2021

# Integer Partitions Under Certain Finiteness Conditions

Tim Wagner
*Michigan Technological University*, tjwagner@mtu.edu

# INTEGER PARTITIONS UNDER CERTAIN FINITENESS CONDITIONS

By

Timothy J. Wagner

A DISSERTATION

Submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

In Mathematical Sciences

MICHIGAN TECHNOLOGICAL UNIVERSITY

2021

This dissertation has been approved in partial fulfillment of the requirements for the Degree of DOCTOR OF PHILOSOPHY in Mathematical Sciences.

Department of Mathematical Sciences

Dissertation Advisor: *Dr. Fabrizio Zanello*

Committee Member: *Dr. William J. Keith*

Committee Member: *Dr. Vladimir D. Tonchev*

Committee Member: *Dr. Bryan J. Freyberg*

Department Chair: *Dr. Jiguang Sun*

# Dedication

To my grandmother Ruth Endean,

who gave constant encouragement and motivation to persevere with this dissertation.

# Contents

# List of Figures

# List of Tables

# Acknowledgments

First and foremost I thank my Savior, the Lord Jesus Christ, for giving me the ability to complete this dissertation.

*For the* Lord *gives wisdom;*

*from his mouth come knowledge and understanding.*

(Proverbs 2:6)

I am particularly grateful to my advisor, Fabrizio Zanello, for all of his guidance, encouragement and patience.

Thanks to my committee members: William Keith, Bryan Freyberg and Vladimir Tonchev. Professor Keith in particular gave very helpful advice during the many meetings where we discussed some of the topics of this dissertation.

I am indebted to Jason Gregersen for using his Mathematica expertise to thoroughly and painstakingly verify the code used in the proof of Lemma 4.3.18. I also appreciate Jason's mentorship over the past several years.

Samuel Judge also kindly checked this Mathematica code. However, I am grateful even more for Sam's friendship and support throughout my time at MTU.

A very special thanks to my incredible wife, Casey, for her unwavering support and her constant encouragement.

Thanks to my children, Levi, Eric and Tessa, for giving me perspective on what really matters.

I am grateful to my parents, Dave and Cathy (and Paul and Cindy), whose support started long before my time as a graduate student.

I would like to also express my gratitude to...

Mark Gockenbach, who provided wise career guidance as well as generous opportunities for professional growth at MTU.

Ann Humes, for her support and advice regarding my teaching duties.

Jeanne Meyers and Kim Puuri, for helping with the logistics of graduate school life.

David Brown and the rest of my professors from BJU, for their continued interest in my academic success.

My uncle, Tim Endean, for the encouragement he often gave during my graduate work.

Tom Kestner, who kindly read a draft of this dissertation and suggested many

helpful corrections.

The many supportive friends I made at MTU, who are too numerous to list.

My church family, for their prayers and encouragement.

# Abstract

This dissertation focuses on problems related to integer partitions under various finiteness restrictions. Much of our work involves the collection of partitions fitting inside a fixed partition $\lambda$, and the associated generating function $G_\lambda$.

In Chapter 2, we discuss the flawlessness of such generating functions, as proved by Pouzet using the Multicolor Theorem [25, 26]. We give novel applications of the Multicolor Theorem to re-prove flawlessness of pure $O$-sequences, and show original flawlessness results for other combinatorial sequences. We also present a linear-algebraic generalization of the Multicolor Theorem that may have far-reaching applications.

In Chapter 3, we extend a technique due to Stanton [37] to prove unimodality of $G_\lambda$ for certain infinite families of partitions $\lambda$ in 5 and 6 parts.

Our most substantial work is presented in Chapter 4, where we initiate the study of the novel poset $P_n = \{G_\lambda \,|\, \lambda \vdash n\}$. We describe some general structural properties of this poset. Of greatest significance is our result that two "balancing" operations on the principal hooks of a partition $\lambda$ produce generating functions at least as large as $G_\lambda$ (in the ordering of $P_n$), hereby imposing a strong necessary condition on the maxima of $P_n$. We conjecture an asymptotic value of $|P_n|$, and show that determining $|P_n|$ exactly appears to be nontrivial. This we demonstrate by providing an infinite

family of non-conjugate pairs of partitions that have the same generating function.

Finally, we prove asymptotic results on the number of maxima in this poset.

# Chapter 1

# Introduction

## 1.1 Partitions

An *integer partition* of a positive integer $n$ is a non-increasing sequence of positive integers that sums to $n$. For example, 4, 2, 1, 1 is a partition of 8 since $4+2+1+1 = 8$. The Swiss mathematician Leonhard Euler was the first to begin a systematic study of partitions, although some problems regarding partitions had been considered before Euler [3]. Despite the fact that partitions are, at face value, a simple number-theoretic concept, their study quickly developed into a rich research area of its own with deep ties to other branches of mathematics such as combinatorics, algebra and analysis. We refer the reader to [3, 4, 11, 33] for fuller historical details and the basics of the

theory of partitions. In this dissertation, we will give some new results on partitions subject to various finiteness restrictions.

We begin by providing some definitions and notation used throughout the work. We denote the positive integers by $\mathbb{P}$, and for $n \in \mathbb{P}$, $[n]$ denotes the set $\{1, 2, \ldots, n\}$. We use the vector notation $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\kappa)$ to represent a partition of a number $n \in \mathbb{P}$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\kappa \geq 1$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_\kappa = n$. For example, the partitions of 1 through 5 are given in the table below.

**Table 1.1**
Partitions of 1 through 5

| $n$ | Partitions of $n$ |
|---|---|
| 1 | (1) |
| 2 | (2), (1, 1) |
| 3 | (3), (2, 1), (1, 1, 1) |
| 4 | (4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1) |
| 5 | (5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1) |

Alternatively, we sometimes use the multiplicity notation $(a_1^{t_1}, a_2^{t_2}, \ldots, a_\ell^{t_\ell})$ (where $a_1 > a_2 > \cdots > a_\ell$) to represent the partition having $t_1$ parts of size $a_1$, $t_2$ parts of size $a_2$, etc.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_\kappa)$, the $\lambda_i$ are called the *parts* of the partition, and the parameter $\kappa$ is called the *length*. We say that $\lambda$ *partitions* $n$, denoted by $\lambda \vdash n$, and that the *size* of $\lambda$ (denoted by $|\lambda|$) is $n$. For example, the partition $(5, 3, 3, 1, 1, 1)$ of $n = 14$ has length 6 and contains parts 1, 3 and 5.

The number of partitions of $n$ is given by the *partition function*, denoted by $p(n)$. Motivated by ring-theoretic considerations, we say $p(0) = 1$, and consider 0 to be partitioned by the "empty" partition, $\varnothing$. Thus from Table 1.1, we see that the first several values of the partition function are 1, 1, 2, 3, 5, 7.

The *Ferrers diagram* (or *Ferrers shape*) of $\lambda = (\lambda_1, \ldots, \lambda_\kappa)$ is a left-justified array of unit boxes (called *cells*) having $\lambda_i$ cells in the $i^{\text{th}}$ row. For example, the Ferrers diagram of the partition $(7, 4, 4, 2, 1)$ is given below:

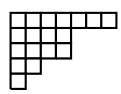


**Figure 1.1:** Ferrers diagram of the partition $(7, 4, 4, 2, 1)$

Reflecting the Ferrers diagram of $\lambda$ about the main (northwest to southeast) diagonal results in a new Ferrers diagram. We call the associated partition the *conjugate* of $\lambda$, denoted $\lambda'$. More formally, $\lambda'$ is the partition having $\lambda_i - \lambda_{i+1}$ parts of size $i$ for $1 \le i \le \kappa$ (where $\lambda_{\kappa+1}$ is defined to be 0).

The conjugate of $(7, 4, 4, 2, 1)$ is $(5, 4, 3, 3, 1, 1, 1)$, as visualized below:

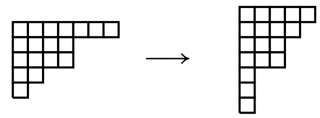


**Figure 1.2:** Ferrers diagrams of (7, 4, 4, 2, 1) and (5, 4, 3, 3, 1, 1, 1)

If $\lambda = \lambda'$, we say that $\lambda$ is *self-conjugate.*

The *Durfee square* is the largest square that can fit inside the Ferrers diagram of $\lambda$. (In the diagram above, the Durfee square has a side length of 3.) The $(i, j)$ *hook* of $\lambda$ is the portion of the Ferrers diagram consisting of the $(i, j)$ cell together with all cells directly to the right of and below it. The *hook length* is the number of cells in the hook. Cells strictly to the right are the *arm* of the hook, while cells strictly below are called the *leg.* For example, the diagram below shows the (3, 1) cell (shaded black) and the corresponding hook of length 6 (in bold).

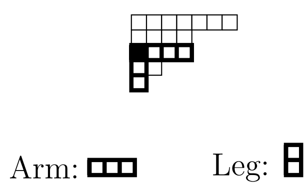



**Figure 1.3:** The (3, 1) hook in (7, 4, 4, 2, 1)

The $(i, i)$ hooks (those along the main diagonal) are called *principal.* It follows that the number of principal hooks of $\lambda$ is equal to the length of the Durfee square. A view of partitions that will prove useful is to regard a partition as the "interlocking"

of its principal hooks. For example, $(7, 4, 4, 2, 1)$ is composed of its principal hooks as seen below:

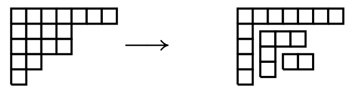


**Figure 1.4:** Principal hooks of $(7, 4, 4, 2, 1)$

Throughout the rest of this dissertation we will assume all hooks are principal, even if not explicitly denoted as such.

An alternative notation for partitions is the *Frobenius* notation. For a partition $\lambda$ having $k$ principal hooks, let $A_k, \ldots, A_1$ be the arm lengths of these hooks (proceeding southeast from the $(1, 1)$ cell), and similarly let $B_k, \ldots, B_1$ be the lengths of the legs. Then the Frobenius notation for $\lambda$ is the following $2 \times k$ array:

$$\left(\begin{smallmatrix} A_k & A_{k-1} & \cdots & A_1 \\ B_k & B_{k-1} & \cdots & B_1 \end{smallmatrix}\right).$$

Note that we will occasionally add commas for clarity, and that decreasing indices are chosen because of a particular representation of partitions used heavily in Chapter 4. As visualized in the decomposition given in Fig. 1.4, the Frobenius notation for $(7, 4, 4, 2, 1)$ is $\left(\begin{smallmatrix} 6 & 2 & 1 \\ 4 & 2 & 0 \end{smallmatrix}\right)$. Note that the $A_i$'s and $B_i$'s are strictly decreasing for all $i$, and furthermore $A_1, B_1 \geq 0$. If $\lambda$ partitions $n$, then we have $\sum_{i=1}^{k}(A_i + B_i + 1) = n$. Also, if $\lambda = \left(\begin{smallmatrix} A_k & A_{k-1} & \cdots & A_1 \\ B_k & B_{k-1} & \cdots & B_1 \end{smallmatrix}\right)$ then $\lambda' = \left(\begin{smallmatrix} B_k & B_{k-1} & \cdots & B_1 \\ A_k & A_{k-1} & \cdots & A_1 \end{smallmatrix}\right)$.

