n/2)'\}$ then xy will appear $n/2 - 1$ times in the first part of the construction and $n/2$ times in the second. This is the same case if $x \in \{n/2+1, n/2+2, \ldots, n\}$ and $y' \in \{(n/1)', (n/2+2)', \ldots, n'\}$. If $x \in 1, 2, \ldots, n/2$ and $y' \in \{(n/2+1)', (n/2+2)', \ldots, n'\}$, then xy' will appear $n/2$ times in the first part and $n/2 - 1$ times in the second part. It is the same case if $y' \in \{1', 2', \ldots, n'\}$ and $x \in \{n/2+1, n/2+2, \ldots, n\}$. Thus $\lambda_2 = n - 1$. □

Note that we have constructed minimal GDDs for $n \equiv 0, 1, 5, 6 \mod 10$ (for all but a few values). Recall that a near-minimal design is one that has exactly twice the minimal indices. By Theorem 2.28, the necessary conditions are sufficient for the existence of a near minimal GDD$(n, 2, 6; \lambda_1, \lambda_2)$ for $n \equiv 2, 3, 4, 7, 8, 9 \mod 10$. We may construct a minimal GDD$(n, 2, 6; \lambda_1, \lambda_2)$ for $n \equiv 3, 7, 9 \mod 10$ given the existence of a 5-resolvable BIBD$(n, 5, 10)$.

Theorem 2.35. *The existence of a* 5*-resolvable BIBD*$(n, 5, 10)$ *implies the existence of a configuration* $(1, 5)$ *GDD*$(n, 2, 6; 5n, 5(n-1)/2)$ *for* $n \equiv 3, 7, 9 \mod 10$.

Proof. Let $n \equiv 3, 7, 9 \mod 10$ and assume there exists a 5-resolvable BIBD $(n, 5, 10)$. Assume the two groups are $A = \{1, 2, 3, \ldots, n\}$ and $A' = \{1', 2', 3', \ldots, n'\}$ and let D be such a design on point set A. Resolve the blocks of D into 5-parallel classes $C_1, C_2, \ldots, C_{n-1/2}$, each having n blocks. Construct a graph G in the following manner. For $j = 1, 2, \ldots, (n-1)/4$, create the complete bipartite graph G_j with bipartitions G_j^1 and G_j^2 where $V(G_j^1)$ are the blocks of C_j and $V(G_j^2)$ are the odd numbers in A'. For $j = (n-1)/4+1, \ldots, (n-1)/2$, create the complete bipartite graph G_j with bipartitions G_j^1 and G_j^2 where $V(G_j^1)$ are the blocks of C_j and $V(G_j^2)$ are the even numbers in A'. This creates half of the desired blocks. To get the other half, let D be a 5-RBIBD$(n, 5, 10)$ on A' and repeat the construction with $V(G_j^2)$ being the even numbers in A for $j = 1, 2, \ldots, (n-1)/4$ and $V(G_j^2)$ being the odd numbers in A for $j = (n-1)/4 + 1, \ldots, (n-1)/2$.

Consider a pair of first associates. It will appear 10 times in D. Therefore, in the given construction it will appear $5n$ times in a block of size 6. Now consider a pair of second associates $\{x, y'\}$. In each part of the construction, this pair appears $5(n-1)/4$ times, thus it appears a total of $5(n-1)/2$ times. □

2.8.1 Summary of Minimality

Table 2.7:
Summary of Constructions and Minimality for Configuration $(1, 5)$

n	λ_1	λ_2	
$n \equiv 0, 10, 11, 15, 6, 16 \mod 20, n \neq 10, 15, 160, 190$	$2n$	$(n-1)$	minimal
$n \equiv 1, 5 \mod 20$	n	$(n-1)/2$	minimal
$n \equiv 2, 4, 8, 12, 14, 18 \mod 20$	$10n$	$5(n-1)$	near-minimal
$n \equiv 3, 7, 9, 13, 17, 19 \mod 20$	$5n$	$5(n-1)/2$	near-minimal

We conclude this section with a summary of the GDDs we have constructed, and their minimality found in Table 2.7.

2.9 Future Work and Open Problems

We conclude this chapter with a summary of open problems and future work regarding fixed block configuration GDDs. A major open problem would be to find a construction for those fixed block configuration GDDs for which there is no construction, or no construction with the minimal indices. Alternatively, it might be the case that no such design exists, in which case a proof of nonexistence is required. We summarize the cases for which there is no minimal index construction, or no construction at all.

Table 2.8:
Open Cases

Configuration	Values of n	Open Cases
$(2,4)$	$n \equiv 0, 16, 24, 32, 40, 48 \mod 56$ or $n \equiv 0, 8 \mod 24$	Construction with minimal indices
$(2,4)$	$n \equiv 8 \mod 56$ or $n \equiv 0, 8, 16 \mod 24$	Construction with near minimal or minimal indices
$(2,4)$	$n \equiv 22, 36, 50 \mod 56$	Construction with near minimal or minimal indices
$(2,4)$	$n \equiv 1 \mod 14, n > 15$	Construction with near minimal or minimal indices
$(1,5)$	$n = 10, 15, 160, 190$	Find a construction
$(1,5)$	$n \equiv 2, 3, 4, 7, 8, 9 \mod 10$	Construction with minimal indices

Notice that for configuration $(1,5)$ and $n \equiv 3, 7, 9 \mod 10$, we have a construction with minimal indices, but it is dependent on the existence of α-resolvable BIBD$(n, 5, \lambda)$. The necessary and sufficient conditions are known for $k = 3, 4$ but to my knowledge there is no such results for block size five.

Finally another possible research project is to find the necessary and sufficient conditions for the existence of higher block sizes or more groups.

Chapter 3

$(3, r)$-Regular Graphs

Let $G = \{V(G), E(G)\}$ be a simple, finite graph. Recall that the neighborhood set of a vertex $v \in V(G)$ is the subset of vertices of $N(v) \subset V(G)$ where for $u \in N(v)$ $uv \in E(G)$. If $\{u, v\} \notin V(G)$ we say the vertices are independent. A graph is (t, r)-regular if the cardinality of the neighborhood set of every t independent vertices is r. Thus an r-regular graph is $(1, r)$-regular. A graph is $(t, 0)$-regular if and only if $G = \overline{K}_n$. In this chapter[1] we survey previous work done in characterizing the structure of (t, r)-regular graphs and give new results on the characterization of $(3, r)$-regular graphs.

We take a moment to remind the reader of some definitions and notation that will be pertinent to this chapter. More information can be found in Chapter 1. If $\{u, v\} \in E(G)$ we will abbreviate the edge uv. Also, we denote $N(u) \cup N(v) \cup \cdots \cup N(z)$ as $N(u, v, \ldots, z)$. The cardinality of the neighborhood set of a vertex $v \in V$ is called the degree and we abbreviate this as $d(v)$. If we are considering the neighbors of v in a specific subset of vertices S, $S \subseteq V(G)$, then we sometimes write $d_S(v)$. The join of two graphs, $G_1 \vee G_2$ is the disjoint union $G_1 \vee G_2$ plus the edges $\{uv : u \in V(G_1), v \in V(G_2)\}$. Finally we say that a (vertex) induced subgraph, $\langle S \rangle$, is a subset $S \subseteq V(G)$ with edges whose endpoints are both in this subset.

3.1 History

There have been several results about $(2, r)$-regular graphs. Faudree and Knisley showed in [8] that if r, s, p are nonnegative integers and G is a $(2, r)$-regular graph of order n, then if n is sufficiently large G is isomorphic to $K_s \vee mK_p$ where $2(p-1) + s = r$. While the Faudree and Knisley characterized the structure for for $(2, r)$-regular graphs with large n,

[1]The results in this chapter are a collaborative effort between myself and Sibel Özkan. The results will be submitted for publication.

Johnson and Morgan [19] established a bound.

Theorem 3.1 (Johnson-Morgan)**.** *Suppose that $r \geq 1$ and G is a $(2, r)$-regular graph on n vertices. Suppose that $n \geq N(2, r)$, where $N(2, 1) = 4$, $N(2, 2) = 6$, $N(2, 3) = 8$, and $N(2, r) = (r-1)^2 + 2$ for $r \geq 4$. Then $G = K_s \vee mK_p$ for some integers $s \geq 0$, $m \geq 2$, $p \geq 1$ satisfying $n = s + mp$ and $r = s + 2(p-1)$.*

In addition, there is been work in characterizing all $(2, r)$-regular graphs for small values of r. Haynes and Knisley [12] determined all $(2, 1)$-regular graphs and $(2, 2)$-regular graphs, as well as some $(2, 3)$-regular graphs. More recently, Bragan and Dooley [4], found all (2,3)-regular graphs on less than $N(2, 3) = 8$ vertices. There have been results on (t, r)-regular graphs. In [18], Jamison and Johnson showed that for sufficiently large order, the structure of (t, r)-regular graphs with $t \geq 3$ is similar to the structure of the graphs described in Faudree and Knisley's paper. In the same paper, the authors discuss the structure of (t, r)-regular graphs for $t \geq 3$ and $r \geq 1$. In particular, they showed that while Faudree and Knisley's theorem does not hold for $t > 2$, we have that for n sufficiently large, if G is (t, r)-regular then G is 'almost' the join of mK_p with a graph H which is 'almost' a clique for $m \geq t$ and p such that $t(p-1) + n(H) = r$ where $n(H)$ denotes the number of vertices in H. We use the notions of t-kernel and t-shell found in [18]. The *t-kernel* of G, denoted $\text{Ker}_t(G)$, as the set of vertices that do not belong to any set of t independent vertices. We define the *t-shell* of G, denoted $\text{Shell}_t(G)$, as $V(G) \backslash \text{Ker}_t(G)$; that is, the *t-shell* is the set of vertices that are in some set of t independent vertices. Using the notions of t-shell and t-kernel, we may now give the main result of [18]:

Theorem 3.2 (Jamison-Johnson)**.** *Let G be a (t, r)-regular graph with order n. Suppose that $t \geq 3$ and $r \geq 1$. For $n \geq N(t, r)$, $\langle \text{Shell}_t(G) \rangle \simeq mK_p$ for some integers $m \geq t$ and $p \geq 1$ such that $r = t(p-1) + |\text{Ker}_t(G)|$. .Furthermore, the smallest*

$N(t, r)$ for which this holds satisfies $N(t, r) \leq \max[N(2, r) + r + t - 2, tr + 3r + t - 1]$.

The remainder of this chapter is devoted to studying $(3, r)$-regular graphs. In this chapter we characterize $(3, r)$-regular graphs where $r \in \{1, 2, 3\}$, and give an analogue of the results in [19] for $(3, r)$-regularity. We give a tighter bound than the one provided in Theorem 3.2. We now present our main result.

Theorem 3.3. *Suppose that $r \geq 1$, G is a $(3, r)$-regular graph of order n. Suppose that $n \geq N(3, r)$ where $N(3, 1) = 5, N(3, 2) = 7, N(3, 3) = 9, N(3, 4) = 16$, and $n \geq N(3, r) = (r-1)^2 + r + 2$ for $r \geq 5$. Then $\langle \text{Shell}_3(G) \rangle \cong mK_p$ for some integers $m \geq 3$ and $p \geq 1$ such that $r = 3(p-1) + |\text{Ker}_3(G)|$.*

For the remainder of this chapter, we will abbreviate $\text{Shell}_3(G)$ as $Shell(G)$ and $\text{Ker}_3(G)$ as $Ker(G)$. Further, we will refer to graphs where $Shell(G) \cong mK_p$ as canonical. Otherwise, we will say that these graphs are sporadic.

3.2 Preliminary Theory

In this section we introduce some basic theorems about $(3, r)$-regular graphs that are useful in proving theorems in later sections.

We begin by repeating a lemma from Johnson and Morgan from [19].

Lemma 3.4 ([19]). *Pairs of distinct non-adjacent vertices in a graph G have no common neighbors if and only if G is a disjoint union of cliques.*

The previous lemma is useful in proving the following.

Lemma 3.5. *Suppose that if $r \geq 1$, and G is a $(3, r)$-regular graph on n vertices, and $\Delta(G) \leq r$.*

(a) If $\Delta(G) = r$ then $n \leq 2r + 2$ with equality only if $G = 2K_1 \cup K_{r,r}$.

(b) If $\Delta(G) = r/3$ then $G = mK_{\frac{r}{3}+1}$ for some integer $m \geq 3$.

Proof. (a) Let $\Delta(G) = r$ and $v \in V(G)$ have maximum degree r, where $r \neq n - 1$ and let $S = V(G)/N_G[v]$. Since G is $(3, r)$-regular, for each $u \in S$, $|N(u, v)| \leq r$ because u, v are not neighbors. But since $|N(v)| = r$, we have $|N(u, v)| = r$, implying that $N(u) \subseteq N(v)$. Since all neighbors of any vertex in S is outside of S, S is an independent set. Let $z \in N(v)$. For any $\{a, b, c\} \in S$, $|N(a, b, c)| = r$ and $N(a, b, c) \subseteq N(v)$ implies that $N(a, b, c) = N(v)$. So z must be in the neighborhood of every 3 set of vertices in S. If z is adjacent to all the vertices of S, then $|S| \leq d(z) - 1 \leq r - 1$, and $n = 1 + r + |S| \leq 2r$. If z is not adjacent to one vertex of S, then $|S| \leq r$ and $n \leq 2r + 1$ and if z is not adjacent to two vertices in S, then $|S| \leq r + 1$ and $n \leq 2r + 2$.

Suppose $n = 2r+2$. Then $|S| \leq r+1$, which implies that z is not adjacent to at least two points in S. Without loss, we may assume z is not adjacent to $\{a, b\}$. Because z is in the neighborhood of every three set of vertices in S, z must be adjacent to all vertices in $S\backslash\{a, b\}$. Thus z has degree r. Furthermore, z is not adjacent to any of the vertices in $N_G[v]$ besides v. Because z is adjacent to all vertices in $S\backslash\{a, b\}$, $|N(z, a, b)| > r$ unless a and b are isolated points. Since v and z are arbitrary, we have $G = 2K_1 \cup K_{r,r}$.

(b) Let G be a $(3, r)$-regular graph with $\Delta(G) = r/3$. Let $\{u, v, w\}$ be an arbitrary independent set. It must be the case that $d(u) = d(v) = d(w)$ since $\Delta(G) = r/3$ and $|N(u, v, w)| = r$. Further, $N(u) \cap N(v) \cap N(w) = \emptyset$. Then u, v, w must be in different components. Further, since $\{u, v, w\}$ are arbitrary, this implies the components are $(r/3)$-regular. Assume the components are not cliques. Then there is a non-adjacent pair $\{x, y\}$ in some component $G_i, 1 \leq i \leq n$ where n is the number of components. Consider a third vertex from a different component, namely $z \in G_j, j \neq i$. Then, $\{x, y, z\}$ is an independent set and $|N(x, y, z)| = r$ since G is $(3, r)$-regular. Then it is true that $d(x) = r/3 = d(y) = d(z)$, implying that $N(x) \cap N(y) = \emptyset$. By Lemma 3.4, G_i, is a disjoint union of cliques. But G_i is a connected component, so it must be a clique, namely $K_{\frac{r}{3}+1}$ since it is $(r/3)$-regular. Thus $Shell(G) \cong mK_{\frac{r}{3}+1}$ for $m \geq 3$.

Finally, we aim to show that the assumptions imply that $Ker(G) = \emptyset$. Again, let $\{u, v, w\} \in Shell(G)$ be a three vertex independent set. Then without loss $u \in G_1, v \in G_2, w \in G_3$ where $G_1 \cong G_2 \cong G_3 \cong K_{\frac{r}{3}+1}$ for different components G_1, G_2, G_3. Since $Ker(G)$ is contained in the neighborhood of every vertex in $\langle Shell(G) \rangle$, $N(u) = N_{G_1}(u) \cup Ker(G)$, $N(v) = N_{G_1}(v) \cup Ker(G)$ and $N(w) = N_{G_1}(w) \cup Ker(G)$. Notice that

$$\begin{aligned} |N(u, v, w)| &= |N_{G_1}(u) \cup N_{G_2}(v) \cup N_{G_3}(w) \cup Ker(G)| \\ &= r/3 + r/3 + r/3 + |Ker(G)| \\ &= r + |Ker(G)| \end{aligned}$$

The above realization implies that $|Ker(G)| = 0$, and thus $G = Shell(G) \cong mK_{\frac{r}{3}+1}$ for $m \geq 3$.

