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## **A HIGH ACCURACY MINIMALLY INVASIVE REGULARIZATION TECHNIQUE FOR NAVIER-STOKES EQUATIONS AT HIGH REYNOLDS NUMBER**

Mustafa Aggul

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A HIGH ACCURACY MINIMALLY INVASIVE REGULARIZATION  
TECHNIQUE FOR NAVIER-STOKES EQUATIONS AT HIGH REYNOLDS  
NUMBER

By

Mustafa Aggul

A THESIS

Submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

In Mathematical Sciences

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This thesis has been approved in partial fulfillment of the requirements for the Degree of  
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# Dedication

*To my mother and the future mother of my kids...*

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Dad! I know you are watching me from somewhere out of this world. No matter where you are now, my heart will always be with you. May you rest in peace!

Love of my life, Deniz Yener. You have always been with me and supporting me with your existence. As we are on the verge of our marriage, my hope is to live happily ever all together.

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# Abstract

A method in this work, that combines the defect and deferred correction approaches to approximate solutions of Navier-Stokes equations at high Reynolds number is presented. This method is of high accuracy in both space and time, and it allows for the usage of legacy codes – a frequent requirement in the simulation of turbulent flows in complex geometries. The two-step method is considered here: in order to obtain a regularization, that is second order accurate in space and time, the method computes a low-order accurate, stable and computationally inexpensive approximation (Backward Euler with artificial viscosity) twice. The results are readily extendable to the higher order accuracy cases by adding more correction steps. Both the theoretical results and the numerical tests are provided to demonstrate that the computed solution is stable and the accuracy in both space and time is improved after the correction step. We also perform a qualitative test to demonstrate that the method is capable of capturing qualitative features of a turbulent flow, even on a very coarse mesh.

# Chapter 1

## Introduction

The motion of incompressible fluid flow in the flow domain  $\Omega = (0, L)^d$  is governed by the Navier-Stokes equations: find the velocity-pressure pair  $u : \Omega \times (0, T] \rightarrow R^d$ , ( $d = 2, 3$ ) and  $p : \Omega \times (0, T] \rightarrow R$  satisfying

$$u_t + u \cdot \nabla u - \nu \Delta u + \nabla p = f, \text{ for } x \in \Omega, 0 < t \leq T \quad (1.0.1)$$

$$\nabla \cdot u = 0, \text{ } x \in \Omega, \text{ for } 0 \leq t \leq T,$$

$$u(x, 0) = u_0(x), \text{ for } x \in \Omega,$$

with the normalization condition  $\int_{\Omega} p(x, t) \, dx = 0$  for  $0 < t \leq T$ . Throughout this paper, we consider the case of homogeneous Dirichlet boundary conditions to simplify the proofs; non-homogeneous Dirichlet boundary conditions can be treated in exactly the same manner and the same results hold.

According to the Kolmogorov theory [24], there exists a continuum of scales in turbulent fluid flow, with the smallest scales (in the case of a 3 –  $D$  flow) being of the order  $O(Re^{-3/4})$ , where the Reynolds number  $Re$  is inverse proportional to the viscosity coefficient  $\nu$ . Thus, capturing all the small structures in a turbulent flow requires the number of mesh points in space for each time step to be  $O(Re^{9/4})$  for

three-dimensional problems. It is not uncommon to have  $Re \sim O(10^8)$  in real-life applications.

Hence, the direct numerical simulation (DNS) of a 3 –  $D$  turbulent flow is often not computationally economical or even feasible. Sometimes it is desirable (especially for turbulent flows in complex geometries) to be able to use pre-existing codes. Thus, we are aiming at constructing a method that would approximate a flow at high Reynolds number, while being computationally attractive, stable and of high accuracy in both space and time.

To that end, we consider a defect correction approach from [15]. Defect correction strategies have been successfully applied to stiff systems [3, 16, 11, 6, 18], and in particular to evolutionary Navier Stokes Equation (NSE) - see, e.g., [15] and references therein.

The general idea of any Defect Correction Method (DCM) can be formulated as follows (see, e.g., [27, 1]):

Find a unique solution of  $Fx = 0$ , by

DCM: Use an approximation  $\tilde{F}$  to build an iterative procedure:

$$\tilde{F}x_1 = 0,$$

$$x_{i+1} = (I - \tilde{F}^{-1}F)x_i, i \geq 1.$$

The choice of a particular approximation  $\tilde{F}$  determines the defect correction method in use. As a result of using the artificial viscosity approximation-based defect correction method of [15], we have an approach that allows for the usage of legacy codes and gives a second order accurate in space approximation of a flow at high Reynolds number. This is obtained by computing two consecutive approximations  $u_1$  and  $u_2$  with Backward Euler method and with exactly the same matrix (the correction step

only modifies the right hand side of the system for  $u_2$  by a function of the previously computed  $u_1$ ). These approximations, however, are first order accurate in time. The question is: can we increase the time accuracy without increasing the computational cost? The answer lies in the temporal counterpart of the defect correction idea, known as deferred correction.

The main advantage of the deferred correction approach is that a simple low-order method can be employed, and the recovered solution is of high-order accuracy, due to a sequence of deferred correction equations. The classical deferred correction approach could be seen, e.g., in [28]. However, in 2000 a modification of the classical deferred correction approach was introduced by Dutt, Greengard and Rokhlin, [25]. This allowed the construction of stable and high-order accurate *spectral deferred correction* methods. In [20], M.L. Minion discusses these spectral deferred correction (SDC) methods in application to an initial value ODE

$$\begin{aligned}\phi'(t) &= F(t, \phi(t)), \quad t \in [a, b] \\ \phi(a) &= \phi_a.\end{aligned}\tag{1.0.2}$$

The solution is written in terms of the Picard integral equation; a polynomial is used to interpolate the subintegrand function and the obtained integral term is replaced by its quadrature approximation. The deferred correction approach was used to improve the temporal accuracy of a turbulence model in [31].

When both the defect and deferred correction are combined into one method, we seek two approximations  $u_1^{h,i}$  and  $u_2^{h,i}$  to the true solution  $u(t_i)$ . Both are computed with the same matrix of the system, but with different right hand sides. The computational attractiveness is due to two important factors. First, the cost of computing each approximation is the cost of solving a Backward Euler method for the NSE with increased viscosity coefficient - a method which is hard to beat in terms of computational cost. Secondly, the defect-deferred correction methods are readily parallelizable, as the solutions  $u_1^{h,i+k}, u_2^{h,i+k-1}, \dots, u_{k+1}^{h,i}$  can be computed simultaneously on  $k+1$

cores to produce a potentially  $(k + 1)$ -order accurate approximation.

We propose the following two-step method that produces a sequence of approximations  $(u_1^h, p_1^h), (u_2^h, p_2^h)$  of the true solution  $(u, p)$ .

$$\begin{aligned} & \left( \frac{u_1^{h,n+1} - u_1^{h,n}}{k}, v^h \right) + (h + \nu)(\nabla u_1^{h,n+1}, \nabla v^h) + b^*(u_1^{h,n+1}, u_1^{h,n+1}, v^h) \\ & - (p_1^{h,n+1}, \nabla \cdot v^h) = (f(t_{n+1}), v^h), \end{aligned} \quad (1.0.3)$$

$$\begin{aligned} & \left( \frac{u_2^{h,n+1} - u_2^{h,n}}{k}, v^h \right) + (h + \nu)(\nabla u_2^{h,n+1}, \nabla v^h) + b^*(u_2^{h,n+1}, u_2^{h,n+1}, v^h) \\ & - (p_2^{h,n+1}, \nabla \cdot v^h) = \left( \frac{f(t_{n+1}) + f(t_n)}{2}, v^h \right) + \frac{\nu}{2} k \left( \nabla \left( \frac{u_1^{h,n+1} - u_1^{h,n}}{k} \right), v^h \right) \\ & + \frac{1}{2} b^*(u_1^{h,n+1}, u_1^{h,n+1}, v^h) - \frac{1}{2} b^*(u_1^{h,n}, u_1^{h,n}, v^h) + h(\nabla u_1^{h,n+1}, \nabla v^h), \end{aligned} \quad (1.0.4)$$

where  $b^*(\cdot, \cdot, \cdot)$  is the explicitly skew-symmetrized trilinear form, defined below.

The remainder of this paper is organized as follows. Section 2 introduces the necessary notation and preliminaries; then follows Section 3 on the accuracy and stability of the defect step approximation. These results come mostly from [15], as the equation for the defect step of the defect-deferred approach is exactly the defect step of the approach in [15]. The novelty of the proposed method appears in Section 4, where stability and increased accuracy (both time and space) of the correction step is studied. The quantitative and qualitative computational tests are presented in Section 5.

## Chapter 2

# Mathematical Preliminaries and Notations

Throughout this paper, the norm  $||\cdot||$  denotes the usual  $L^2(\Omega)$  norm of scalars, vectors and tensors, induced by the usual  $L^2$  inner-product, denoted by  $(\cdot, \cdot)$ . The space in which velocity sought(at time  $t$ ) is

$$X = H_0^1(\Omega)^d = \{v \in L^2(\Omega)^d : \nabla v \in L^2(\Omega)^{dx d} \text{ and } v = 0 \text{ on } \partial\Omega\}.$$

with the norm  $||v||_X = ||\nabla v||$ . The space dual to  $X$  is equipped with the norm

$$||f||_{-1} = \sup_{v \in X} \frac{(f, v)}{||\nabla v||}.$$

The space that velocity (at time  $t$ ) belongs to is

$$Q = L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_{\Omega} q(x) dx = 0\}.$$

Introduce the space of weakly divergence-free functions

$$X \supset V = \{v \in X : (\nabla \cdot v, q) = 0, \forall q \in Q\}.$$

For measurable  $v : [0, T] \rightarrow X$ , we define

$$\|v\|_{L^p(0,T;X)} = \left( \int_0^T \|v\|_X^p dt \right)^{\frac{1}{p}}, 1 \leq p < \infty,$$

and

$$\|v\|_{L^\infty(0,T;X)} = \operatorname{ess\,sup}_{0 \leq t \leq T} \|v(t)\|_X.$$

Define the trilinear form on  $X \times X \times X$

$$b(u, v, w) = \int_{\Omega} u \cdot \nabla v \cdot w dx.$$

The following lemma is also necessary for the analysis.

