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Uniformly resolvable decompositions of K_v in 1-factors and 4-stars

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Abstract

If X is a connected graph, then an X -factor of a larger graph is a spanning subgraph in which all of its components are isomorphic to X . A *uniformly resolvable $\{X, Y\}$ -decomposition* of the complete graph K_v is an edge decomposition of K_v into exactly r X -factors and s Y -factors. In this article we determine necessary and sufficient conditions for when the complete graph K_v has a uniformly resolvable decompositions into 1-factors and $K_{1,4}$ -factors.

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1 Introduction and definitions

For any graph G , let $V(G)$ and $E(G)$ be the vertex-set and the edge-set of G , respectively. Throughout the paper K_v will denote the complete graph on v vertices, while $K_v \setminus K_h$ will denote the graph with $V(K_v)$ as vertex-set and $E(K_v) \setminus E(K_h)$ as edge-set (this graph is sometimes referred to as a complete graph of order v with a *hole* of size h).

Given a set \mathcal{H} of pairwise non-isomorphic graphs, an \mathcal{H} -*decomposition* (or \mathcal{H} -*design*) of a graph G is a decomposition of the edge-set of G into subgraphs (called *blocks*) isomorphic to some element of \mathcal{H} . An \mathcal{H} -*factor* of G is a spanning subgraph of G whose components are isomorphic to a members of \mathcal{H} . If $X \in \mathcal{H}$, then an X -*factor* is a spanning subgraph whose components are isomorphic to X . An \mathcal{H} -decomposition of G is *resolvable* if its blocks can be partitioned into \mathcal{H} -factors and is called an \mathcal{H} -*factorization* of G . An \mathcal{H} -factorization \mathcal{F} of G is called *uniform* if each factor of \mathcal{F} is an X -factor for some graph $X \in \mathcal{H}$. A K_2 -factorization of G is known as a 1-*factorization* and its factors are called 1-*factors*; it is well known that a 1-factorization of K_v exists if and only if v is even ([18]).

An \mathcal{H} -isofactorization of G is an \mathcal{H} -factorization with isomorphic factors. If \mathcal{H} is the set of all possible cycles of K_v , then determining the existence of possible \mathcal{H} -isofactorizations of K_v , v odd is known as the *Oberwolfach Problem*. It was first posed in 1967 by Gerhard Ringel and asks whether it is possible to seat an odd number v of mathematicians at n round tables in $(v - 1)/2$ meals so that each mathematician sits next to everyone else exactly once. If the n round tables are of sizes p_1, p_2, \dots, p_n (with $p_1 + p_2 + \dots + p_n = v$), the Oberwolfach Problem asks for an isofactorization of K_v with factors isomorphic to the 2-factor with components isomorphic to cycles of length p_1, p_2, \dots, p_n . The uniform Oberwolfach problem (all cycles of the 2-factor have the same size) has been completely solved by Alspach and Häggkvist [4] and Alspach, Schellenberg, Stinson and Wagner [5].

Additional existence problems for \mathcal{H} -factorizations of K_v have been studied and many results have been obtained, especially on uniformly resolvable \mathcal{H} -decompositions: when \mathcal{H} is a set of two complete graphs of order at most five in [8, 21, 22, 24]; when \mathcal{H} is a set of two or three paths on two, three or four vertices in [11, 12, 17]; for $\mathcal{H} = \{P_3, K_3 + e\}$ in [10]; for $\mathcal{H} = \{K_3, K_{1,3}\}$ in [14]; for $\mathcal{H} = \{C_4, P_3\}$ in [19]; for $\mathcal{H} = \{K_3, P_3\}$ in [20]. And most famous is the variation of the Oberwolfach problem known as the Hamilton-Waterloo problem. In this problem the meals for the dinning mathematicians take place at two different venues. Hence a decomposition of K_v is sought where the factors can be either one of two types. In particular the uniform case asks for a decomposition of K_v into C_p -factors and C_q -factors. Thus the round tables in one venue sit p mathematicians whereas the tables in the other venue each sit q . Of course in this case p and q must divide v , v must be odd and $\mathcal{H} = \{C_p, C_q\}$.

A uniformly resolvable $\{X, Y\}$ -decomposition of K_v into exactly r X -factors and s Y -factors, is abbreviated (X, Y) -URD($v; r, s$). The uniform case of the Hamilton-Waterloo problem is the existence problem for (C_p, C_q) -URD($v; r, s$).

In this paper, we focus on the case $\mathcal{H} = \{K_2, K_{1,n}\}$. The resulting uniformly resolvable problem, affectionally known as the stars and stripes problem, can be seen as the attendance at a conference of v participants that has $v/(n+1)$ parallel sessions and in which during the breaks the participants pair up for one on one discussions. The parallel sessions are $K_{1,n}$ -factors and are also known as star-factors: the one on one discussions are K_2 -factors and are the stripes.

The existence of a $(K_2, K_{1,n})$ -URD($v; r, s$) was studied and completely solved for $n = 3$ in [6] and [13]. Here we concentrate on the case $n = 4$ and, because the results for the extremal cases $s = 0$ and $r = 0$ are known, i.e.:

- a $(K_2, K_{1,n})$ -URD($v; r, 0$) exists if and only if v is even;
- if n is even, a $(K_2, K_{1,n})$ -URD($v; 0, s$) exists is and only $v \equiv 1 \pmod{2n}$ and $v \equiv 0 \pmod{n+1}$ ([25]);

we deal with $(K_2, K_{1,4})$ -URD($v; r, s$) where $r, s > 0$ and so $v \equiv 0 \pmod{10}$ and $r = v - 1 - \frac{8s}{5}$.

For $v \equiv 0 \pmod{10}$, define $J(v)$ according to the following table:

v	$J(v)$
0 (mod 40)	$\{(v - 1 - 8x, 5x), x = 0, 1, \dots, \frac{v-8}{8}\}$
10 (mod 40)	$\{(v - 1 - 8x, 5x), x = 0, 1, \dots, \frac{v-2}{8}\}$
20 (mod 40)	$\{(v - 1 - 8x, 5x), x = 0, 1, \dots, \frac{v-4}{8}\}$
30 (mod 40)	$\{(v - 1 - 8x, 5x), x = 0, 1, \dots, \frac{v-6}{8}\}$

Table 1: The set $J(v)$

In this paper we completely solve the existence problem of a $(K_2, K_{1,4})$ -URD($v; r, s$) by proving the following result.

Main Theorem. *For any $v \equiv 0 \pmod{10}$, there exists a $(K_2, K_{1,4})$ -URD($v; r, s$) if and only if $(r, s) \in J(v)$.*

2 Necessary conditions

In this section we will give necessary conditions for the existence of a $(K_2, K_{1,4})$ -URD($v; r, s$).

Lemma 2.1. *Let $v \equiv 0 \pmod{10}$. If there exists a $(K_2, K_{1,4})$ -URD($v; r, s$), then $(r, s) \in J(v)$.*

Proof. Assume that there exists a $(K_2, K_{1,4})$ -URD($v; r, s$). By resolvability, it follows that

$$\frac{rv}{2} + \frac{4sv}{5} = \frac{v(v-1)}{2}$$

and hence

$$5r + 8s = 5(v-1). \quad (1)$$

Denote by R the set of r K_2 -factors and by S the set of s $K_{1,4}$ -factors. Since the factors of R are regular of degree 1, every vertex of K_v is incident to r edges in R and $(v-1) - r$ edges in S . Assume that the any fixed vertex appears in x $K_{1,4}$ -factors with degree 4 and in y $K_{1,4}$ -factors with degree 1. Since

$$x + y = s \quad \text{and} \quad 4x + y = v - 1 - r,$$

the equality (1) gives

$$5(v-1-4x-y) + 8(x+y) = 5(v-1),$$

which implies $y = 4x$ and so $s = 5x$. Further, replacing $s = 5x$ in Equation (1) provides $r = v - 1 - 8x$, where $x \leq \frac{v-1}{8}$ (because r is a non-negative integer). \square

3 General constructions and related structures

An \mathcal{H} -decomposition of $K_{u(g)}$, the complete multi-partite graph with u parts of size g , is known as a *group divisible decomposition* (\mathcal{H} -GDD, in short) of type g^u ; the parts of size g are called the *groups*. (If \mathcal{H} consists of complete subgraphs, then a GDD is called a *group divisible design*.) When $\mathcal{H} = \{H\}$ we simply write H -GDD and when $H = K_n$ we refer to such a group divisible design as an n -GDD. We denote a (uniformly) resolvable \mathcal{H} -GDD by \mathcal{H} -(U)RGDD. Specifically, a (X, Y) -URGDD with r X -factors and s Y -factors is denoted by (X, Y) -URGDD(r, s). It is easy to deduce that the number of H -factors of a H -RGDD is $\frac{g(u-1)|V(H)|}{2|E(H)|}$.