A major focus of this dissertation is the study of collections of partitions satisfying certain finiteness conditions. In particular, we are interested in the set of partitions whose Ferrers diagrams are restricted to fitting inside the Ferrers diagram of some fixed partition. We say that a partition $\mu = (\mu_1, \ldots, \mu_\eta)$ *fits inside* a partition $\lambda = (\lambda_1, \ldots, \lambda_\kappa)$, denoted $\mu \leq \lambda$, if $\eta \leq \kappa$ and $\mu_i \leq \lambda_i$ for all $i$. The relation $\leq$ indicates the so-called *Young order* (see [43, p. 597]). If $\mu \leq \lambda$ and $\mu \neq \lambda$, we write $\mu < \lambda$.

This can be visualized as the Ferrers diagram for $\mu$ fitting entirely inside that of $\lambda$. For example, $(3,3,1)$ and $(5,4,4,1,1)$ are two (of many) partitions that fit inside $(7,4,4,2,1)$, as illustrated below.

$(3,3,1) < (7,4,4,2,1)$ $(5,4,4,1,1) < (7,4,4,2,1)$

**Figure 1.5:** Examples of inclusion in (7, 4, 4, 2, 1)

The relation $\leq$ on the set of all partitions that fit inside some fixed partition $\lambda$ produces a *partially ordered set*, or *poset*. In general, a poset is a nonempty set together with a relation defined on the set that is reflexive, antisymmetric and transitive. (For more background on the theory of posets, see [33, Ch. 3].) We denote by $P_\lambda$ the poset of partitions fitting inside $\lambda$ and ordered by $\leq$.

For example, the Ferrers diagrams of all partitions in $P_{(4,2,1)}$ are shown below:

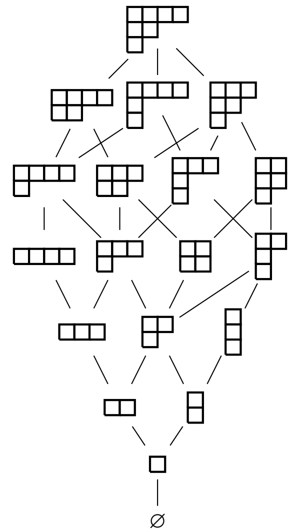


**Figure 1.6:** The Hasse diagram for $P_{(4,2,1)}$

Note in the diagram above that partitions are arranged vertically by size, and a connecting line drawn between a partition $\mu$ of $i$ and a partition $\lambda$ of $i+1$ indicates $\mu \prec \lambda$. This is called the *Hasse diagram* for the poset $P_{(4,2,1)}$.

A *chain* in a poset is a subset of totally ordered elements. For example the following is a chain in $P_{(4,2,1)}$:

$$\varnothing \prec (1) \prec (1,\,1) \prec (2,\,1) \prec (3,\,1) \prec (3,\,1,\,1) \prec (3,\,2,\,1) \prec (4,\,2,\,1).$$

A chain is called *saturated* if there is no element that can be inserted into the chain (except possibly at the beginning or end). The chain shown above is clearly saturated.

A poset is *graded* (or *ranked*) if every maximal chain has the same number of elements, or equivalently, if every saturated chain between two fixed elements has the same length. This means that each element in the poset is some distance (rank) above the smallest element(s).

It is not hard to see that the poset $P_\lambda$ is graded for any partition $\lambda$. In this context specifically, let $a_i$ be the number of partitions $\mu$ of $i$ that fit inside $\lambda$. Then the *generating function* for $\lambda$ is given by

$$G_\lambda = G_\lambda(q) = \sum_{\mu \leq \lambda} q^{|\mu|} = \sum_{i=0}^{n} a_i q^i.$$

For example, the generating function for (4, 2, 1) is

$$G_{(4,2,1)} = 1 + q + 2q^2 + 3q^3 + 4q^4 + 4q^5 + 3q^6 + q^7.$$

Note that the coefficient of degree $i$ is the number of Ferrers diagrams in row $i$ in Fig. 1.6. Since $\mu \leq \lambda$ if and only if $\mu' \leq \lambda'$, we see that $G_\lambda = G_{\lambda'}$. Thus in studying these generating functions, it suffices to consider partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\kappa)$ where $\lambda_1 \geq \kappa$.

We will extensively study the following properties that are potentially relevant for the

coefficients of $G_\lambda$ and other related sequences. A sequence $a_0, a_1, \dots, a_n$ is *symmetric* if $a_i = a_{n-i}$ for all $i$, where $0 \leq i \leq n$. A sequence is *unimodal* if it is of the form $a_0 \leq a_1 \leq \cdots \leq a_j \geq \cdots \geq a_n$ for some $j$ (that is, the sequence never strictly increases after a strict decrease).

For example, the following sequence is unimodal:

$$1, 2, 4, 5, 6, 6, 5, 3.$$

The following sequence is symmetric but not unimodal:

$$2, 3, 4, 3, 5, 3, 4, 3, 2.$$

Unimodality is a well-studied property in combinatorics. For an exposition on the importance and ubiquity of unimodality in combinatorics, algebra and related subjects, see R. Stanley's survey [31], and F. Brenti's update [9]. The most desirable method for proving unimodality (which is often very difficult to apply) is to give a combinatorial proof. More generally, finding a combinatorial argument of some result (even if it has already been verified with, say, an algebraic argument) remains one of the most important types of open problems in combinatorics. For a comprehensive survey (updated to 2003) about combinatorial techniques in partition theory specifically, see

[24].

We define a sequence to be *flawless* if $a_i \le a_{i+j}$ for any nonnegative integers $i$ and $j$ such that $2i + j \le n$. We note that our definition of flawless is stronger than what has been commonly given in the literature (e.g., [7], where this property is defined as $a_i \le a_{n-i}$ for all $i \le \lfloor \frac{n}{2} \rfloor$). This stronger definition has been referred to as *strongly* flawless [19]. However, the two definitions are equivalent if the sequence is increasing throughout the "first half" (that is, $a_0 \le a_1 \le \cdots \le a_{\lfloor \frac{n}{2} \rfloor}$). Many well-studied and significant sequences for which flawlessness holds in the weaker sense also increase throughout the first half. Thus it seems natural to define the flawless property as we do here.

For example, the following sequence is flawless:

$$1, 2, 4, 5, 8, 8, 6, 5, 2, 1.$$

The following sequence is unimodal but not flawless:

$$1, 3, 4, 5, 5, 4, 4, 3, 2.$$

Observe that if a flawless sequence is non-unimodal, then this pathology must occur in the second half. Also, proving that a symmetric sequence is flawless immediately

implies unimodality.

Flawlessness – though not as well-studied as unimodality – is a property enjoyed by many important sequences in combinatorics and algebra. Motivation for studying flawlessness typically comes from commutative algebra [15, 17, 29]. As an aside, we remark that highlighting the presence of flawlessness in this area of combinatorics (see Section 2.4.4) may hopefully lead to the discovery of more connections between the subject areas. In general, flawlessness of a sequence seems to suggest the existence of an underlying algebraic structure, or at least a linear-algebraic explanation [15, 34].

Notably, as we are going to re-prove in Chapter 2, pure $O$-sequences (the Hilbert functions of pure monomial order ideals), and also in particular the $f$-vectors of pure simplicial complexes (or equivalently, square-free monomial order ideals whose maximal monomials all have the same degree) are flawless [7, 15, 17]. Certain other Artinian level Hilbert functions of interest specifically in commutative algebra also satisfy flawlessness. Also, $h$-vectors of matroid complexes are flawless, as conjectured by T. Hibi [18] and proved by M. Chari [10] and T. Hausel [15]. For any unexplained terminology from commutative algebra, we refer the reader to [5, 16, 22, 32].

In general, Hilbert functions that are not flawless suggest the existence of an underlying algebraic pathology in those algebras which have that Hilbert function. In particular, it is believed that all Artinian level Hilbert functions in 3 variables satisfy flawlessness. Note that, should that fail, any level algebra with such a Hilbert

function has a Gorenstein quotient that fails the Strong Lefschetz Property [8]. In characteristic 0, this would contradict a famous conjecture in this area: that every Gorenstein algebra in three variables has the Strong Lefschetz Property.

A *q-analogue* of a mathematical quantity is an expression in the variable $q$ that simplifies to the original quantity when $q \to 1$. The standard $q$-analogue of $n \in \mathbb{P}$, denoted $[n]_q$, is $1 + q + q^2 + \ldots + q^{n-1} = \frac{1-q^n}{1-q}$. (Note that $\lim_{q\to 1}[n]_q = n$.) Consequently, the *q-factorial* of $n$ is $[n]_q! = [1]_q[2]_q\cdots[n]_q$. We can then define the *q-binomial*, or *Gaussian coefficient*, $\left[{n \atop k}\right]_q$, by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

The $q$-binomial coefficients appear in many combinatorial contexts, but our interest is in their connection to partitions. A standard result in partition theory is that $\left[{a+b \atop b}\right]_q$ is the generating function enumerating all partitions fitting inside the "rectangle" partition $(b^a) = (\overbrace{b, \ldots, b}^{a})$ (see e.g. [33] for a proof of this result). From this fact, it is easy to see that any $q$-binomial is a symmetric polynomial with coefficients in $\mathbb{P}$.

It is a well-known and important result in combinatorics that the coefficients of $\left[{n \atop k}\right]_q$ form a unimodal sequence. This was first shown by J. Sylvester using the theory of invariants [38], and proved in simpler linear-algebraic terms by R. Proctor [27]. Stanley (crediting the result to A. Iarrobino) also noted that unimodality is implied

by the Hard Lefschetz Theorem [30]. Then in 1990, K. O'Hara provided a celebrated combinatorial proof of this result [23, 43].

## 1.2 Outline of the Dissertation

Chapter 2 of this dissertation focuses on applications of a result (originally given in the language of relation theory) due to M. Pouzet [12, 25], called the Multicolor Theorem. We will show how this theorem brings under the same umbrella the flawlessness of a broad range of combinatorial sequences. In particular, we show how flawlessness can be proved using the Multicolor Theorem in each of the following settings:

1. The generating function of the order ideal generated by a single monic monomial (known trivially by other techniques).

2. Pure $O$-sequences (first shown by Hibi [17], then Hausel [15] and re-proven here using the results from the first application).

3. The generating function of compositions embedded inside a fixed composition (our result).

4. The generating function of partitions fitting inside a fixed Ferrers diagram (first observed by Pouzet in [26]).

5. The generating function of the poset of partitions fitting inside *several* fixed

Ferrers diagrams of the same size (our result).

In particular, we will note that the Multicolor Theorem places the proof of the unimodality of the $q$-binomial coefficients in a much broader context.

The background of Chapter 3 is a paper by D. Stanton [37], in which he initiated the study of the generating functions $G_\lambda$ for arbitrary $\lambda$. His main interest was to answer questions regarding the unimodality of these generating functions. We extend a technique used by Stanton to prove unimodality for some novel infinite families of partitions.