**Lemma 2.0.1** *There exist finite constant  $M = M(d)$  and  $N = N(d)$  s.t.  $M \geq N$  and*

$$M = \sup_{u,v,w \in X} \frac{b(u, v, w)}{\|u\| \|v\| \|w\|} < \infty, N = \sup_{u,v,w \in V} \frac{b(u, v, w)}{\|u\| \|v\| \|w\|} < \infty.$$

The proof can be found in [29]. The corresponding constants  $M^h$  and  $N^h$  are defined by replacing  $X$  by the finite element space  $X^h \subset X$  and  $V$  by  $V^h \subset X$ . Note that  $M \geq \max(M^h, N, N^h)$  and that as  $h \rightarrow 0$ ,  $N^h \rightarrow N$  and  $M^h \rightarrow M$  (see [29]).

Throughout the paper, we shall assume that the velocity-pressure finite element spaces  $X^h \subset X$  and  $Q^h \subset Q$  are conforming, have typical approximation properties of finite element spaces commonly in use, and satisfy the discrete inf-sup, or  $LBB^h$ , condition

$$\inf_{q^h \in Q^h} \sup_{v^h \in X^h} \frac{(q^h, \nabla \cdot v^h)}{\|\nabla v^h\| \|q^h\|} \geq \beta^h > 0, \quad (2.0.1)$$

where  $\beta^h$  is bounded away from zero uniformly in  $h$ . Examples of such spaces can be found in [29]. We shall consider  $X^h \subset X$ ,  $Q^h \subset Q$  to be spaces of continuous

piecewise polynomials of degree  $m$  and  $m - 1$ , respectively, with  $m \geq 2$ . The case of  $m = 1$  is not considered, because the optimal error estimate (of the order  $h$ ) is obtained after the first step of the method, and therefore the DCM in this case is reduced to the artificial viscosity approach.

The space of discretely divergence-free functions is defined as follows

$$V^h = \{v^h \in X^h : (q^h, \nabla \cdot v^h) = 0, \forall q^h \in Q^h\}.$$

In the analysis we use the properties of the following Modified Stokes Projection

**Definition 2.0.2 (Modified Stokes Projection)** *Define the Stokes projection operator  $P_S: (X, Q) \rightarrow (X^h, Q^h)$ ,  $P_S(u, p) = (\tilde{u}, \tilde{p})$ , satisfying*

$$\begin{aligned} (h + \nu)(\nabla(u - \tilde{u}), \nabla v^h) - (p - \tilde{p}, \nabla \cdot v^h) &= 0, \\ (\nabla \cdot (u - \tilde{u}), q^h) &= 0, \end{aligned} \tag{2.0.2}$$

for any  $v^h \in V^h, q^h \in Q^h$ .

In  $(V^h, Q^h)$  this formulation reads: given  $(u, p) \in (X, Q)$ , find  $\tilde{u} \in V^h$  satisfying

$$(h + \nu)(\nabla(u - \tilde{u}), \nabla v^h) - (p - q^h, \nabla \cdot v^h) = 0, \tag{2.0.3}$$

for any  $v^h \in V^h, q^h \in Q^h$ .

Define the explicitly skew-symmetrized trilinear form

$$b^*(u, v, w) := \frac{1}{2}(u \cdot \nabla v, w) - \frac{1}{2}(u \cdot \nabla w, v).$$

The following estimate is easy to prove (see, e.g., [29]): there exists a constant  $C = C(\Omega)$  such that

$$|b^*(u, v, w)| \leq C(\Omega) \|\nabla u\| \|\nabla v\| \|\nabla w\|. \tag{2.0.4}$$

The proofs will require the sharper bound on the nonlinearity. This upper bound is improvable in  $R^2$ .

**Lemma 2.0.3 (The sharper bound on the nonlinear term)** *Let  $\Omega \subset R^d$ ,  $d = 2, 3$ . For all  $u, v, w \in X$*

$$|b^*(u, v, w)| \leq C(\Omega) \sqrt{\|u\| \|\nabla u\|} \|\nabla v\| \|\nabla w\|.$$

**Proof 2.0.1** *See [29].*

We will also need the following inequalities: for any  $u \in V$

$$\inf_{v \in V^h} \|\nabla(u - v)\| \leq C(\Omega) \inf_{v \in X^h} \|\nabla(u - v)\|, \quad (2.0.5)$$

$$\inf_{v \in V^h} \|u - v\| \leq C(\Omega) \inf_{v \in X^h} \|\nabla(u - v)\|, \quad (2.0.6)$$

The proof of (2.0.5) can be found, e.g., in [29], and (2.0.6) follows from the Poincare-Friedrich's inequality and (2.0.5).

We will also assume that the inverse inequality holds: there exists a constant  $C$  independent of  $h$ , such that

$$\|\nabla v\| \leq Ch^{-1} \|v\|, \quad \forall v \in X^h. \quad (2.0.7)$$

Define also the number of time steps  $N := \frac{T}{k}$ .

We will use the error decomposition

$$\begin{aligned} e_\ell^i &= u^i - u_\ell^{h,i} = u^i - \tilde{u}^i + \tilde{u}^i - u_\ell^{h,i} = \eta_\ell^i - \phi_\ell^{h,i}, \\ \text{where } \tilde{u}^i &\in V^h \text{ is some projection of } u^i \text{ onto } V^h, \\ \text{and } \eta_\ell^i &= u^i - \tilde{u}^i, \phi_\ell^{h,i} = u_\ell^{h,i} - \tilde{u}^i, \phi_\ell^{h,i} \in V^h, \forall i, \forall \ell = 1, 2. \end{aligned} \quad (2.0.8)$$

We conclude the preliminaries by formulating the discrete Gronwall's lemma, see, e.g. [30]

**Lemma 2.0.4** *Let  $k, B$ , and  $a_\mu, b_\mu, c_\mu, \gamma_\mu$ , for integers  $\mu \geq 0$ , be nonnegative numbers such that:*

$$a_n + k \sum_{\mu=0}^n b_\mu \leq k \sum_{\mu=0}^n \gamma_\mu a_\mu + k \sum_{\mu=0}^n c_\mu + B \text{ for } n \geq 0.$$

*Suppose that  $k\gamma_\mu < 1$  for all  $\mu$ , and set  $\sigma_\mu = (1 - k\gamma_\mu)^{-1}$ . Then*

$$a_n + k \sum_{\mu=0}^n b_\mu \leq e^{k \sum_{\mu=0}^n \sigma_\mu \gamma_\mu} \cdot [k \sum_{\mu=0}^n c_\mu + B].$$

Upon giving the relationship between Reynolds Number and the kinetic viscosity ( $\nu$ ) below, we will use  $\nu$  instead of  $Re^{-1}$ .

$$Re = \frac{\rho v L}{\mu} = \frac{v L}{\nu},$$

where  $v$  is the maximum velocity of the object relative to the fluid,  $L$  is a characteristic linear dimension,  $\mu$  is the dynamic viscosity of the fluid and  $\rho$  is the density of the fluid.

# Chapter 3

## AV Approximation

In this section we prove the unconditional stability and error estimate of the discrete artificial viscosity approximation  $u_1^h$  and use this result to prove an error estimate of its time derivative  $\frac{de_1}{dt}$ . Over  $0 \leq t \leq T < \infty$  the approximations  $u_1^h$  is bounded uniformly in  $\nu$ .

Hence, the formulation (1.0.3) gives  $O(h + k)$  accurate, unconditionally stable extension of the artificial viscosity approximation to the time-dependent Navier-Stokes equations.

We start by giving stability and error estimate of the modified Stokes Projection, that we use as the approximation  $\tilde{u}^0$  to the initial velocity  $u_0$ .

### 3.1 Stokes Projection

**Proposition 3.1.1 (Stability of the Stokes projection)** *Let  $u, \tilde{u}$  satisfy (2.0.3).*

*The following bound holds*

$$\begin{aligned} (h + \nu) \|\nabla \tilde{u}\|^2 &\leq 2(h + \nu) \|\nabla u\|^2 \\ &\quad + 2d(h + \nu)^{-1} \inf_{q^h \in Q^h} \|p - q^h\|^2, \end{aligned} \tag{3.1.1}$$

where  $d$  is the dimension,  $d = 2, 3$ .

**Proposition 3.1.2 (Error estimate for Stokes Projection).** *Suppose the discrete inf-sup condition (2.0.1) holds. Then the error in Stokes Projection satisfies*

$$\begin{aligned} (h + \nu) \|\nabla(u - \tilde{u})\|^2 &\leq C[(h + \nu) \inf_{v^h \in V^h} \|\nabla(u - v^h)\|^2 \\ &\quad + (h + \nu)^{-1} \inf_{q^h \in Q^h} \|p - q^h\|^2], \end{aligned} \tag{3.1.2}$$

where  $C$  is a constant independent of  $h$  and  $Re$ .

**Proof 3.1.1** *Proofs can be found in [15]*

### 3.2 Stability of the AV approximation

**Lemma 3.2.1** *Let  $u_1^h$  satisfy the equation (1.0.3). Let  $f \in L^2(0, T; H^{-1}(\Omega))$ . Then for  $n = 0, \dots, N - 1$*

$$\begin{aligned} \|u_1^{h, n+1}\|^2 + k \sum_{i=1}^{n+1} (h + \nu) \|\nabla u_1^{h, i}\|^2 &\leq \|u_0^s\|^2 \\ &\quad + \frac{1}{h + \nu} k \sum_{i=1}^{n+1} \|f(t_i)\|_{-1}^2. \end{aligned}$$

Also, if  $f \in L^2(0, T; L^2(\Omega))$  and the time constraint  $T$  is finite, then there exists a constant  $C = C(T)$  such that

$$\begin{aligned} & \|u_1^{h,n+1}\|^2 + k \sum_{i=1}^{n+1} (h + \nu) \|\nabla u_1^{h,i}\|^2 \\ & \leq C(\|u_0^s\|^2 + k \sum_{i=1}^{n+1} \|f(t_i)\|^2). \end{aligned} \quad (3.2.1)$$

**Proof 3.2.1** Can be found in [15].

### 3.3 Error Estimates of AV Approximation

**Definition 3.3.1** Let

$$\begin{aligned} C_u &:= \|u(x, t)\|_{L^\infty(0, T; L^\infty(\Omega))}, \\ C_{\nabla u} &:= \|\nabla u(x, t)\|_{L^\infty(0, T; L^\infty(\Omega))}, \\ \tilde{C}_u &:= \|u(x, t)\|_{L^\infty(0, T; L^2(\Omega))}, \\ \tilde{C}_{\nabla u} &:= \|\nabla u(x, t)\|_{L^\infty(0, T; L^2(\Omega))}, \end{aligned}$$

and introduce  $\tilde{C}$ , satisfying

$$\inf_{v \in V^h} \|\nabla(u - v)\| \leq C_1 \inf_{v \in X^h} \|\nabla(u - v)\| \leq C_2 h^m \|u\|_{H^{m+1}} \leq \tilde{C} h^m \quad (3.3.1)$$

Also, using the constant  $C(\Omega)$  from Lemma 2.3, we define  $\bar{C} := 1728C^4(\Omega)$ .