$\square$

Next we make the seemingly simple but important following two observations. First note that $|Ker(G)| \leq r$, since $Ker(G) \subseteq N(u, v, w)$ for all 3 independent vertices $\{u, v, w\} \in Shell(G)$. Next, for $u \in Shell(G)$, it must be that $d(u) \leq r$, else there would be a 3 vertex independent set with more than r neighbors.

Proposition 3.6. *Let G be a $(3, r)$-regular graph. Let $u \in Shell(G)$, and $|N(u)| = k$. Then $G \backslash N[u]$ is a $(2, r - k)$-regular graph.*

Proof. Let G be a $(3, r)$-regular graph and $u \in Shell(G)$ with $|N(u)| = k$. Any pair of independent vertices $v, w \in G \backslash N[u]$ forms a 3 independent set with u. Thus, $|N(u, v, w)| = r$. When we consider $G \backslash N[u]$, we have removed all k neighbors of u, so it follows that $|N(v, w)| = r - k$ for any pair of independent vertices in $G \backslash N[u]$. So $G \backslash N[u]$ is a $(2, r - k)$-regular graph. $\square$

Corollary 3.7. *Let G be a $(3, r)$-regular graph on $n \geq (r - 1)^2 + r + 3$ vertices. Let $u \in Shell(G)$ have degree k. Then $G \backslash N[u]$ must be of the form $K_s \vee mK_p$ where $s + 2(p - 1) = r - k$.*

Proof. Let G be a $(3, r)$-regular graph on $n \geq (r-1)^2 + r + 3$ vertices. Let $u \in Shell(G)$ have degree k. Consider $S = G \backslash N[u]$. By Proposition 3.6, S is a $(2, r-k)$-regular graph. Notice that $|S| = n - k - 1 \geq (r-1)^2 + (r-k) + 2$. But since $r \geq k$, we have that $|S| \geq (r-k-1)^2 + 2$. By the Johnson-Morgan bound, this implies S is of the form $K_s \vee mK_p$ where $s + 2(p-1) = r - k$. $\square$

3.3 $(3, 1)$-regular graphs

Suppose that $r \geq 1$, G is a $(3, r)$-regular graph of order n and $n \geq N(3, r)$. Then $\langle Shell(G) \rangle \cong mK_p$ for some integers $m \geq 3$ and $p \geq 1$ such that $r = 3(p-1) + |Ker(G)|$. In this section we characterize all $(3, 1)$-regular graphs and show that $N(3, 1) = 5$.

Theorem 3.8. *A graph is $(3, 1)$-regular if and only if $G = K_2 \cup \overline{K}_2$, $G = K_1 \cup P_2$, and $G = \alpha K_1 \cup K_{1,m}$ where $m \geq 3$ and $\alpha = \{0, 1\}$.*

Proof. It is clear that the graphs listed are $(3, 1)$-regular. Assume G is a $(3, 1)$-regular graph, and let $\{u, v, w\}$ be an independent vertex set. Then $|N(u, v, w))| = 1$. Let $x \in N(u, v, w)$. First assume u, v, w are all adjacent to x. If there are no other vertices then $G = K_{1,3}$. If there is another vertex $y \in V(G)$, then $\{u, v, y\}$, $\{u, w, y\}$, and $\{v, w, y\}$ are independent sets and $x \in N(u, v, y), x \in N(u, w, y)$ and $x \in N(v, w, y)$. So, y is adjacent to either x or y is isolated. If y is adjacent to x then $\{u, y\}$, $\{v, y\}$ and $\{w, y\}$ do not form an edge and all vertices are connected to x, so $G = K_{1,4}$. We may continue to add vertices in this manner, so we will get $K_1 \cup K_{m+1}$. If we add an isolated vertex to G, we will have $G' = K_1 \cup K_4$. If we add more vertices to $G' = K_1 \cup K_{1,m+1}$, there may be only 1 more isolated vertex, so we get either $\overline{K}_2 \cup K_{1,m}$ or $K_1 \cup K_{1,1+m}$ for $m \geq 2$.

Now assume without loss that u and v are adjacent to x but w is not. Since G is $(3, 1)$-regular, $|N(u, v, w)| = 1$. Thus w must be an isolated vertex and $N(u) = N(v) = \{x\}$. If there are no other vertices then $G = K_1 \cup P_2$. If there is another vertex $y \in V(G)$, by the same argument as above y is either isolated or adjacent to x. If y is isolated then $\{x, y, w\}$ form an independent set with $x, y,$ and w, so $N(x, y, w) = N(x) = \{u, v\}$, so $|N(x, y, w)| = 2$. which is a contradiction. Thus y cannot be isolated, and has to be adjacent to x. Then by the same argument as above, $G = K_{1,m}$.

Finally without loss, assume u is adjacent to x but v and w are not. Since G is (3,1)-regular, $|N(u, v, w)| = |N(u, w, x)| = 1$. Thus v, w must be isolated vertices and $N(x) = \{u\}$ and $N(u) = \{x\}$. If there are no other vertices in G then $G = K_2 \cup \overline{K}_2$. If there is another vertex $y \in V(G)$, by the same argument as above y is either isolated or adjacent to x. If y is isolated then $|N(y, v, w)| = 0$, which contradicts the (3,1)-regularity of G. Further, if y is not isolated then $|N(y, v, w)| \geq 2$. Thus there are no other vertices and $G = K_2 \cup \overline{K}_2$. $\square$

From the above characterization, the following result is apparent.

Corollary 3.9. *The only sporadic* $(3,1)$*-regular graph is* $K_2 \cup \overline{K}_2$*. Thus* $N(3,1) = 5$*.*

3.4 $(3,2)$-regular graphs

We now characterize some $(3,2)$-regular graphs and show that $N(3,2) = 7$. The following lemma will be useful.

Lemma 3.10. *Let* G *be a* $(3,r)$*-regular graph. Then* $|Ker(G)| \leq r$*.*

Proof. Let G be a $(3,r)$-regular graph. Let $x \in Ker(G)$ be contained in no neighborhood of an independent set $\{u,v,w\}$. Then $\{u,v,w\}$ creats an independent set that contradicts the fact that x is in the kernel. So $Ker(G)$ must be contained in the neighborhood of every independent set of size 3, and since the degree of any vertex in $Shell(G)$ must be less than or equal to r, $Ker(G)$ may not have more than r vertices. □

Theorem 3.11. *If* G *is a* $(3,2)$*-regular graph on* $n \geq 7$ *vertices then* $\langle Shell(G) \rangle \cong mK_1$ *for some integer* $m \geq 3$ *and* $|Ker(G)| = 2$*.*

Proof. Let G be a $(3,2)$-regular graph on $n \geq 7$ vertices, and let $e = uv \in \langle Shell(G) \rangle$. Since $u \in Shell(G)$, $d(u) \leq 2$. First, assume $d(u) = 1$, so v is the only neighbor of u. Now let $S = G \backslash N[u]$, that is, $S = G \backslash \{u,v\}$.

By Proposition 3.6, S is a $(2,1)$-regular graph. Since $|V(G)| = n \leq 7$, $|V(S)| \geq 5$. Then by Theorem 3.1, S is of the form $K_s \vee mK_p$ where $1 = s + 2(p-1)$. The only way this equation holds is when $s = 1$ and $p = 1$. So S is a star, $K_1 \vee mK_1$, where $m \geq 4$. Let y be the center vertex of the star. Since every 3 independent $\{x,w,z\}$ has y as the only neighbor in S, every independent set $\{x,w,z\}$ should also have v as their neighbor in G since G is $(3,2)$-regular. Because $\{u,v\} \in Shell(G)$, $d(v) \leq 2$ and we know $u \in N(v)$ it follows v may only be neighbors with one vertex in $\{x,w,z\}$. Without loss, assume $v \notin N(w)$ and $v \notin N(z)$. Since $n \geq 7$, there is another vertex $t \notin \{u,v,y,w,x,z\}$ and since $t \neq y$, t is independent with $\{w,x,z\}$. And we know that $\{v,y\}$ is the neighborhood of every independent set by the above argument.

So, $N(w,z,t) = \{v,y\}$ but $v \notin N(w)$ and $v \notin N(z)$ by assumption. But then $v \in N(t)$ implying $d(v) \geq 3$. This is not possible since $v \in Shell(G)$.

Now assume $d(u) = 2$ and let y be the other neighbor of u. By Proposition 3.6, $S' = G \backslash N[u]$ is a $(2,0)$-regular graph, containing any isolated vertices. Since $n \geq 7$, $|V(S')| \geq 4$. So for any $\{x,w,z\} \in S'$, $\{v,y\} = N(x,w,z)$ in G. As in the first case, $d(v) \leq 2$ and without loss assume $v \notin N(w)$ and $v \notin N(z)$ and that we can find another t that is not

u, v, y, x, z. Now, we must have $\{v, y\} = N(w, z, t)$ as well. But we have $v \notin N(w)$ and $v \notin N(z)$. So, $v \in N(t)$ implying $d(v) \geq 3$ since $\{u, x, t\} \subseteq N(v)$. This is a contradiction, so $d(u) \neq 2$.

Combining the above two cases, we have $d(u) = 0$ for all $u \in Shell(G)$. So, $\langle Shell(G) \rangle \cong mK_1$ for some m. Now let $\{u, w, x\}$ be a 3 vertex independent set in $Shell(G)$. It follows that

$$\begin{aligned} |N(u, w, x)| &= |N_{Shell(G)}(u, w, x)| + |Ker(G)| \\ 2 &= 0 + |Ker(G)| \\ 2 &= |Ker(G)| \end{aligned}$$

□

Some canonical $(3, 2)$-regular graphs include $G = K_{2,m}, m \geq 3$, $G = K_2 \vee mK_1, m \geq 2$; $G = K_2 \vee mK_1 \cup K_1, m \geq 2$, and the graphs in Figure 3.1

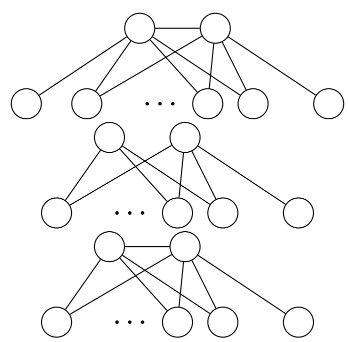


Figure 3.1: Some canonical $(3, 2)$-regular graphs.

In light of the last theorem, we now characterize all low order $(3, 2)$-regular graphs.

Theorem 3.12. *If G is $(3, 2)$-regular on 5 or 6 vertices, then G must be one of the following graphs: $K_{2,3}, K_{2,4}, K_{2,3} - e, K_{2,4} - e$ where e is an edge, $K_2 \vee \overline{K}_3, (K_2 \vee \overline{K}_3) \cup K_1, K_2 \vee \overline{K}_4, (H \vee \overline{K}_3) - e^*, (H \vee \overline{K}_4) - e^*$ where $e^* \not\subseteq E(H)$ and $H = K_2$, $C_4 \cup K_1, C_4 \cup \overline{K}_2, P_4, P_3 \cup K_1, P_2 \cup K_2, (K_4 - e) \cup K_1, K_3 \cup \overline{K}_2, 2K_2 \cup K_1$ or a graph $G_i, 1 \leq i \leq 8$ listed in the proof.*

Proof. The graphs listed are clearly $(3, 2)$-regular. Assume $G = (V, E)$ is a $(3, 2)$-regular graph and let $\{u, v, w\} \in V$ be an independent set of 3 vertices. Thus $|N(u, v, w)| = 2$. Let $\{x, y\} = N(u, v, w)$. We consider the cases depending on the degrees of $d(u), d(v), d(w)$. Note that $d(u) \leq 2, d(v) \leq 2$ and $d(w) \leq 2$. In the following cases, let $z \in V(G)$ and

assume z to be independent of u, v, w. Then $N(z) \subseteq N(u, v, w)$. We consider the possible degrees of u, v, w, whether or not x, y form an edge, and possible degrees of z.
First assume $d(u) = d(v) = d(w) = 2$.

1. First consider the case where x and y do not form an edge. If there are no additional vertices in G, then G must be $K_{2,3}$. Now assume there is another vertex $z \in V(G)$. Then $d(z) \in \{0, 1, 2\}$ and $N(z) \subseteq \{x, y\}$. If $d(z) = 0$, then $\{x, y, z\}$ form an independent set that is collectively adjacent to 3 vertices, implying this graph is not $(3, 2)$-regular. If $d(z) = 1$ then without loss, assume z is adjacent to x. This graph is $K_{2,4} - e$ where e is a single edge. If $d(z) = 2$ then z must be connected to both x and y. Thus $G = K_{2,4}$.

2. Now assume x and y are adjacent. If there are no additional vertices in G, then G must be $K_2 \vee \overline{K}_3$. Consider another vertex z. Then $d(z) \in \{0, 1, 2\}$ and $N(z) \subseteq \{x, y\}$. If $d(z) = 0$ then G is $(K_2 \vee \overline{K}_3) \cup K_1$. If $d(z) = 1$ then without loss let z be adjacent to y and G is $(K_2 \vee \overline{K}_4) - e^*$ where $e^* \in E(G)$ and $e^* \neq xy$. Finally if $d(z) = 2$ then G is $K_2 \vee \overline{K}_4$.

Let $d(u) = d(v) = 2$ and $d(w) = 1$ and without loss let wy be an edge.

1. We first consider the case where x and y are not adjacent. If there are no other vertices G is $K_{2,3} - e$ where e is an edge . Now assume there is another vertex z; $d(z) \in \{0, 1, 2\}$ and $N(z) \subseteq \{x, y\}$. If $d(z) = 0$ then G is $K_{2,3} - e \cup K_1$, which is not $(3, 2)$-regular since $|N(x, y, z)| = 3$. If $d(z) = 1$ then z may be adjacent to x or z may be adjacent to y. If z is adjacent to x then G is the graph (G_1) given in Figure 3.2. If z is adjacent to y then $|N(x, y, z)| = 3$, implying this graph is not

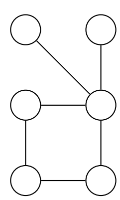


Figure 3.2: $(3, 2)$-regular graph G_1

$(3, 2)$-regular. Finally if $d(z) = 2$, then z is adjacent to both x, y and $G = K_{2,4} - e$.

2. Assume x and y are adjacent. If there are no other vertices in G then $G = (K_2 \vee \overline{K}_3) - e^*$ where $e^* \in E(G)$ and $e^* \neq xy$. Now assume there is another vertex z; $d(z) \in \{0, 1, 2\}$ and $N(z) \subseteq \{x, y\}$. If $d(z) = 0$ then $\{x, z, w\}$ form a 3 vertex independent set and G is not $(3, 2)$-regular. If $d(z) = 1$ then z is adjacent to either x or y. If z is adjacent to x, G is the graph (G_2) illustrated in Figure 3.3.

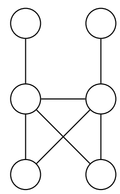


Figure 3.3: $(3, 2)$-regular graph G_2

If z is adjacent to y then $|N(x, w, z)| > 2$ implying that this graph is not $(3, 2)$-regular. If $d(z) = 2$ then z is adjacent to both x and y which is $G = (K_2 \vee \overline{K}_4) - e^*$ where $e^* \in E(G)$ and $e^* \neq xy$.

Assume $d(u) = d(v) = 2$ and $d(w) = 0$.

1. First assume x and y are not adjacent. If there are no other vertices in G, then $G = C_4 \cup K_1$. Now assume there is another vertex $z \in V(G)$. Then $d(z) \in \{0, 1, 2\}$ and $N(z) \subseteq \{x, y\}$. If $d(z) = 0$ then $G = C_4 \cup \overline{K}_2$. If $d(z) = 1$ then without loss assume z is adjacent to x. This graph is not $(3, 2)$-regular since $|N(z, w, y)| = 3$. Finally if $d(z) = 2$, $G = K_{2,3} \cup K_1$, and this is not $(3, 2)$-regular.