If the blocks of an n -GDD of type g^u can be partitioned into *partial* factors, each of which contains all vertices except those of one group, we refer to such a decomposition as a *n-frame*. It is easy to deduce that the number of partial factors missing a specified group is $\frac{g}{n-1}$ (see [9]). It is well known that a 2-frame of type g^u exists if and only if $u \geq 3$ and $g(u-1) \equiv 0 \pmod{2}$; and a 3-frame of type g^u exists if and only if $u \geq 4$, g is even and $g(u-1) \equiv 0 \pmod{3}$ (see [7]).

An \mathcal{H} -decomposition of $K_{v+h} \setminus K_h$ is known as an *incomplete \mathcal{H} -design of order $v+h$ with a hole of size h* . We are interested in incomplete resolvable \mathcal{H} -designs, which will be used in the “Filling” and “Frame”-Constructions of this section. These designs have two types factors: *partial* factors, which cover every vertex except the ones in the hole; and *full* factors, which cover every vertex of K_{v+h} .

Specifically, a (X, Y) -IURD($v+h, h; [r', s'], [r, s]$) is a uniformly resolvable (X, Y) -decomposition of $K_{v+h} \setminus K_h$ with r' partial X -factors and s' partial Y -factors which

cover every vertex not in the hole, and r X -factors and s Y -factors which cover every point of K_{v+h} .

Given a graph G and a positive integer t , then $G_{(t)}$ will denote the graph on $V(G) \times \mathbb{Z}_t$ with edge-set $\{\{x_i, y_j\} : \{x, y\} \in E(G), i, j \in \mathbb{Z}_t\}$, where the subscript notation a_i is used to denote the pair (a, i) . The graph $G_{(t)}$ is said to be obtained from G by expanding each vertex t times. When $G = K_n$, the graph $G_{(t)}$ is the complete equipartite graph $K_{\underbrace{t, t, \dots, t}_{n \text{ times}}}$ with n parts of size t and will be denoted by

$K_{n(t)}$; while $C_{n(t)}$ will denote the graph $G_{(t)}$ where G is an n -cycle.

Remark 3.1. Note that the graph $G_{(t)}$ admits t 1-factors corresponding to each 1-factor of G ; for instance, because a $2m$ -cycle has two 1-factors, $C_{2m(t)}$ admits $2t$ 1-factors.

For any two pairs of non-negative integers (r, s) and (r', s') , define $(r, s) + (r', s') = (r + r', s + s')$. If X and X' are two sets of pairs of non-negative integers and a is a positive integer, then $X + X'$ will denote the set $\{(r, s) + (r', s') : (r, s) \in X, (r', s') \in X'\}$ and $a * X$ will denote the set of all pairs of non-negative integers which can be obtained by adding any a pairs of X together (repetitions of elements of X are allowed).

Construction 3.2. (GDD-construction) Let t be a positive integer and \mathcal{G} be an H -RGDD of type g^u , where H is a graph with $n \geq 2$ vertices and m edges. If there exists a (X, Y) -URD (\bar{r}, \bar{s}) of $H_{(t)}$ for each $(\bar{r}, \bar{s}) \in J$, then so does a (X, Y) -URGDD (r, s) of type $(gt)^u$ for each $(r, s) \in \alpha * J$, where $\alpha = \frac{ng(u-1)}{2m}$.

Proof. Let $G_i, i = 1, 2, \dots, u$, be the groups and $F_1, F_2, \dots, F_\alpha$ an H -factorization of \mathcal{G} , where $\alpha = \frac{ng(u-1)}{2m}$. Expand each vertex t times, and for each block B of the H -factor F_j , for $j = 1, 2, \dots, \alpha$, place a copy of a (X, Y) -URD (r_j, s_j) of $H_{(t)}$ with $(r_j, s_j) \in J$ on $V(B) \times \mathbb{Z}_t$. Thus we obtain a (X, Y) -URGDD (r, s) of type $(gt)^u$ with $r = \sum_{j=1}^\alpha r_j$ and $s = \sum_{j=1}^\alpha s_j$, and so $(r, s) \in \alpha * J$. □

Construction 3.3. (Filling Construction) Suppose there exists a (X, Y) -URGDD (r, s) of type g^u for each $(r, s) \in J$. If there exists a (X, Y) -URD $(g; r', s')$, for each $(r', s') \in J'$, then so does:

- (i) a (X, Y) -IURD $(ug, g; [r', s'], [r, s])$ for each $(r', s') \in J'$ and $(r, s) \in J$;
- (ii) a (X, Y) -URD $(ug; \bar{r}, \bar{s})$, for each $(\bar{r}, \bar{s}) \in J' + J$.

Proof. Fix any pairs $(r, s) \in J$ and $(r', s') \in J'$, and start with a (X, Y) -URGDD (r, s) with u groups of size $g, G_i, i = 1, 2, \dots, u$. For every $i = 2, 3, \dots, u$, place a copy of a (X, Y) -URD $(g; r', s')$ on G_i to obtain a (X, Y) -IURD $(gu, g; [r', s'], [r, s])$ with G_1 as the hole. Finally, on G_1 place a copy of a (X, Y) -URD $(g; r', s')$ to obtain a (X, Y) -URD $(gu; r' + r, s' + s)$. □

Remark 3.4. Note that the “filling” technique allows us to construct a (X, Y) -URD($v + h; r' + r, s' + s$) whenever a (X, Y) -IURD($v + h, h; [r', s'], [r, s]$) and a (X, Y) -URD($h; r', s'$) are given.

Construction 3.5. (Frame-construction) Let v, g, t, h and u be positive integers such that $v = gtu + h$. If there exists

- (i) a n -frame \mathcal{F} of type $g^u, n \geq 2$;
- (ii) a (X, Y) -URGDD(\bar{r}, \bar{s}) of type t^n for each $(\bar{r}, \bar{s}) \in J$;
- (iii) a (X, Y) -IURD($gt + h, h; [r', s'], [\bar{r}, \bar{s}]$) for each $(r', s') \in J'$ and $(\bar{r}, \bar{s}) \in \alpha * J$, where $\alpha = \frac{g}{n-1}$;
- (iv) a (X, Y) -URD($h; r', s'$) for each $(r', s') \in J'$;

then so does a (X, Y) -URD($v; r, s$) for each $(r, s) \in J' + u\alpha * J$.