**Theorem 3.3.2** Let  $f \in L^2(0, T; H^{-1})$ , let  $u_1^h, u_2^h$  satisfy (1.0.3) and (1.0.4), respectively,

$$\begin{aligned} k & \leq \frac{h + \nu}{4C_u^2 + 2(h + \nu)C_{\nabla u} + 2\bar{C}\tilde{C}^4(h + \nu)^{-2}h^{4m}}, \\ u & \in L^2(0, T; H^{m+1}(\Omega)) \cap L^\infty(0, T; L^\infty(\Omega)), \nabla u \in L^\infty(0, T; L^\infty(\Omega)), \\ u_t & \in L^2(0, T; H^{m+1}(\Omega)), u_{tt} \in L^2(0, T; L^2(\Omega)), p \in L^2(0, T; H^m(\Omega)). \end{aligned}$$

Then there exist a constant  $C = C(\Omega, T, u, p, f, h + \nu)$ , such that

$$\max_{1 \leq i \leq N} \|u(t_i) - u_1^{h,i}\| + \left( k \sum_{i=1}^{n+1} (h + \nu) \|\nabla(u(t_i) - u_1^{h,i})\|^2 \right)^{1/2} \leq C(h^m + h + k)$$

**Proof 3.3.1** Can be found [15].

We will need the following lemma in the proof of Theorem (3.3.4).

**Lemma 3.3.3** Let  $f \in L^2(0, T; H^{-1}(\Omega))$ . Suppose  $\phi^0$  and  $\phi^1$  to be the Stokes projections of the initial velocity and velocity at the first time level, respectively. Let  $m \geq 2$  and

$$k < \frac{4(h + \nu)}{13(4(h + \nu)C_{\nabla u} + 3C_u^2)}.$$

Then there exist a constant  $C = C(\Omega, T, u, p, f, h + \nu)$ , such that

$$\left\| \frac{\phi^1 - \phi^0}{k} \right\|^2 + \frac{13}{2}(h + \nu)k \left\| \nabla \frac{\phi^1 - \phi^0}{k} \right\|^2 \leq C(kh^{2m} + h^2 + k^2 + k^2h^{2m-3}) \quad (3.3.2)$$

**Proof 3.3.2** From the Stokes Projection(2.0.2) and error decomposition(2.0.8), we have

$$(h + \nu)(\nabla \phi^0, \nabla v) - (h + \nu)(\nabla \eta^0, \nabla v) - (p^0 - q, \nabla \cdot v) = 0 \quad (3.3.3)$$

On the other hand the solution at the first time level satisfies the following

$$\begin{aligned}
& \|\frac{\phi^1 - \phi^0}{k}\|^2 + (h + \nu)(\nabla \phi^1, \nabla \frac{\phi^1 - \phi^0}{k}) + b^*(u^1, u^1, \frac{\phi^1 - \phi^0}{k}) \\
& \quad - b^*(u_1^1, u_1^1, \frac{\phi^1 - \phi^0}{k}) + (p^1, \nabla \cdot \frac{\phi^1 - \phi^0}{k}) \\
& \quad = h(\nabla u^1, \nabla \frac{\phi^1 - \phi^0}{k}) + k(\rho^1, \frac{\phi^1 - \phi^0}{k}) \\
& \quad + (\frac{\eta^1 - \eta^0}{k}, \frac{\phi^1 - \phi^0}{k}) + (h + \nu)(\nabla \eta^1, \frac{\phi^1 - \phi^0}{k}), \\
& \text{where } k\rho^1 = \frac{u^1 - u^0}{k} - u_t^1 = ku_{tt}^\theta, \text{ for some } \theta \in (0, k).
\end{aligned} \tag{3.3.4}$$

Subtracting equation 3.3.3 from equation 3.3.4 for  $v = \frac{\phi^1 - \phi^0}{k}$ , we have

$$\begin{aligned}
& \|\frac{\phi^1 - \phi^0}{k}\|^2 + k(h + \nu)\|\nabla \frac{\phi^1 - \phi^0}{k}\|^2 \\
& + b^*(u^1, u^1, \frac{\phi^1 - \phi^0}{k}) - b^*(u_1^1, u_1^1, \frac{\phi^1 - \phi^0}{k}) \\
& \quad - k(\frac{p^1 - p^0}{k} - q, \nabla \cdot \frac{\phi^1 - \phi^0}{k}) \\
& = h(\nabla u^1, \nabla \frac{\phi^1 - \phi^0}{k}) + (\rho^1, \frac{\phi^1 - \phi^0}{k}) + (\frac{\eta^1 - \eta^0}{k}, \frac{\phi^1 - \phi^0}{k}) \\
& \quad + k(h + \nu)(\nabla \frac{\eta^1 - \eta^0}{k}, \nabla \frac{\phi^1 - \phi^0}{k})
\end{aligned} \tag{3.3.5}$$

Adding and subtracting  $b^*(u_1^1, u^1, \frac{\phi^1 - \phi^0}{k})$  to the nonlinear terms in equation (3.3.5) together with error decomposition (2.0.8) gives

$$\begin{aligned}
& b^*(u^1, u^1, \frac{\phi^1 - \phi^0}{k}) - b^*(u_1^1, u_1^1, \frac{\phi^1 - \phi^0}{k}) \\
& = b^*(e_1^1, u^1, \frac{\phi^1 - \phi^0}{k}) + b^*(u_1^1, e_1^1, \frac{\phi^1 - \phi^0}{k}) \\
& = b^*(\phi^1, u^1, \frac{\phi^1 - \phi^0}{k}) - b^*(\eta^1, u^1, \frac{\phi^1 - \phi^0}{k}) \\
& \quad + b^*(u_1^1, \phi^1, \frac{\phi^1 - \phi^0}{k}) - b^*(u_1^1, \eta^1, \frac{\phi^1 - \phi^0}{k})
\end{aligned} \tag{3.3.6}$$

Adding and subtracting  $\phi^0$  to the first component of the first nonlinear term in the equation (3.3.6) gives

$$b^*(\phi^1, u^1, \frac{\phi^1 - \phi^0}{k}) = kb^*(\frac{\phi^1 - \phi^0}{k}, u^1, \frac{\phi^1 - \phi^0}{k}) + b^*(\phi^0, u^1, \frac{\phi^1 - \phi^0}{k}) \quad (3.3.7)$$

In the first nonlinear term of (3.3.7), applying Cauchy-Schwarz and Young's inequalities together with the regularity assumption of  $u$  and bound 2.0.4 gives

$$\begin{aligned} k|b^*(\frac{\phi^1 - \phi^0}{k}, u^1, \frac{\phi^1 - \phi^0}{k})| &\leq kC_{\nabla u} \|\frac{\phi^1 - \phi^0}{k}\|^2 \\ &+ k\mu^*(h + \nu) \|\nabla \frac{\phi^1 - \phi^0}{k}\|^2 + k \frac{C_u^2}{16(h + \nu)\mu^*} \|\frac{\phi^1 - \phi^0}{k}\|^2 \end{aligned} \quad (3.3.8)$$

In the second nonlinear term of (3.3.7), applying Cauchy Schwarz and Young's inequalities together with bound 2.0.4 and inverse inequality (2.0.7) gives

$$|b^*(\phi^0, u^1, \frac{\phi^1 - \phi^0}{k})| \leq \mu \|\frac{\phi^1 - \phi^0}{k}\|^2 + \frac{Ch^{-2}}{4\mu} \|\nabla \phi^0\|^2 \quad (3.3.9)$$

In the second nonlinear term of (3.3.6), applying Cauchy Schwarz and Young's inequalities together with bound 2.0.4 and inverse inequality (2.0.7) gives

$$|b^*(\eta^1, u^1, \frac{\phi^1 - \phi^0}{k})| \leq \mu \|\frac{\phi^1 - \phi^0}{k}\|^2 + \frac{Ch^{-2}}{4\mu} \|\nabla \eta^1\|^2 \quad (3.3.10)$$

For the third nonlinear term of equation (3.3.6), applying error decomposition (2.0.8) gives

$$\begin{aligned} |b^*(u_1^1, \phi^1, \frac{\phi^1 - \phi^0}{k})| &\leq |b^*(u^1, \phi^1, \frac{\phi^1 - \phi^0}{k})| + |b^*(\phi^1, \phi^1, \frac{\phi^1 - \phi^0}{k})| \\ &\quad + |b^*(\eta^1, \phi^1, \frac{\phi^1 - \phi^0}{k})| \end{aligned} \quad (3.3.11)$$

Since nonlinear form is skew-symmetric in the second and third entry, we can replace terms like the first nonlinear term in the inequality (3.3.11) with terms like  $|b^*(u^1, \phi^0, \frac{\phi^1 - \phi^0}{k})|$ . Applying Cauchy-Schwarz and Young's inequalities together with the regularity assumption of  $u$  and inverse inequality gives

$$|b^*(u^1, \phi^0, \frac{\phi^1 - \phi^0}{k})| \leq 2\mu \|\frac{\phi^1 - \phi^0}{k}\|^2 + \frac{C_u^2}{4\mu} (\|\nabla \phi^0\|^2 + h^{-2} \|\phi^0\|^2) \quad (3.3.12)$$