2. Now let xy be an edge. If there are no other vertices in G, then $G = (K_4 - e) \cup K_1$. If $d(z) = 0$, then $|N(w, z, y)| = 3$. If $d(z) = 1$ then without loss assume xz is an edge. Then $|N(z, w, y)| = 3$. Finally let $d(z) = 2$. Then G is $(K_2 \vee \overline{K}_3) \cup K_1$.

Assume $d(u) = 2$ and $d(v) = d(w) = 1$. It is possible for v and w to be adjacent to the same vertex or to two different ones, so without loss we will consider cases (a) vx and wx are edges and (b) vx and wy are edges.

1. Assume further that x and y are not adjacent.

   (a) Now without loss, consider the case where vx and wx are edges. Then, if there are no other vertices, G is the graph (G_3) illustrated in Figure 3.4. Now

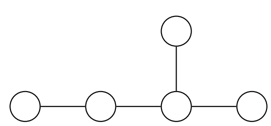


Figure 3.4: $(3, 2)$-regular graph G_3

assume that there is another vertex $z \in V(G)$. If z is an isolated vertex, then

$G = G_3 \cup K_1$ which is not $(3,2)$-regular since $\{v, w, z\}$ are independent vertices and $|N(v, w, z)| = 1$. Now assume that $d(z) = 1$. If xz is an edge, then xz, vx, wx are all adjacent to only x implying this graph is not $(3,2)$-regular. If yz is an edge then G is the graph (G_4) illustrated in Figure 3.5. Finally let

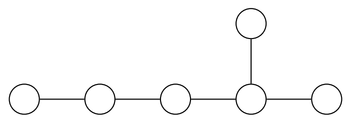


Figure 3.5: $(3,2)$-regular graph G_4

$d(z) = 2$. Then $|N(y, v, w)| = 3$ and this graph is not $(3,2)$-regular.

(b) First, without loss let vx and wy be edges. If there are no other vertices then $G = P_4$. Now assume that there is another vertex $z \in V(G)$. If z is an isolated vertex, then $G = P_4 \cup K_1$, however, this graph is not $(3,2)$-regular. Now assume that $d(z) = 1$, and without loss let xz be an edge. Then $|N(z, u, v)| = 3$ and since $\{u, v, z\}$ is an independent set this graph is not $(3,2)$-regular. Finally, let $d(z) = 2$. Then G is the graph (G_5) illustrated in Figure 3.6.

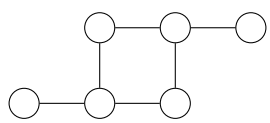


Figure 3.6: $(3,2)$-regular graph G_5

2. Now assume that x and y are adjacent.

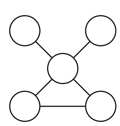


Figure 3.7: $(3,2)$-regular graph G_6

(a) Let vx and wx be edges. If there are no other edges then G is the following graph (G_6) illustrated in Figure 3.7.

Now assume there is another vertex $z \in V(G)$. If z is an isolated vertex, then $\{z, v, w\}$ form an independent set and $|N(v, w, z)| = 1$, so this graph cannot be $(3,2)$-regular. Consider $d(z) = 1$. It may be that z is adjacent to x or y. If xz is an edge, then $\{z, v, w\}$ are an independent set whose neighbourhood has one vertex, and if yz is an edge then $\{v, w, y\}$ are an independent set whose neighborhood set has cardinality three. Thus neither of these graphs are $(3, 2)$-regular. If $d(u) = 2$, then $\{v, w, y\}$ are an independent set and the cardinality of their neighborhood set is 3, so this graph is not $(3, 2)$-regular.

(b) Let vx and wy be edges. If there are no other edges then G is the graph (G_7) given in Figure 3.8. Now assume there is another vertex $z \in V(G)$. If $d(z) = 0$

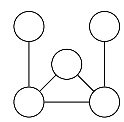


Figure 3.8: $(3, 2)$-regular graph G_7

then $\{y, v, z\}$ form an independent set and $|N(v, y, z)| = 3$, so this graph cannot be $(3, 2)$-regular. Consider $d(z) = 1$. It may be that z is adjacent to x or y. If xz is an edge, then $\{w, x, z\}$ are an independent set whose neighbourhood has three vertices, and if yz is an edge then $\{v, w, y\}$ are an independent set whose neighborhood set has cardinality three. Thus neither of these graphs are $(3, 2)$-regular. Finally assume $d(z) = 2$. Then G is the graph G_2.

Assume $d(u) = 2$, $d(v) = 1$, and $d(w) = 0$. Without loss, assume vx is an edge.

1. First assume that x and y are not adjacent. If there are no other vertices in G, then $G = P_3 \cup K_1$. Assume there is another vertex $z \in V(G)$. If $d(z) = 0$ then $G = P_3 \cup \overline{K}_2$, which is not $(3, 2)$-regular. Let $d(z) = 1$. Then if xz is an edge, then $G = G_3 \cup K_1$ which is not $(3, 2)$-regular. Further, if yz is an edge, then $G = P_4 \cup K_1$ which is also not $(3, 2)$-regular since $\{w, x, y\}$ form an independent set that has 3 vertices in its neighbourhood set. Finally assume $d(z) = 2$. In this case, $\{w, x, y\}$ are an independent set that have three vertices in their neighborhood set, so this graph is not $(3, 2)$-regular.

2. Now assume xy is an edge. If there are no other vertices in G, then it is the graph (G_8) given in Figure 3.9 Now assume there is another vertex $z \in V(G)$. If $d(z) = 0$,

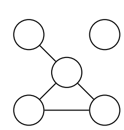


Figure 3.9: $(3, 2)$-regular graph G_8

then $G = G_8 \cup K_1$, but $\{v, w, z\}$ form an independent set with only one vertex in its neighborhood set, so this is not $(3,2)$-regular. Now assume $d(z) = 1$. If xz forms an edge then $G = G_6 \cup K_1$ which is not $(3,2)$-regular since $|N(v,w,z)| = 1$. If yz is an edge then $G = G_7 \cup K_1$ which is not $(3,2)$-regular. Finally if $d(z) = 2$, then $|N(v,w,y)| = 3$, implying this graph is not $(3,2)$-regular.

Assume $d(u) = 2$ and $d(v) = d(w) = 0$.

1. Let x and y be nonadjacent. If there are no additional vertices, then $G = P_2 \cup \overline{K}_2$, which is not $(3,2)$-regular. If $d(z) = 0$, then $G = P_2 \cup \overline{K}_3$ which is obviously not $(3,2)$-regular. If $d(z) = 1$, then without loss assume xz is an edge. Then $G = P_3 \cup \overline{K}_2$ which is not $(3,2)$-regular. Finally if $d(z) = 2$, then $G = C_4 \cup \overline{K}_2$ which is $(3,2)$-regular.

2. Now assume x and y are adjacent. If there are no other vertices, then $G = K_3 \cup \overline{K}_2$. If there is another isolated vertex z, then $\{v,w,z\}$ are three isolated vertices and G is obviously not $(3,2)$-regular. If $d(z) = 1$, then without loss assume xz is an edge. Then $\{v,w,z\}$ again are an independent set but since there is only one vertex in their open neighbourhood this graph cannot be $(3,2)$-regular. Finally if $d(z) = 2$ then $G = (K_2 \vee \overline{K}_2) \cup \overline{K}_2$ which is not $(3,2)$-regular since $|N(v,w,y)| = 3$ and $\{v,w,y\}$ are independent.

Now assume $d(u) = d(v) = d(w) = 1$. Notice that we have two subcases (up to isomorphism) where u,v,w are all connected to either x or y, and where u,v are adjacent to x and w is adjacent to y. However, if u,v,w are all connected to one vertex, then this graph cannot be $(3,2)$-regular, so we consider only the latter case.

1. First assume that xy is not an edge, and there are no other vertices in G. If ux, vx and wy are edges then $G = P_2 \cup K_2$. Now assume there is another vertex, z. First consider $d(z) = 0$. Then $\{u,v,z\}$ are independent and $N(z,u,v) = x$, so this graph is not $(3,2)$-regular. Now assume $d(z) = 1$, and z is adjacent to x. Then we have that $N(u,v,z) = x$ and $N(w) = y$ so this graph is clearly not $(3,2)$-regular. Now consider the case when yz is an edge. Then $N(u,v) = x$ and $N(w,z) = y$. This graph is $2P_2$ which is not $(3,2)$-regular. Finally assume that $d(z) = 2$. This graph has the property that $|N(u,v,y)| = 3$ and thus is not $(3,2)$-regular.

2. Now assume that xy form an edge. If there are no additional vertices then $G = G_3$. Assume there is another vertex $z \in V(G)$. First assume z is an isolated vertex, then G is $G_3 \cup K_1$. However, this graph is not $(3,2)$-regular. If $d(z) = 1$, then either xz or yz can be an edge. If xz is an edge then $N(z,u,v) = \{x\}$, which combined with the fact $\{u,v,z\}$ is an independent set means this graph is not $(3,2)$-regular. If yz is an edge then $|N(y,u,v)| = 3$ which by the same reason as before implies this graph

is not $(3,2)$-regular. If $d(z) = 2$ then $|N(y,u,v)| = 3$ which again implies the graph is not $(3,2)$-regular.

Finally assume $d(u) = d(v) = 1$ and $d(w) = 0$. If u, v were adjacent to the same vertex, the graph would not be $(3,2)$-regular. So, without loss assume ux, vy are edges.

1. Assume x and y are not adjacent. If there are no additional vertices, then $G = 2K_2 \cup K_1$. Now assume there is another vertex $z \in V(G)$. If $d(z) = 0$, then $G = 2K_2 \cup \overline{K}_2$, but $|N(z,w,u)| = 1$, so this graph is not $(3,2)$-regular. Now let $d(z) = 1$, and without loss let zx be an edge. Then $G = P_2 \cup K_2 \cup K_1$ which is not $(3,2)$-regular as $|N(z,u,w)| = 1$. If $d(z) = 2$ and xz, yz are edges, then $G = P_4 \cup K_1$ which is not $(3,2)$-regular.

2. Assume x and y are adjacent. If there are no additional vertices then $G = P_3 \cup K_1$. Assume there is another vertex $z \in V(G)$. If $d(z) = 0$, then $G = P_3 \cup \overline{K}_2$ which is not $(3,2)$-regular. Let $d(z) = 1$ and without loss let xz be an edge. Then $G = G_3 \cup K_1$ which is not $(3,2)$-regular. Finally if $d(z) = 2$, then $G = G_7 \cup K_1$ which is also not $(3,2)$-regular.

$\square$

From the above characterization, we know all $(3,2)$-regular graphs on 5 or 6 vertices. We may list sporadic $(3,2)$-regular graphs by simply determining which have $\langle Shell(G) \rangle \cong mK_1$.

Corollary 3.13. *The only sporadic $(3,2)$-regular graphs are G_1, G_3, G_4 and G_8.*

3.5 A bound on $(3,3)$-regular graphs

In this section we reexamine the Jamison-Johnson bound (Theorem 3.2) for $(3,3)$-regular graphs. By the Jamison-Johnson bound $N(3,3) \leq 12$. We sharpen this bound in the next theorem.

Theorem 3.14. *Let G be a $(3,3)$-regular graph with order n. For $n \geq 9$, $\langle Shell(G) \rangle \cong mK_p$, for $m \geq 3$ and $3 = 3(p-1) + |Ker(G)|$.*

Proof. Let G be a $(3,3)$-regular graph with order $n \geq 9$. First assume $|Ker(G)| = 0$. Then, by assumption $|Shell(G)| \geq 9$ and every vertex is in some 3 vertex independent set since $V(G) = Shell(G)$. We aim to show that $d(u) = 1$ for all $u \in V(G)$.

We first make the observation that $\Delta(G) \leq 3$ since every vertex is in some 3 vertex independent set. Trivially, $\Delta(G)$ cannot be 0 since we assumed our graph to be $(3,3)$-regular. Further, $\Delta(G) \neq 3$ because by Theorem 3.5 this would imply that $n \leq 8$. But we assumed the order of the graph was at least 9, so this may not be the case. Now assume $\Delta(G) = 2$ and let $d(u) = 2$ for some $u \in V(G)$. Suppose $N(u) = \{x, y\}$, and let $G' = V(G) \backslash N[u]$. G' is $(2,1)$-regular by Proposition 3.6. Since in G' has more than 4 vertices by Theorem 3.2 it follows G' is a star with at least 6 leaves. But the center of the star has degree 6, which is a contradiction since we assumed $\Delta(G) = 2$. So $\Delta(G) \neq 2$. Because $\Delta(G)$ is not $0, 2$ or 3, and $\Delta(G) \leq 3$, this means that $\Delta(G) = 1$. By 3.5, this means that G is mK_2. So $G = Shell(G) = mK_2$ for $m \geq 3$.

Now assume that the kernel is not empty. It still is the case that $\Delta(G) \leq 3$. Let $\{u, v, w\} \in Shell(G)$, and without loss let $\{uw, vw\} \in E(G)$. Now consider $a \in Shell(G) \backslash \{u, v, w\}$, and let a have degree k. Next let $G' = V(G) \backslash N[a]$, which by Proposition 3.6 must be a $(2, 3-k)$-regular graph. Since $a \in Shell(G)$, $d(a) \leq 3$. If $d(a) = 0$, then G' is a $(2,3)$-regular graph on at least 8 vertices, if $d(a) = 1$, then G' is a $(2,2)$-regular graph on at least 7 vertices, if $d(a) = 2$ then G' is a $(2,1)$-regular graph on at least 6 vertices and finally if $d(a) = 3$ then G' is a $(2,0)$-regular graph on at least 5 vertices. By the Johnson-Morgan theorem, we have that $N(2,1) = 4, N(2,2) = 6, N(2,3) = 8$ so if $0 \leq d(a) \leq 2$, then G' is a $(2, 3-k)$-regular graph of the form $K_{s'} \vee mK_{p'}, s' + mp' = |G'|, 3 - k = s' + 2(p' - 1), s' \geq 0, m \geq 2, p' \geq 1$. Note that $d(a) \neq 3$ since that would imply G' is a $(2,0)$-regular graph, that is isolated vertices, and we assumed that uw, vw are edges in the the subgraph induced by $Shell(G)$. We make the following observations: since u, v are not adjacent, they cannot be in the same clique, and since w is adjacent to both it must be in the $K_{s'}$ component of G'. However, if w is a vertex in $K_{s'}$, then it must be adjacent to everything in G', and its degree is $|G'| - 1$. In particular, $d(w) \geq 5$. Since we assumed w to be in $Shell(G)$, this is not possible. Therefore, it cannot be that for $u, v, w \in Shell(G)$, $uw, vw \in E(G)$ while uv is not. Thus the subgraph generated by $Shell(G)$ is a disjoint union of cliques mK_p.

Finally consider a set X of three indpendent vertices in $Shell(G)$. They are in 3 different K_p's and thus connected to $3(p-1)$ vertices in $Shell(G)$. Further, every X must be connected to every element of the kernel so $3 = 3(p-1) + |Ker(G)|$. The last equation implies that $p = 1$ and $|Ker(G)| = 3$ or $p = 2$ and $|Ker(G)| = 0$. $\square$

In the appendix, we have included $(3,3)$-regular graphs on 10 or less vertices found by computational search.