Chapter 4 contains the most significant and novel results of the dissertation. In that chapter, we introduce the study of a new poset,

$$P_n = \{G_\lambda \,|\, \lambda \vdash n\},$$

the set of distinct generating functions of partitions of $n$ ordered in the following natural manner: $G_\eta \leq G_\lambda$ if and only if $G_\lambda - G_\eta$ has nonnegative coefficients. We discuss some general structural properties of $P_n$. Most significantly, we prove that a certain two "balancing" procedures on a partition $\lambda$ produce partitions whose generating functions are at least as great (in the ordering of $P_n$) as $G_\lambda$. This gives a necessary condition for the maximal elements in $P_n$. We then give a family of pairs of non-conjugate partitions that have the same generating function, and conjecture the

existence of more general families. Finally, we prove that the number of maxima in $P_n$ tends to infinity as $n$ grows, and provide both a nontrivial lower and upper bound (the latter assuming a certain conjecture) on this quantity.

# Chapter 2

# Combinatorial Applications of the Multicolor Theorem

## 2.1 Introduction

In this chapter we consider some results stemming from work by Pouzet [25], written in French. This topic was subsequently presented in English by R. Fraïssé [12].

Our interest in this area was motivated by a letter written by Pouzet to Stanton ([26], reproduced in Appendix D with permission from Stanton). In this letter, Pouzet explained a combinatorial application of his results, which were phrased in [25] using the

language of relation theory. We generalize this application to several other combinatorial settings. This gives a powerful new method of showing flawlessness for certain sequences. (Recall that a sequence $a_0, \ldots, a_n$ is called *flawless* if $a_i \leq a_{i+j}$ whenever $2i + j \leq n$.) In particular, while some of the flawlessness results we discuss are already known by other means, we demonstrate that they may be regarded as falling under the same overarching umbrella.

Perhaps more significantly, we translate the key principle behind Pouzet's result and extend it to a simple yet crucial linear algebra framework. This is discussed in detail in the next section.

In the last chapter of this dissertation, we mention some significant open problems related to our work. There is hope that modifying the techniques discussed in this chapter may lead to progress on those important questions.

## 2.2 The Full Rank Lemma

In this section, we present the general linear-algebraic idea that we distilled from Pouzet's work, which will be discussed in the next section. For any unexplained linear algebra concepts, we refer the reader to [13, Ch. 3].

Consider two positive integers $a$ and $b$ where $a \leq b$. Let $U$ and $V$ be vector spaces of

dimensions $a$ and $b$, respectively, over some field $k$. Then, given any injective linear map $T : U \to V$ and bases $\mathcal{U}$ for $U$ and $\mathcal{V}$ for $V$, we can create the $a \times b$ matrix $M$ corresponding to $T$ with respect to bases $\mathcal{U}$ and $\mathcal{V}$. (Note a slight deviation from [13]; we adopt the convention of representing $T$ by an $a \times b$ matrix operating on elements of $U$ by multiplying them on the left of the matrix, whereas conventional notation represents $T$ by a $b \times a$ matrix with elements in $U$ multiplied on the right.) Since $T$ is injective, it follows that $M$ has full rank (that is, $\mathrm{Rank}(M) = a$), or equivalently (under our convention), that the rows of $M$ are linearly independent. We now state our generalization of Pouzet's theorem.

**Lemma 2.2.1** (Full Rank Lemma)**.** *Let $a \leq b$, and let $M$ be an $a \times b$ matrix having full rank. Consider taking a collection of $t$ rows of $M$, and replace one of these rows with a nonzero linear combination of the $t$ rows, while eliminating the rest. Then the resulting $(a - t + 1) \times b$ matrix has full rank.*

Equivalently, we may say that the rows of the resulting matrix retain linear independence. The proof of this lemma follows easily from basic linear algebra. Clearly this procedure of combining/replacing rows can be repeated multiple times with other collections of rows. At the end of this process, we are left with a full rank matrix, which we call $M'$.

From this very general observation from linear algebra, we consider the following application to combinatorics. Suppose we have a collection of combinatorial objects,

to each of which we can apply some statistic (call it *degree* as a generic term for simplicity of notation) that yields a sequence: $a_0$, $a_1, \ldots, a_n$. Suppose moreover that this sequence is flawless.

As we saw in the previous discussion, we can create an $a_i \times a_{i+j}$ matrix $M$ with linearly independent rows. (Ideally, we wish to exploit some relationship between the two sets of combinatorial objects counted by $a_i$ and $a_{i+j}$ that gives us a natural way of forming this full rank matrix $M$ as a sort of "incidence matrix" induced by the underlying structure.)

Suppose furthermore that we have some "coarser" way to view this collection of combinatorial objects. That is, consider partitioning the objects into equivalence classes "modulo" some condition. This in turn yields a natural combining of rows of $M$ (as discussed above) to form a new matrix $M'$. The number of rows would then be the number of equivalence classes of degree $i$. If it can be demonstrated that the number of columns of $M'$ is no more than the number of equivalence classes of degree $i + j$, then we have successfully shown that the generating function corresponding to this new collection of objects is flawless.

## 2.3 The Multicolor Theorem

All of the applications that we present of the concept discussed in the previous section will begin with taking our collection of combinatorial objects to be the boolean algebra $B_n$: all subsets of the set $S = [n] = \{1, 2, \ldots, n\}$. The sequence $a_0, a_1, \ldots, a_n$ is the binomial coefficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$. We follow Fraïssé's [12] terminology in the following definition.

For positive integers $i$ and $j$ where $2i + j \leq n$, define the *adjacency matrix*, $M$, to be the $\binom{n}{i} \times \binom{n}{i+j}$ matrix with rows labeled by the $i$-subsets of $S$ and columns labeled by the $(i + j)$-subsets of $S$, where $M_{s,t}$ is 1 if the $i$-subset corresponding to row $s$ is contained inside the $(i + j)$-subset corresponding to row $t$, and 0 otherwise.

It is well known that the sequence of binomial coefficients is symmetric and unimodal, and thus in particular flawless [31]. The next lemma implies this result by claiming the stronger fact that the associated adjacency matrix has full rank. The proof of this lemma can be found in [12].

**Lemma 2.3.1** (Linear Independence Lemma)**.** *The rows of the adjacency matrix $M$ are linearly independent.*

Lemma 2.3.1 implies that $M$ produces an injection from the collection of $i$-subsets of

$S$ into the collection of $(i+j)$-subsets. In particular, we have the following:

**Corollary 2.3.2.** *$M$ determines an injection that maps each $i$-subset $X$ to an $(i+j)$-subset, $Y$ of $S$ that contains $X$.*

*Proof.* Take an $\binom{n}{i} \times \binom{n}{i}$ square submatrix of $M$ having full rank, or equivalently, whose determinant is nonzero. Such a square submatrix exists since $M$ itself has full rank. We assert that, for any $r$, the $r^{\text{th}}$ row of $M$ intersects some $c^{\text{th}}$ column in a nonzero entry such that deleting the $r^{\text{th}}$ row and $c^{\text{th}}$ column yields an $\binom{n}{i} - 1 \times \binom{n}{i} - 1$ submatrix with nonzero determinant. For if this were not the case, then finding $\det(M)$ using the recursive determinant formula with co-factor expansion along the $r^{\text{th}}$ row would yield $\det(M) = 0$, a contradiction.

Thus in the injection, we may take the subset $X$ that labels the $r^{\text{th}}$ row to be mapped to the subset $Y$ which labels the $c^{\text{th}}$ column, where $M_{r,c} \neq 0$. Recall that by definition, $M_{r,c} = 1$ exactly when the $i$-subset labeling row $c$ is contained in the $(i+j)$-subset labeling column $c$. Thus we have that $X \subseteq Y$. The rest of this subset-preserving injection can be completed inductively. $\square$

We can now begin discussing more precisely the concept of "coarsening" the collection of subsets of $S$ to some other collection of combinatorial objects of interest. Recall that $S$ has size $n$, and always in the following discussion we have arbitrary (but fixed) degrees $i$ and $i+j$ such that $2i+j \leq n$. We use the notation $\binom{S}{i}$ to denote the collection

of $i$-subsets of $S$.

Fix a set $U = \{u_1, u_2, \ldots, u_m\}$ of indeterminants, called colors, where $m \leq \binom{n}{i}$. These "colors" correspond to the collection of new combinatorial objects of degree $i$. Let $c_i : \binom{S}{i} \to U$ be any surjective function (called the *$i$-color function*). Note that $c_i$ induces a partitioning of the collection of $i$-subsets of $S$ into $m$ nonempty classes labeled by the colors in $U$.

Let $Y$ be a subset of $S$ of size $i + j$. Given an $i$-color function $c_i$, the *$i$-multicolor* of $Y$, denoted by $mult_i(Y)$, is the $m$-tuple $\mathbf{y} = (y_1, y_2, \ldots, y_m)$, where $y_\ell$ is the number of $i$-subsets of $Y$ having color $u_\ell$. We say $u_\ell$ *figures* in the multicolor $\mathbf{y}$ if $y_\ell \neq 0$. Note that $c_i$ uniquely determines $mult_i$, and the sum of the elements of $mult_i$ is $\binom{i+j}{i}$.

*Example* 2.3.3. Consider $S = \{1, 2, 3, 4, 5\}$, and fix colors $U = \{1, 2, 3, 4\}$. Fix $i = 2$ and $j = 2$ (following the notation above). For any 2-subset $\{a, b\}$ of $S$, suppose we define the color function $c_2$ to be $c_2(\{a, b\}) = |a - b|$. (That is, the "color" of a subset is the absolute value of the difference of its elements.) To see how the multicolor function $mult_2$ acts, consider the specific subset $Y = \{1, 3, 4, 5\}$. The 2-subsets of $Y$ are $\{1, 3\}$, $\{1, 4\}$, $\{1, 5\}$, $\{3, 4\}$, $\{3, 5\}$, $\{4, 5\}$ having colors 2, 3, 4, 1, 2, 1, respectively. Hence $mult_2(Y) = (2, 2, 1, 1)$.

We can now state Pouzet's original theorem, as presented in Fraïssé's work [12].

**Theorem 2.3.4** (The Multicolor Theorem)**.** *Let $S = [n]$ and let $i$ and $j$ be positive*

*integers where $2i + j \leq n$. Fix any surjective $i$-color function*

$$c_i : \binom{S}{i} \to U = \{u_1, u_2, \ldots, u_m\}.$$

*Then the number of $i$-multicolors of $(i+j)$-subsets of $S$ is greater than or equal to the number of colors of $i$-subsets of $S$.*

*More specifically, there exists an injection which associates each color $u_\ell$ of $i$-subsets to a multicolor of $(i+j)$-subsets in which $u_\ell$ figures (and moreover there is an $(i+j)$-subset having this multicolor).*

We note that the first part of this theorem follows from Lemma 2.2.1, as applied to the adjacency matrix $M$ to form a new matrix $M'$. The rows of this matrix are labeled by the $i$-colors, and the columns are exactly the $i$-multicolors. Note that $M'$ may have duplicate columns arising from multiple $(i + j)$-subsets having the same $i$-multicolor. However, it is a simple result from linear algebra that deleting any subsequently repeated columns does not change the linear independence of the rows. After deleting these, we obtain a full rank matrix with rows labeled by the $i$-colors and distinct columns equal to the $i$-multicolors. Thus we have at least as many $i$-multicolors as $i$-colors. We will refer to this matrix as the *reduced adjacency matrix*.