Applying Young's inequality together with the sharper bound (2.0.3) and inverse inequality (2.0.7) in the second nonlinear term of 3.3.11 gives

$$|b^*(\phi^1, \phi^1, \frac{\phi^1 - \phi^0}{k})| = |b^*(\phi^1, \phi^0, \frac{\phi^1 - \phi^0}{k})| \leq \mu \|\frac{\phi^1 - \phi^0}{k}\|^2 + \frac{Ch^{-3}}{4\mu} \|\phi^1\|^2 \|\nabla \phi^0\|^2 \quad (3.3.13)$$

For the last nonlinear term in the inequality (3.3.11), we can apply 2.0.4 and inverse inequality followed by Young's inequality to have

$$\begin{aligned} |b^*(\eta^1, \phi^1, \frac{\phi^1 - \phi^0}{k})| &= |b^*(\eta^1, \phi^0, \frac{\phi^1 - \phi^0}{k})| \\ &\leq \mu \|\frac{\phi^1 - \phi^0}{k}\|^2 + \frac{Ch^{-2}}{4\mu} \|\nabla \eta^1\|^2 \|\nabla \phi^0\|^2 \end{aligned} \quad (3.3.14)$$

For the forth nonlinear term of equation (3.3.6), applying error decomposition gives

$$\begin{aligned} |b^*(u_1^1, \eta^1, \frac{\phi^1 - \phi^0}{k})| &\leq |b^*(u^1, \eta^1, \frac{\phi^1 - \phi^0}{k})| + |b^*(\phi^1, \eta^1, \frac{\phi^1 - \phi^0}{k})| \\ &\quad + |b^*(\eta^1, \eta^1, \frac{\phi^1 - \phi^0}{k})| \end{aligned} \quad (3.3.15)$$

For all the nonlinear terms in the inequality (3.3.15), we can apply bound 2.0.4 and inverse inequality followed by Young's inequality to have

$$|b^*(u^1, \eta^1, \frac{\phi^1 - \phi^0}{k})| \leq \mu \|\frac{\phi^1 - \phi^0}{k}\|^2 + \frac{Ch^{-2}}{4\mu} \|\nabla \eta^1\|^2 \quad (3.3.16)$$

$$|b^*(\phi^1, \eta^1, \frac{\phi^1 - \phi^0}{k})| \leq \mu \|\frac{\phi^1 - \phi^0}{k}\|^2 + Ch^{-4} \|\nabla \eta^1\|^2 \|\phi^1\|^2 \quad (3.3.17)$$

$$|b^*(\eta^1, \eta^1, \frac{\phi^1 - \phi^0}{k})| \leq \mu \|\frac{\phi^1 - \phi^0}{k}\|^2 + Ch^{-2} \|\nabla \eta^1\|^4 \quad (3.3.18)$$

Apply Cauchy-Schwarz and Young's inequalities to (3.3.5).

Since  $\|\nabla \cdot \frac{\phi^1 - \phi^0}{k}\| \leq d \|\nabla \frac{\phi^1 - \phi^0}{k}\|$ ,

$$\begin{aligned} & (1 - 12\mu - (\frac{C_{\nabla u}}{2} + \frac{C_u^2}{16(h+\nu)\mu^*})k) \|\frac{\phi^1 - \phi^0}{k}\|^2 \\ & + (1 - 3\mu^*)(h+\nu)k \|\nabla \frac{\phi^1 - \phi^0}{k}\|^2 \\ \leq & \frac{dk}{4\mu^*(h+\nu)} \inf_{q \in Q^h} \|\frac{p^1 - p^0}{k} - q\|^2 + \frac{h^2}{4\mu} \|\Delta u^1\|^2 + \frac{k^2}{4\mu} \|\rho^1\|^2 + \frac{1}{4\mu} \|\frac{\eta^1 - \eta^0}{k}\|^2 \\ & + \frac{k(h+\nu)}{4\mu^*} \|\nabla \frac{\eta^1 - \eta^0}{k}\|^2 + \frac{Ch^{-2}}{4\mu} \|\nabla \phi^0\|^2 + \frac{C_u^2}{4\mu} \|\nabla \phi^0\|^2 + \frac{C_u^2 h^{-2}}{4\mu} \|\phi^0\|^2 \\ & + \frac{Ch^{-3}}{4\mu} \|\phi^1\|^2 \|\nabla \phi^0\|^2 + \frac{Ch^{-2}}{4\mu} \|\nabla \eta^1\|^2 \|\nabla \phi^0\|^2 \\ & + \frac{Ch^{-2}}{2\mu} \|\nabla \eta^1\|^2 + Ch^{-4} \|\phi^1\|^2 \|\nabla \eta^1\|^2 + Ch^{-2} \|\nabla \eta^1\|^4 \end{aligned} \quad (3.3.19)$$

Use the approximation properties of  $X^h, Q^h$ . Since the mesh nodes do not depend upon the time level, it follows from (2.0.5), (2.0.6) that

$$\begin{aligned}
\inf_{q \in Q} \left\| \frac{p^1 + p^0}{k} - q \right\|^2 &\leq Ch^{2m}, \\
\left\| \frac{\eta_2^1 - \eta_2^0}{k} \right\|^2 &\leq Ch^{2m+2}, \\
\|\eta_2^1\|^2 &\leq Ch^{2m+2}.
\end{aligned} \tag{3.3.20}$$

Taking  $\mu = 1/13$  and  $\mu^* = 1/6$  and using bounds (3.3.20) for each term, it follows from the regularity assumption of  $u$  that

$$\begin{aligned}
&\left( \frac{1}{13} - \left( \frac{C_{\nabla u}}{2} + \frac{3C_u^2}{8(h+\nu)} \right) k \right) \left\| \frac{\phi^1 - \phi^0}{k} \right\|^2 + \frac{1}{2}(h+\nu)k \left\| \nabla \frac{\phi^1 - \phi^0}{k} \right\|^2 \\
&\leq C(h^{2m-2} + h^2 + k^2 + k^2 h^{2m-3})
\end{aligned} \tag{3.3.21}$$

The last inequality implies the lemma statement.

**Theorem 3.3.4** *Let the assumptions of Lemma (3.3.3) and Theorem (3.3.2) be satisfied.*

Let  $k \leq \min\left\{ \frac{h+\nu}{2CC_{\nabla u}(h+\nu)+2CC_u^2}, C(h+\nu)^{\frac{5}{3}}, C(h+\nu)^3 \right\}$

Then

$$\left\| \frac{e_1^{n+1} - e_1^n}{k} \right\|^2 + k \sum_{i=0}^n (h+\nu) \left\| \nabla \frac{e_1^{i+1} - e_1^i}{k} \right\|^2 \leq C[h^{2m} + h^2 + k^2]$$

**Proof 3.3.3** *Start with the proof of the bound for  $\left\| \frac{\phi^{n+1} - \phi^n}{k} \right\|$ .*

*From the inequality (5.14) in [15], we have*

$$\begin{aligned}
& \|s^{h,n+1}\|^2 + k(h+\nu) \sum_{i=1}^n \|\nabla s^{h,i+1}\|^2 \\
& \leq \|s^{h,1}\|^2 + C[h^{2m} + h^2 + k^2] \\
& + Ck \sum_{i=1}^n (C_{\nabla u} + \frac{C_u^2}{h+\nu} + \frac{1}{(h+\nu)^3} \|\nabla e_1^i\|^4) \|s^{h,i+1}\|^2, \\
& \text{where } s^{h,n+1} = \frac{\phi^{n+1} - \phi^n}{k}
\end{aligned} \tag{3.3.22}$$

In order to apply Gronwall's Lemma 2.0.4 in the inequality 3.3.22, we have to verify that

$$Ck(C_{\nabla u} + \frac{C_u^2}{h+\nu} + \frac{1}{(h+\nu)^3} \|\nabla e_1^i\|^4) < 1.$$

To this end, we can first assume

$$Ck(C_{\nabla u} + \frac{C_u^2}{h+\nu}) < \frac{1}{2} \text{ and } \frac{Ck}{(h+\nu)^3} \|\nabla e_1^i\|^4 < \frac{1}{2}.$$

Due to the first inequality, we have a bound on  $k$  in the form

$$k < \frac{h+\nu}{CC_{\nabla u}(h+\nu) + CC_u^2}.$$

For the second inequality we investigate case by case.

For  $k \leq h$ , it follows from the inverse inequality and theorem (3.3.2) that

$$\begin{aligned}
\frac{Ck}{(h+\nu)^3} \|\nabla e_1^i\|^4 & \leq \frac{Ckh^{-4}}{(h+\nu)^3} \|e_1^i\|^4 \leq \frac{Ck}{(h+\nu)^3} (1 + \frac{k}{h})^4 \\
& \leq \frac{Ck}{(h+\nu)^3} < \frac{1}{2}.
\end{aligned}$$

Thus, we have a bound on  $k$  in the form  $k < C(h+\nu)^3$ .

For  $h \leq k$ , it follows from the theorem (3.3.2) that

$$\frac{Ck}{(h+\nu)^3} \|\nabla e_1^i\|^4 \leq \frac{Ck^{-1}}{(h+\nu)^5} (h^4 + k^4) \leq \frac{2Ck^3}{(h+\nu)^5} < \frac{1}{2}.$$

It follows from the above calculations and theorem statement that

$$(C_{\nabla u} + \frac{C_u^2}{h+\nu} + \frac{1}{(h+\nu)^3} \|\nabla e_1^i\|^4)k < 1.$$

Now, we can apply discrete Gronwall's Lemma in the inequality (3.3.22) to have following bound

$$\|\frac{\phi^{n+1} - \phi^n}{k}\|^2 + (h+\nu)k \sum_{i=1}^n \|\nabla \frac{\phi^{i+1} - \phi^i}{k}\|^2 \leq C[h^{2m} + h^2 + k^2] \quad (3.3.23)$$

Using the triangle inequality in the error decomposition (2.0.8), we obtain

$$\|\frac{e_1^{n+1} - e_1^n}{k}\|^2 + k \sum_{i=0}^n (h+\nu) \|\nabla \frac{e_1^{i+1} - e_1^i}{k}\|^2 \leq C[h^{2m} + h^2 + k^2] \quad (3.3.24)$$

This result proves the theorem.