3.6 Sharpening the Jamison-Johnson Bound on $(3, r)$-regularity

Theorem 3.15. *Suppose that G is a $(3, r)$-regular graph of order $n \geq 16$ for $r = 4$ and $n \geq (r-1)^2 + r + 2$ for $r \geq 5$. Then $\langle ShellG \rangle \cong mK_p$ for some integers $m \geq 3$ and $p \geq 1$ such that $r = 3(p-1) + |Ker(G)|$.*

Proof. First let G be a $(3, 4)$-regular graph and suppose there are $n \geq 16$ vertices. vertices. Let $u, v, w \in Shell(G)$ such that $vw, uw \in E(G)$ but u and v do not form an edge. Since $u, v, w \in Shell(G)$ and no vertex in $Shell(G)$ has more than degree 4, it must be the case that $|N[u, v, w]| \leq 11$ since each vertex is connected to at most 4 other vertices and vw and uw are edges. We make the following observations using the fact that $|Ker(G)| \leq 4$.

$$\begin{aligned} |Shell(G)| &= n - |Ker(G)| \\ &\geq 16 - |Ker(G)| \\ &\geq 12 \end{aligned}$$

For $r = 4$, $|N[u, v, w]| \leq 11 < 12 \leq |Shell(G)|$. So $|N[u, v, w]| < |Shell(G)|$, meaning there is another vertex, $z \in Shell(G) \backslash N[u, v, w]$. Let z have degree k for some k and let $S = G \backslash N[z]$. Then

$$\begin{aligned} |S| &= n - (k+1) \\ &\geq 15 - k \\ &\geq 11 \end{aligned}$$

By Proposition 3.6 and Proposition 3.7, S is a $(2, 4-k)$-regular graph of the form $K_s \vee mK_p$. Now assume G is a $(3, r)$-regular graph for $r > 4$, we make a similar observation.

$$\begin{aligned} |Shell(G)| &= n - |Ker(G)| \\ &\geq (r-1)^2 + r + 3 - |Ker(G)| \\ &\geq (r-1)^2 + 2 \end{aligned}$$

Then for $r \geq 4$, $|N[u, v, w]| \leq 3r - 1 < (r-1)^2 + 2 \leq |Shell(G)|$. So $|N[u, v, w]| < |Shell(G)|$, meaning there is another vertex, $z' \in Shell(G) \backslash N[u, v, w]$. Let z' have degree k for some k and let $S = G \backslash N[z']$. Then

$$\begin{aligned} |S| &= n - (k+1) \\ &\geq (r-1)^2 + r + 2 - k - 1 \\ &= (r-1)^2 + r - k + 1 \\ &\geq (r-1)^2 + 1 \end{aligned}$$

The last inequality comes from the fact that $k \leq r$. If $r = k$, then S is $(2,0)$-regular implying that it is isomorphic to mK_1. So, without loss, assume that $r - k > 0$. Thus $|S| \geq (r-1)^2 + 2$ and, again, by Proposition 3.6 and Proposition 3.7, S is a $(2, r-k)$-regular graph of the form $K_s \vee mK_p$.

Since u and v are not adjacent, they cannot be in K_s and must be in different copies of K_p. Further, since w is a common neighbor of u and v it must be the case that w is in K_s.

Consider the degree of w, $d(w)$. Since S is of the form $K_s \vee mK_p$, w is connected to every vertex of S, or $|S| = n - (k+1)$. Thus

$$\begin{aligned} d(w) &= |S| - 1 \\ &= n - (k+1) - 1 \\ &= (r-1)^2 + (r-k) + 1 \\ &\geq (r-1)^2 + 1 \end{aligned}$$

Where the last inequality comes from the fact that $r \geq k$. For $r \geq 2$, $(r-1)^2 + 1 > r$. This implies that w is not in any 3 vertex independent set, or that w cannot be in $Shell(G)$ which is a contradiction. So, we cannot have three vertices $\{u, v, w\} \subset Shell(G)$ so that uw and vw are edges and uv are not an edge. Thus the subgraph generated by $Shell(G)$ is a disjoint union of cliques mK_p.

Finally consider a set X of three independent vertices in $Shell(G)$. They need to be in 3 different K_p's and thus connected to $3(p-1)$ vertices in $Shell(G)$. Further every X must be connected to every element of the $Ker(G)$ so $r = 3(p-1) + |Ker(G)|$. □

We conclude by repeating Theorem 3.3 which summarizes the results of this chapter.

Theorem 3.16. *Suppose that $r \geq 1$, G is a $(3,r)$-regular graph of order n. Suppose that $n \geq N(3,r)$ where $N(3,1) = 5, N(3,2) = 7, N(3,3) = 9, N(3,4) = 16$, and $n \geq (r-1)^2 + r + 2$ for $r \geq 5$. Then $\langle Shell(G) \rangle \cong mK_p$ for some integers $m \geq 3$ and $p \geq 1$ such that $r = 3(p-1) + |Ker(G)|$.*

3.7 Open Problems

The open problems relating to (t,r)-regular graphs mostly involve reducing previously known bounds, or characterizing low order graphs of given values of t and r. A fun computational exercise would be to find all $(2,4)$-regular or $(3,4)$-regular graphs, for example. However, the most interesting results would be to sharpen the Jamison-Johnson bound on (t,r)-regular graphs, or even reduce the Johnson-Morgan bound on $(2,r)$-regularity. Also, one could show that these bounds are the lowest possible, although the author suspects this is not the case. With regard to $(3, r)$-regularity, future work lies in sharpening the Jamison-Johnson bound for $n \geq 6$. In particular, it would be nice to prove one of the following conjectures.

1. Let G be $(3, r)$-regular with order n. For $n \geq N(3, r)$, where $r \geq 4$, $\langle Shell(G) \rangle \cong mK_p$ for some integers $m \geq t$ and $p \geq 1$ such that $r = 3(p-1) + |Ker(G)|$. The smallest such $N(3, r) \leq \min[N(2, r) + r + 3, 6r + 2]$.

2. Let G be $(3, r)$-regular with order n and suppose $r \geq 4$. Let $n \geq N(3, r)$ where $N(3, r) = (r-1)^2 + 3$. Then $\langle Shell(G) \rangle \cong K_p$ for some integers $m \geq t$ and $p \geq 1$ such that $r = 3(p-1) + |Ker(G)|$.

For $4 \leq r \leq 6$, Conjecture 1 holds. It remains to be shown that it is true for $r > 6$. Conjecture 2 stems from obtaining values of $(3, r)$-regular graphs for small values of r and comparing it to that of $(2, r)$-regular graphs. Here, the observation was made that the values of $N(2, r)$ and $N(3, r)$ differ by 1 for respective values of r (for $r \leq 3$).

Bibliography

[1] R. J. R. ABEL, G. GE, AND J. YIN, *Resolvable and near-resolvable designs*, in Handbook of Combinatorial Designs, C. Colbourn and J. Dinitz, eds., vol. 2, CRC Press, 2007.

[2] R. BOSE AND W. CONNOR, *Combinatorial properties of group divisible incomplete block designs*, Annals of Mathematical Statistics., 23 (1952), pp. 367–383.

[3] R. BOSE AND T. SHIMAMOTO, *Classification and analysis of partially balanced incomplete block designs with two associate classes*, Journal of the American Statistical Association, 47 (1952), pp. 151–184.

[4] K. BRAGAN AND C. DOOLEY, *The characterization of all $(2,3)$-regular graphs*, Congressus Numeratium, (2010).

[5] R. BRUALDI, *Introductory combinatorics*, Pearson Education, Inc., 4th ed., 2004.

[6] W. CLATWORTHY, *Tables of two-associate-class partially balanced designs*, National Bureau of Standards (U.S.) Applied Mathematics Series, 63 (1973).

[7] C. COLBOURN AND J. DINITZ, eds., *Handbook of Combinatorial Designs*, Chapman & Hall, 2nd ed., 2006.

[8] R. FAUDREE AND D. KNISLEY, *The characterization of large $(2,r)$-regular graphs*, Congressus Numeratium, 121 (1996), pp. 105–108.

[9] H. FU AND C. RODGER, *Group divisible designs with two associate classes: $n = 2$ or $m = 2$*, Combinatorial Theory (A), 83 (1998), pp. 94–117.

[10] H. FU, C. RODGER, AND D. SARVATE, *The existence of group divisible designs with first and second associates*, Ars Combinatoria, 54 (2000), pp. 33–50.

[11] H. HANANI, *Balanced incomplete block designs and related designs*, Discrete Mathematics, 11 (1975), pp. 255–289.

[12] T. HAYNES AND D. KNISLEY, *Generalized maximum degree and totally regular graphs*, Utilitas Mathematica, 54 (1998), pp. 211–221.

[13] D. HENSON, S. HURD, AND D. SARVATE, *Group divisible designs with three groups and block size four*, Discrete Mathematics, 307 (2007), pp. 1693–1706.

[14] S. HURD, N. MISHRA, AND D. SARVATE, *Group divisible designs with two groups and block size five with fixed block configuration*, Journal of Combinatorial Mathematics and Combintorial Computing, 70 (2009), pp. 15–31.

[15] S. HURD AND D. SARVATE, *Group divisible designs with two groups and block configuration* $(1, 4)$, Journal of Combinatorics, Information and System Sciences.

[16] ——, *Odd and even group divisible designs with two groups and block size four*, Discrete Mathematics, 284 (2004), pp. 189–196.

[17] ——, *Group divisible designs with block size four and two groups*, Discrete Mathematics, 308 (2008), pp. 2663–2673.

[18] R. JAMISON AND P. JOHNSON, *The structure of* (t, r)*-regular graphs of large order*, Discrete Mathematics, 272 (2003), pp. 297–300.

[19] P. JOHNSON AND E. MORGAN, *Sharpening the Faudree-Knisley theorem on* $(2, r)$*-regularity*, Bulletin of the Institute of Combinatorics, 39 (2003), pp. 21–26.

[20] D. JUNGNICKEL, R. MULLIN, AND S. VANSTONE, *The spectrum of* α*-resolvable block designs with block size 3*, Discrete Mathematics, 97 (1991), pp. 269–277.

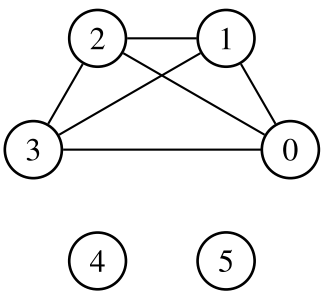
[21] D. STINSON, *Combinatorial Designs: Construction and Analysis*, Springer, 1st ed., 2003.

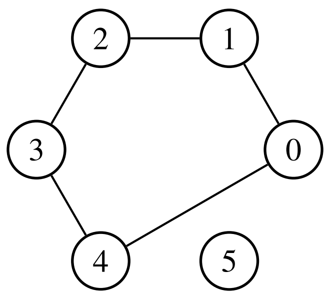
[22] T. VASIGA, S. FURINO, AND A. LING, *The spectrum of* α*-resolvable block designs with block size four*, Journal of Combinatorial Designs, 9 (2001), pp. 1–16.

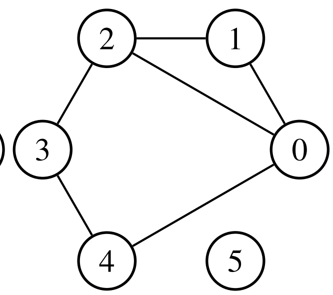
[23] D. B. WEST, *Introduction to Graph Theory*, Prentice Hall, 2nd ed., 2000.

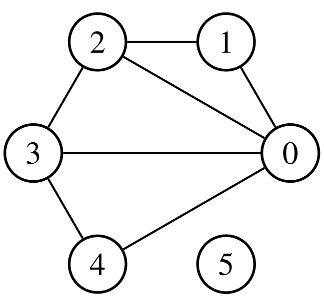
Appendix

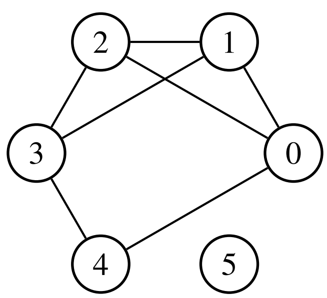
$(3,3)$**-regular graphs on 6 to 10 vertices.** The following graphs are the result of a computer search using Sage and *nauty* for all $(3,3)$-regular graphs on 10 vertices or less.