The second part of the theorem follows from a similar argument as that used in the proof of Corollary 2.3.2. Obviously a color $u$ (labeling a row) figures in a multicolor

(a column) if and only if the row and column in question intersect in a nonzero entry.

We can summarize the usefulness of the Multicolor Theorem in combinatorial settings as follows (note that this is a specialization of the discussion from Section 2.2): Let $\mathcal{C}$ be a collection of combinatorial objects with some statistic, $\deg : \mathcal{C} \to [n]$. Let $\mathcal{C}_l = \{c \in \mathcal{C} \mid \deg(c) = l\}$. Assume that we suspect a priori that the sequence $|\mathcal{C}_1|, |\mathcal{C}_2|, \ldots, |\mathcal{C}_n|$ is flawless. Again take $S = [n]$. Our goal is to find a surjective color function

$$c_i : \binom{S}{i} \to \mathcal{C}_i$$

which uniquely defines the $i$-multicolor function

$$mult_i : \binom{S}{i+j} \to \mathbb{P}^{|\mathcal{C}_i|},$$

and also to find an $(i+j)$-color function

$$\tilde{c}_{i+j} : \binom{S}{i+j} \to \mathcal{C}_{i+j}$$

which together satisfy the following property: if for any two $(i+j)$-subsets $Y$ and $Y'$ we have $\tilde{c}_{i+j}(Y) = \tilde{c}_{i+j}(Y')$, then $mult_i(Y) = mult_i(Y')$. We note that this property could be verified by exhibiting an $i$-color preserving bijection between $\binom{Y}{i}$ and $\binom{Y'}{i}$. We also note that $\tilde{c}_{i+j}$ acts analogously to $c_i$ in the applications we consider.

Alternatively, this condition can be stated as follows: if we know $\tilde{c}_{i+j}(Y)$ for some unknown subset $Y$, then we can uniquely determine $mult_i(Y)$. That is, there are at least as many $(i+j)$-colors as $i$-multicolors of $(i+j)$-subsets. Then if $2i+j \leq n$, we have an injection from $\mathcal{C}_i$ into $\mathcal{C}_{i+j}$ by the Multicolor Theorem. As a consequence, this would show that $|\mathcal{C}_1|$, $|\mathcal{C}_2|, \ldots, |\mathcal{C}_n|$ is flawless.

Recall that the reduced adjacency matrix has linearly independent rows labeled by the $i$-colors, with distinct columns that are equal to the $i$-multicolors. Each column is labeled by some $(i+j)$-subset, $Y$, as inherited from the original adjacency matrix. Consider re-labeling each column by $\tilde{c}_{i+j}(Y)$. Then if the condition discussed in the previous paragraph holds, these $(i+j)$-color labels are all distinct. It will be useful in the applications given in the next section to regard the columns as labeled in this way.

## 2.4 Combinatorial Applications

### 2.4.1 The Order Ideal Generated by a Single Monomial

Borrowing language from combinatorial commutative algebra, we define a *monomial order ideal* in the variables $x_1$, $x_2, \ldots, x_k$ to be a set of monic monomials of non-negative degree which is closed with respect to taking division. For the definitions

and general theory of such structures, see [7, 22].

Let $\boldsymbol{X} = x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ be a monomial of degree $n = n_1 + n_2 + \cdots + n_k$. Our set of combinatorial objects, $\mathcal{C}$ is the order ideal *generated by* $\boldsymbol{X}$: the collection of all (monic) monomials that divide $\boldsymbol{X}$. This is denoted by $\mathcal{C} = \langle \boldsymbol{X} \rangle$. We define $\mathcal{C}_i = \langle \boldsymbol{X} \rangle_i \subset \langle \boldsymbol{X} \rangle$ to be the collection of divisors of $\boldsymbol{X}$ having degree $i$.

The generating function for $\langle \boldsymbol{X} \rangle$ is

$$F_{\langle \boldsymbol{X} \rangle}(q) = \prod_{i=1}^{k} \left(1 + q + q^2 + \ldots + q^{n_i}\right).$$

It is a well-known fact in combinatorics (see e.g. [31]) that the product of symmetric unimodal polynomials (with nonnegative coefficients) is again symmetric and unimodal. Thus $F_{\langle \boldsymbol{X} \rangle}(q)$ is symmetric unimodal, and in particular we trivially have that $F_{\langle \boldsymbol{X} \rangle}(q)$ is flawless.

Despite this simple proof, we demonstrate how the flawlessness of $F_{\langle \boldsymbol{X} \rangle}(q)$ can be shown using the Multicolor Theorem 2.3.4. Indeed, as a consequence of this we will have the stronger result that the associated injection from $\langle \boldsymbol{X} \rangle_i$ to $\langle \boldsymbol{X} \rangle_{i+j}$ preserves divisibility. That is, a degree $i$ monomial $\boldsymbol{A}$ is mapped to a degree $i + j$ monomial $\boldsymbol{B}$ where $\boldsymbol{A} | \boldsymbol{B}$. Furthermore, in the next section we will use the corresponding reduced adjacency matrices to show the non-trivial result that the generating function of the order ideal generated by multiple monic monomials of the same degree is also flawless.

This constitutes an original approach to proving this well-known result, which was first demonstrated by Hibi [17], then Hausel [15].

We proceed to apply the Multicolor Theorem to re-prove the single monomial case. Given the generating monomial $\boldsymbol{X} = x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$, partition $S = [n]$ into $k$ subsets, $A_1, A_2, \ldots, A_k$, where $A_1 = \{1, 2, \ldots, n_1\}$, $A_2 = \{n_1 + 1, \ldots, n_1 + n_2\}$, etc. For any subset $W$ of $S$ and any $\ell \in \{1, \ldots, k\}$, denote

$$W_\ell = W \cap A_\ell.$$

We "color" the $i$-subsets of $S$ using monomials of degree $i$ as follows: for any $i$-subset $X$ of $S$, define $c_i(X)$ to be the monomial

$$c_i(X) = x_1^{|X_1|} x_2^{|X_2|} \cdots x_k^{|X_k|}$$

dividing $\boldsymbol{X}$. Similarly for any $(i + j)$-subset $Y$ of $[n]$, define

$$\tilde{c}_{i+j}(Y) = x_1^{|Y_1|} x_2^{|Y_2|} \cdots x_k^{|Y_k|}.$$

Suppose that for two $(i+j)$-subsets $Y$ and $Y'$ we have $\tilde{c}_{i+j}(Y) = \tilde{c}_{i+j}(Y')$. This implies $|Y_t| = |Y'_t|$ for all $t \in \{1, \ldots, k\}$. Then for any monomial $u = x_1^{b_1} \cdots x_k^{b_k}$ of degree $i$ that

divides $\boldsymbol{X}$, the component corresponding to $u$ in the multicolor of both $Y$ and $Y'$ is

$$\binom{|Y_1|}{b_1}\binom{|Y_2|}{b_2}\cdots\binom{|Y_k|}{b_k}=\binom{|Y_1'|}{b_1}\binom{|Y_2'|}{b_2}\cdots\binom{|Y_k'|}{b_k}.$$

Thus $mult_i(Y) = mult_i(Y')$.

Therefore, knowing $\tilde{c}_{i+j}(Y)$ (that is, the degree $i+j$ monomial corresponding to $Y$) in turn uniquely determines $mult_i(Y)$. Importantly, note that this determination is independent of the starting monomial $\boldsymbol{X}$.

Thus there are at least as many $(i+j)$-colors as $i$-multicolors, and so by the Multicolor Theorem 2.3.4 (and our discussion at the end of Section 2.3), the generating function $F_{\langle \boldsymbol{X} \rangle}(q)$ is flawless.

### 2.4.2 An Extension to Pure $O$-sequences of Arbitrary Type

We now consider the order ideal generated by several monomials,

$$\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_t,$$

of the same degree; that is, the collection of all monomials that divide any of the $\boldsymbol{X}_\ell$. Such an order ideal is called *pure* of *type* $t$, and the coefficients of the corresponding

generating function form a *pure $O$-sequence*, first introduced by Stanley [29]. We refer the reader to [7] for the most comprehensive study of pure $O$-sequences. As mentioned above, it is well known that pure $O$-sequences enjoy the flawless property. We re-prove this fact here using the full rank reduced adjacency matrices given by the Multicolor Theorem 2.3.4. Our proof shares some similarity to Hibi's [17] in that we also use induction on the number of monomial generators. However, the real essence of our proof - showing the existence of injective linear maps - is closer in flavor to Hausel's technique from [15]. Using the Hard Lefschetz Theorem, Hausel showed injectivity between graded pieces of a suitable quotient ring (namely, the Artinian monomial level algebra corresponding via Macaulay's inverse systems to our order ideal; see [7, 15]).

We will illustrate the proof in the case of 2 and 3 generators, and from this it will be easy to see that the general case holds by induction. Let $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ be monic monomials of degree $n$. For degrees $i$ and $i+j$ where $2i+j \leq n$, let $M_{\boldsymbol{X}_1}$ and $M_{\boldsymbol{X}_2}$ be the corresponding reduced adjacency matrices for $\langle \boldsymbol{X}_1 \rangle$ and $\langle \boldsymbol{X}_2 \rangle$, respectively. Form the block matrix

$$N = \begin{bmatrix} M_{\boldsymbol{X}_1} & P \\ O & M_{\boldsymbol{X}_2} \end{bmatrix},$$

where $P$ and $O$ represent zero matrices of appropriate sizes.

Note that the rows of $N$ are linearly independent since $M_{\boldsymbol{X}_1}$ and $M_{\boldsymbol{X}_2}$ have full rank.

For every degree $i$ monomial $u$ in $\langle \boldsymbol{X}_1 \rangle \cap \langle \boldsymbol{X}_2 \rangle$, replace the row of $P$ labeled by $u$ to be the row of $M_{\boldsymbol{X}_2}$ labeled by $u$, and then delete the row labeled by $u$ in $\begin{bmatrix} O & M_{\boldsymbol{X}_2} \end{bmatrix}$.

After this procedure we have a new matrix,

$$N' = \begin{bmatrix} M_{\boldsymbol{X}_1} & P' \\ O' & M'_{\boldsymbol{X}_2} \end{bmatrix},$$

where $O'$ is a zero matrix, and $P'$ is now not a zero matrix unless $\langle \boldsymbol{X}_1 \rangle \cap \langle \boldsymbol{X}_2 \rangle = \varnothing$. The matrix $M'_{\boldsymbol{X}_2}$ is simply $M_{\boldsymbol{X}_2}$ with some rows deleted, and $M_{\boldsymbol{X}_1}$ is unchanged. It is easy to see that the rows of $N'$ are linearly independent.