Proof. Let \mathcal{F} be an n -frame of type g^u with groups $G_i, i = 1, 2, \dots, u$. Expand each vertex t times and add a set $H = \{a_1, a_2, \dots, a_h\}$. For $j = 1, 2, \dots, \alpha = \frac{g}{n-1}$, let F_{ij} be the j -th partial factor which misses the group G_i . For each block $B \in F_{ij}$, on $B \times \mathbb{Z}_t$ place a copy, $\mathcal{D}_{ij}(B)$, of a (X, Y) -URGDD(r_{ij}, s_{ij}) of type t^n with $(r_{ij}, s_{ij}) \in J$. For $i = 1, 2, \dots, u$, on $H \cup (G_i \times \mathbb{Z}_t)$ place a copy \mathcal{D}_i of a (X, Y) -IURD($gt + h, h; [r', s'], [r_i, s_i]$) with $(r', s') \in J'$ and $(r_i, s_i) = \sum_{j=1}^{\alpha} (r_{ij}, s_{ij}) \in \alpha * J$. For every $i = 1, 2, \dots, u$, combine all together the factors of $\mathcal{D}_{ij}(B), B \in F_{ij}$, along with the full factors of \mathcal{D}_i so to obtain \bar{r} X -factors and \bar{s} Y -factors, where $(\bar{r}, \bar{s}) = \sum_{i=1}^u (r_i, s_i) \in u\alpha * J$. Now, fill the hole H with a copy \mathcal{D} of a (X, Y) -URD($h; r', s'$) with $(r', s') \in J'$. Combine the factors of \mathcal{D} with the partial factors of \mathcal{D}_i to obtain further r' X -factors and s' Y -factors with $(r', s') \in J'$. The result is a (X, Y) -URD($v; r, s$) where $(r, s) = (r' + \bar{r}, s' + \bar{s}) \in J' + u\alpha * J$. □

4 Small cases

In what follows, we will denote by $(a_1; a_2, a_3, a_4, a_5)$ the graph $K_{1,4}$ on the vertex-set $\{a_1, a_2, a_3, a_4, a_5\}$ with edge-set $\{\{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}, \{a_1, a_5\}\}$; and by (a_1, a_2, \dots, a_n) the n -cycle on $\{a_1, a_2, \dots, a_n\}$ with edge-set $\{\{a_1, a_2\}, \{a_2, a_3\}, \dots, \{a_{n-1}, a_n\}, \{a_n, a_1\}\}$. If the vertices of $B = (a; b, c, d, e)$ belong to \mathbb{Z}_n , then we will say *orbit of B under \mathbb{Z}_n* the set $\{(a + i; b + i, c + i, d + i, e + i) : i \in \mathbb{Z}_n\}$.

For any positive integer n , let $I(n)$ be the set of pairs of non-negative integers

$$I(n) = \{(n - 8x, 5x) : x = 0, 1, \dots, \lfloor \frac{n}{8} \rfloor\}.$$

By induction it is easy to prove the following lemma.

Lemma 4.1. *If $n \equiv 0 \pmod{8}$, then $\alpha * I(n) = I(\alpha n)$ for any positive integer α .*

Lemma 4.2. *A $(K_2, K_{1,4})$ -URD(r, s) of $C_{2m(5)}$ exists for every $(r, s) \in I(10)$.*

Proof. The case $(r, s) = (10, 0)$ follows by Remark 3.1. For the case $(r, s) = (2, 5)$, let $C_{2m(5)}$ be the graph obtained by starting with the cycle $C = (0, 1, \dots, 2m - 1)$ on \mathbb{Z}_{2m} and taking the five $K_{1,4}$ -factors

$$F_j = \{(i_j; (1 + i)_{j+1}, (1 + i)_{j+2}, (1 + i)_{j+3}, (1 + i)_{j+4}) : i \in \mathbb{Z}_{2m}\}, \quad j \in \mathbb{Z}_5.$$

The two 1-factors are easily obtainable by decomposing the remaining set of edges, which can be considered as the disjoint union of the five $2m$ -cycles $C_j = (0_j, 1_j, \dots, (2m - 1)_j)$, $j \in \mathbb{Z}_5$. □

Lemma 4.3. *A $(K_2, K_{1,4})$ -URGDD (r, s) of type 2^5 exists for every $(r, s) \in I(8)$.*

Proof. The case $(r, s) = (8, 0)$ corresponds to a 1-factorization of $K_{5(2)}$, which is known to exist ([7]). To settle the case $(r, s) = (0, 5)$, take the orbit of $B = (0; 1, 2, 3, 4)$ under \mathbb{Z}_{10} , which can be decomposed into the five $K_{1,4}$ -factors:

$$F_j = \{B + j + 5i : i = 0, 1\}, \quad j = 0, 1, 2, 3, 4.$$

The groups are the cosets $H, H + 1, H + 2, H + 3, H + 4$ of $H = 5\mathbb{Z}_{10}$ in \mathbb{Z}_{10} . □

Lemma 4.4. *A $(K_2, K_{1,4})$ -URD $(10; r, s)$ exists for every $(r, s) \in J(10)$.*

Proof. The case $(r, s) = (9, 0)$ corresponds to a 1-factorization of the complete K_{10} , which is known to exist ([7]). For the case $(r, s) = (1, 5)$, apply the Filling Construction to a $(K_2, K_{1,4})$ -URGDD $(0, 5)$ of type 2^5 , which is given by Lemma 4.3. □

Lemma 4.5. *A $(K_2, K_{1,4})$ -URGDD (r, s) of type 10^2 exists for every $(r, s) \in I(10)$.*

Proof. Apply the GDD-construction with $t = 5$ to a trivial C_4 -RGDD of type 2^2 , where $\alpha = 1$. The input designs are given by Lemma 4.2. □

Lemma 4.6. *A $(K_2, K_{1,4})$ -URD $(20; r, s)$ exists for every $(r, s) \in J(20)$.*

Proof. The Filling Construction applied to a $(K_2, K_{1,4})$ -URGDD (\bar{r}, \bar{s}) of type 10^2 from Lemma 4.5 (with input designs given by Lemma 4.4) gives a $(K_2, K_{1,4})$ -URD $(20; r, s)$ for each $(r, s) \in J(10) + I(10) = J(20)$. □

Lemma 4.7. *A $(K_2, K_{1,4})$ -URD $(40; r, s)$ exists for every $(r, s) \in J(40)$.*

Proof. Applying the GDD-construction with $t = 10$ to a 2-RGDD of type 2^2 (where $\alpha = 2$) gives a $(K_2, K_{1,4})$ -URGDD (\bar{r}, \bar{s}) of type 20^2 for each $(\bar{r}, \bar{s}) \in 2 * I(10)$ (the input designs are given by Lemma 4.5). Now filling the groups with designs given by Lemma 4.6 gives a $(K_2, K_{1,4})$ -URD $(40; r, s)$ for each $(r, s) \in J(20) + 2 * I(20) = J(40)$. □

Lemma 4.8. *A $(K_2, K_{1,4})$ -URD $(0, 25)$ of $C_{m(20)}$ exists for every $m \geq 3$.*

Proof. Let $C_m = (1, 2, \dots, m)$. For $i = 1, 2, \dots, m$, let $X^{(i)} = \{i\} \times \mathbb{Z}_{20} = \bigcup_{k=0}^4 X_k^{(i)}$, where $X_k^{(i)} = \{i_{4k}, i_{4k+1}, i_{4k+2}, i_{4k+3}\}$, and for every $r, s \in \mathbb{Z}_5$ let $R_{rs}^{(i)}$ denote the set of the following four copies of $K_{1,4}$

$$\begin{aligned} & (i_{4r}; (i + 1)_{4s}, (i + 1)_{4s+1}, (i + 1)_{4s+2}, (i + 1)_{4s+3}), \\ & (i_{4r+1}; (i + 1)_{4s+4}, (i + 1)_{4s+5}, (i + 1)_{4s+6}, (i + 1)_{4s+7}), \\ & (i_{4r+2}; (i + 1)_{4s+8}, (i + 1)_{4s+9}, (i + 1)_{4s+10}, (i + 1)_{4s+11}), \\ & (i_{4r+3}; (i + 1)_{4s+12}, (i + 1)_{4s+13}, (i + 1)_{4s+14}, (i + 1)_{4s+15}), \end{aligned}$$

where $m + 1 = 1$. If $m = 2n$, $n \geq 2$, take the five $K_{1,4}$ -factors

$$\begin{aligned} F_1 &= \bigcup_{i=0}^{n-1} \left(R_{01}^{(2i+1)} \cup R_{01}^{(2i+2)} \right), \\ F_2 &= \bigcup_{i=0}^{n-1} \left(R_{02}^{(2i+1)} \cup R_{11}^{(2i+2)} \right), \\ F_3 &= \bigcup_{i=0}^{n-1} \left(R_{03}^{(2i+1)} \cup R_{21}^{(2i+2)} \right), \\ F_4 &= \bigcup_{i=0}^{n-1} \left(R_{04}^{(2i+1)} \cup R_{31}^{(2i+2)} \right), \\ F_5 &= \bigcup_{i=0}^{n-1} \left(R_{00}^{(2i+1)} \cup R_{41}^{(2i+2)} \right), \end{aligned}$$