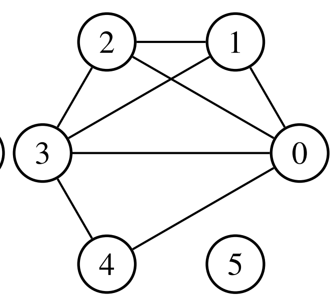


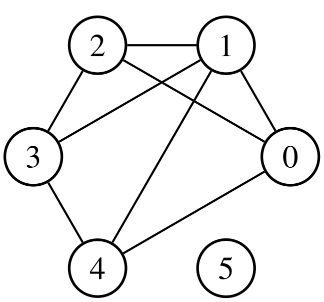


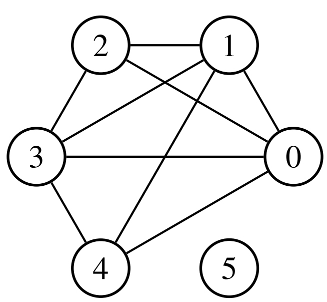


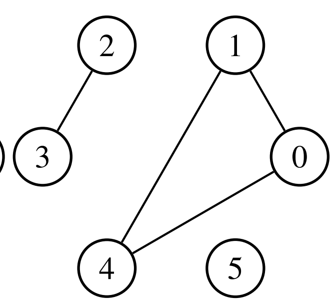


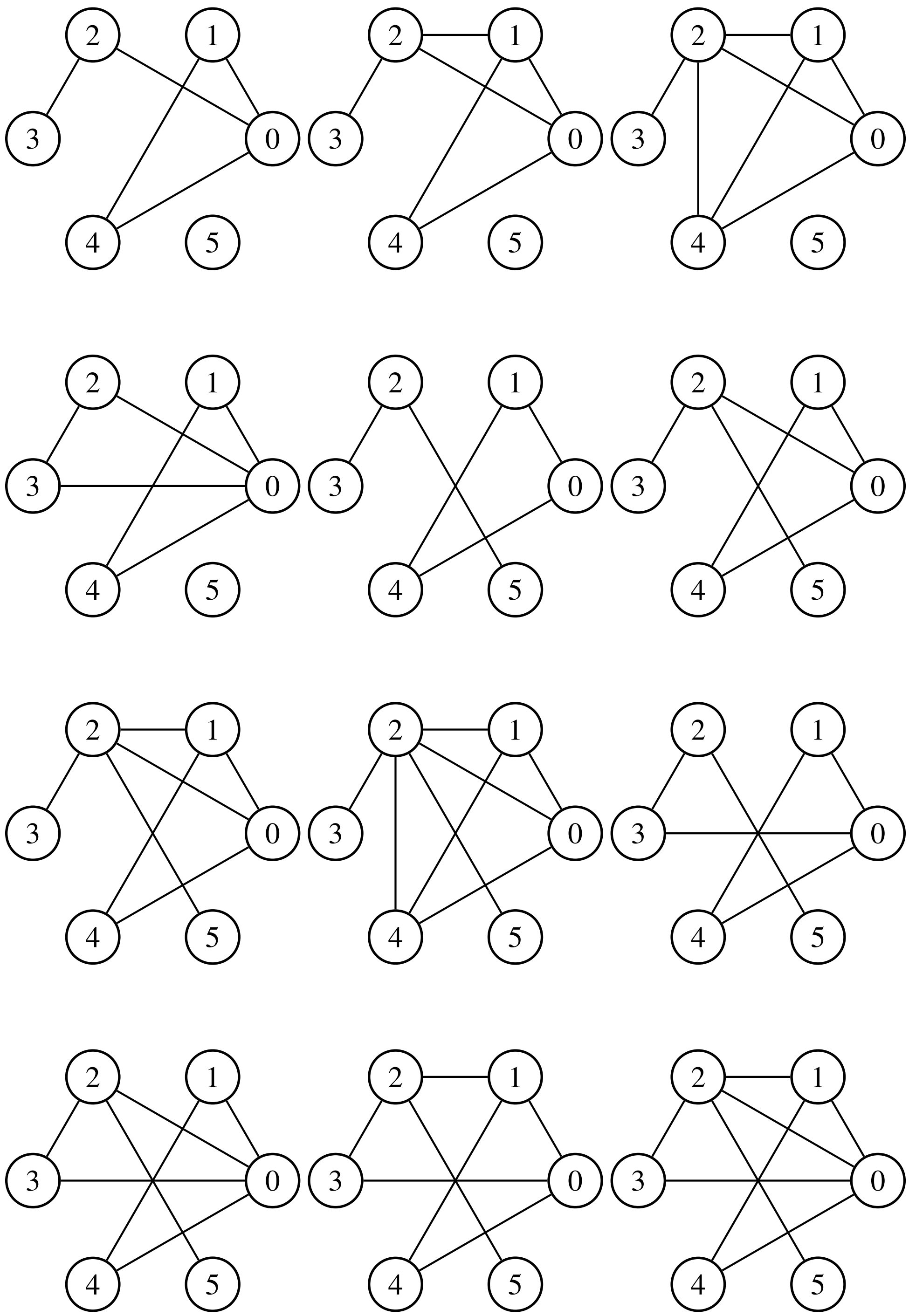


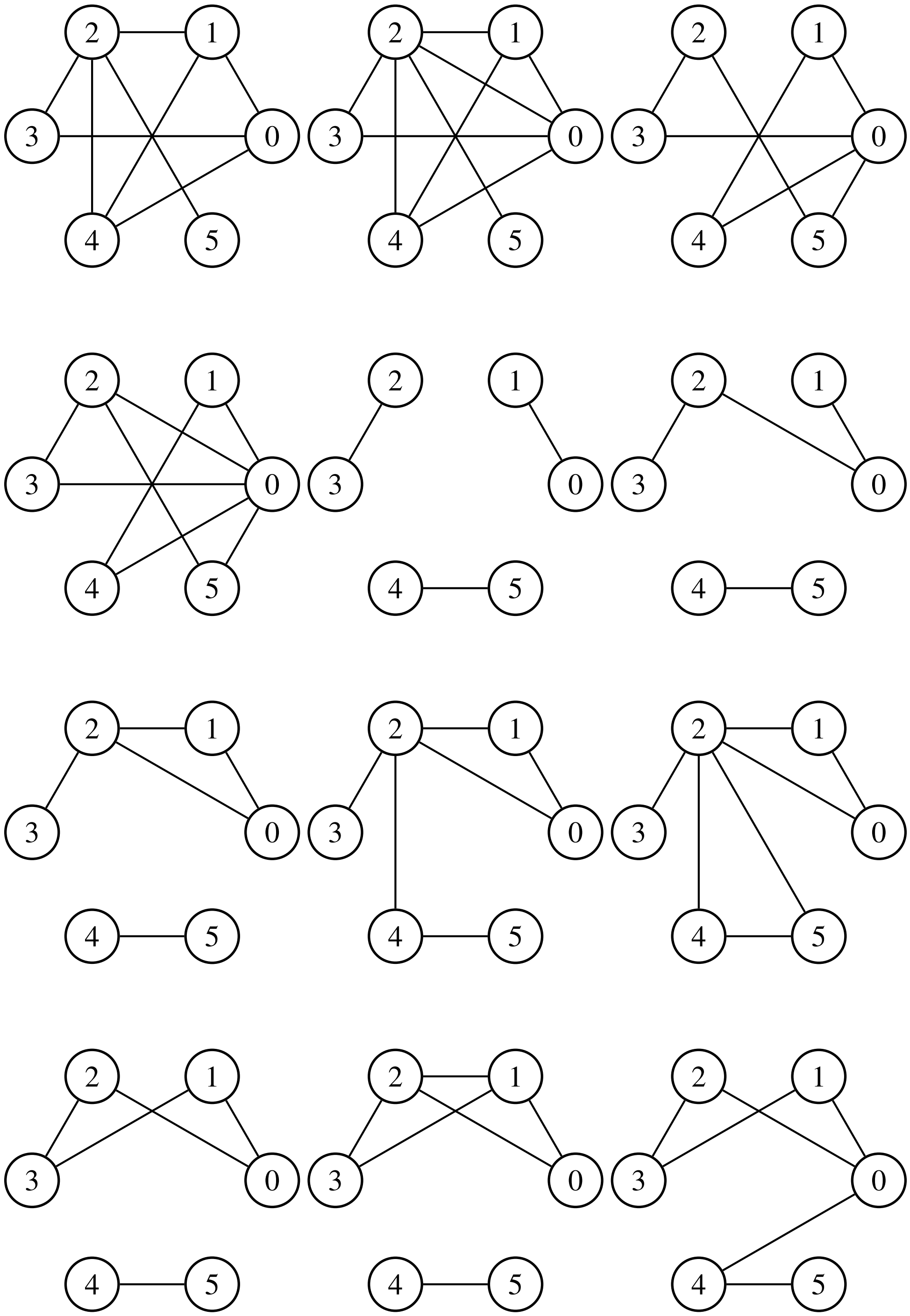


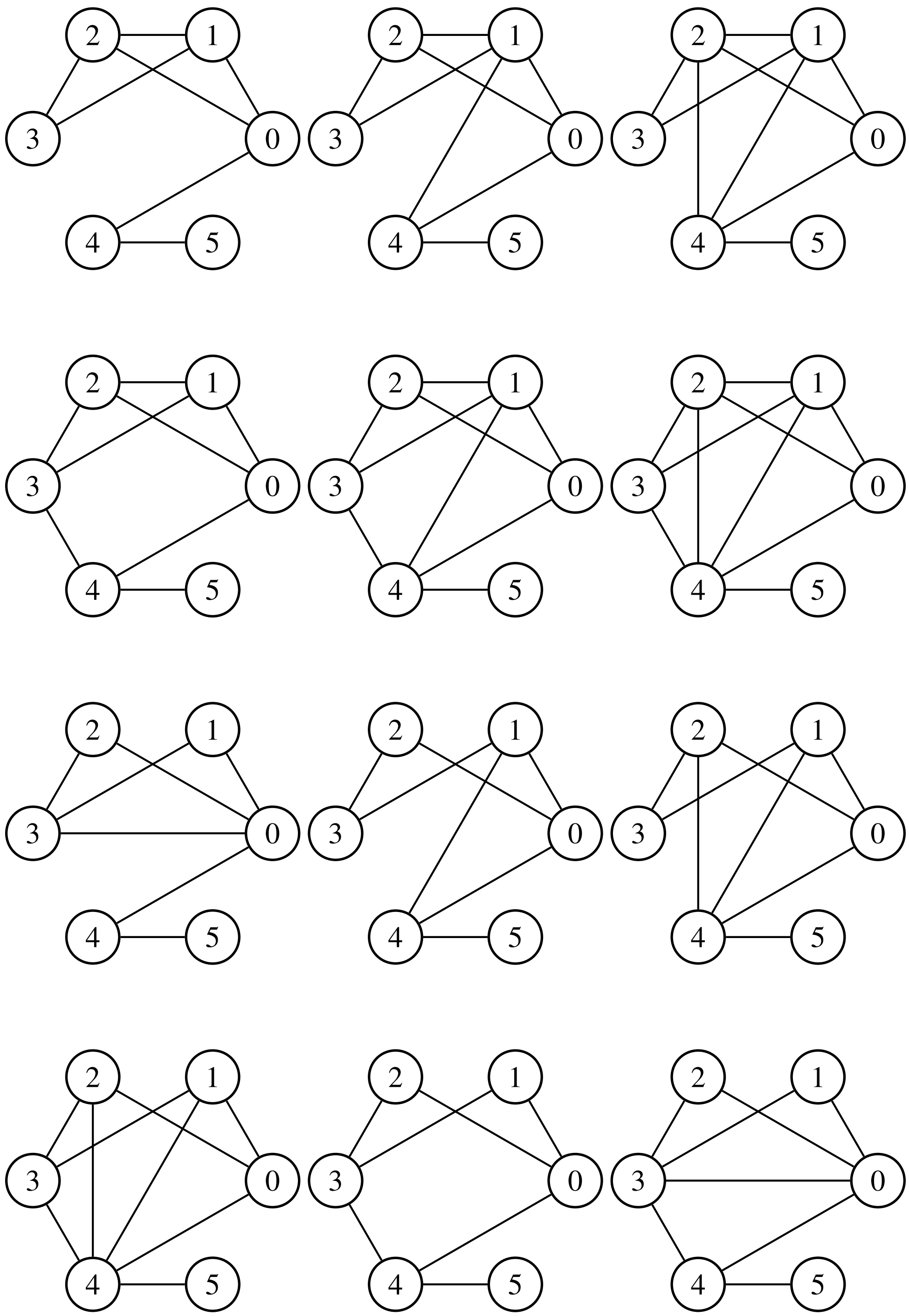


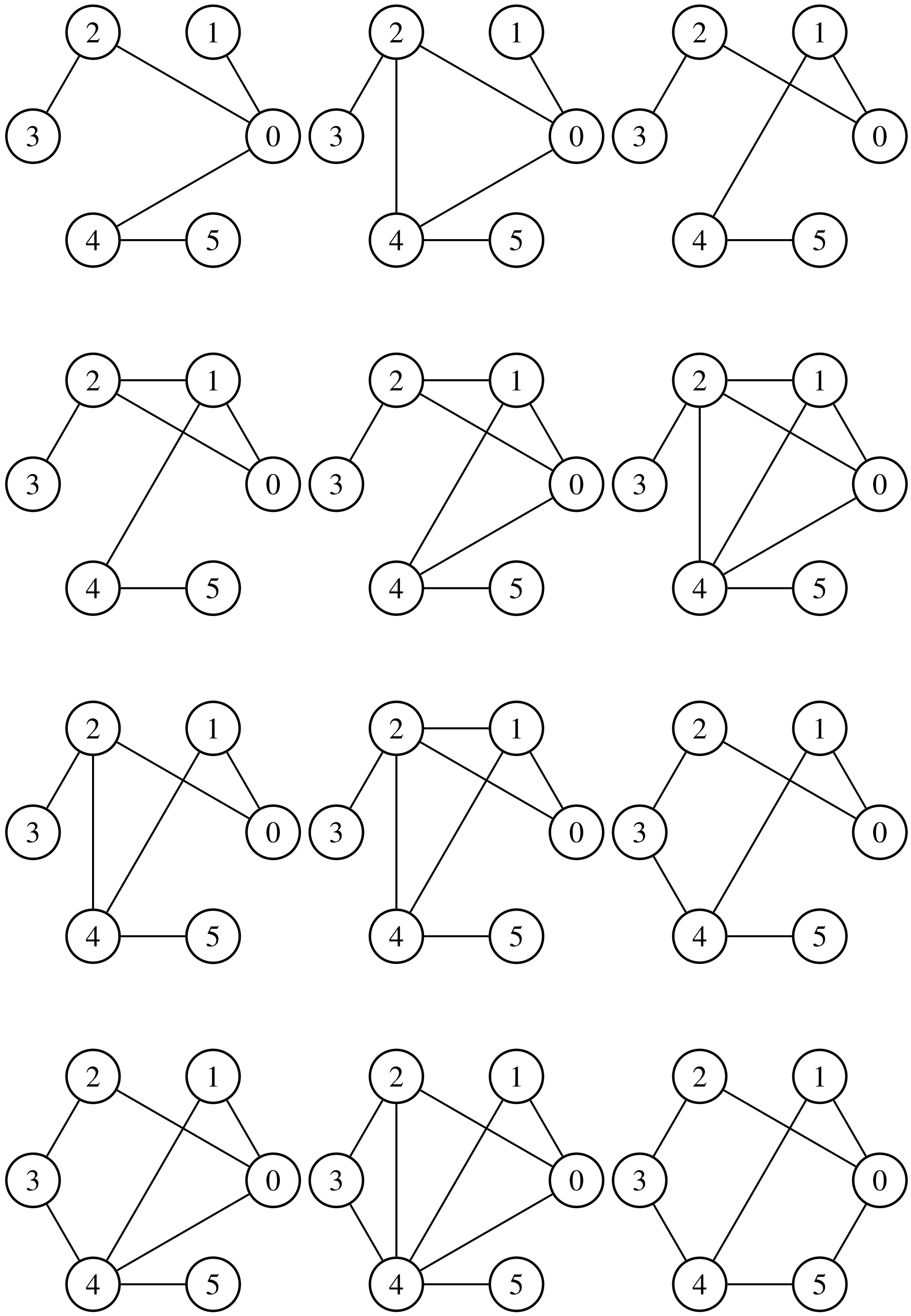


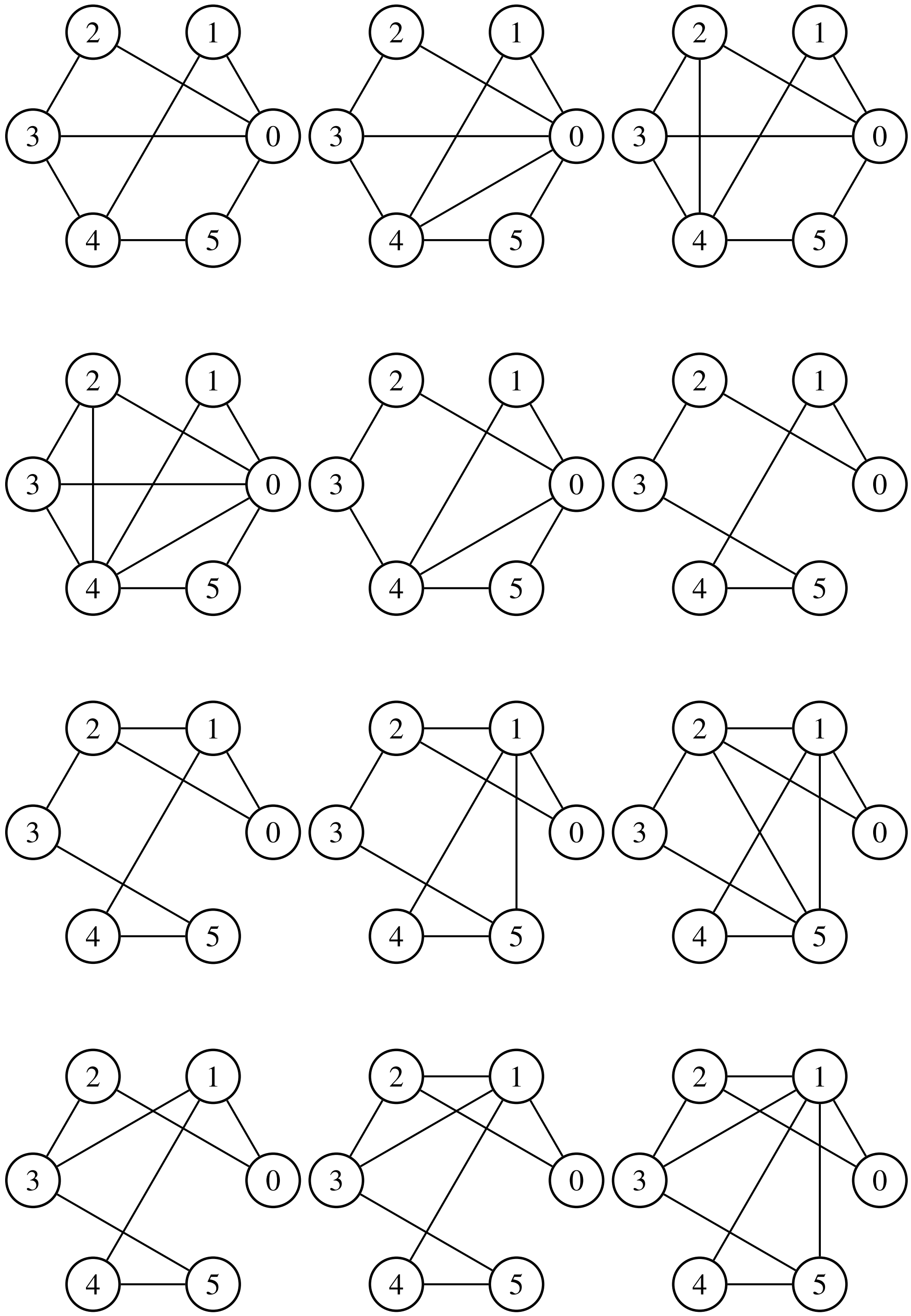


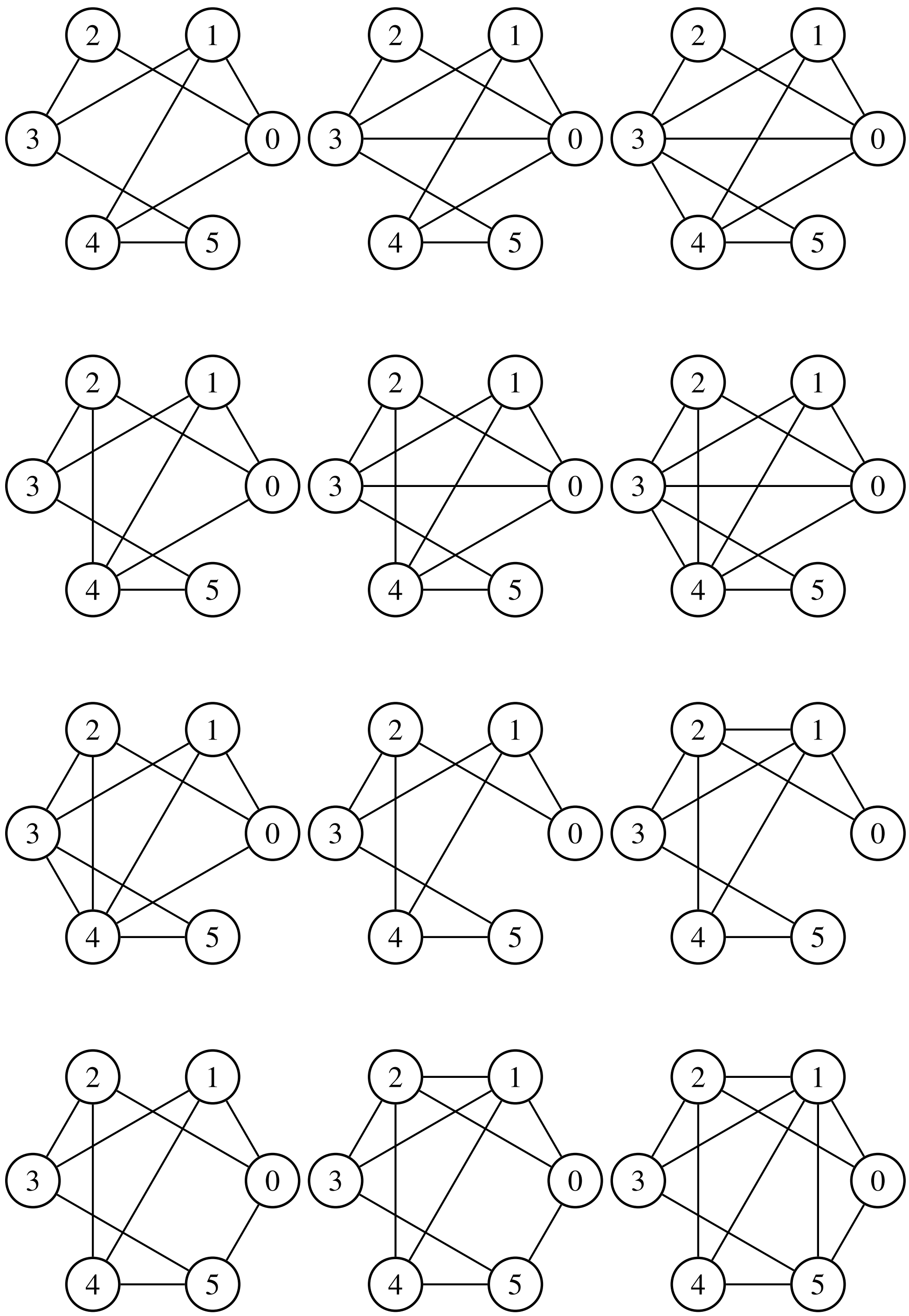


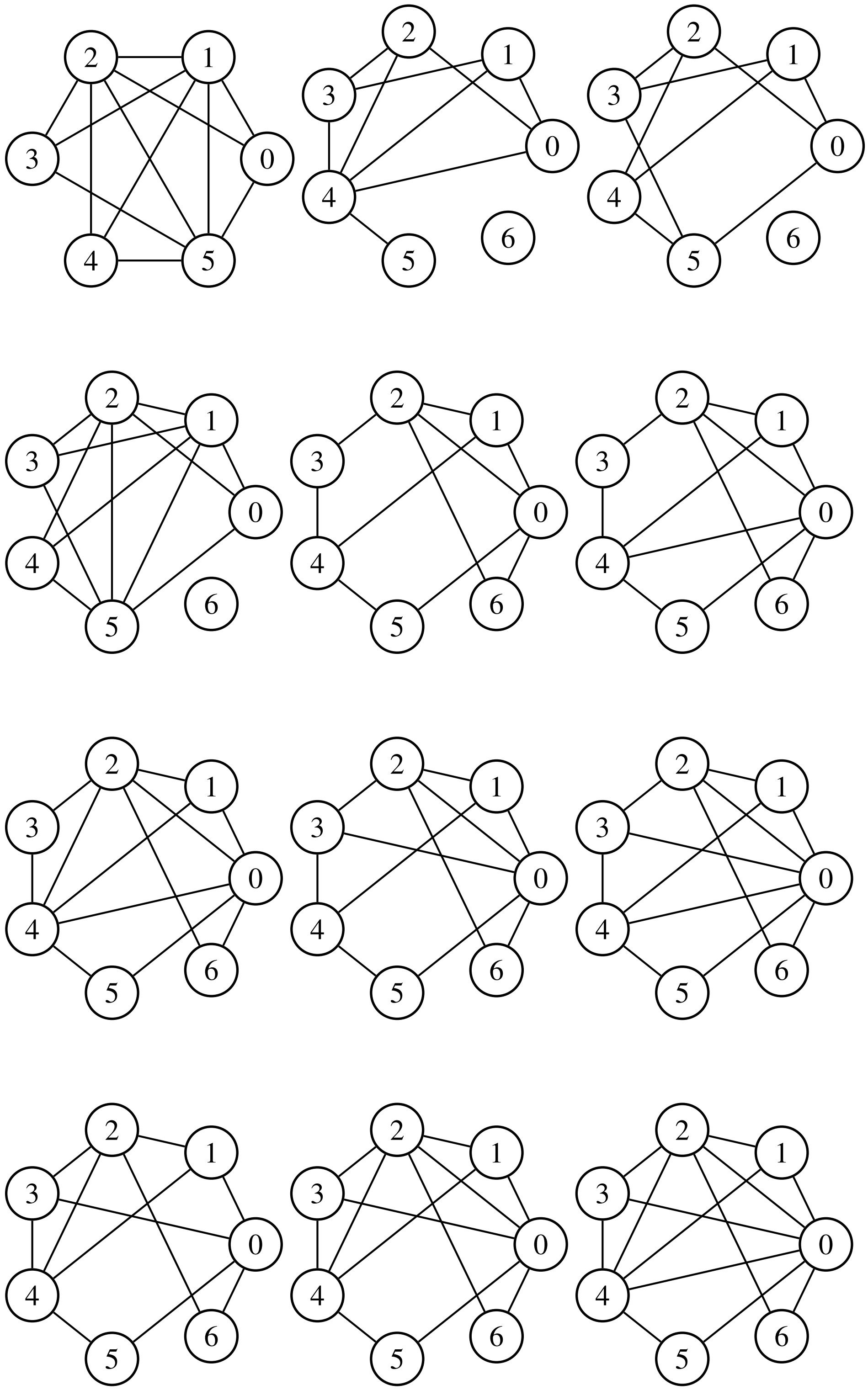


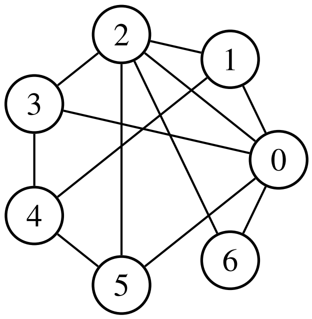
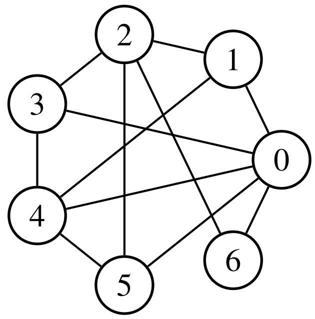
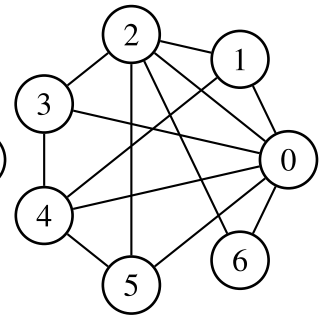
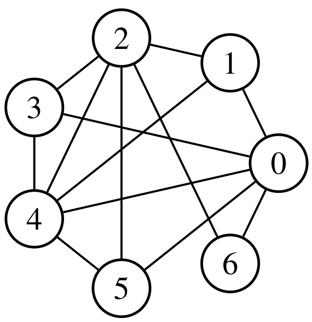
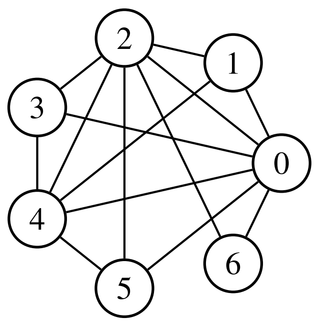
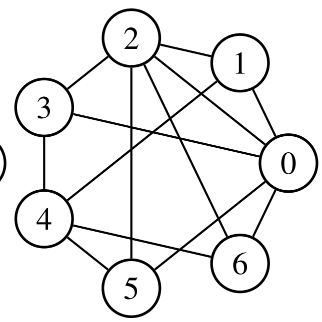
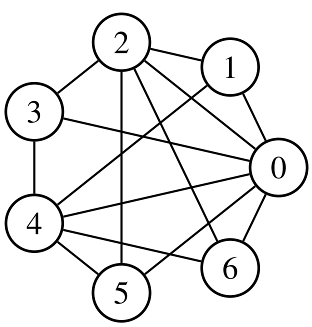
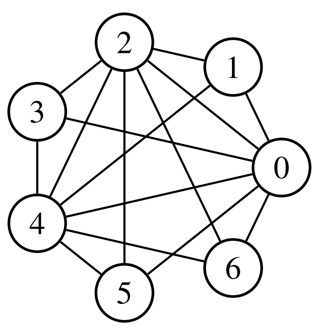