It is important to note that an $(i + j)$-color (that is, a monomial of degree $i + j$) has exactly the same degree $i$ monomial divisors ($i$-colors), regardless of the degree $n$ generating monomial. Recall also that the $(i + j)$-color label, $\tilde{c}_{i+j}(Y)$, of a column in turn uniquely determines the multicolor $mult_i(Y)$, and that this determination is again independent of the generating monomial.

This implies that any two columns $c_1$ and $c_2$ of $\begin{bmatrix} M_{\boldsymbol{X}_1} \\ O' \end{bmatrix}$ and $\begin{bmatrix} P' \\ M'_{\boldsymbol{X}_2} \end{bmatrix}$, respectively, labeled by the same $(i + j)$-color are identical. Deleting these duplicate columns in $\begin{bmatrix} P' \\ M'_{\boldsymbol{X}_2} \end{bmatrix}$ preserves the linear independence of the rows of $N'$. Let us call the resulting matrix $M_{\boldsymbol{X}_1, \boldsymbol{X}_2}$. Then $M_{\boldsymbol{X}_1, \boldsymbol{X}_2}$ is a matrix whose rows are labeled by the degree $i$

monomials in $\langle \boldsymbol{X}_1, \boldsymbol{X}_2 \rangle$ and whose columns are labeled by (at least some of) the degree $i+j$ monomials in $\langle \boldsymbol{X}_1, \boldsymbol{X}_2 \rangle$. Since $M_{\boldsymbol{X}_1, \boldsymbol{X}_2}$ has full rank, we have at least as many degree $i+j$ as degree $i$ monomials. Hence a pure $O$-sequence generated by two monomials is flawless.

Suppose we have a third monomial, $\boldsymbol{X}_3$, of degree $n$ with associated reduced adjacency matrix $M_{\boldsymbol{X}_3}$ (using the same $i$ and $j$). Form the block matrix

$$L = \begin{bmatrix} M_{\boldsymbol{X}_1, \boldsymbol{X}_2} & P \\ O & M_{\boldsymbol{X}_3} \end{bmatrix},$$

which has full rank as before. Repeat an analogous procedure as discussed above: for every degree $i$ monomial $u$ in $\langle \boldsymbol{X}_1, \boldsymbol{X}_2 \rangle \cap \langle \boldsymbol{X}_3 \rangle$, replace the row of $P$ labeled by $u$ to be the row of $M_{\boldsymbol{X}_3}$ labeled by $u$, and then delete the row labeled by $u$ in $\begin{bmatrix} O & M_{\boldsymbol{X}_3} \end{bmatrix}$.

After this procedure we have a new matrix,

$$L' = \begin{bmatrix} M_{\boldsymbol{X}_1, \boldsymbol{X}_2} & P' \\ O' & M'_{\boldsymbol{X}_3} \end{bmatrix},$$

with linearly independent rows. Again deleting duplicate-labeled columns, we obtain a matrix $M_{\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3}$ with linearly independent rows. The rows are labeled by the degree $i$ monomials, and the columns by (at least some of) the degree $i+j$ monomials of $\langle \boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3 \rangle$.

Clearly we can continue to add monomials of degree $n$ and repeat this procedure in a similar fashion. Thus inductively we conclude that the generating function of any pure monomial order ideal, i.e., any pure $O$-sequence, is flawless.

*Example* 2.4.1. Consider the three monomials $\boldsymbol{X}_1 = x^2y^2z^2$, $\boldsymbol{X}_2 = x^2y^2t^2$ and $\boldsymbol{X}_3 = y^3w^3$. By simple computations, the pure $O$-sequence for the order ideal $\langle \boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3 \rangle$ is $h_0, h_1, \ldots, h_6 = 1, 5, 11, 16, 14, 8, 3$. Take $i = 2$ and $j = 1$ (so $2i+j = 5 < n = 6$). Then the corresponding reduced adjacency matrices as given by the Multicolor Theorem 2.3.4 (with rows and columns labeled for clarity) are the following:

$$M_{\boldsymbol{X}_1} = \begin{array}{c} \\ x^2 \\ y^2 \\ z^2 \\ xy \\ xz \\ yz \end{array} \!\!\begin{array}{c} \begin{array}{ccccccc} xyz & x^2y & x^2z & xy^2 & y^2z & xz^2 & yz^2 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 2 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 2 \end{pmatrix} \end{array}$$

$$M_{\boldsymbol{X}_2} = \begin{array}{c} \\ x^2 \\ y^2 \\ t^2 \\ xy \\ xt \\ yt \end{array} \!\!\! \begin{array}{c} \begin{array}{ccccccc} xyt & x^2y & x^2t & xy^2 & y^2t & xt^2 & yt^2 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 2 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 2 \end{pmatrix} \end{array}$$

$$M_{\boldsymbol{X}_3} = \begin{array}{c} \\ y^2 \\ yw \\ w^2 \end{array} \!\!\! \begin{array}{c} \begin{array}{cccc} y^3 & y^2w & yw^2 & w^3 \end{array} \\ \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix} \end{array}$$

The initial block matrix $N$ for $M_{\boldsymbol{X}_1}$ and $M_{\boldsymbol{X}_2}$ is

$$N = \begin{array}{cc} & \begin{array}{cccccccccccccc} xyz & x^2y & x^2z & xy^2 & y^2z & xz^2 & yz^2 & xyt & x^2y & x^2t & xy^2 & y^2t & xt^2 & yt^2 \end{array} \\ \begin{array}{c} x^2 \\ y^2 \\ z^2 \\ xy \\ xz \\ yz \\ x^2 \\ y^2 \\ t^2 \\ xy \\ xt \\ yt \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 2 \end{pmatrix} \end{array}.$$

After changing rows of $P$ and deleting rows of $\begin{bmatrix} O & M_{\boldsymbol{X}_2} \end{bmatrix}$, we get

$$N' = \begin{array}{r} \\ x^2 \\ y^2 \\ z^2 \\ xy \\ xz \\ yz \\ t^2 \\ xt \\ yt \end{array} \begin{array}{c} \begin{array}{cccccccccccccc} xyz & x^2y & x^2z & xy^2 & y^2z & xz^2 & yz^2 & xyt & x^2y & x^2t & xy^2 & y^2t & xt^2 & yt^2 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 2 & 0 & 0 & 0 & 1 & 2 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 2 \end{pmatrix} \end{array}.$$

After deleting duplicate-labeled columns, we finally obtain

$$M_{\boldsymbol{X}_1,\boldsymbol{X}_2} = \begin{array}{c} \\ x^2 \\ y^2 \\ z^2 \\ xy \\ xz \\ yz \\ t^2 \\ xt \\ yt \end{array} \begin{array}{c} \begin{array}{cccccccccccc} xyz & x^2y & x^2z & xy^2 & y^2z & xz^2 & yz^2 & xyt & x^2t & y^2t & xt^2 & yt^2 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 2 \end{pmatrix} \end{array}.$$

Adding in $\boldsymbol{X}_3$ and repeating the above procedure, we obtain $M_{\boldsymbol{X}_1,\boldsymbol{X}_2,\boldsymbol{X}_3}$:

$$
\begin{array}{c}
\\
\begin{array}{c} x^2 \\ y^2 \\ z^2 \\ xy \\ xz \\ yz \\ t^2 \\ xt \\ yt \\ yw \\ w^2 \end{array}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccccc} xyz & x^2y & x^2z & xy^2 & y^2z & xz^2 & yz^2 & xyt & x^2t & y^2t & xt^2 & yt^2 & y^3 & y^2w & yw^2 & w^3 \end{array} \\
\begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 3 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3
\end{pmatrix}
\end{array}.
$$

Thus $h_2 = 11$ is less than or equal to $h_3 = 16$, which illustrates flawlessness of the pure $O$-sequence between degrees 2 and 3.

### 2.4.3 Compositions

A *composition* of a positive integer $n$ is a vector $\boldsymbol{c} = (c_1, c_2, \ldots, c_k) \in \mathbb{P}^k$ where $c_1 + c_2 + \ldots + c_k = n$. We will say that a composition $\boldsymbol{b} = (b_1, b_2, \ldots, b_m)$ is *embedded* in $\boldsymbol{c}$ if

$\boldsymbol{c}$ has $m$ (not necessarily consecutive) parts $c_{\ell_1}, c_{\ell_2}, \dots, c_{\ell_m}$ with $\ell_1 < \ell_2 < \dots < \ell_m$ such that $b_1 \le c_{\ell_1}, b_2 \le c_{\ell_2}, \dots, b_m \le c_{\ell_m}$.

For this application, we take $\mathcal{C}$ to be the collection of all compositions embedded inside $\boldsymbol{c} = (c_1, c_2, \dots, c_k)$. The set $\mathcal{C}_i$ is the set of compositions of $i$ embedded inside $\boldsymbol{c}$. We proceed to use the Multicolor Theorem 2.3.4 to show that the generating function $F_{\mathcal{C}}(q) = \sum_{\ell=1}^{n} |\mathcal{C}_\ell| q^\ell$ is flawless.

(We note that a different definition of inclusion of compositions is used by G. Andrews [2] and B. Sagan [28], namely, that $\boldsymbol{b}$ fits inside $\boldsymbol{c}$ if $m \le k$ and $b_\ell \le c_\ell$ for all $\ell$. It remains an open problem to apply the Multicolor Theorem to this scenario.)

First, partition $S = [n]$ into $k$ classes, $A_1, A_2, \dots, A_k$, where $A_1 = \{1, 2, \dots, c_1\}$, $A_2 = \{c_1 + 1, \dots, c_1 + c_2\}$, etc. For any $W \subseteq S$, denote $W_\ell = W \cap A_\ell$. Then for any $i$-subset $X$, define $c_i(X)$ by

$$c_i(X) = (a_1, a_2, \dots, a_p)$$

if there are exactly $p$ nonempty $X_\ell$'s: $X_{\ell_1}, X_{\ell_2}, \dots, X_{\ell_p}$ with $\ell_1 < \ell_2 < \dots < \ell_p$ such that $a_r = |X_{\ell_r}|$ for all $r \in \{1, \dots, p\}$.

Similarly for any $(i + j)$-subset $Y$, define $\tilde{c}_{i+j}(Y)$ to be

$$\tilde{c}_{i+j}(Y) = (b_1, b_2, \dots, b_s)$$

if there are exactly $s$ nonempty $Y_\ell$'s: $Y_{\ell_1}, Y_{\ell_2}, \dots, Y_{\ell_s}$ with $\ell_1 < \ell_2 < \cdots < \ell_s$ such that $b_t = |Y_{\ell_t}|$ for all $t \in \{1, \dots, s\}$.

Suppose for two $(i+j)$-subsets $Y$ and $Y'$ we have $\tilde{c}_{i+j}(Y) = \tilde{c}_{i+j}(Y')$. This implies that there are indices $\ell_1 < \ell_2 < \cdots < \ell_s$ and $\ell'_1 < \ell'_2 < \cdots < \ell'_s$ such that $|Y_{\ell_t}| = |Y'_{\ell'_t}|$ for all $t \in \{1, \dots, s\}$.