while if $m = 2n + 1$, $n \geq 1$, take the five $K_{1,4}$ -factors

$$\begin{aligned} F'_1 &= \left(R_{01}^{(1)} \cup R_{01}^{(2)} \cup R_{01}^{(3)} \right) \cup \left[\bigcup_{i=2}^n \left(R_{01}^{(2i)} \cup R_{01}^{(2i+1)} \right) \right], \\ F'_2 &= \left(R_{02}^{(1)} \cup R_{13}^{(2)} \cup R_{21}^{(3)} \right) \cup \left[\bigcup_{i=2}^n \left(R_{02}^{(2i)} \cup R_{11}^{(2i+1)} \right) \right], \\ F'_3 &= \left(R_{03}^{(1)} \cup R_{20}^{(2)} \cup R_{41}^{(3)} \right) \cup \left[\bigcup_{i=2}^n \left(R_{03}^{(2i)} \cup R_{21}^{(2i+1)} \right) \right], \\ F'_4 &= \left(R_{04}^{(1)} \cup R_{32}^{(2)} \cup R_{11}^{(3)} \right) \cup \left[\bigcup_{i=2}^n \left(R_{04}^{(2i)} \cup R_{31}^{(2i+1)} \right) \right], \\ F'_5 &= \left(R_{00}^{(1)} \cup R_{44}^{(2)} \cup R_{31}^{(3)} \right) \cup \left[\bigcup_{i=2}^n \left(R_{00}^{(2i)} \cup R_{41}^{(2i+1)} \right) \right]. \end{aligned}$$

The required 25 star factors are

$$F_{k,j} = (F_k) + j = \{R_{r+j,s+j}^{(i)} : R_{rs}^{(i)} \in F_k, r, s \in \mathbb{Z}_5\}, j \in \mathbb{Z}_5, k = 1, 2, 3, 4, 5$$

when $m = 2n$ and

$$F'_{k,j} = (F'_k) + j = \{R_{r+j,s+j}^{(i)} : R_{rs}^{(i)} \in F'_k, r, s \in \mathbb{Z}_5\}, j \in \mathbb{Z}_5, k = 1, 2, 3, 4, 5$$

when $m = 2n + 1$. □

Lemma 4.9. *A C_{2m} -decomposition of $C_{m(2)}$ exists for any $m \geq 3$.*

Proof. Let $C_{m(2)}$ be the graph obtained by expanding twice the vertices of the cycle $(1, 2, \dots, m)$, $m \geq 3$. If m is even, take the two $2m$ -cycles

$$C_1 = (1_0, 2_1, 3_0, 4_1, \dots, m_1, 1_1, 2_0, 3_1, 4_0, \dots, m_0),$$

$$C_2 = (1_0, 2_0, 3_0, 4_0, \dots, m_0, 1_1, 2_1, 3_1, 4_1, \dots, m_1),$$

while if m is odd, take the following ones:

$$C'_1 = (1_0, 2_1, 3_0, 4_1, \dots, m_0, 1_1, m_1, (m-1)_0, (m-2)_1, (m-3)_0, \dots, 2_0),$$

$$C'_2 = (1_0, m_0, (m-1)_0, (m-2)_0, (m-3)_0, \dots, 2_0, 1_1, 2_1, 3_1, 4_1, \dots, m_1). \quad \square$$

Lemma 4.10. *A $(K_2, K_{1,4})$ -URD(r, s) of $C_{m(20)}$, $m \geq 3$, exists for every $(r, s) \in I(40)$.*

Proof. The case $(r, s) = (0, 25)$ follows by Lemma 4.8. For any $(r, s) \in I(40) \setminus \{(0, 25)\}$, start from the C_{2m} -decomposition of $C_{m(2)}$ of Lemma 4.9, which admits $\alpha = 4$ 1-factors (each $2m$ -cycle gives two 1-factors). Expand each vertex 10 times. For each edge e of a given 1-factor, place on $e \times \mathbb{Z}_{10}$ a copy of a $(K_2, K_{1,4})$ -URGDD(\bar{r}, \bar{s}) of type 10^2 with $(\bar{r}, \bar{s}) \in I(10)$ (given by Lemma 4.5) so to obtain a $(K_2, K_{1,4})$ -URD(r, s) of $C_{m(20)}$ with $(r, s) \in 4 * I(10) = I(40) \setminus \{(0, 25)\}$. \square

Lemma 4.11. *A $(K_2, K_{1,4})$ -URD(r, s) of $C_{m(10)}$, $m \geq 3$, exists for every $(r, s) \in I(20)$.*

Proof. Start with a C_{2m} -decomposition of $C_{m(2)}$, which is given by Lemma 4.9 and is trivially resolvable with $\alpha = 2$ factors (i.e., the two $2m$ -cycles). Expand each vertex 5 times. For each cycle C , place on $V(C) \times \mathbb{Z}_5$ a copy of a $(K_2, K_{1,4})$ -URD(\bar{r}, \bar{s}) of $C_{2m(5)}$ with $(\bar{r}, \bar{s}) \in I(10)$ given by Lemma 4.2 so to obtain a $(K_2, K_{1,4})$ -URD(r, s) of $C_{m(10)}$ with $(r, s) \in 2 * I(10) = I(20)$. \square

Lemma 4.12. *A $(K_2, K_{1,4})$ -URGDD(r, s) of type 40^2 exists for every $(r, s) \in I(40)$.*

Proof. Apply the GDD-construction with $t = 20$ to a trivial C_4 -RGDD of type 2^2 , where $\alpha = 1$. The input designs are given by Lemma 4.10 for $m = 4$. \square

Lemma 4.13. *A $(K_2, K_{1,4})$ -URD($30; r, s$) exists for every $(r, s) \in J(30)$.*

Proof. The Filling Construction applied to a $(K_2, K_{1,4})$ -RGDD(\bar{r}, \bar{s}) of type 10^3 with $(\bar{r}, \bar{s}) \in I(20)$ (from Lemma 4.11) gives a $(K_2, K_{1,4})$ -URD($30; r, s$) for each $(r, s) \in J(10) + I(20) = J(30)$. The input designs are given by Lemma 4.4. \square

Lemma 4.14. *There exists a $(K_2, K_{1,4})$ -URGDD($0, 25$) of type 10^5 .*

Proof. The union of the orbits of $B_i = (0; 1 + 5i, 2 + 5i, 3 + 5i, 4 + 5i)$, $i = 0, 1, 2, 3, 4$, under \mathbb{Z}_{50} gives the block set of a GDD of type 10^5 , whose groups are the cosets $H, H + 1, H + 2, H + 3, H + 4$ of $H = 5\mathbb{Z}_{50}$ in \mathbb{Z}_{50} . For every $i = 0, 1, 2, 3, 4$, the orbit of B_i can be decomposed into five $K_{1,4}$ -factors:

$$F_{ij} = \{B_i + j + 5k, k = 0, 1, \dots, 9\}, \quad j = 0, 1, 2, 3, 4.$$

\square

Lemma 4.15. *A $(K_2, K_{1,4})$ -URGDD(r, s) of type 10^5 exists for every $(r, s) \in I(40)$.*

Proof. The case $(r, s) = (0, 25)$ follows by Lemma 4.14. For any $(r, s) \in I(40) \setminus \{(0, 25)\}$, the GDD-Construction applied with $t = 10$ to a trivial C_5 -RGDD of type 1^5 (where $\alpha = 2$) gives a $(K_2, K_{1,4})$ -URGDD(r, s) of type 10^5 for each $(r, s) \in 2 * I(20) = I(40) \setminus \{(0, 25)\}$. The input designs are given by Lemma 4.11. \square

Lemma 4.16. *A $(K_2, K_{1,4})$ -IURD($50, 10; [r', s'], [r, s]$) exists for every $(r', s') \in J(10)$ and $(r, s) \in I(40)$.*

Proof. Apply the Filling Construction to a $(K_2, K_{1,4})$ -URGDD(r, s) of type 10^5 with $(r, s) \in I(40)$ from by Lemma 4.15 (the input designs are given by Lemma 4.4). \square

Let $S \subset \mathbb{Z}_n$ be such that if $s \in S$, then $-s \notin S$ and set $B = \{0, s\} : s \in S\}$, then the orbit of B is the circulant graph with edges $\{x, y\}$ where either $x - y$ or $y - x \in S$. The edge $\{x, y\}$ has even order if $s = y - x$ has even additive order modulo n . In the next Lemma we use the following famous result of Stern and Lenz.