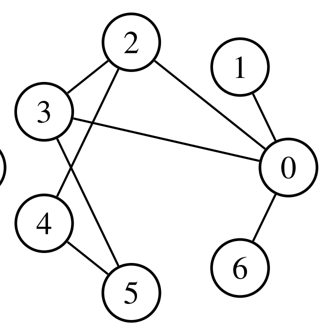
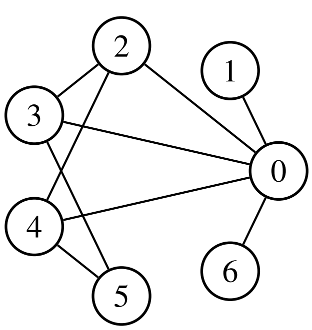
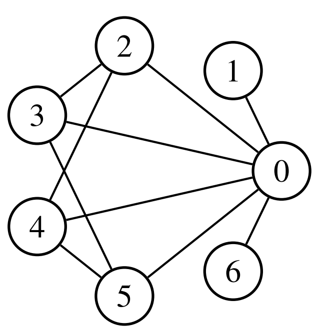
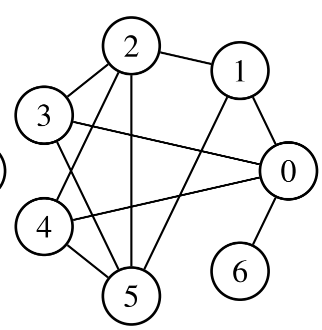

2 1 3 0 4 5
2 1 3 0 4 5
2 1 3 0 4 5
2 1 3 0 4 5
2 1 3 0 4 5
2 1 3 0 4 5
2 1 3 0 4 5
2 1 3 0 4 5
2 1 3 0 4 5
2 1 3 0 4 5
2 1 3 0 4 5
2 1 3 0 4 5

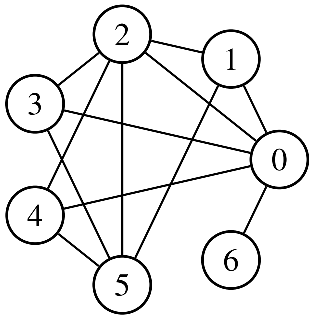
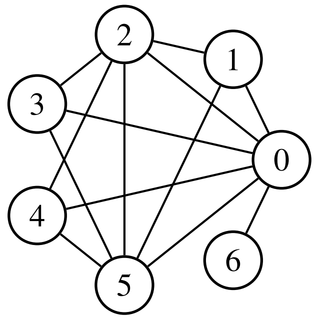
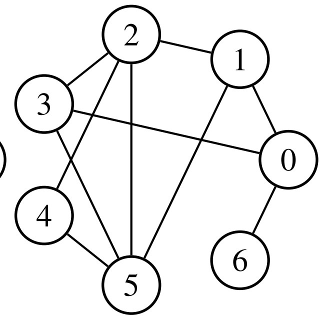
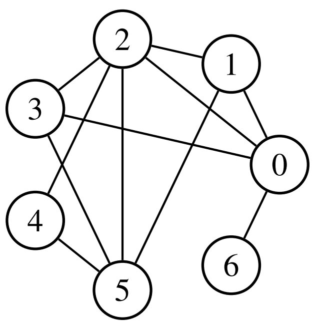
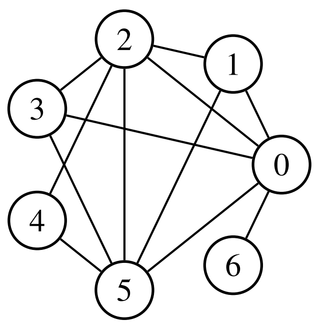
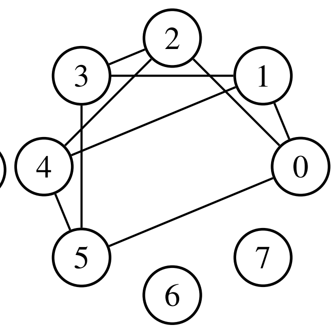
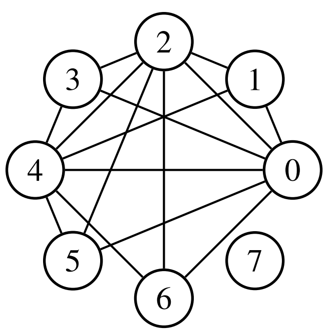
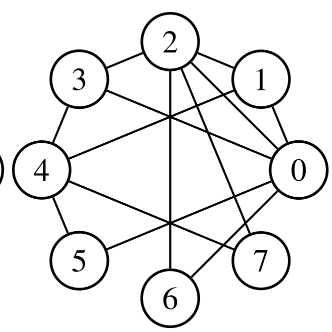
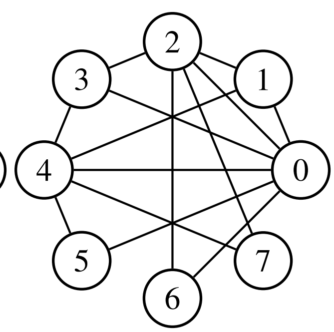
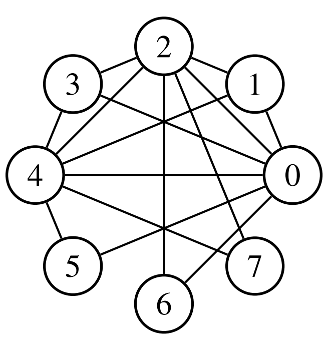
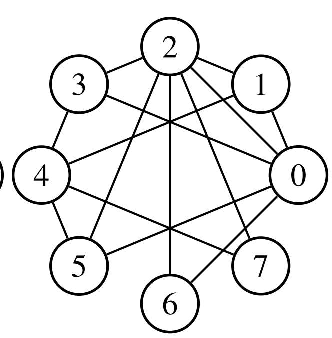
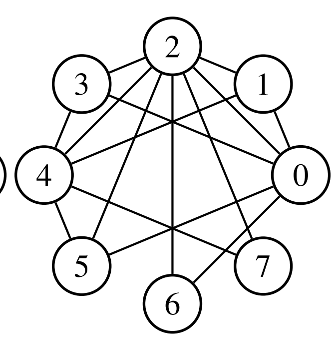