(Note that if $\ell_{i_1} < \ell_{i_2}$, then $\ell'_{i_1} < \ell'_{i_2}$.) Let $\phi : Y \to Y'$ be any bijection satisfying

$$\phi(Y_{\ell_t}) = Y'_{\ell'_t}$$

for all $t \in \{1, \dots, s\}$. Clearly such a bijection can be created since $|Y_{\ell_t}| = |Y'_{\ell'_t}|$. Now let $X$ be any $i$-subset of $Y$ having color $c_i(X) = (a_1, a_2, \dots, a_p)$.

This means that there are indices $\ell_{i_1}, \ell_{i_2}, \dots, \ell_{i_p} \in \{\ell_1, \ell_2, \dots, \ell_s\}$ such that $\ell_{i_1} < \ell_{i_2} < \cdots < \ell_{i_p}$ and $a_r = |X_{\ell_{i_r}}|$ for all $r \in \{1, \dots, p\}$. But then since $X_{\ell_{i_r}} \subseteq Y_{\ell_{i_r}}$, we have that

$$\phi(X_{\ell_{i_r}}) \subseteq Y'_{\ell'_{i_r}}$$

for all $r \in \{1, \dots, p\}$. Since $\ell_{i_1} < \ell_{i_2} < \cdots < \ell_{i_p}$, we have $\ell'_{i_1} < \ell'_{i_2} < \cdots < \ell'_{i_p}$. Also, clearly

$$|\phi(X_{\ell_{i_r}})| = |X_{\ell_{i_r}}| = a_r$$

for all $r \in \{1, \dots, p\}$.

Thus we conclude that $c_i(\phi(X)) = (a_1, \ldots, a_p)$, so $\phi$ is an $i$-color preserving bijection between $Y$ and $Y'$, and thus $mult_i(Y) = mult_i(Y')$, as desired.

### 2.4.4 Partitions

The result of this section is due to Pouzet. Indeed, the discussion of this application of the Multicolor Theorem 2.3.4 was the very content of the seminal letter written by Pouzet to Stanton [26]. We rephrase Pouzet's results in more combinatorial terms, and add several original observations and implications.

Let $n$ be a positive integer and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of $n$. Our set of combinatorial objects, $\mathcal{C}$, is the collection of partitions fitting inside $\lambda$. Recall that a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ fits inside $\lambda$ if $m \leq k$ and $\mu_\ell \leq \lambda_\ell$ for all $\ell$. The set $\mathcal{C}_\ell$ is the collection of partitions of $\ell$ fitting inside $\lambda$. We will show that the generating function $G_\lambda = \sum_{i=1}^{n} |C_\ell| q^\ell$ is flawless.

We again partition $S = [n]$ into $k$ classes, $A_1, A_2, \ldots, A_k$, where $A_1 = \{1, 2, \ldots, \lambda_1\}$, $A_2 = \{\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2\}$, etc. For any $W \subseteq S$, denote $W_\ell = W \cap A_\ell$. For any $i$-subset $X$ of $S$, define $c_i(X)$ to be

$$c_i(X) = (\eta_1, \eta_2, \ldots, \eta_p)$$

if there are exactly $p$ nonempty $X_\ell$'s: $X_{\ell_1}, X_{\ell_2}, \ldots, X_{\ell_p}$ where $|X_{\ell_1}| \geq |X_{\ell_2}| \geq \cdots \geq |X_{\ell_p}|$ with $\eta_r = |X_{\ell_r}|$ for all $r \in \{1, \ldots, p\}$.

Similarly for any $(i+j)$-subset $Y$, define $\tilde{c}_{i+j}(Y)$ to be

$$\tilde{c}_{i+j}(Y) = (\zeta_1, \zeta_2, \ldots, \zeta_s)$$

if there are exactly $s$ nonempty $Y_\ell$'s: $Y_{\ell_1}, Y_{\ell_2}, \ldots, Y_{\ell_s}$ where $|Y_{\ell_1}| \geq |Y_{\ell_2}| \geq \cdots \geq |Y_{\ell_s}|$ with $\zeta_t = |Y_{\ell_t}|$ for all $t \in \{1, \ldots, s\}$

Suppose for two $(i+j)$-subsets $Y$ and $Y'$ we have $\tilde{c}_{i+j}(Y) = \tilde{c}_{i+j}(Y')$. This implies that there are indices $\ell_1, \ell_2, \ldots, \ell_s$ and $\ell'_1, \ell'_2, \ldots, \ell'_s$ such that $|Y_{\ell_t}| = |Y'_{\ell'_t}|$ for all $t \in \{1, \ldots, s\}$.

Let $\phi : Y \to Y'$ be a bijection satisfying

$$\phi(Y_{\ell_t}) = Y'_{\ell'_t}$$

for all $t \in \{1, \ldots, s\}$. Clearly such a bijection can be defined since $|Y_{\ell_t}| = |Y'_{\ell'_t}|$. Now let $X$ be any $i$-subset of $Y$ having color $c_i(X) = (\eta_1, \ldots, \eta_p)$. This means that there are indices $\ell_{i_1}, \ell_{i_2}, \ldots, \ell_{i_p} \in \{\ell_1, \ell_2, \ldots, \ell_s\}$ such that $\eta_r = |X_{\ell_{i_r}}|$ for all $r \in \{1, \ldots, p\}$. But then since $X_{\ell_{i_r}} \subseteq Y_{\ell_{i_r}}$, we have that

$$\phi(X_{\ell_{i_r}}) \subseteq Y'_{\ell'_{i_r}}$$

for all $r \in \{1, \ldots, p\}$. Also, clearly

$$|\phi(X_{\ell_{i_r}})| = |X_{\ell_{i_r}}| = \eta_r$$

for all $r \in \{1, \ldots, p\}$.

Thus we conclude that $c_i(\phi(X)) = (\eta_1, \ldots, \eta_p)$, so $\phi$ is an $i$-color preserving bijection between $Y$ and $Y'$, and thus $mult_i(Y) = mult_i(Y')$.

This shows that the generating function $G_\lambda$ for the poset of all partitions contained inside some fixed partition $\lambda$ is flawless. Note how this was essentially the same proof as the one given for compositions in the previous section.

Recall that a flawless sequence weakly increases throughout the "first half". Thus, although $G_\lambda$ may in general be non-unimodal (see [37]), unimodality may only possibly fail in the second half.

Consider the special case when $\lambda$ is a rectangle, that is, when $\lambda = (b^a)$ where $n = ab$. The poset, or more precisely, the lattice associated with this partition is denoted by $L(a, b)$ [34]. The corresponding generating function is obviously symmetric by the structure of the $b \times a$ rectangle. Since it is also flawless as we have shown, we note in particular that the generating function is unimodal.

Recall that these generating functions are the ubiquitous $q$-binomial coefficients.

Hence, the result of this section re-proves the well-known fact that the $q$-binomial coefficients are unimodal [23, 27, 38, 43], as mentioned in Chapter 1.

Furthermore, the injective maps from the Multicolor Theorem 2.3.4 assert some strong structural properties of the poset $L(a, b)$. As implied by the Multicolor Theorem, a partition $\mu$ of $i$ inside $\lambda$ is mapped to a partition, $\eta$, of $i+j$. Furthermore, the "color" $\mu$ figures in the multicolor of $i+j$. This implies that the partition $\mu$ fits inside $\eta$. In other words, we have a Young order-preserving injection from degree $i$ to $i+j$.

This observation gives a sort of "half-way" chain decomposition of $L(a, b)$. In other words, it is possible to partition the elements in $L(a, b)$ between degrees 1 and $\lfloor \frac{n}{2} \rfloor$ into saturated chains that respect the Young order inclusion. Taking complements of these chains in the rectangle also gives a "mirror image" chain decomposition of the partitions in $L(a, b)$ from degrees $\lceil \frac{n}{2} \rceil$ to $n$. It is not hard to see that this "upper half" decomposition again preserves the Young order.

It is an outstanding open problem in this area to find a *symmetric chain decomposition* of $L(a, b)$ [30]. Such decompositions have been given for $a = 3$ (see [20]) and $a = 4$ (see [39]), while the result is trivial for $a \leq 2$. The problem remains open for $a \geq 5$. The chains we discussed in the previous paragraph do not imply the existence of such a decomposition because they are not symmetrically connected in the middle. That is, we cannot guarantee that chains in the "first half" from degree $i$ to degree $\lfloor \frac{n}{2} \rfloor$ will always connect with chains in the "second half", from degree $\lceil \frac{n}{2} \rceil$ to $n-i$.

Manipulating these chains to construct a symmetric decomposition appears to be difficult, if not impossible. However, this "half-way" decomposition so far appears to be the strongest possible result short of actually proving that $L(a, b)$ has a symmetric chain decomposition. Note that this half-way decomposition also exists in the lower half of the poset $P_\lambda$ for any arbitrary partition $\lambda$.

### 2.4.5 Multiple Generators

In this section, we discuss an original result that is analogous to the flawlessness of pure $O$-sequences. Note that the argument employed in Section 2.4.2 to extend flawlessness from one to many generating monomials can be used in other combinatorial settings as well. The crux of the proof in that section relied on the following property: Regardless of the reduced adjacency matrix in which a particular $(i+j)$-color may appear (as a column label), that $(i+j)$-color gives rise to exactly the same $i$-colors of the same multiplicity. That is, the nonzero entries of the $i$-multicolor vector are the same *and* labeled by the same $i$-colors.

It is not hard to see that this same property holds in the case of partitions. For instance, if

$$\Lambda_1, \Lambda_2, \ldots, \Lambda_s$$

are all partitions of $n$, then the generating function enumerating all partitions $\lambda$ fitting

inside *any* of the $\Lambda_t$ is also flawless. We shall call the sequence of these coefficients a *pure* $P$*-sequence*. In any other combinatorial setting that has this property, we may extend the flawlessness of the single "generator" case to that of multiple generators of the same degree.

# Chapter 3

# Some New Families of Unimodal Partitions

## 3.1 Introduction

The background of this brief chapter is Stanton's paper [37], in which he initiated the study of the generating function $G_\lambda$ for an arbitrary partition $\lambda = (\lambda_1, \ldots, \lambda_\kappa)$. His main interest was to answer questions regarding the unimodality of these generating functions. (For brevity we will say that $\lambda$ is unimodal if the coefficients of $G_\lambda$ are unimodal.) Note that in this chapter, when discussing $G_\lambda$ (where $\lambda$ has $\kappa$ parts), we always implicitly assume that $\lambda_1 \geq \kappa$ since clearly $G_\lambda = G_{\lambda'}$.

Stanton showed that partitions having $\kappa \le 3$ parts are always unimodal. He gave several infinite classes of non-unimodal partitions with $\kappa = 4$ parts. He found no examples of non-unimodal partitions having 5 or more than 6 parts, and discovered only a finite number of non-unimodal examples in 6 parts. Stanton also conjectured that all self-conjugate partitions are unimodal.

S. Zbarsky [42] gave more families of non-unimodal partitions. In fact, he proved that a positive density (if such density exists) of partitions with $\kappa = 4$ parts are non-unimodal. He also anticipated that Stanton's non-unimodality findings were in some sense comprehensive by conjecturing that, apart from finitely many examples with $\kappa = 6$, there are no non-unimodal partitions with $\kappa > 4$ parts (up to conjugation).