# Chapter 4

## Correction Step Approximation

In this section we prove the unconditional stability and error estimate of the correction step approximation  $u_2^h$ . Over  $0 \leq t \leq T < \infty$  the approximations  $u_2^h$  is bounded uniformly in  $Re$ .

Hence, the formulation (1.0.4) gives  $O(h^2 + k^2)$  accurate, unconditionally stable extension of correction step approximation to the time-dependent Navier-Stokes equations.

We start by proving stability of correction step approximation.

### 4.1 Stability of the CS Approximation

**Theorem 4.1.1** *Let  $f \in L^2(0, T; H^{-1}(\Omega))$ , let  $u_1^h, u_2^h$  satisfy (1.0.3) and (1.0.4), respectively. Then for  $n=0, \dots, N-1$ ,*

$$\begin{aligned}
& \|u_2^{h,n+1}\|^2 + 5h^2(h+\nu)^{-2}\|u_1^{h,n+1}\|^2 + k \sum_{i=1}^{n+1} (h+\nu) \|\nabla u_2^{h,i}\|^2 \\
& \leq C[\|u_0^s\|^2 + (h+\nu)^{-1} k \sum_{i=1}^{n+1} \|f(t_i)\|_{-1}^2].
\end{aligned}$$

**Proof 4.1.1** Take  $v^h = u_2^{h,n+1} \in V^h$  in the equation (1.0.4). This gives with Cauchy-Schwarz and Young's inequality that

$$\begin{aligned}
& \frac{1}{2k}(\|u_2^{h,n+1}\|^2 - \|u_2^{h,n}\|^2) + (h+\nu) \|\nabla u_2^{h,n+1}\|^2 \\
& \leq \left( \frac{f(t_{n+1}) + f(t_n)}{2}, u_2^{h,n+1} \right) + \frac{\nu}{2} k \left( \nabla \left( \frac{u_1^{n+1} - u_1^n}{k} \right), \nabla u_2^{h,n+1} \right) \quad (4.1.1) \\
& + \frac{1}{2} b^*(u_1^{h,n+1}, u_1^{h,n+1}, u_2^{h,n+1}) - \frac{1}{2} b^*(u_1^{h,n}, u_1^{h,n}, u_2^{h,n+1}) + h(\nabla u_1^{h,n}, \nabla u_2^{h,n})
\end{aligned}$$

It follows from Cauchy-Schwarz, Young's and triangle inequalities with the error estimate  $e_1^i = u(t_i) - u_1^i$  that

$$\begin{aligned}
& \frac{\nu}{2} k \left( \nabla \left( \frac{u_1^{n+1} - u_1^n}{k} \right), \nabla u_2^{h,n+1} \right) \leq \mu(h+\nu) \|\nabla u_2^{h,n+1}\|^2 \\
& + \frac{\nu^2 k^2}{8\mu(h+\nu)} \left\| \nabla \left( \frac{u^{n+1} - u^n}{k} \right) \right\|^2 + \frac{\nu^2 k}{8\mu(h+\nu)} k \left\| \nabla \left( \frac{e_1^{n+1} - e_1^n}{k} \right) \right\|^2. \quad (4.1.2)
\end{aligned}$$

Adding and subtracting  $\frac{1}{2} b^*(u_1^{h,n+1}, u_1^{h,n}, u_2^{h,n+1})$  to the nonlinear terms and applying the bound (2.0.4) followed by Cauchy-Schwarz, Young's and triangle inequalities with regularity assumption of  $u$ , we have

$$\begin{aligned}
& \frac{1}{2}b^*(u_1^{h,n+1}, u_1^{h,n+1}, u_2^{h,n+1}) - \frac{1}{2}b^*(u_1^{h,n}, u_1^{h,n}, u_2^{h,n+1}) \\
\leq & \frac{1}{2}[kb^*(u_1^{h,n+1}, \frac{u_1^{h,n+1} - u_1^{h,n}}{k}, u_2^{h,n+1}) + kb^*(\frac{u_1^{h,n+1} - u_1^{h,n}}{k}, u_1^{h,n}, u_2^{h,n+1})] \\
& \leq 2\mu(h+\nu)||\nabla u_2^{h,n+1}||^2 \\
& + \frac{1}{16\mu(h+\nu)^3}(h+\nu)k||\nabla(\frac{u_1^{h,n+1} - u_1^{h,n}}{k})||^2[(h+\nu)k||\nabla u_1^{h,n+1}||^2 \\
& \quad + (h+\nu)k||\nabla u_1^{h,n}||^2] \\
& \leq 2\mu(h+\nu)||\nabla u_2^{h,n+1}||^2 \\
& + \frac{1}{8\mu(h+\nu)^3}(h+\nu)k||\nabla(\frac{e_1^{h,n+1} - e_1^{h,n}}{k})||^2[(h+\nu)k||\nabla u_1^{h,n+1}||^2 \\
& \quad + (h+\nu)k||\nabla u_1^{h,n}||^2] \\
& + \frac{1}{8\mu(h+\nu)^2}kC_{\nabla u_t}^2[(h+\nu)k||\nabla u_1^{h,n+1}||^2 + (h+\nu)k||\nabla u_1^{h,n}||^2], \\
& \text{where } C_{\nabla u_t} = ||\nabla(\frac{u^{n+1} - u^n}{k})||^2
\end{aligned} \tag{4.1.3}$$

Cauchy-Schwarz and Young's inequalities with  $\mu = 1/10$  give

$$\begin{aligned}
& \frac{1}{2k}(|u_2^{h,n+1}|^2 - |u_2^{h,n}|^2) + \frac{1}{2}(h+\nu)||\nabla u_2^{h,n+1}||^2 \\
& \leq \frac{5}{2(h+\nu)}||\frac{f(t_{n+1}) - f(t_n)}{2}||_{-1}^2 \\
& + \frac{5\nu^2 k^2}{4(h+\nu)}C_{\nabla u_t}^2 + \frac{5\nu^2 k}{4(h+\nu)^2}k(h+\nu)||\nabla(\frac{e_1^{n+1} - e_1^n}{k})||^2 \\
& \quad + \frac{5h^2}{2(h+\nu)^2}(h+\nu)||\nabla u_1^{h,n+1}||^2 \\
& + \frac{5}{4(h+\nu)^3}(h+\nu)k||\nabla(\frac{e_1^{h,n+1} - e_1^{h,n}}{k})||^2[(h+\nu)k||\nabla u_1^{h,n+1}||^2 \\
& \quad + (h+\nu)k||\nabla u_1^{h,n}||^2] \\
& + \frac{5}{4(h+\nu)^2}kC_{\nabla u_t}^2[(h+\nu)k||\nabla u_1^{h,n+1}||^2 + (h+\nu)k||\nabla u_1^{h,n}||^2]
\end{aligned} \tag{4.1.4}$$

Multiplying inequality by  $2k$  and summing over all time levels followed by Lemma (3.2.1) and Theorem (3.3.4) give

$$\begin{aligned}
& \|u_2^{h,n+1}\|^2 + \sum_{i=1}^{n+1} (h+\nu) \|\nabla u_2^{h,i}\|^2 \\
& \leq \|u_0^s\|^2 + \frac{5}{(h+\nu)} k \sum_{i=1}^{n+1} \left\| \frac{f(t_i) - f(t_{i-1})}{2} \right\|_{-1}^2 \\
& \quad + \frac{5\nu^2 k^3}{2(h+\nu)} C_{\nabla u_t}^2 + \frac{5\nu^2 k^2}{2(h+\nu)^2} C(h^{2m} + h^2 + k^2) \\
& \quad + \frac{5h^2}{(h+\nu)^2} (\|u_0^s\|^2 - \|u_1^{h,n+1}\|^2 + \frac{1}{h+\nu} k \sum_{i=1}^{n+1} \|f(t_i)\|_{-1}^2) \\
& \quad + \frac{5}{2(h+\nu)^2} \left( \frac{(h^{2m} + h^2 + k^2)}{(h+\nu)} + k^2 C_{\nabla u_t}^2 \right) [2\|u_0^s\|^2 \\
& \quad + \frac{1}{h+\nu} k \sum_{i=1}^{n+1} \|f(t_i)\|_{-1}^2 + \frac{1}{h+\nu} k \sum_{i=1}^{n+1} \|f(t_i)\|_{-1}^2]
\end{aligned} \tag{4.1.5}$$

After some algebraic manipulation, we have the following inequality

$$\begin{aligned}
& \|u_2^{h,n+1}\|^2 + \frac{5h^2}{(h+\nu)^2} \|u_1^{h,n+1}\|^2 + \sum_{i=1}^{n+1} (h+\nu) \|\nabla u_2^{h,i}\|^2 \\
& \leq \|u_0^s\|^2 + \frac{5}{(h+\nu)} k \sum_{i=1}^{n+1} \left\| \frac{f(t_i) - f(t_{i-1})}{2} \right\|_{-1}^2 \\
& \quad + \frac{5\nu^2 k^3}{2(h+\nu)} C_{\nabla u_t}^2 + \frac{5\nu^2 k^2}{2(h+\nu)^2} C(h^{2m} + h^2 + k^2) \\
& \quad + C(\|u_0^s\|^2 + \frac{1}{h+\nu} k \sum_{i=1}^{n+1} \|f(t_i)\|_{-1}^2)
\end{aligned} \tag{4.1.6}$$

The last inequality implies the theorem statement.

The result of Theorem (4.1.1), combined with the result Proposition (3.1.1), proves the unconditional stability of both  $u_1^{h,i}$  and  $u_2^{h,i}$  for any  $i \geq 0$ .

Next we will prove the error estimate of correction step approximation.