Theorem 4.17. *(Theorem of Stern and Lenz [23]) Every circulant graph containing an edge of even order has a one-factorization.*

Lemma 4.18. *A $(K_2, K_{1,4})$ -URGDD(r, s) of type 10^9 exists for every $(r, s) \in \{(8, 45), (0, 50)\}$*

Proof. On \mathbb{Z}_{90} let:

$$F = \{(89; 0, 1, 18, 19), (52; 53, 54, 71, 72), (2; 40, 67, 87, 85), \\ (3; 41, 68, 88, 86), (4; 65, 73, 77, 81), (5; 66, 74, 78, 82), \\ (6; 46, 48, 50, 61), (7; 47, 49, 51, 62), (8; 38, 42, 59, 83), \\ (9; 39, 43, 60, 84), (10; 32, 34, 36, 63), (11; 33, 35, 37, 64), \\ (12; 20, 69, 24, 79), (13; 21, 70, 25, 80), (14; 28, 30, 55, 57), \\ (15; 29, 31, 56, 58), (16; 44, 22, 26, 75), (17; 45, 23, 27, 76)\}.$$

and $B = (0; 3, 4, 11, 32)$. Take the forty-five $K_{1,4}$ -factors $F + 2i$, for $i = 0, 1, \dots, 44$, and partition the orbit of B under \mathbb{Z}_{90} into the five $K_{1,4}$ -factors:

$$F_j = \{B + j + 5k, k = 0, 1, \dots, 17\}, \quad j = 0, 1, 2, 3, 4.$$

The resulting design is a $(K_2, K_{1,4})$ -URGDD($0, 50$) of type 10^9 , whose groups are the cosets of $H = 9\mathbb{Z}_{90}$ in \mathbb{Z}_{90} , i.e., $H + h$, for $h = 0, 1, \dots, 8$.

For the case $(r, s) = (8, 45)$, remove the $K_{1,4}$ -factors obtained from the orbit of B and decompose the graph whose edges cover the differences of B into 1-factors by using the theorem of Stern and Lenz. \square

Lemma 4.19. *A $(K_2, K_{1,4})$ -URGDD(r, s) of type 10^9 exists for every $(r, s) \in I(80)$.*

Proof. The cases $(r, s) = (0, 50), (8, 45)$ follow by Lemma 4.18. To settle the remaining cases, apply the GDD-construction with $t = 10$ to a C_9 -RGDD of type 1^9 (where $\alpha = 4$) to get a $(K_2, K_{1,4})$ -URGDD(r, s) of type 10^9 for each $(r, s) \in 4 * I(20) = I(80) \setminus \{(0, 50), (8, 45)\}$. The input designs are given by Lemma 4.11. \square

Lemma 4.20. *A $(K_2, K_{1,4})$ -IURD(90, 10; $[r', s'], [r, s]$) exists for every $(r', s') \in J(10)$ and $(r, s) \in I(80)$.*

Proof. Apply the Filling Construction to a $(K_2, K_{1,4})$ -URGDD(r, s) of type 10^9 with $(r, s) \in I(80)$ from by Lemma 4.19 (the input designs are given by Lemma 4.4). \square

Lemma 4.21. *A $(K_2, K_{1,4})$ -IURD(70, 30; $[r', s'], [0, 25]$) exists for every $(r', s') \in J(30)$.*

Proof. Let $V = \mathbb{Z}_{40} \cup \{a_1, a_2, \dots, a_{30}\}$ be the vertex set, where $\{a_1, a_2, \dots, a_{30}\}$ is the hole. Consider the five $K_{1,4}$ -factors on V :

$$F_1 = \{(a_1; 16, 17, 18, 19), (a_2; 20, 21, 22, 23), (a_3; 24, 25, 26, 27), \\ (a_4; 28, 29, 30, 31), (a_5; 32, 33, 34, 35), (a_6; 36, 37, 38, 39), \\ (0; 8, a_{14}, a_{16}, a_{18}), (1; 9, a_{26}, a_{28}, a_{30}), (2; 10, a_{20}, a_{22}, a_{24}), \\ (3; 11, a_8, a_{10}, a_{12}), (4; 12, a_{13}, a_{15}, a_{17}), (5; 13, a_{25}, a_{27}, a_{29}), \\ (6; 14, a_{19}, a_{21}, a_{23}), (7; 15, a_7, a_9, a_{11})\};$$

$$F_2 = \{(a_7; 8, 9, 10, 11), (a_8; 12, 13, 14, 15), (a_9; 24, 25, 26, 27), \\ (a_{10}; 28, 29, 30, 31), (a_{11}; 32, 33, 34, 35), (a_{12}; 36, 37, 38, 39), \\ (0; 16, a_{26}, a_{28}, a_{30}), (1; 17, a_2, a_4, a_6), (2; 18, a_{14}, a_{16}, a_{18}), \\ (3; 19, a_{20}, a_{22}, a_{24}), (4; 20, a_1, a_3, a_5), (5; 21, a_{19}, a_{21}, a_{23}), \\ (6; 22, a_{13}, a_{15}, a_{17}), (7; 23, a_{25}, a_{27}, a_{29})\};$$

$$F_3 = \{(a_{13}; 8, 9, 10, 11), (a_{14}; 20, 21, 22, 23), (a_{15}; 24, 25, 26, 27), \\ (a_{16}; 28, 29, 30, 31), (a_{17}; 32, 33, 34, 35), (a_{18}; 36, 37, 38, 39), \\ (0; 12, a_8, a_{10}, a_{12}), (1; 13, a_{20}, a_{22}, a_{24}), (2; 14, a_2, a_4, a_6), \\ (3; 15, a_{26}, a_{28}, a_{30}), (4; 16, a_7, a_9, a_{11}), (5; 17, a_1, a_3, a_5), \\ (6; 18, a_{25}, a_{27}, a_{29}), (7; 19, a_{19}, a_{21}, a_{23})\};$$

$$F_4 = \{(a_{19}; 8, 9, 10, 11), (a_{20}; 12, 13, 22, 23), (a_{21}; 24, 25, 26, 27), \\ (a_{22}; 28, 29, 30, 31), (a_{23}; 32, 33, 34, 35), (a_{24}; 36, 37, 38, 39), \\ (0; 14, a_2, a_4, a_6), (1; 15, a_8, a_{10}, a_{12}), (2; 16, a_{26}, a_{28}, a_{30}), \\ (3; 17, a_{14}, a_{16}, a_{18}), (4; 18, a_{25}, a_{27}, a_{29}), (5; 19, a_7, a_9, a_{11}), \\ (6; 20, a_1, a_3, a_5), (7; 21, a_{13}, a_{15}, a_{17})\};$$

$$F_5 = \{(a_{25}; 8, 9, 10, 11), (a_{26}; 12, 13, 14, 15), (a_{27}; 16, 17, 26, 27), \\ (a_{28}; 28, 29, 30, 31), (a_{29}; 32, 33, 34, 35), (a_{30}; 36, 37, 38, 39), \\ (0; 18, a_{20}, a_{22}, a_{24}), (1; 19, a_{14}, a_{16}, a_{18}), (2; 20, a_8, a_{10}, a_{12}), \\ (3; 21, a_2, a_4, a_6), (4; 22, a_{19}, a_{21}, a_{23}), (5; 23, a_{13}, a_{15}, a_{17}), \\ (6; 24, a_7, a_9, a_{11}), (7; 25, a_1, a_3, a_5)\}.$$

For each $j = 1, 2, 3, 4, 5$, take the five $K_{1,4}$ -factors $F_j + 8i$, for $i = 0, 1, 2, 3, 4$, where $a_k + x = a_k$ for every $x \in \mathbb{Z}_{40}$ and for every $k = 1, 2, \dots, 30$. Let $D = \{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 15, 17, 19, 20\}$, i.e. the set of the 15 differences not covered by the 25 above factors, and decompose the graph consisting of the edges $\{i, d + i\}$, $i \in \mathbb{Z}_{40}$ and $d \in D$, as follows.