Stanley and F. Zanello studied the analogous question of partitions with *distinct* parts fitting inside a fixed Ferrers diagram [35]. Their findings regarding unimodality in this framework strikingly paralleled Stanton's results. Further investigation in this area was done by L. Alpoge [1] and Zbarsky [42]. We note that Zbarsky's conjecture mentioned in the previous paragraph was given for both the arbitrary and distinct part cases.

Returning to the context of Stanton's original work [37] on partitions with arbitrary parts fitting inside a fixed Ferrers diagram, our contribution in this area consists of extending one of Stanton's techniques for showing unimodality. In particular, we provide two interesting new families of unimodal partitions. We believe that more

can be done in this direction.

## 3.2 Results

For any generating function $G_\lambda$, we define the first difference, $G_\lambda^* = (1-q)G_\lambda$, the polynomial whose coefficients are the consecutive differences of the coefficients of $G_\lambda$. It is clear that $G_\lambda$ is unimodal if and only if the coefficients of $G_\lambda^*$ are nonnegative up to some degree $t$ and then nonpositive after degree $t$.

**Theorem 3.2.1.** *For any $b \in \mathbb{P}$, partitions with $\kappa = 5$ or 6 parts having all parts of size $b$ or $2b$ are unimodal.*

**Theorem 3.2.2.** *In $\kappa = 5$ or 6 parts, if $\lambda_1 - \lambda_\kappa \leq 1$ then $(\lambda_1, \ldots, \lambda_\kappa)$ is unimodal, except for $(10, 9, 9, 9, 9, 9)$.*

*Proof.* We indicate the proof of Theorem 3.2.2 for the partition $(b+1, b, b, b, b)$. The verifications for the rest of the cases and for Theorem 3.2.1 are performed analogously.

The generating function for $\lambda = (b+1, b, b, b, b)$ is

$$G_\lambda = \begin{bmatrix} b+5 \\ 5 \end{bmatrix}_q + q^{b+1}\begin{bmatrix} b+4 \\ 4 \end{bmatrix}_q$$

$$= \frac{(1-q^{b+1})(1-q^{b+2})(1-q^{b+3})(1-q^{b+4})(1+q^{b+1}-q^{b+5}-q^{b+6})}{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)}. \tag{3.1}$$

As we have previously observed, it is well known that the $q$-binomial $\begin{bmatrix} b+a \\ a \end{bmatrix}_q$ is a symmetric and unimodal polynomial, thus having its highest coefficient (the peak) at degree(s) $\left\lfloor \frac{ab}{2} \right\rfloor$ and $\left\lceil \frac{ab}{2} \right\rceil$ [23, 27, 38, 43]. Since (3.1) is the sum of two $q$-binomials (one shifted) with different peaks, it is easy to see that (3.1) is non-decreasing from degree 0 to $\left\lceil \frac{5b}{2} \right\rceil$ and then non-increasing from $3b+1$ to $5b+1$. Therefore the only interval in which unimodality could possibly fail is between degrees $\left\lceil \frac{5b}{2} \right\rceil$ and $3b+1$.

Recall that $G_\lambda$ is unimodal if and only if the coefficients of $G_\lambda^* = (1-q)G_\lambda$ are nonnegative up to some degree $t$ and then nonpositive after degree $t$. Based on what we observed above, we know that the coefficients of $G_\lambda^*$ are nonnegative for degrees $t \le \left\lceil \frac{5b}{2} \right\rceil$ and then nonpositive for all degrees $t \ge 3b+2$.

Notice that we may rewrite (3.1) as:

$$G_\lambda = f_5(q)\left(1-q^{b+1}\right)\left(1-q^{b+2}\right)\left(1-q^{b+3}\right)\left(1-q^{b+4}\right)\left(1+q^{b+1}-q^{b+5}-q^{b+6}\right), \qquad (3.2)$$

where $f_5(q) = \prod_{i=1}^{5} \frac{1}{1-q^i}$ is the generating function for partitions with all parts less than or equal to 5.

We relied heavily on Mathematica to symbolically prove the desired result (the full code is provided in Appendix C.2). Using the **SeriesCoefficient**[] command, it is possible to explicitly write the $i^{\text{th}}$ coefficient of $f_5(q)$ as a quartic polynomial in $i$ (dependent on the value of $i \pmod{60}$). For example, if $i = 1 \pmod{60}$, the $i^{\text{th}}$

coefficient is

$$\frac{i^4 + 30i^3 + 310i^2 + 1230i + 1309}{2880}.$$

Multiplying (3.2) by $(1-q)$ and expanding gives

$$\begin{aligned}G_\lambda^* = f_5(q)\Big(&1 - q - q^{b+2} + q^{b+7} - q^{2b+2} + q^{2b+3} + q^{2b+5} + q^{2b+6} + q^{2b+7} - q^{2b+8} - q^{2b+10}\\ &-q^{2b+11} + q^{3b+4} - q^{3b+7} - q^{3b+8} - 2q^{3b+9} + q^{3b+12} + q^{3b+13} + q^{3b+14} - q^{4b+7}\\ &+q^{4b+10} + q^{4b+11} + q^{4b+12} - q^{4b+15} - q^{4b+16} + q^{5b+11} - q^{5b+12} - q^{5b+15} + q^{5b+17}\Big).\end{aligned}$$

Since unimodality might only possibly fail between degrees $\left\lceil \frac{5b}{2} \right\rceil$ and $3b+1$, we may truncate the expression in the parenthesis above and need only consider coefficients of the series

$$f_5(q)\Big(1 - q - q^{b+2} + q^{b+7} - q^{2b+2} + q^{2b+3} + q^{2b+5} + q^{2b+6} + q^{2b+7} - q^{2b+8} - q^{2b+10} - q^{2b+11}\Big). \quad (3.3)$$

By the definition of series multiplication, the coefficients of the series (3.3) will agree with $G_\lambda^*$ in all degrees $i$ for $i \leq 3b+3$. (Note that we assume $b \geq 8$ in order not to conflate the terms $-q^{2b+11} + q^{3b+4}$.) In particular, notice that the coefficients of (3.3) will be correct from degrees $\left\lceil \frac{5b}{2} \right\rceil$ and $3b+1$. This is the only interval where we need to show that $G_\lambda^*$ does not have a positive term after a negative one in order to prove unimodality of $G_\lambda$.

Knowing the $i^{\text{th}}$ coefficient of $f_q(6)$, we used Mathematica to find the coefficients of (3.3). The $k^{\text{th}}$ coefficient in this expression is a polynomial in $b$ and $k$ that depends on the value of both $b$ and $k \pmod{60}$. For instance, if $b = k = 1 \pmod{60}$, the $k^{\text{th}}$ coefficient is

$$-\frac{1}{240}\left(61 - 100b + 55b^2 + 25b^3 + \left(33 - 70b - 35b^2\right)k + (18 + 15b)k^2 + 2k^3\right). \qquad (3.4)$$

For example, if $b = 61$ and $k = 181$, the expression above yields a coefficient of -1001. Computationally, one can independently verify that the corresponding terms of the generating function for $(62, 61, 61, 61, 61)$ are $75831q^{180}$ and $74830q^{181}$, which yield the term $-1001q^{181}$ in the first difference. This coefficient matches the output from (3.4) using $k = 181$ and $b = 61$.

There are potentially 3600 such expressions as (3.4), but with duplicates there happen to be only 72 that are unique.

We used Mathematica to symbolically verify that the coefficient expressions found in the previous paragraph are nonnegative on the interval from $\left\lfloor \frac{5b}{2} \right\rfloor$ to $\left\lfloor \frac{5b+5}{2} \right\rfloor$ and nonpositive in the interval from $\left\lfloor \frac{5b+7}{2} \right\rfloor$ to $3b + 1$.

Perhaps rather surprisingly, this did not require a significant breakdown into many cases. We merely had to consider separate cases for the parity of $b$, and assumed $b \geq 33$

in order for the program to symbolically verify the desired result. The statement of the theorem for $b \leq 32$ is easily verified by direct computation. $\square$

*Remark* 3.2.3.

1. Note that this proof also yields the degrees of the peaks in the associated generating functions. Once again, we mention that the approach employed in our argument is based on the technique Stanton developed in [37] for some of his unimodality results. We are particularly grateful to him for a personal correspondence in which he explained this technique in great detail.

2. The proof of Theorem 3.2.1 is completed by naturally extending the method discussed above. We took the expressions for the $k^{\text{th}}$ coefficients of $G^*_\lambda$ (of a form similar to (3.4)) and, viewing them as cubics in $k$, determined the roots. These cubics gave exactly one root in the relevant intervals. These roots correspond to the degree in which $G^*_\lambda$ switched from nonnegative to nonpositive in the sense that the root and the actual peak differ by less than 1. Although these roots were not generally identical for the various values of $b$ and $k \,(\mathrm{mod}\ 60)$, they appeared to be asymptotically equal, and linear in $b$. To find this slope, we differentiated the appropriate root of one of these cubics (with respect to $b$) and then took the limit of the root as $b \to \infty$. This indeed produced a desired constant slope, $r$. For example, with the partition $(2b, b, b, b, b)$, the slope $r$ is

$$\frac{45}{13} + \frac{15}{13}\sin\left(\frac{1}{3}\tan^{-1}\left(\frac{13\sqrt{11}}{4}\right)\right) - \frac{5}{13}\sqrt{3}\cos\left(\frac{1}{3}\tan^{-1}\left(\frac{13\sqrt{11}}{4}\right)\right) = 3.42046\ldots$$

Having this *precise* slope for this asymptotic linear expression was requisite for Mathematica to symbolically check that the coefficients for $G_\lambda^*$ were nonnegative up to $rb$ and nonpositive after $rb+1$. Using even a "close" decimal approximation such as 3.4205 would not produce the correct root for a sufficiently large value of $b$. Using this precise linear asymptotic for the root, Mathematica showed that $G_\lambda^*$ can never have a positive value followed by a negative one, thus proving unimodality of $G_\lambda$.

3. We feel completely confident that Theorems 3.2.1 and 3.2.2 also hold for $\kappa \geq 7$ parts. Applying the method discussed in the proof above may be possible for smaller values of $\kappa$, but quickly becomes computationally infeasible as $\kappa$ grows.

# Chapter 4

# The Poset of Generating Functions of Partitions of $n$

## 4.1 Introduction

The research presented in this chapter was inspired by an intriguing question posed by F. Bergeron [6], and first related to the author by Stanley. Given integers $a \leq b \leq c \leq d$ such that $ad = bc$, Bergeron conjectured that the polynomial $\begin{bmatrix} b+c \\ c \end{bmatrix}_q - \begin{bmatrix} a+d \\ d \end{bmatrix}_q$ (which is clearly symmetric) has nonnegative coefficients. It furthermore appears that the coefficients of this polynomial form a unimodal sequence. This was conjectured by Stanley and Zanello and first formally stated in [41] (see also [36]). Special cases of

Bergeron's conjecture have been proven by Stanley (for $q = 1$) and Zanello (for $a \le 3$) (see [41], where in fact unimodality is also shown). We have found another simple yet neat family for which both positivity and unimodality clearly hold.