## 4.2 Error Estimate of CS Approximation

**Theorem 4.2.1** *Let the assumptions of Theorem (3.3.4) be satisfied. Let*

$$k < \frac{h + \nu}{(h + \nu)C_{\nabla u} + 2C_u^2 + (h + \nu)Ch^{m-1} + 2Ch^{2m}}.$$

*Then there exists a constant  $C = C(\Omega, T, u, p, f, h + \nu)$ , such that*

$$\begin{aligned} \max_{1 \leq i \leq N} \|u(t_i) - u_2^{h,i}\| + (k \sum_{i=0}^n (h + \nu) \|\nabla(u(t_i) - u_2^{h,i})\|^2)^{1/2} \\ \leq C(h^m + h^2 + k^2 + hk). \end{aligned}$$

**Proof 4.2.1** *By Taylor expansion around  $t = \frac{t_{n+1} + t_n}{2}$ , we have  $\frac{u^{n+1} - u^n}{k} - \frac{u_t^{n+1} + u_t^n}{2} = k^2 \rho^{n+1}$ , where  $\rho^{n+1} = \frac{u_{ttt}^{n+\frac{1}{2}}}{8}$ .*

*Summing variational formulations of NSE at  $t = t_n$  and at  $t = t_{n+1}$ , and then, dividing by 2, we have the following equation.*

$$\begin{aligned} & \left( \frac{u^{n+1} - u^n}{k}, v \right) + \frac{\nu}{2} \left( \nabla \left( \frac{u^{n+1} + u^n}{2} \right), \nabla v \right) + \frac{1}{2} b^*(u^{n+1}, u^{n+1}, v) \\ & \quad + \frac{1}{2} b^*(u^n, u^n, v) - \left( \frac{p^{n+1} + p^n}{2}, \nabla \cdot v \right) \\ & = \left( \frac{f(t_{n+1}) + f(t_n)}{2}, v \right) - \left( \frac{u_t^{n+1} + u_t^n}{2}, v \right) + \left( \frac{u^{n+1} - u^n}{k}, v \right) \end{aligned} \quad (4.2.1)$$

*Subtracting (1.0.4) from the equation (4.2.1) and using error decomposition (2.0.8), we have*

$$\begin{aligned}
& \left( \frac{\phi_2^{h,n+1} - \phi_2^{h,n}}{k}, \phi_2^{h,n+1} \right) + (h + \nu)(\nabla \phi_2^{h,n+1}, \nabla \phi_2^{h,n+1}) \\
&= \frac{\nu}{2} k (\nabla (\frac{e_1^{h,n+1} - e_1^{h,n}}{k}), \nabla \phi_2^{h,n+1}) + (\frac{p^{h,n+1} + p^{h,n}}{2} - p_2^{n+1}, \nabla \phi_2^{h,n+1}) \\
&\quad - b^*(u^{n+1}, \phi_2^{h,n+1}, \phi_2^{h,n+1}) + b^*(u^{n+1}, \eta_2^{n+1}, \phi_2^{h,n+1}) \\
&\quad - b^*(\phi_2^{h,n+1}, u_2^{h,n+1}, \phi_2^{h,n+1}) + b^*(\eta_2^{n+1}, u_2^{h,n+1}, \phi_2^{h,n+1}) \\
&\quad + \frac{1}{2} k b^*(\frac{u^{n+1} - u^n}{k}, e_1^{h,n}, \phi_2^{h,n+1}) + \frac{1}{2} k b^*(u^{n+1}, \frac{e_1^{h,n+1} - e_1^{h,n}}{k}, \phi_2^{h,n+1}) \\
&+ \frac{1}{2} k b^*(e_1^{h,n+1}, \frac{u_1^{h,n+1} - u_1^{h,n}}{k}, \phi_2^{h,n+1}) + \frac{1}{2} k b^*(\frac{e_1^{h,n+1} - e_1^{h,n}}{k}, u_1^{h,n}, \phi_2^{h,n+1}) \\
&\quad h(\nabla e_1^{h,n+1}, \nabla \phi_2^{h,n+1}) + k^2(\rho^{n+1}, \phi_2^{h,n+1}) + (\frac{\eta_2^{n+1} - \eta_2^n}{k}, \phi_2^{h,n+1}) \\
&\quad + (h + \nu)(\nabla \eta_2^{n+1}, \nabla \phi_2^{h,n+1})
\end{aligned} \tag{4.2.2}$$

We bound the nonlinear terms on the right hand side of (4.2.2), starting now with the second, fifth and sixth terms. Use the bound (2.0.4), regularity assumption of  $u$  and Young's inequality to obtain

$$\begin{aligned}
|b^*(u^{n+1}, \eta_2^{n+1}, \phi_2^{h,n+1})| &\leq \mu(h + \nu) \|\nabla \phi_2^{h,n+1}\|^2 \\
&\quad + \frac{C_{\nabla u}^2}{4\mu(h + \nu)} \|\nabla \eta_2^{n+1}\|^2
\end{aligned} \tag{4.2.3}$$

$$\begin{aligned}
|\frac{1}{2} k b^*(\frac{u^{n+1} - u^n}{k}, e_1^{h,n}, \phi_2^{h,n+1})| &\leq \mu(h + \nu) \|\nabla \phi_2^{h,n+1}\|^2 \\
&\quad + \frac{k^2 C_{\nabla u_t}^2}{16\mu(h + \nu)} \|\nabla e_1^{h,n}\|^2
\end{aligned} \tag{4.2.4}$$

$$\begin{aligned}
|\frac{1}{2} k b^*(u^{n+1}, \frac{e_1^{h,n+1} - e_1^{h,n}}{k}, \phi_2^{h,n+1})| &\leq \mu(h + \nu) \|\nabla \phi_2^{h,n+1}\|^2 \\
&\quad + \frac{k^2 C_{\nabla u}^2}{16\mu(h + \nu)} \|\nabla (\frac{e_1^{h,n+1} - e_1^{h,n}}{k})\|^2
\end{aligned} \tag{4.2.5}$$

In order to obtain bounds on the third and the fourth terms, we use the error decomposition (2.0.8), triangle inequality, bound (2.0.3), regularity assumption of  $u$  and Young's inequality

$$\begin{aligned}
|b^*(\phi_2^{h,n+1}, u_2^{h,n+1}, \phi_2^{h,n+1})| &\leq |b^*(\phi_2^{h,n+1}, u^{h,n+1}, \phi_2^{h,n+1})| \\
&\quad + |b^*(\phi_2^{h,n+1}, \eta_2^{n+1}, \phi_2^{h,n+1})| \\
&\leq 2\mu(h + \nu) \|\nabla \phi_2^{h,n+1}\|^2 \\
&\quad + \|\phi_2^{h,n+1}\|^2 \left( \frac{C_{\nabla u}}{2} + \frac{C_u^2}{16\mu(h + \nu)} + \frac{1}{2} \|\nabla \eta_2^{n+1}\| + \frac{1}{16\mu(h + \nu)} \|\nabla \eta_2^{n+1}\|^2 \right)
\end{aligned} \tag{4.2.6}$$

$$\begin{aligned}
|b^*(\eta_2^{n+1}, u_2^{h,n+1}, \phi_2^{h,n+1})| &\leq |b^*(\eta_2^{n+1}, u^{h,n+1}, \phi_2^{h,n+1})| + |b^*(\eta_2^{n+1}, \eta_2^{n+1}, \phi_2^{h,n+1})| \\
&\leq 2\mu(h + \nu) \|\nabla \phi_2^{h,n+1}\|^2 \\
&\quad + \frac{1}{4\mu(h + \nu)} \|\nabla \eta_2^{n+1}\|^2 (C_{\nabla u}^2 + \|\nabla \eta_2^{n+1}\|^2)
\end{aligned} \tag{4.2.7}$$

For the bounds on the seventh and the eighth terms, we use the error decomposition  $u_1^{h,n} = u^n - e_1^{h,n}$ , triangle inequality, bound (2.0.4), regularity assumptions of  $u$  and Young's inequality

$$\begin{aligned}
\left| \frac{1}{2} k b^*(e_1^{h,n+1}, \frac{u_1^{h,n+1} - u_1^{h,n}}{k}, \phi_2^{h,n+1}) \right| &\leq \left| \frac{1}{2} k b^*(e_1^{h,n+1}, \frac{e_1^{h,n+1} - e_1^{h,n}}{k}, \phi_2^{h,n+1}) \right| \\
&\quad + \left| \frac{1}{2} k b^*(e_1^{h,n+1}, \frac{u^{n+1} - u^n}{k}, \phi_2^{h,n+1}) \right| \leq 2\mu(h + \nu) \|\nabla \phi_2^{h,n+1}\|^2 \\
&\quad + \frac{1}{16\mu(h + \nu)} \|\nabla e_1^{h,n+1}\|^2 (k^2 C_{\nabla u_t}^2 + k^2 \|\nabla(\frac{e_1^{h,n+1} - e_1^{h,n}}{k})\|^2)
\end{aligned} \tag{4.2.8}$$

$$\begin{aligned}
\left| \frac{1}{2} k b^*(\frac{e_1^{h,n+1} - e_1^{h,n}}{k}, u_1^{h,n}, \phi_2^{h,n+1}) \right| &\leq \left| \frac{1}{2} k b^*(\frac{e_1^{h,n+1} - e_1^{h,n}}{k}, u^n, \phi_2^{h,n+1}) \right| \\
&\quad + \left| \frac{1}{2} k b^*(\frac{e_1^{h,n+1} - e_1^{h,n}}{k}, e_1^{h,n}, \phi_2^{h,n+1}) \right| \leq 2\mu(h + \nu) \|\nabla \phi_2^{h,n+1}\|^2 \\
&\quad + \frac{1}{16\mu(h + \nu)} \|\nabla(\frac{e_1^{h,n+1} - e_1^{h,n}}{k})\|^2 (k^2 C_{\nabla u}^2 + k^2 \|\nabla e_1^{h,n}\|^2)
\end{aligned} \tag{4.2.9}$$