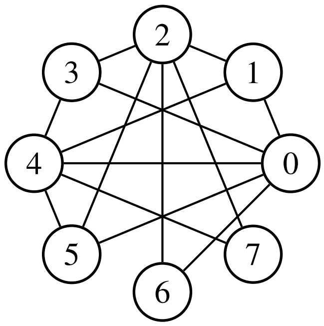
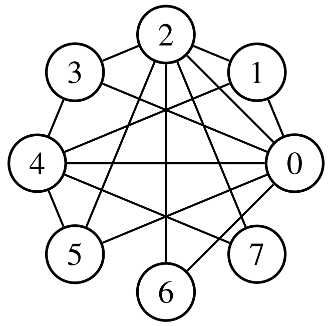
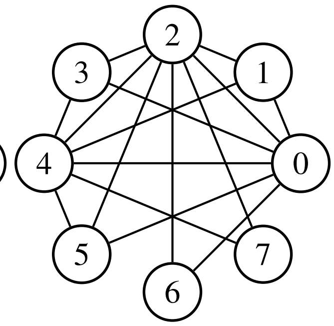
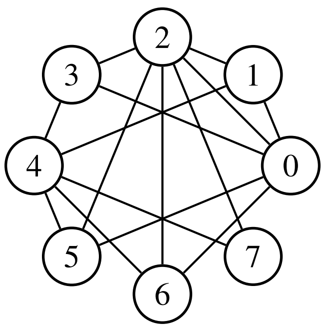
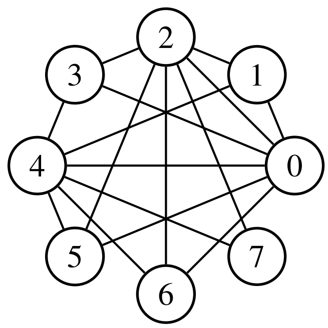
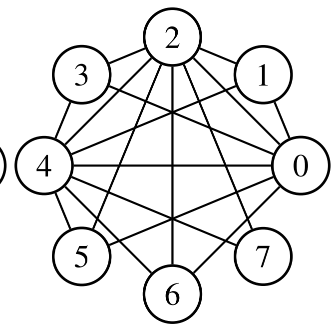
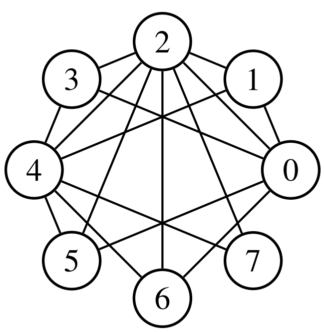
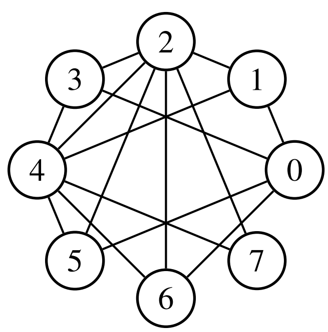
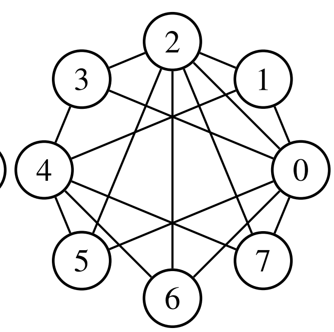

2 1 3 0 4 5
2 1 3 0 4 5
2 1 3 0 4 5
2 1 3 0 4 5
2 1 3 0 4 5
2 1 3 0 4 5
2 1 3 0 4 5
2 1 3 0 4 5
2 1 3 0 4 5
2 1 3 0 4 5
2 1 3 0 4 5
2 1 3 0 4 5

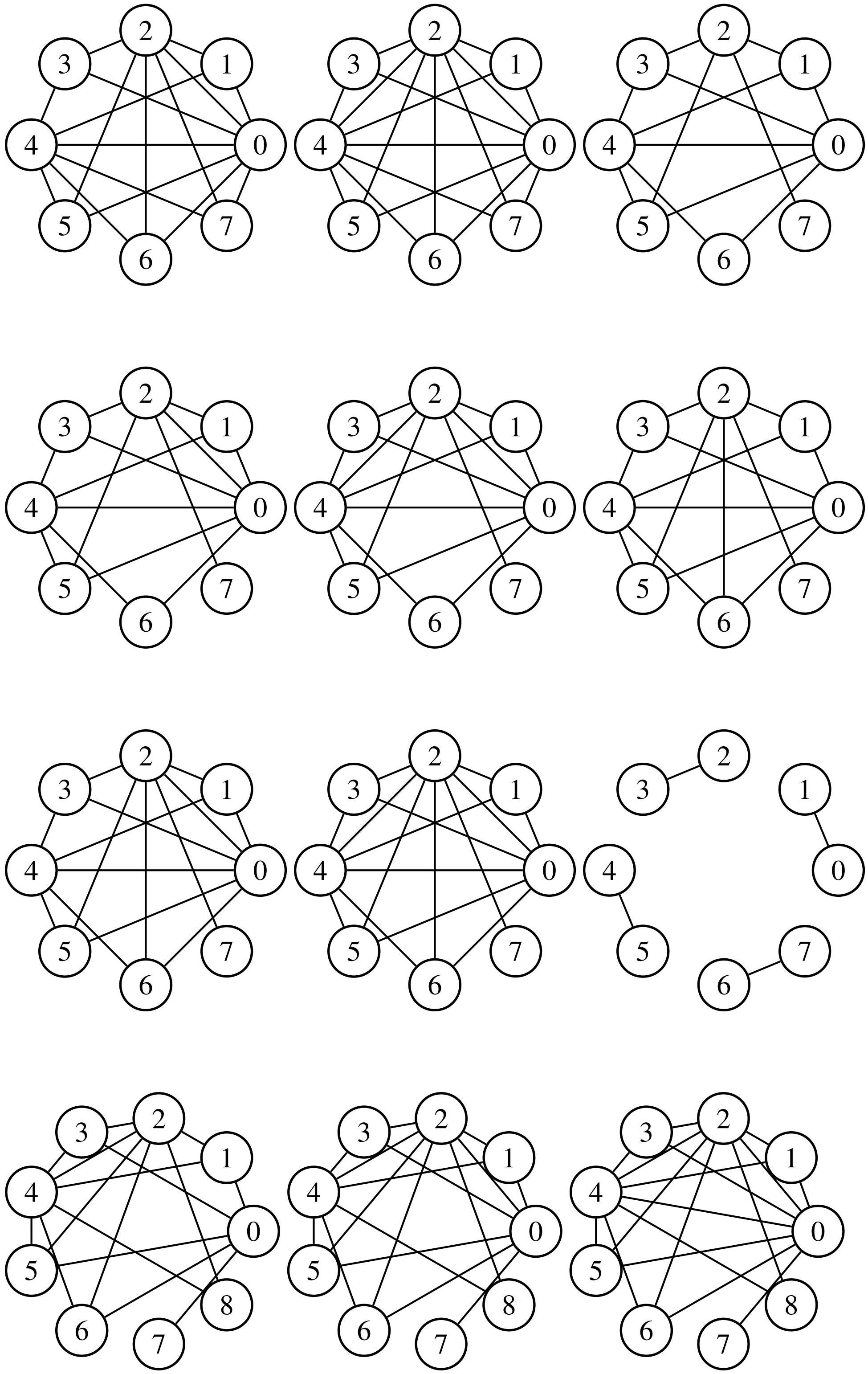


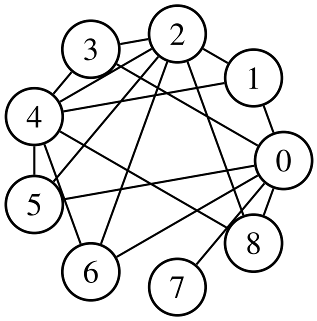
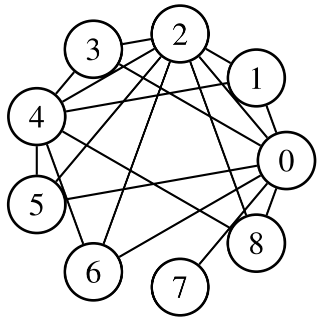
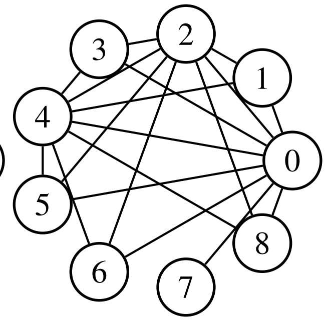
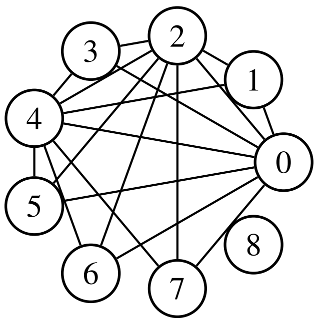
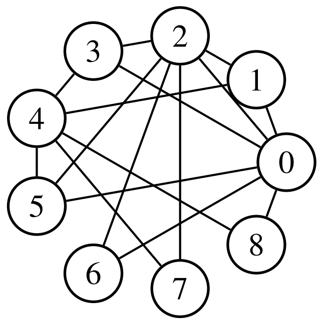
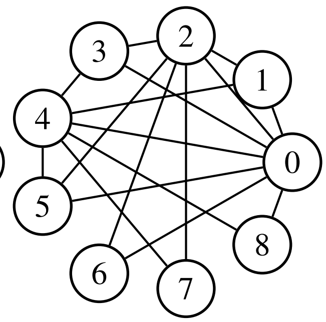
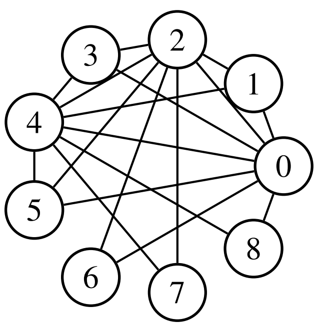
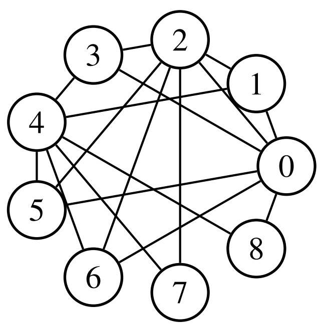
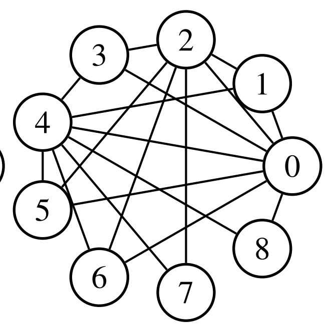
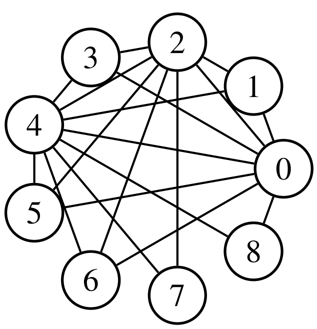
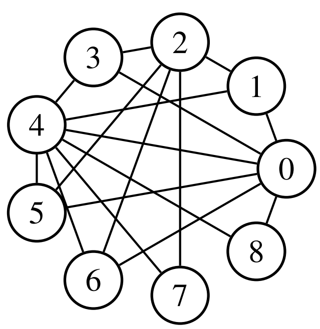
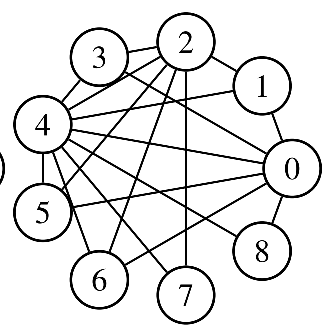


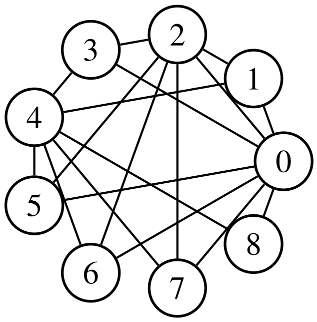
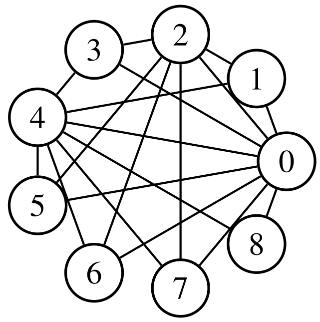
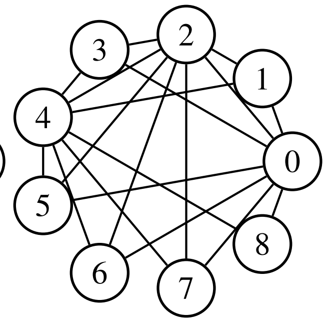
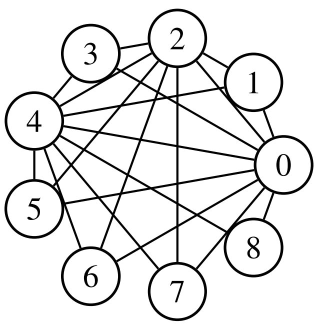
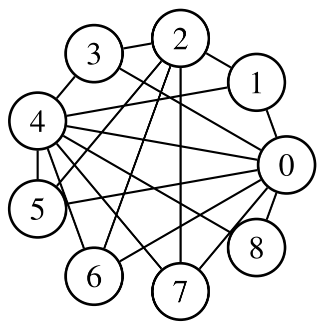
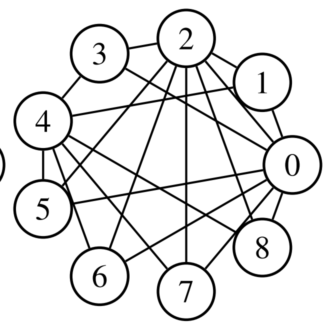
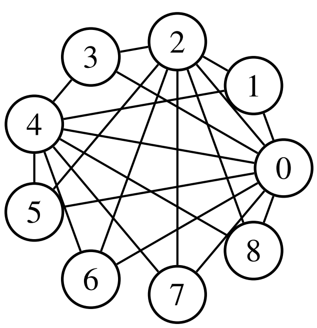
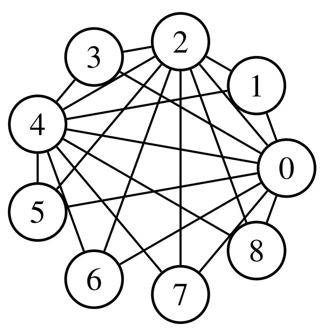
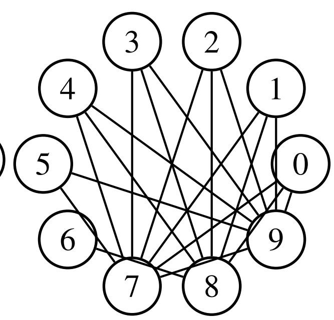
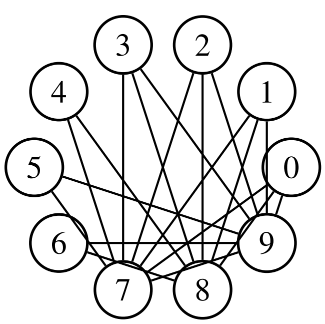
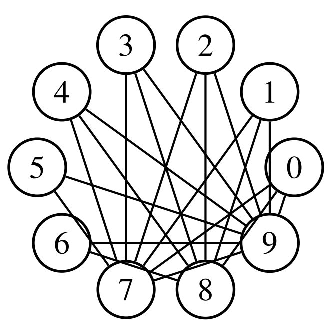
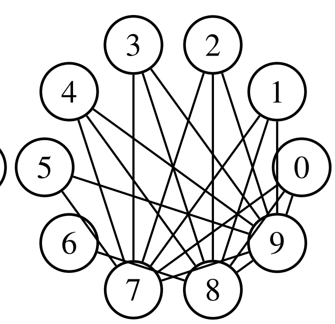