For $(r', s') = (29, 0)$, apply the theorem of Stern and Lenz to decompose the graph consisting of the edges $\{i, d + i\}$, $i \in \mathbb{Z}_{40}$ and $d \in D$, into 29 partial 1-factors on \mathbb{Z}_{40} .

For $(r', s') = (21, 5)$, take the base block $B = (0; 7, 9, 11, 13)$, whose orbit modulo 40 can be decomposed into five $K_{1,4}$ -factors on \mathbb{Z}_{40} :

$$F'_j = \{B + j + 5k : k = 0, 1, \dots, 7\}, \quad j = 0, 1, 2, 3, 4.$$

Decompose the graph whose edges cover the remaining differences of D into 1-factors by using the theorem of Stern and Lenz.

For $(r', s') = (13, 10)$, take the base blocks

$$B = (0; 7, 9, 11, 13) \text{ and } B_1 = (0; 1, 2, 3, 4),$$

which give in total ten $K_{1,4}$ -factors on \mathbb{Z}_{40} . Then decompose the graph whose edges cover the differences of D into 1-factors by using the theorem of Stern and Lenz.

For $(r', s') = (5, 15)$, take the base blocks $B = (0; 7, 9, 11, 13)$, $B_2 = (0; 3, 4, 5, 19)$ and $B_3 = (6; 7, 8, 12, 21)$. Obtain five $K_{1,4}$ -factors on \mathbb{Z}_{40} from B , and ten $K_{1,4}$ -factors from B_2 and B_3 as follows:

$$F''_j = \{B_2 + j + 10k, B_3 + j + 10k : k = 0, 1, 2, 3\}, \quad j = 0, 1, \dots, 9.$$

Decompose the graph whose edges cover the remaining differences 10, 17 and 20 of D into 1-factors by using the theorem of Stern and Lenz. □

Lemma 4.22. *A $(K_2, K_{1,4})$ -IURD(70, 30; $[r', s'], [r, s]$) exists for every $(r', s') \in J(30)$ and $(r, s) \in I(40)$.*

Proof. The case $(r, s) = (0, 25)$ follows by Lemma 4.21. For any $(r, s) \in I(40) \setminus \{(0, 25)\}$, start from the decomposition of the graph $K_7 \setminus K_3$ on $X = \{x, y, z\} \cup \{a_1, a_2, a_3, a_4\}$ into one 4-cycle $C_0 = (a_1, a_2, a_3, a_4)$ and two hamiltonian cycles $C_1 = (a_1, a_3, y, a_2, z, a_4, x)$ and $C_2 = (a_1, z, a_3, x, a_2, a_4, y)$. Expand each vertex 10 times and on $V(C_j) \times \mathbb{Z}_{10}$, for $j = 0, 1, 2$, place a copy of a $(K_2, K_{1,4})$ -URD(r_j, s_j) of $C_{m(10)}$ ($m = 4$ or 7) with $(r_j, s_j) \in I(20)$ (given by Lemma 4.11). It follows that, corresponding to the hamiltonian cycles C_1 and C_2 , there are r full 1-factors and s full $K_{1,4}$ -factors, where $(r, s) = (r_1 + r_2, s_1 + s_2) \in 2 * I(20) = I(40) \setminus \{(0, 25)\}$; while C_0 provides r_0 partial 1-factors and s_0 partial $K_{1,4}$ -factors missing the hole $\{x, y, z\} \times \mathbb{Z}_{10}$. Now, placing on each set $\{a_i\} \times \mathbb{Z}_{10}$, $i = 1, 2, 3, 4$, a copy of a $(K_2, K_{1,4})$ -URD(10, r'', s'') with $(r'', s'') \in J(10)$ (from Lemma 4.4) gives further r'' partial 1-factors and s'' partial $K_{1,4}$ -factors so that the resulting design is a $(K_2, K_{1,4})$ -IURD(70, 30; $[r', s'], [r, s]$), where $(r', s') = (r_0 + r'', s_0 + s'') \in I(20) + J(10) = J(30)$. □

Lemma 4.23. *A $(K_2, K_{1,4})$ -IURD(110, 30; $[r', s'], [r, s]$) exists for every $(r', s') \in J(30)$ and $(r, s) \in \{(8, 45), (0, 50)\}$.*

Proof. Let $V = \mathbb{Z}_{80} \cup \{a_1, a_2, \dots, a_{30}\}$ be the vertex set, where $\{a_1, a_2, \dots, a_{30}\}$ is the hole. Consider the two $K_{1,4}$ -factors on V :

$$F_1 = \{(a_{25}; 16, 18, 20, 22), (a_{26}; 24, 26, 28, 30), (a_{27}; 32, 34, 36, 38), \\ (a_{28}; 17, 19, 21, 23), (a_{29}; 25, 27, 29, 31), (a_{30}; 33, 35, 37, 39), \\ (0; 14, a_{22}, a_{23}, a_{24}), (1; 15, a_{19}, a_{20}, a_{21}), (2; 8, a_{16}, a_{17}, a_{18}), \\ (3; 9, a_{13}, a_{14}, a_{15}), (4; 12, a_{10}, a_{11}, a_{12}), (5; 13, a_7, a_8, a_9), \\ (6; 10, a_4, a_5, a_6), (7; 11, a_1, a_2, a_3), \\ (40; 67, 68, 69, 70), (41; 62, 64, 66, 72), (42; 63, 65, 73, 78), \\ (43; 54, 58, 60, 79), (44; 55, 56, 59, 61), (45; 48, 50, 52, 57), \\ (46; 49, 51, 53, 71), (47; 74, 75, 76, 77)\};$$

$$F_2 = \{(a_{25}; 4, 14, 24, 34), (a_{26}; 2, 12, 22, 32), (a_{27}; 0, 10, 20, 30), \\ (a_{28}; 5, 15, 25, 35), (a_{29}; 3, 13, 23, 33), (a_{30}; 1, 11, 21, 31), \\ (46; 56, a_{22}, a_{23}, a_{24}), (47; 57, a_{19}, a_{20}, a_{21}), (44; 54, a_{16}, a_{17}, a_{18}), \\ (45; 55, a_{13}, a_{14}, a_{15}), (42; 52, a_{10}, a_{11}, a_{12}), (43; 53, a_7, a_8, a_9), \\ (40; 50, a_4, a_5, a_6), (41; 51, a_1, a_2, a_3), \\ (6; 8, 48, 64, 66), (7; 9, 49, 65, 67), (16; 18, 74, 76, 62), \\ (17; 19, 75, 77, 63), (26; 28, 68, 60, 72), (27; 29, 69, 61, 73), \\ (36; 38, 78, 70, 58), (37; 39, 79, 71, 59)\};$$

and the sets of pairs $A_1 = \{\{0, 20\}, \{1, 21\}, \{6, 26\}, \{7, 27\}\}$ and $A_2 = \{\{2, 40\}, \{3, 41\}, \{6, 28\}, \{7, 29\}\}$.