Apply the Cauchy-Schwarz and Young's inequality to (4.2.2). Since  $\|\nabla \cdot \phi_2^{h,n+1}\|^2 \leq d\|\nabla \phi_2^{h,n+1}\|^2$  for all  $\mu > 0$

$$\begin{aligned}
& \frac{\|\phi_2^{h,n+1}\|^2 - \|\phi_2^{h,n}\|^2}{2k} + (1 - 16\mu)(h + \nu)\|\nabla \phi_2^{h,n+1}\|^2 \\
& \leq \frac{d}{4\mu(h + \nu)} \inf_{q^h \in Q^h} \left\| \frac{p^{h,n+1} + p^{h,n}}{2} - q^{h,n+1} \right\|^2 \\
& \quad + \frac{\nu^2 k^2}{16\mu(h + \nu)} \left\| \nabla \left( \frac{e_1^{h,n+1} - e_1^{h,n}}{k} \right) \right\|^2 \\
& \quad + \frac{h^2}{4\mu(h + \nu)} \|\nabla e_1^{h,n+1}\|^2 + \frac{k^4}{4\mu(h + \nu)} \|\rho^{n+1}\|_{-1}^2 \\
& + \frac{1}{4\mu(h + \nu)} \left\| \frac{\eta_2^{n+1} - \eta_2^n}{k} \right\|_{-1}^2 + \frac{h + \nu}{4\mu} \|\nabla \eta_2^{n+1}\|^2 + \frac{C_{\nabla u}^2}{4\mu(h + \nu)} \|\nabla \eta_2^{n+1}\|^2 \\
& \quad + \frac{k^2 C_{\nabla u_t}^2}{16\mu(h + \nu)} \|\nabla e_1^{h,n}\|^2 + \frac{k^2 C_{\nabla u}^2}{16\mu(h + \nu)} \left\| \nabla \left( \frac{e_1^{h,n+1} - e_1^{h,n}}{k} \right) \right\|^2 \\
& + \|\phi_2^{h,n+1}\|^2 \left( \frac{C_{\nabla u}}{2} + \frac{C_u^2}{16\mu(h + \nu)} + \frac{1}{2} \|\nabla \eta_2^{n+1}\| + \frac{1}{16\mu(h + \nu)} \|\nabla \eta_2^{n+1}\|^2 \right) \\
& \quad + \frac{1}{4\mu(h + \nu)} \|\nabla \eta_2^{n+1}\|^2 (C_{\nabla u}^2 + \|\nabla \eta_2^{n+1}\|^2) \\
& \quad + \frac{k^2}{16\mu(h + \nu)} \|\nabla e_1^{h,n+1}\|^2 (C_{\nabla u_t}^2 + \left\| \nabla \left( \frac{e_1^{h,n+1} - e_1^{h,n}}{k} \right) \right\|^2) \\
& \quad + \frac{k^2}{16\mu(h + \nu)} \left\| \nabla \left( \frac{e_1^{h,n+1} - e_1^{h,n}}{k} \right) \right\|^2 (C_{\nabla u}^2 + \|\nabla e_1^{h,n}\|^2)
\end{aligned} \tag{4.2.10}$$

Take  $\mu = 1/32$ , multiply (4.2.10) by  $2k$  and sum over all time levels. It follows from the regularity assumptions of theorem that

$$k \sum_{i=0}^n \|\rho^{i+1}\|_{-1}^2 k^4 \leq Ck \sum_{i=0}^n \|\rho^{i+1}\|^2 k^4 \leq Ck^4$$

Therefore we obtain

$$\begin{aligned}
& \|\phi_2^{h,n+1}\|^2 + (h + \nu)k \sum_{i=0}^n \|\nabla \phi_2^{h,i+1}\|^2 \\
& \leq \frac{C}{h + \nu} k \sum_{i=0}^n \left[ \inf_{q^h \in Q^h} \left\| \frac{p^{h,i+1} + p^{h,i}}{2} - q^{h,i+1} \right\|^2 \right. \\
& \quad k^2 \left\| \nabla \left( \frac{e_1^{i+1} - e_1^{i+1}}{k} \right) \right\|^2 + h^2 \|\nabla e_1^{i+1}\|^2 + k^4 + \left\| \frac{\eta_2^{i+1} - \eta_2^i}{k} \right\|_{-1}^2 \\
& \quad + \|\nabla \eta_2^{i+1}\|^2 + k^2 \|\nabla e_1^{i+1}\|^2 + \|\nabla \eta_2^{i+1}\|^4 \\
& \quad + k \left\| \nabla \left( \frac{e_1^{i+1} - e_1^{i+1}}{k} \right) \right\|^2 (k \|\nabla e_1^{i+1}\|^2 + k \|\nabla e_1^i\|^2) \\
& \quad + k \sum_{i=0}^n \|\phi_2^{h,i+1}\|^2 \left[ \frac{C_{\nabla u}}{2} + \frac{2C_u^2}{(h + \nu)} + \frac{1}{2} \|\nabla \eta_2^{i+1}\| \right. \\
& \quad \left. + \frac{2}{h + \nu} \|\nabla \eta_2^{i+1}\|^2 \right] + \|\phi_2^{h,0}\|^2
\end{aligned} \tag{4.2.11}$$

Take  $\tilde{u}^i$  in the error decomposition (2.0.8) to be the  $L^2$ -projection onto  $V^h$ , for  $i \geq 1$ . Take  $\tilde{u}^0$  to be  $u_0^s$ . This gives  $\phi_2^{h,0} = 0$  and  $e_1^0 = \eta_2^0$ . Also it follows from the Proposition (3.1.2) that  $\|\eta_2^0\| \leq Ch^m$ ; under the assumption of the theorem applying the discrete Gronwall's lemma (2.0.4) and using bounds in theorems (3.3.2), (3.3.4), give

$$\begin{aligned}
& \|\phi_2^{h,n+1}\|^2 + (h + \nu)k \sum_{i=0}^n \|\nabla \phi_2^{h,i+1}\|^2 \\
& \leq \frac{C}{h + \nu} k \sum_{i=0}^n \left[ \inf_{q^h \in Q^h} \left\| \frac{p^{h,i+1} + p^{h,i}}{2} - q^{h,i+1} \right\|^2 \right. \\
& \quad + \frac{k^2}{h + \nu} (h^2 + k^2) + \frac{h^2}{h + \nu} (h^2 + k^2) + k^4 \\
& \quad + \left\| \frac{\eta_2^{i+1} - \eta_2^i}{k} \right\|_{-1}^2 + \|\nabla \eta_2^{i+1}\|^2 + \|\nabla \eta_2^{i+1}\|^4 \\
& \quad \left. + \frac{k}{(h + \nu)^2} (h^2 + k^2)(h^2 + k^2) \right] + Ch^{2m}
\end{aligned} \tag{4.2.12}$$

Use the approximation properties of  $X^h, Q^h$ . Since the mesh nodes do not depend upon the time level, it follows from (2.0.5), (2.0.6) that

$$\begin{aligned}
k \sum_{i=0}^n \inf_{q^h \in Q^h} \left\| \frac{p^{h,i+1} + p^{h,i}}{2} - q^{h,i+1} \right\|^2 &\leq Ch^{2m}, \\
k \sum_{i=0}^n \left\| \frac{\eta_2^{i+1} - \eta_2^i}{k} \right\|_{-1}^2 &\leq Ck \sum_{i=0}^n \left\| \frac{\eta_2^{i+1} - \eta_2^i}{k} \right\|^2 \leq Ch^{2m}, \\
k \sum_{i=0}^n \|\eta_2^{i+1}\|^2 &\leq Ch^{2m}.
\end{aligned} \tag{4.2.13}$$

Bounds (4.2.12) and (4.2.13) give the following result

$$\begin{aligned}
&\|\phi_2^{h,n+1}\|^2 + (h + \nu)k \sum_{i=0}^n \|\nabla \phi_2^{h,i+1}\|^2 \\
&\leq \frac{C}{(h + \nu)^2} (h^{2m} + h^4 + k^4 + h^2 k^2).
\end{aligned} \tag{4.2.14}$$

Using the error decomposition and triangle inequality with (4.2.14), we obtain

$$\begin{aligned}
&\|e_2^{h,n+1}\| + ((h + \nu)k \sum_{i=0}^n \|\nabla e_2^{h,i+1}\|^2)^{\frac{1}{2}} \\
&\leq \frac{C}{(h + \nu)} (h^m + h^2 + k^2 + hk).
\end{aligned} \tag{4.2.15}$$

This proves the Theorem (4.2.1). Thus, we derived the error estimates, that agree with the general theory of the defect and deferred correction methods. Briefly, the Correction Step approximation  $u_2^h$  is improved by an order of  $h$  in space and of  $k$  in time, compared to the Artificial Viscosity approximation  $u_1^h$ .

Next, we will give some computational results.

# Chapter 5

## Computational Tests

We perform one quantitative and one qualitative test of the proposed regularization procedure. In both tests the non-homogeneous Dirichlet boundary conditions are implemented, and the computational results support the theoretical findings.

### 5.1 Quantitative Test

For the quantitative assessment, consider a two-dimensional problem with a known exact solution. The traveling wave solution of the NSE in  $\Omega = [0.5, 1]^2$  is given by

$$\begin{aligned} u &= \begin{pmatrix} 0.75 + 0.25 \cos(2\pi(x-t)) \sin(2\pi(y-t)) \exp(-8\pi^2 t\nu) \\ 0.75 - 0.25 \sin(2\pi(x-t)) \cos(2\pi(y-t)) \exp(-8\pi^2 t\nu) \end{pmatrix}, \\ p &= -\frac{1}{64}(\cos(4\pi(x-t)) + \cos(4\pi(y-t))) \exp(-16\pi^2 t\nu), \end{aligned} \quad (5.1.1)$$

and the right-hand side  $f$  and initial condition  $u_0$  are computed so that (5.1.1) satisfies (1.0.1). The final time in the computations is taken to be  $T = 1$ .

In order to verify the theoretical claims on the convergence rates, we take the time

step equal to the mesh diameter,  $\Delta t = h$ .

For  $\nu = \frac{1}{100}$  the calculated convergence rates in Tables 5.1 and 5.2 confirm what is predicted by Theorems (3.3.2) and (4.2.1) for  $(P_2, P_1)$  Taylor-Hood finite elements: the convergence rates are doubled after the correction step. Notice also the asymptotic character of convergence, typical of the defect correction methods.

Table 5.1: AV approximation,  $\nu = 0.01$ .

N	$\ u - u_1^h\ _{L^2(0,T;L^2(\Omega))}$	rate	$\ u - u_1^h\ _{L^2(0,T;H^1(\Omega))}$	rate
8	0.0139742	-	0.23282	-
16	0.00945258	0.56	0.179798	0.37
32	0.00580328	0.70	0.123682	0.54
64	0.00331349	0.81	0.0766837	0.69
128	0.00178142	0.90	0.0433087	0.82
256	0.000922772	0.95	0.0228883	0.92

Table 5.2: Correction step approximation,  $\nu = 0.01$ .