2
1
3
0
4
6
5

2
1
3
0
4
6
5

2
1
3
0
4
6
5

2
1
3
0
4
6
5

2
1
3
0
4
6
5

2
1
3
0
4
6
5

2
1
3
0
4
6
5

2
1
3
0
4
6
5

2
1
3
0
4
6
5

2
1
3
0
4
6
5

2
1
3
0
4
6
5

2
1
3
0
4
6
5

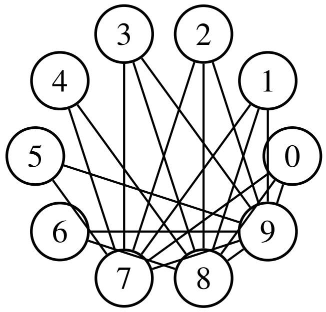
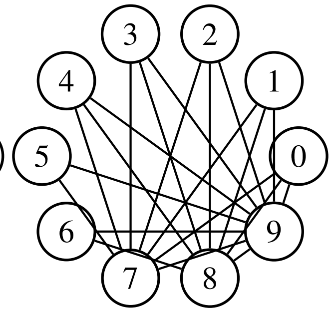
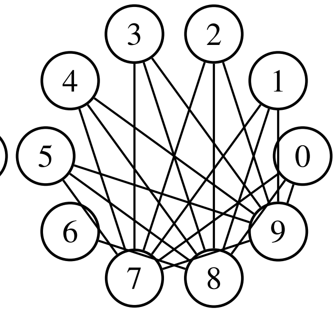
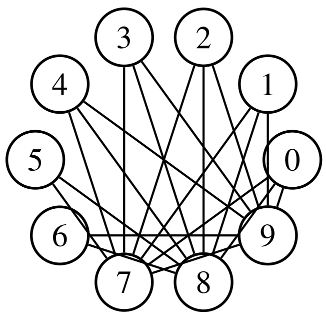
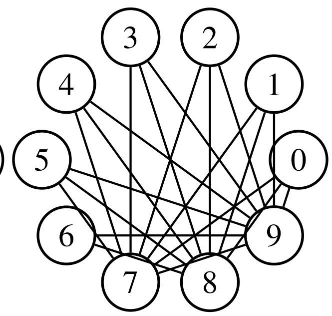
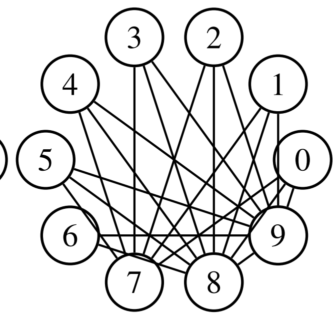
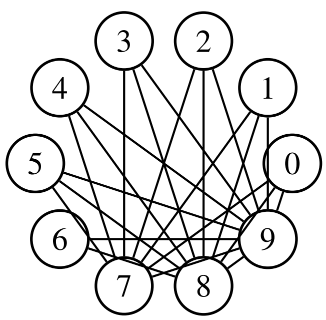
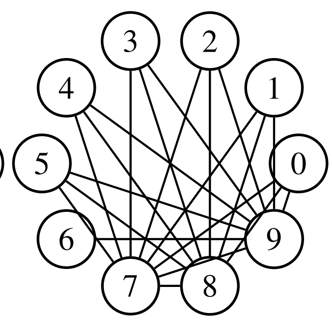
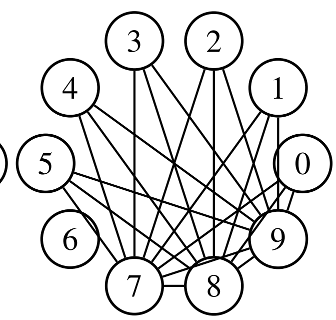
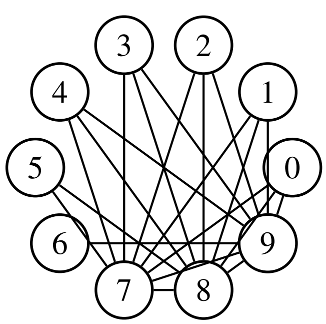
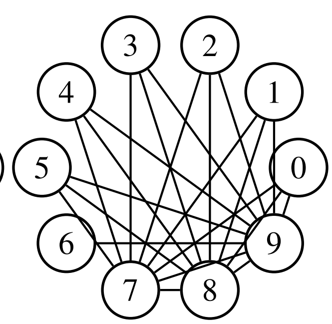
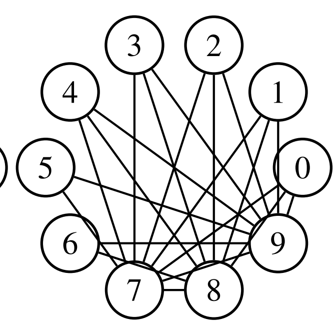

2
1
3
0
4
6
5

2
1
3
0
4
6
5

2
1
3
0
4
6
5

2
1
3
0
4
6
5

2
1
3
0
4
6
5

2
3
1
4
0
5
7
6

2
3
1
4
0
5
7
6

2
3
1
4
0
5
7
6

2
3
1
4
0
5
7
6

2
3
1
4
0
5
7
6

2
3
1
4
0
5
7
6

2
3
1
4
0
5
7
6

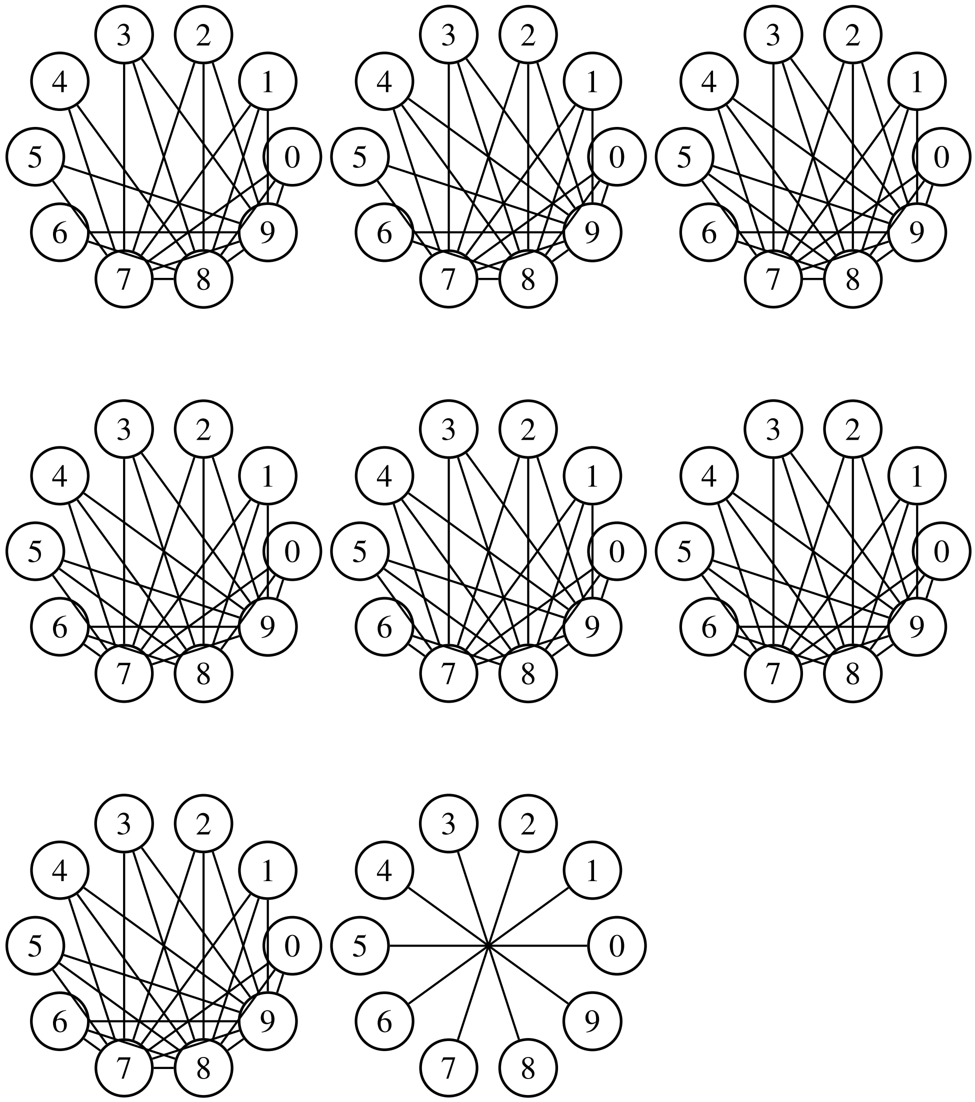

2
3
1
4
0
5
7
6

2
3
1
4
0
5
7
6

2
3
1
4
0
5
7
6

2
3
1
4
0
5
7
6

2
3
1
4
0
5
7
6

2
3
1
4
0
5
7
6

2
3
1
4
0
5
7
6

2
3
1
4
0
5
7
6

2
3
1
4
0
5
7
6




3
2
1
4
0
5
8
6
7

3
2
1
4
0
5
8
6
7

3
2
1
4
0
5
8
6
7

3
2
1
4
0
5
8
6
7

3
2
1
4
0
5
8
6
7

3
2
1
4
0
5
8
6
7

3
2
1
4
0
5
8
6
7

3
2
1
4
0
5
8
6
7

3
2
1
4
0
5
8
6
7

3
2
1
4
0
5
8
6
7

3
2
1
4
0
5
8
6
7

3
2
1
4
0
5
8
6
7


0
1
2
3
4
5
6
7
8

0
1
2
3
4
5
6
7
8

0
1
2
3
4
5
6
7
8

0
1
2
3
4
5
6
7
8

0
1
2
3
4
5
6
7
8

0
1
2
3
4
5
6
7
8

0
1
2
3
4
5
6
7
8

0
1
2
3
4
5
6
7
8

0
1
2
3
4
5
6
7
8
9

0
1
2
3
4
5
6
7
8
9

0
1
2
3
4
5
6
7
8
9

0
1
2
3
4
5
6
7
8
9


3 2 4 1 5 0 6 9 7 8

3 2 4 1 5 0 6 9 7 8

3 2 4 1 5 0 6 9 7 8

3 2 4 1 5 0 6 9 7 8

3 2 4 1 5 0 6 9 7 8

3 2 4 1 5 0 6 9 7 8

3 2 4 1 5 0 6 9 7 8

3 2 4 1 5 0 6 9 7 8

3 2 4 1 5 0 6 9 7 8

3 2 4 1 5 0 6 9 7 8

3 2 4 1 5 0 6 9 7 8

3 2 4 1 5 0 6 9 7 8



3

Search Code and Sage Output

The following is a patch to *nauty* that finds $(3, 3)$-regular graphs. The algorithm used is similar to NUMTRIANGLES1 but instead we consider independent triples and not triangles. It was further modified to find independent triples that have three vertices in their collective neighborhood. A better explanation of this code can be found here:

```
 diff -r 9ec2ed8b4aa8 gutil1.c
--- a/gutil1.c Sun Sep 19 07:01:08 2010 -0400
+++ b/gutil1.c Sun Sep 19 07:16:01 2010 -0400
@@ -67,28 +67,66 @@ degstats(graph *g, int m, int n, unsigne
```

```
 boolean
 isconnected1(graph *g, int n)
 /* test if g is connected (m=1) */
 {
         setword seen,expanded,toexpand;
         int i;

         seen = bit[0];
-        expanded = 0;
+        expanded = 0;// use?
+
+ // why not for all i, != g[i]?
+ // because we are doing a breadth first search
+ // not just tallying all things adjacent
+ // |= for all g[i] would only get you
+ // all isolated verts

         while ((toexpand = (seen & ~expanded)) != 0)
         {
             i = FIRSTBIT(toexpand);
             expanded |= bit[i];
             seen |= g[i];
         }

         return  POPCOUNT(seen) == n;
 }

+
+ /*********************************************/
+boolean
+is33regular1(graph *g, int n)
+/* test if a graph is 33regular */
+{
+  setword v,t;
+  int i,j,k;
+  setword gi,w;
+  long total;
+
+  total = 0;
+  for (i = 0; i < n-2; ++i)
+  {
```

```
+    gi = ((~(g[i]))^BITMASK(n-1)) & BITMASK(i);//
+ everything greater and not
adj
+    while (gi)
+    {
+      TAKEBIT(j,gi);// j is first not adj to i
+      w = (~(g[j])) & gi;// not adj to either j or i
+      if (w) total++;// we have seen at least one 3 ind set
+      while(w)
+      {
+ TAKEBIT(k,w); // i,j,k is ind set
+ if (POPCOUNT(g[i]|g[j]|g[k]) !=3) return FALSE;
+      }
+    }
+  }
+
+  return !(!(total));
+}
+
+
 /********************************************/

 boolean
 isconnected(graph *g, int m, int n)
 /* Test if g is connected */
 {
  int i,head,tail,w;
  set *gw;
diff -r 9ec2ed8b4aa8 testg.c
--- a/testg.c Sun Sep 19 07:01:08 2010 -0400
+++ b/testg.c Sun Sep 19 07:16:01 2010 -0400
@@ -30,17 +30,17 @@
      Constraints are applied to all
 input graphs, and only those\n\
      which match all constraints are counted or selected.\n\
 \n\
     -n#  number of vertices
-e#  number of edges\n\
      -d#  minimum degree
 -D#  maximum degree\n\
      -r   regular
```

```
 -b   bipartite\n\
      -z#  radius
 -Z#  diameter\n\
      -g#  girth (0=acyclic)
   -Y#  total number of cycles\n\
-     -T#  number of triangles\n\
+     -T#  number of triangles
-R 33 regular\n\
      -a#  group size
  -o# orbits  -F# fixed points
 -t vertex-transitive\n\
      -c#  connectivity (only implemented for 0,1,2).\n\
      -i#  min common nbrs of adjacent vertices;
 -I# maximum\n\
      -j#  min common nbrs of non-adjacent vertices;
-J# maximum\n\
 \n\
   Sort keys:\n\
      Counts are made for all graphs passing the constraints.
Counts\n\
@@ -145,20 +145,22 @@ static struct constraint_st
 /* Table
 #define I_j 18
    {'j',0,FALSE,FALSE,CMASK(I_i),-NOLIMIT,NOLIMIT,
"minnoncn",INTTYPE,0},
 #define I_J 19
   {'J',0,FALSE,FALSE,CMASK(I_i),-NOLIMIT,NOLIMIT,
"maxnoncn",INTTYPE,0},
 #define I_T 20
   {'T',0,FALSE,FALSE,0,-NOLIMIT,NOLIMIT,"triang",
INTTYPE,0},
 #define I_Q 21
 #ifdef USERDEF
-   {'Q',0,FALSE,FALSE,0,-NOLIMIT,NOLIMIT,
USERDEFNAME,INTTYPE,0}
+   {'Q',0,FALSE,FALSE,0,-NOLIMIT,NOLIMIT,
USERDEFNAME,INTTYPE,0},
 #else
-   {' ',0,FALSE,FALSE,0,-NOLIMIT,NOLIMIT,
USERDEFNAME,INTTYPE,0}
+   {' ',0,FALSE,FALSE,0,-NOLIMIT,NOLIMIT,
```

```
USERDEFNAME,INTTYPE,0},
 #endif
+#define I_R 22
+   {'R',0,FALSE,FALSE,0,-NOLIMIT,NOLIMIT,"33regular",
BOOLTYPE,0}
 };

 #define NUMCONSTRAINTS
(sizeof(constraint)/sizeof(struct constraint_st))
 #define SYMBOL(i) (constraint[i].symbol)
 #define ISNEEDED(i) (constraint[i].needed > 0)
 #define NEEDED(i) (constraint[i].needed)
 #define ISKEY(i) ((constraint[i].needed & 1) != 0)
 #define ISCONSTRAINT(i) (constraint[i].needed > 1)
@@ -486,16 +488,20 @@ compute(graph *g, int m, int n, int code
          COMPUTED(I_E) = COMPUTED(I_r) = TRUE;
  break;

      case I_b:
  VAL(I_b) = isbipartite(g,m,n);
  COMPUTED(I_b) = TRUE;
  break;

+     case I_R:
+       VAL(I_R) = is33regular1(g,n);
+       COMPUTED(I_R) = TRUE;
+       break;
      case I_g:
  VAL(I_g) = girth(g,m,n);
  COMPUTED(I_g) = TRUE;
  break;

      case I_z:
      case I_Z:
  diamstats(g,m,n,&rad,&diam);
```

Next we made the graphs found in Section 1 of the appendix by porting the graphs found above to SAGE.

```
 def print_graph(g):
```

```
    print(r"\begin{tikzpicture}[scale=1.7]")
    print("\GraphInit[vstyle=Normal]")
    ast = str(g.vertices())
    ast = ast.replace("[","")
    ast = ast.replace(", ",",")
    ast = ast.replace("]","")
    print(join(["\Vertices*{circle}{",ast,"}"],""))
    for e in g.edges():
        print("\Edge({0[0]})({0[1]})".format(e))
    print(r"\end{tikzpicture}")
    print("%")
i = 0
for g in itertools.chain(graphs633,graphs733,graphs833,
graphs933,graphs1033):
    i = i + 1
    print_graph(g)
    if (i%4) == 0:
        print("\\\\")
```