Take the 50 full $K_{1,4}$ -factors $F_1 + 2i$, for $i = 0, 1, \dots, 39$, and $F_2 + 8i$, for $i = 0, 1, \dots, 9$, where $a_k + x = a_{k+3x}$ for every $x \in \mathbb{Z}_{80}$ and $k = 1, 2, \dots, 30$; and the two partial 1-factors $(A_j) = \{\{x + 8i, y + 8i\} : \{x, y\} \in A_j, i = 0, 1, \dots, 9\}$, $j = 1, 2$. Let $D = \{1, 9, 13, 16, 18, 19, 24, 26, 32, 33, 35, 37, 39, 40\}$, i.e. the set of the 14 differences not covered by (A_1) , (A_2) and the above 50 full $K_{1,4}$ -factors. Decompose the graph consisting of the edges $\{i, d + i\}$, $i \in \mathbb{Z}_{80}$ and $d \in D$, as follows:

For $(r', s') = (29, 0)$, apply the theorem of Stern and Lenz to decompose the graph consisting of the edges $\{i, d + i\}$, $i \in \mathbb{Z}_{80}$ and $d \in D$, into 27 partial 1-factors on \mathbb{Z}_{80} so to obtain 29 partial 1-factors along with (A_1) and (A_2) .

For $(r', s') = (21, 5)$, take the base block $B = (0; 26, 32, 33, 39)$, whose orbit can be decomposed into five $K_{1,4}$ -factors on \mathbb{Z}_{80} :

$$F'_j = \{B + j + 5k : k = 0, 1, \dots, 15\}, \quad j = 0, 1, 2, 3, 4.$$

Decompose the graph whose edges cover the remaining differences of D into 1-factors by using the theorem of Stern and Lenz.

For $(r', s') = (13, 10)$, take the base blocks $B = (0; 26, 32, 33, 39)$ and $B_1 = (0; 1, 13, 24, 37)$, which give in total ten $K_{1,4}$ -factors on \mathbb{Z}_{80} , while decomposing the graph whose edges cover the differences of D into 1-factors by using the theorem of Stern and Lenz.

For $(r', s') = (5, 15)$, take the base blocks $B = (0; 26, 32, 33, 39)$, $B_2 = (0; 1, 16, 19, 35)$ and $B_3 = (4; 13, 17, 22, 28)$. Obtain five $K_{1,4}$ -factors on \mathbb{Z}_{80} from B , and ten

$K_{1,4}$ -factors from B_2 and B_3 as follows:

$$F''_j = \{B_2 + j + 10k, B_3 + j + 10k : k = 0, 1, \dots, 7\}, \quad j = 0, 1, \dots, 9.$$

Decompose the graph whose edges cover the remaining differences 37 and 40 of D into 1-factors by using the theorem of Stern and Lenz.

Finally, to prove the existence of a $(K_2, K_{1,4})$ -IURD(110, 30; $[r', s'], (8, 45)$), with $(r', s') \in J(30)$, it will be sufficient to start from the constructed $(K_2, K_{1,4})$ -IURD(110, 30; $[r, s], (0, 50)$). Destroy the 5 full $K_{1,4}$ -factors $F_1 + 16i$, for $i = 0, 1, 2, 3, 4$, and rearrange the resulting edges into the 8 full 1-factors $(M_j) = \{\{x + 16i, y + 16i\} : \{x, y\} \in M_j, i = 0, 1, 2, 3, 4\}, j = 1, 2, \dots, 8$, where:

$$\begin{aligned} M_1 &= \left\{ \begin{array}{l} \{7, a_3\}, \{6, a_5\}, \{5, a_7\}, \{2, a_{18}\}, \{1, a_{20}\}, \{0, a_{22}\}, \\ \{4, 12\}, \{40, 67\}, \{41, 62\}, \{43, 58\}, \{47, 77\} \end{array} \right\}; \\ M_2 &= \left\{ \begin{array}{l} \{6, a_6\}, \{5, a_8\}, \{4, a_{10}\}, \{1, a_{21}\}, \{0, a_{23}\}, \{18, a_{25}\}, \\ \{7, 11\}, \{42, 63\}, \{44, 56\}, \{45, 57\}, \{46, 51\} \end{array} \right\}; \\ M_3 &= \left\{ \begin{array}{l} \{5, a_9\}, \{4, a_{11}\}, \{3, a_{13}\}, \{0, a_{24}\}, \{30, a_{26}\}, \{17, a_{28}\}, \\ \{6, 10\}, \{41, 72\}, \{43, 79\}, \{44, 55\}, \{45, 50\} \end{array} \right\}; \\ M_4 &= \left\{ \begin{array}{l} \{7, a_1\}, \{4, a_{12}\}, \{3, a_{14}\}, \{2, a_{16}\}, \{38, a_{27}\}, \{29, a_{29}\}, \\ \{40, 69\}, \{41, 64\}, \{44, 59\}, \{46, 49\}, \{47, 74\} \end{array} \right\}; \\ M_5 &= \left\{ \begin{array}{l} \{7, a_2\}, \{6, a_4\}, \{3, a_{15}\}, \{2, a_{17}\}, \{1, a_{19}\}, \{37, a_{30}\}, \\ \{0, 14\}, \{40, 68\}, \{42, 73\}, \{44, 61\}, \{47, 75\} \end{array} \right\}; \\ M_6 &= \left\{ \begin{array}{l} \{20, a_{25}\}, \{28, a_{26}\}, \{32, a_{27}\}, \{23, a_{28}\}, \{31, a_{29}\}, \{33, a_{30}\}, \\ \{2, 8\}, \{3, 9\}, \{5, 13\}, \{42, 78\}, \{43, 54\} \end{array} \right\}; \\ M_7 &= \left\{ \begin{array}{l} \{16, a_{25}\}, \{26, a_{26}\}, \{34, a_{27}\}, \{19, a_{28}\}, \{25, a_{29}\}, \{39, a_{30}\}, \\ \{1, 15\}, \{40, 70\}, \{45, 52\}, \{46, 53\}, \{43, 60\} \end{array} \right\}; \\ M_8 &= \left\{ \begin{array}{l} \{22, a_{25}\}, \{24, a_{26}\}, \{36, a_{27}\}, \{21, a_{28}\}, \{27, a_{29}\}, \{35, a_{30}\}, \\ \{41, 66\}, \{42, 65\}, \{45, 48\}, \{46, 71\}, \{47, 76\} \end{array} \right\}. \end{aligned}$$

□

Lemma 4.24. *A $(K_2, K_{1,4})$ -IURD(110, 30; $[r', s'], [r, s]$) exists for every $(r', s') \in J(30)$ and $(r, s) \in I(80)$.*

Proof. The cases $(r, s) = (8, 45), (0, 50)$ follow by Lemma 4.21. For any $(r, s) \in I(80) \setminus \{(8, 45), (0, 50)\}$, start from the decomposition of the graph $K_{11} \setminus K_3$ on $X = \{x, y, z\} \cup \{a_1, a_2, \dots, a_8\}$ into one 8-cycle $C_0 = (a_1, a_2, \dots, a_8)$ and four hamiltonian cycles

$$\begin{aligned} C_1 &= (a_1, x, a_3, y, a_5, z, a_7, a_2, a_6, a_8, a_4), \\ C_2 &= (a_2, x, a_4, y, a_6, z, a_8, a_3, a_1, a_7, a_5), \\ C_3 &= (a_5, x, a_7, y, a_1, z, a_3, a_6, a_4, a_2, a_8), \\ C_4 &= (a_6, x, a_8, y, a_2, z, a_4, a_7, a_3, a_5, a_1). \end{aligned}$$

Expanding each vertex 10 times and using similar arguments to the proof of Lemma 4.22 gives a $(K_2, K_{1,4})$ -IURD $(110, 30; [r', s'], [r, s])$ for each $(r', s') \in I(20) + J(10) = J(30)$ and $(r, s) \in 4 * I(20) = I(80) \setminus \{(8, 45), (0, 50)\}$. \square

Lemma 4.25. *Let $v = 50, 70, 90, 110$. A $(K_2, K_{1,4})$ -URD (v, r, s) exists for every $(r, s) \in J(v)$.*

Proof. Apply the Filling Construction to the IURDs of Lemmas 4.16, 4.20, 4.22 and 4.24. The input designs are given by Lemmas 4.4 and 4.13. \square

The next three results are all obtained by applying the Frame-Construction with various parameters. We leave them as separate results so that it is easier for the reader to find each case.