N	$\ u - u_2^h\ _{L^2(0,T;L^2(\Omega))}$	rate	$\ u - u_2^h\ _{L^2(0,T;H^1(\Omega))}$	rate
8	0.0106313	-	0.189918	-
16	0.0060028	0.83	0.128519	0.56
32	0.00272105	1.14	0.0710604	0.86
64	0.000993846	1.45	0.0314236	1.18
128	0.000302142	1.72	0.0111824	1.49
256	0.0000817667	1.89	0.00336761	1.73

As the viscosity coefficient  $\nu$  decreases, the convergence rates improve slower - see the results for the flow at  $\nu = \frac{1}{2000}$  in Tables 5.3 and 5.4.

Table 5.3: AV approximation,  $\nu = 0.0005$ .

N	$\ u - u_1^h\ _{L^2(0,T;L^2(\Omega))}$	rate	$\ u - u_1^h\ _{L^2(0,T;H^1(\Omega))}$	rate
8	0.0262208	-	0.439399	-
16	0.0188948	0.47	0.367997	0.26
32	0.0125722	0.59	0.291022	0.34
64	0.00776946	0.69	0.2206	0.40
128	0.00443914	0.81	0.159449	0.49
256	0.00237518	0.9	0.108957	0.55

Table 5.4: Correction step approximation,  $\nu = 0.0005$ .

N	$\ u - u_2^h\ _{L^2(0,T;L^2(\Omega))}$	rate	$\ u - u_2^h\ _{L^2(0,T;H^1(\Omega))}$	rate
8	0.0217697	-	0.396863	-
16	0.0141143	0.63	0.32536	0.29
32	0.00777988	0.86	0.249133	0.39
64	0.0036525	1.09	0.178223	0.48
128	0.00152888	1.26	0.118064	0.59
256	0.000597594	1.36	0.0709845	0.73

To further comment on the asymptotic nature of convergence of defect correction methods, notice that the a priori error estimates have the term  $(h + \nu)^{-1}$  in the right hand side. This decreases the convergence rates on the coarse meshes, where  $h \gg \nu$ . The term that contains  $\nabla(u - u_i^h)$ ,  $i = 1, 2$  in the left hand side is also proportional to  $(h + \nu)$ , which further decreases the convergence rates in the  $H^1$ -seminorm on coarse meshes for problems with high Reynolds number. We also ran the same tests (not shown here) with  $\Delta t = h^2$  and obtained the convergence rates very similar to those presented above, which indicates that the reduced convergence rates are due to the

asymptotic behaviour of the defect correction, and not the deferred correction part of the error.

## 5.2 Qualitative Test

For the qualitative assessment, consider the 2-D flow past an obstacle, at high Reynolds number  $Re = 600$ . The von Karman vortex street is expected to be seen for a fully resolved flow; on a coarse mesh with  $h \sim \frac{1}{32}$  the true solution demonstrates the oscillatory behavior past the obstacle (Figure 5.1). Note that the solution is known to depend on the Reynolds number in the following manner: for  $1 < Re < 10$  the flow is no longer symmetric behind the obstacle, for  $10 < Re < 100$  re-circulation areas appear in the wake behind the obstacle and, as the Reynolds number grows beyond  $Re = 100$ , these vortices develop and start to oscillate. Roughly at  $Re = 1000$  turbulence develops and the coherent structures in the flow disappear.

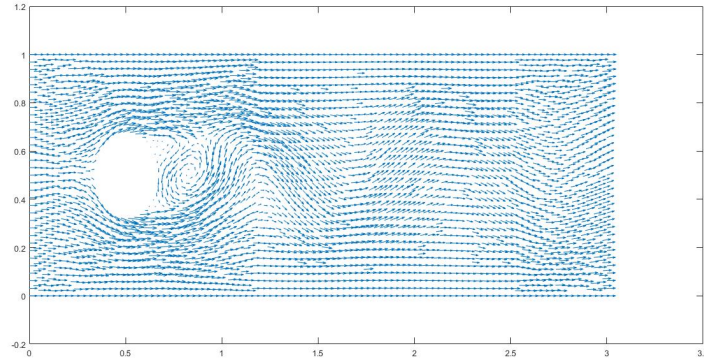


Figure 5.1: DNS velocity field  $u$

We compute the defect step solution  $u_1$  and the corrected solution  $u_2$  on the same coarse mesh with 32 nodes per unit boundary ( $h \sim \frac{1}{32}$ ). The domain is  $\Omega =$

$[0, 1] \times [0, 3]$  with a circle of radius 0.15, centered at  $(0.5, 0.5)$ , cut out of  $\Omega$ . The parabolic inflow on the left boundary is introduced, with zero forcing. The results were computed with  $Re = 600$ ,  $T = 20$ ,  $\Delta t = h$ .

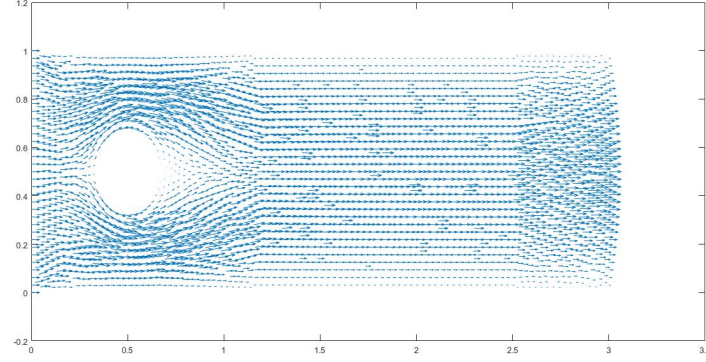


Figure 5.2: AV Approximation  $u_1^h$

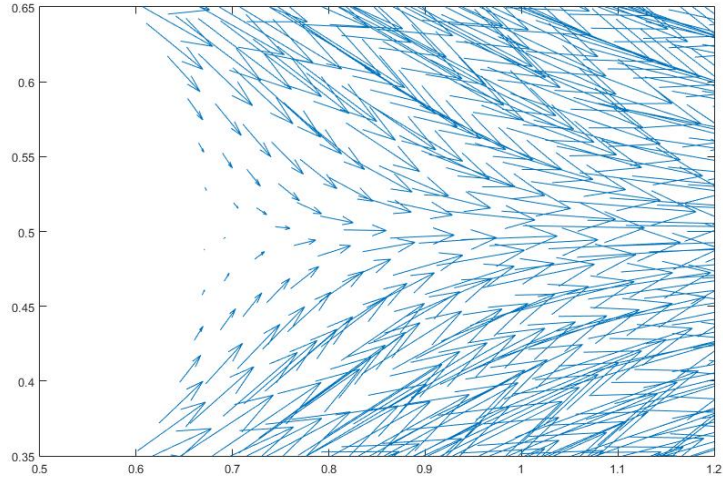


Figure 5.3: AV Approximation zoomed in

As seen in Figures 5.2 and 5.3, artificial viscosity approximation gives a result that cannot capture the flow pattern due to high viscosity coefficient and low accuracy of the AV approximation.

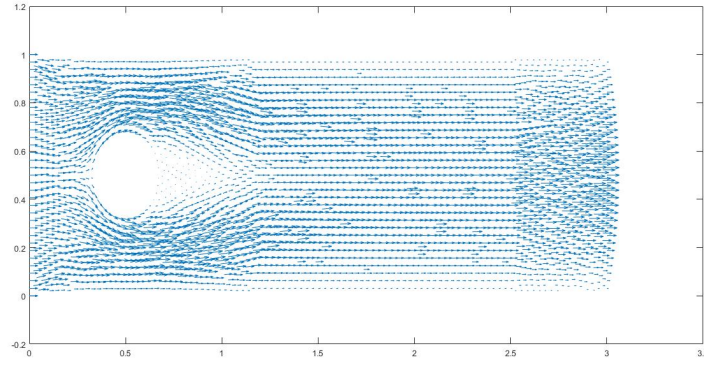


Figure 5.4: CS Approximation  $u_2^h$

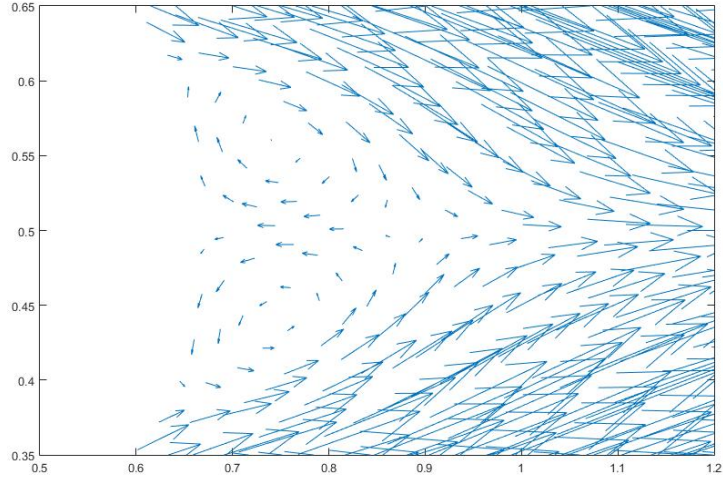


Figure 5.5: CS Approximation Zoomed in

Although the correction solution is computed with the same viscosity coefficient as the AV approximation, it gives some qualitative features of the flow pattern even on the coarse mesh - one can clearly see the re-circulation regions in the wake. This demonstrates the qualitative behavior of the correction step solution: it behaves as if the Reynolds number of the flow was increased, although the matrix of the system remains the same as in the AV case. Thus, the benefits of using the correction procedure

are clear: for virtually no extra cost (when the parallelization is implemented) one can model turbulent flows at increasing Reynolds numbers (the interesting possibility that two or three correction steps would deepen this effect is yet to be explored).

# Chapter 6

## Conclusions

The method for solving nearly singular, time-dependent problems is presented. It combines both deferred correction method for the time derivative and the defect correction method for the spatial operator. The method is applied to the Navier-Stokes equations, and the stability and the error estimate results for velocity is given. As observed in both theoretical and numerical results, the method presented is high accurate in both time and space.

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