Lemma 4.26. *A $(K_2, K_{1,4})$ -URD $(190; r, s)$ exists for every $(r, s) \in J(190)$.*

Proof. Apply the Frame-Construction with $t = 20$ and $h = 30$ to a 3-frame of type 2^4 (where $\alpha = 1$) to obtain a $(K_2, K_{1,4})$ -URD $(190; r, s)$ for each $(r, s) \in J(30) + 4 * I(40) = J(30) + I(160) = J(190)$ (where the first equality follows by Lemma 4.1). The input designs are given by Lemmas 4.10, 4.13 and 4.22. \square

Lemma 4.27. *Let $v = 690, 930$. A $(K_2, K_{1,4})$ -URD $(v; r, s)$ exists for every $(r, s) \in J(v)$.*

Proof. Apply the Frame-Construction with $t = 40$ and $h = 10$ to a 2-frame of type $1 \frac{v-10}{40}$ (where $\alpha = 1$) to obtain a $(K_2, K_{1,4})$ -URD $(v; r, s)$ for each $(r, s) \in J(10) + \frac{v-10}{40} * I(40) = J(10) + I(v-10) = J(v)$ (where the first equality follows by Lemma 4.1). The input designs are given by Lemmas 4.4, 4.12 and 4.16. \square

Lemma 4.28. *A $(K_2, K_{1,4})$ -URD $(1290; r, s)$ exists for every $(r, s) \in J(1290)$.*

Proof. Apply the Frame-Construction with $t = 40$ and $h = 10$ to a 2-frame of type 2^{16} (where $\alpha = 2$) to obtain a $(K_2, K_{1,4})$ -URD $(1290; r, s)$ for each $(r, s) \in J(10) + 32 * I(40) = J(10) + I(1280) = J(1290)$ (where the first equality follows by Lemma 4.1). The input designs are given by Lemmas 4.4, 4.12 and 4.20. \square

5 The main result

In the proof of the following lemmas we make use of the equality $\alpha * I(n) = I(\alpha n)$, which holds by Lemma 4.1 when n is a multiple of 8.

Lemma 5.1. *Let $v \equiv 0 \pmod{40}$. Then a $(K_2, K_{1,4})$ -URD $(v; r, s)$ exists for every $(r, s) \in J(v)$.*

Proof. Let $v = 40k$, $k \geq 1$. The case $v = 40$ follows by Lemma 4.7. For $k > 1$, applying the GDD-Construction with $t = 20$ to a C_{2k} -RGDD of type 2^k , i.e., a

decomposition of $K_{k(2)}$ into $\alpha = k - 1$ hamiltonian cycles (see [15]), gives a $(K_2, K_{1,4})$ -URGDD (\bar{r}, \bar{s}) of type 40^k for each $(\bar{r}, \bar{s}) \in (k - 1) * I(40)$. (The input designs are given by Lemma 4.10.) Filling each group with a $(K_2, K_{1,4})$ -URD $(40; r', s')$ with $(r', s') \in J(40)$ (from Lemma 4.7) gives a $(K_2, K_{1,4})$ -URD $(v; r, s)$ for each $(r, s) \in J(40) + (k - 1) * I(40) = J(40) + \frac{v-40}{40} * I(40) = J(40) + I(v - 40) = J(v)$. \square

Lemma 5.2. *Let $v \equiv 10 \pmod{40}$. Then a $(K_2, K_{1,4})$ -URD $(v; r, s)$ exists for every $(r, s) \in J(v)$.*

Proof. Let $v = 40k + 10, k \geq 0$. The cases $v = 10, 50, 90, 690, 930, 1290$ follow by Lemmas 4.4, 4.25, 4.27, and 4.28. For $k > 2, k \neq 17, 23, 32$, start from a 5-RGDD of type 1^{20k+5} (where $\alpha = 5k + 1$, see [1, 2, 3, 9, 26]). Applying the GDD-construction with $t = 2$ gives a $(K_2, K_{1,4})$ -URGDD (\bar{r}, \bar{s}) of type 2^{20k+5} for each $(\bar{r}, \bar{s}) \in (5k + 1) * I(8)$ (the input designs are given by Lemma 4.3). Now fill the groups with a trivial $(K_2, K_{1,4})$ -URD $(2; 1, 0)$ to get a $(K_2, K_{1,4})$ -URD $(v; r, s)$ for each $(r, s) \in \{(1, 0)\} + (5k + 1) * I(8) = \{(1, 0)\} + \frac{v-2}{8} * I(8) = \{(1, 0)\} + I(v - 2) = J(v)$. \square

Lemma 5.3. *Let $v \equiv 20 \pmod{40}$. Then a $(K_2, K_{1,4})$ -URD $(v; r, s)$ exists for every $(r, s) \in J(v)$.*

Proof. Let $v = 40k + 20, k \geq 0$. The case $v = 20$ follows by Lemma 4.6. For $k > 0$, applying the GDD-Construction with $t = 20$ to a C_{2k+1} -cycle system of order $2k + 1$, i.e. a decomposition of K_{2k+1} into $\alpha = k$ hamiltonian cycles (see [3, 16]), gives a $(K_2, K_{1,4})$ -URGDD (\bar{r}, \bar{s}) of type 20^{2k+1} for each $(\bar{r}, \bar{s}) \in k * I(40)$. (The input designs are given by Lemma 4.10.) Filling each group with a copy of a $(K_2, K_{1,4})$ -URD $(20; r', s')$, with $(r', s') \in J(20)$ (from Lemma 4.6) gives a $(K_2, K_{1,4})$ -URD $(v; r, s)$ for each $(r, s) \in J(20) + k * I(40) = J(20) + \frac{v-20}{40} * I(40) = J(20) + I(v - 20) = J(v)$. \square

Lemma 5.4. *Let $v \equiv 30 \pmod{80}$. Then a $(K_2, K_{1,4})$ -URD $(v; r, s)$ exists for every $(r, s) \in J(v)$.*

Proof. Let $v = 80k + 30, k \geq 0$. The cases $v = 30, 110, 190$ follow by Lemmas 4.13, 4.25 and 4.26. For $k > 2$, applying the Frame-Construction with $t = 40$ and $h = 30$ to a 2-frame of type 2^k (where $\alpha = 2$) gives a $(K_2, K_{1,4})$ -URD $(v; r, s)$ for each $(r, s) \in J(30) + 2k * I(40) = J(30) + \frac{v-30}{40} * I(40) = J(30) + I(v - 30) = J(v)$. (The input designs are given by Lemmas 4.12, 4.13 and 4.24.) \square

Lemma 5.5. *Let $v \equiv 70 \pmod{80}$. Then a $(K_2, K_{1,4})$ -URD $(v; r, s)$ exists for every $(r, s) \in J(v)$.*

Proof. Let $v = 80k + 70, k \geq 0$. The case $v = 70$ follows by Lemma 4.25. For $k > 0$, apply the Frame-Construction with $t = 40$ and $h = 30$ to a 2-frame of type 1^{2k+1} (where $\alpha = 1$) to obtain a $(K_2, K_{1,4})$ -URD $(v; r, s)$ for each $(r, s) \in J(30) + (2k + 1) * I(40) = J(30) + \frac{v-30}{40} * I(40) = J(30) + I(v - 30) = J(v)$. (The input designs are given by Lemmas 4.12, 4.13 and 4.22.) \square

As consequence of Lemmas 2.1, 5.1–5.5, our main result immediately follows.

Theorem 5.6. *A $(K_2, K_{1,4})$ -URD $(v; r, s)$, with $r, s > 0$, exists if and only if $v \equiv 0 \pmod{10}$ and $(r, s) \in J(v)$.*

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