**Proposition 4.1.1.** *Taking $a = m$, $b = m+1$, $c = 2m$ and $d = 2m+2$ (so $ad = bc = 2m^2 + 2m$), we have*

$$\begin{bmatrix} 3m+1 \\ 2m \end{bmatrix}_q - \begin{bmatrix} 3m+2 \\ 2m+2 \end{bmatrix}_q = q^{m+1} \begin{bmatrix} 3m+1 \\ m-1 \end{bmatrix}_q .$$

*Proof.* This can be proved by a straightforward application of the definition of the $q$-binomial coefficients. □

*Remark* 4.1.2. It is natural to wonder if there are any other such elegant cases where $\begin{bmatrix} b+c \\ c \end{bmatrix}_q - \begin{bmatrix} a+d \\ d \end{bmatrix}_q$ is a shifted $q$-binomial. We note the following trivial case:

$$\begin{bmatrix} n+2 \\ 2 \end{bmatrix}_q - \begin{bmatrix} 2n+1 \\ 1 \end{bmatrix}_q = q^2 \begin{bmatrix} n \\ 2 \end{bmatrix}_q .$$

Computationally, we found no such example up to $ad = bc = 1000$ that was not a case of either the preceding identity or the one given in Proposition 4.1.1.

In this chapter, we introduce the study of a poset that naturally places Bergeron's positivity problem in a broader context. Recall that we define the poset $P_n = \{G_\lambda \mid \lambda \vdash n\}$, the distinct generating functions of partitions of $n$. This set is partially ordered in the following manner: $G_\eta \le G_\lambda$ if and only if $G_\lambda - G_\eta$ has nonnegative coefficients in

every degree. We will see that Bergeron's conjecture is equivalent to showing that the generating functions of rectangles of size $n$ form a chain in this poset, where the chain increases to progressively more "square-like" rectangles.

We will give a number of results on this poset. Our main result is to prove that a certain "balancing" procedure on the principal hooks of a partition yields a new partition with a generating function at least as large (in the ordering on $P_n$) as that of the original partition. This result agrees with the spirit of Bergeron's conjecture, which remains unproven. We will give a second balancing procedure of a slightly different flavor which also yields a "better" generating function.

As noted before, for any partition $\lambda$ of $n$, we have $G_\lambda = G_{\lambda'}$ (that is, $\lambda$ and $\lambda'$ correspond to the same element of $P_n$). We will give some families of non-conjugate pairs of partition having the same generating function. Despite these results, we conjecture that the size of $P_n$ is approximately $\frac{p(n)}{2}$, which is equivalent to saying that "almost all" non-conjugate partitions have different generating functions.

We will also give some constructions showing that the number of maxima in $P_n$ tends to infinity as $n$ grows. The balancing procedures mentioned previously also give a strong restriction on the structure of partitions corresponding to maxima in $P_n$. This restriction reveals a curious connection to the first Rogers-Ramanujan identity from partition theory [3]. In turn, we obtain an upper bound on the number of maxima in $P_n$. Conditional to our conjecture on $|P_n|$, we show that the number of maxima is

negligible with respect to the entire size of the poset.

## 4.2 Discussion of the General Structure of $P_n$

Our goal in this section is to give an overview of some structural properties of $P_n$. We encourage the reader to refer to the diagram provided at the end of this section to aid in an intuitive understanding of the concepts discussed.

First, we observe that $P_n$ is *not* graded. For example, in $P_9$, the following are both saturated chains having the same first and last elements:

$$G_{(9)} \leq G_{(3,3,3)} \leq G_{(7,1,1)}$$

$$G_{(9)} \leq G_{(8,1)} \leq G_{(5,4)} \leq G_{(7,2)} \leq G_{(7,1,1)}$$

Although the poset is not graded, there is still a heuristic sense of generating functions being "large" (or "small") relative to the whole, in the sense of having "many" generating functions above (or below) them in the poset. It is in this sense that we use these and similar terms for the remainder of this section.

Given some partition $\lambda$ of $n$, we proceed to define two "balancing" operations on $\lambda$. In the next section, we will prove the two Balancing Theorems (4.3.2 and 4.3.25) which

assert that these balancing operations produce partitions with generating functions at least as great as that of $\lambda$. In this sense, we have a method for "moving up" in the poset $P_n$ starting at the generating function corresponding to $\lambda$. This also allows us to describe the location of generating functions associated with certain Ferrers shapes in $P_n$. Furthermore, as mentioned above, the Balancing Theorems give a strong restriction on the structure of partitions corresponding to maxima in $P_n$, which we will discuss in more detail in Section 4.5.

Given a partition $\lambda = \left(\begin{smallmatrix} A_k & A_{k-1} & \cdots & A_1 \\ B_k & B_{k-1} & \cdots & B_1 \end{smallmatrix}\right)$ (in Frobenius notation), we define

$$\tilde{\lambda} = \left(\begin{smallmatrix} \tilde{A}_k & \tilde{A}_{k-1} & \cdots & \tilde{A}_1 \\ \tilde{B}_k & \tilde{B}_{k-1} & \cdots & \tilde{B}_1 \end{smallmatrix}\right), \text{ where } \tilde{A}_i + \tilde{B}_i = A_i + B_i, \text{ and } \tilde{A}_i = \begin{cases} A_i - 1 & \text{if } A_i > B_i + 1 \\ A_i + 1 & \text{if } A_i + 1 < B_i \\ A_i & \text{otherwise.} \end{cases}$$

Note that the definition of $\tilde{A}_i$ determines the values of $\tilde{B}_i$. This map from $\lambda$ to $\tilde{\lambda}$ "balances" any principal hooks of the Ferrers diagram of $\lambda$ that are "off-center" about the main diagonal. We say that $\lambda$ is *balanced* to $\tilde{\lambda}$.

*Example* 4.2.1. If $\lambda = \left(\begin{smallmatrix} 9 & 5 & 1 \\ 5 & 4 & 3 \end{smallmatrix}\right)$, then $\tilde{\lambda} = \left(\begin{smallmatrix} 8 & 5 & 2 \\ 6 & 4 & 2 \end{smallmatrix}\right)$ and $\tilde{\tilde{\lambda}} = \left(\begin{smallmatrix} 7 & 5 & 2 \\ 7 & 4 & 2 \end{smallmatrix}\right)$. The Ferrers diagrams are:

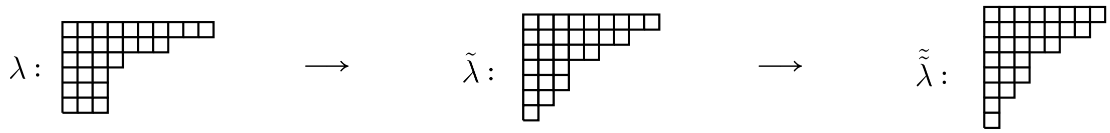


**Figure 4.1:** Illustration of the first balancing operation $\lambda \to \tilde{\lambda}$

We now introduce a second balancing operation on partitions. Note that even in a *completely balanced* partition (i.e., $\lambda = \tilde{\lambda}$), hooks of even length cannot be evenly balanced, but will remain offset by one in either the arm or the leg. For example, the partition $\left(\begin{smallmatrix} 9 & 4 \\ 8 & 5 \end{smallmatrix}\right)$ is completely balanced, but the first hook is offset by one in the arm, while the second hook is offset by one in the leg. We now consider offsetting all hooks in the *same* direction.

For a partition $\lambda = \begin{pmatrix} A_k & A_{k-1} & \cdots & A_1 \\ B_k & B_{k-1} & \cdots & B_1 \end{pmatrix}$, define $\overline{\lambda} = \begin{pmatrix} \overline{A}_k & \overline{A}_{k-1} & \cdots & \overline{A}_1 \\ \overline{B}_k & \overline{B}_{k-1} & \cdots & \overline{B}_1 \end{pmatrix}$, where $\overline{A}_i = \max(A_i, B_i)$ and $\overline{B}_i = \min(A_i, B_i)$. This map from $\lambda$ to $\overline{\lambda}$ takes all principal hooks and offsets them in the same direction.

*Example* 4.2.2. If $\lambda = \left(\begin{smallmatrix} 12 & 4 & 3 & 0 \\ 8 & 7 & 2 & 1 \end{smallmatrix}\right)$, then $\overline{\lambda} = \left(\begin{smallmatrix} 12 & 7 & 3 & 1 \\ 8 & 4 & 2 & 0 \end{smallmatrix}\right)$. The Ferrers diagrams are:

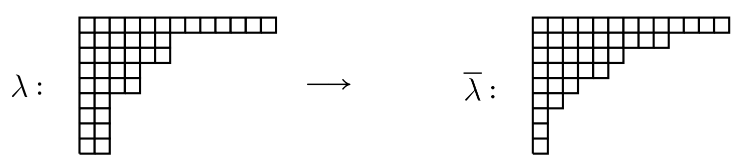


**Figure 4.2:** Illustration of the second balancing operation $\lambda \to \overline{\lambda}$

We re-emphasize that the usefulness of these two balancing operations on $\lambda$ is that

they produce generating functions at least as large as $G_\lambda$ in the ordering of $P_n$. (For brevity, we say that they "improve" the generating function, even if they yield the same function.) This is the assertion of the two Balancing Theorems, which we will prove in the following section. In other words, these theorems claim that

$$G_\lambda \le G_{\tilde{\lambda}} \qquad \text{(Theorem 4.3.2)}$$

and

$$G_\lambda \le G_{\overline{\lambda}} \qquad \text{(Theorem 4.3.25).}$$

The Balancing Theorems give a necessary condition that maxima in $P_n$ must correspond to partitions whose principal hooks are completely balanced and then offset in the same direction. However, this condition is by no means sufficient to guarantee that a partition yields a maximum in $P_n$, as we show in the following example.

*Example* 4.2.3. The partition $\binom{m+1}{m}$ (in Frobenius notation) of $2m+2$ is completely balanced. However, for $m \ge 2$ its generating function in $P_{2m+2}$ is not a maximum: the partition $\left(\begin{smallmatrix} m & 0 \\ m & 0 \end{smallmatrix}\right)$ yields a greater one. Indeed, standard calculations show that

$$G_{\left(\begin{smallmatrix} m & 0 \\ m & 0 \end{smallmatrix}\right)} - G_{\binom{m+1}{m}} = q^4[m]_q[m-2]_q + q^{m+3}[m-1]_q$$

(recall that $[m]_q = 1 + q + \ldots + q^{m-1}$ is the standard $q$-analogue of $m$).

Similarly in the odd case for $P_{2m+1}$, we have

$$G_{\binom{m \quad 0}{m-1 \ 0}} - G_{\binom{m}{m}} = q^4[m]_q[m-3]_q + q^{m+2}[m-1]_q.$$

Next, we consider the $q$-binomials in $P_n$, which are the generating functions corresponding to rectangles. For a partition $\lambda$, rectangle or otherwise, note that the penultimate coefficient in $G_\lambda$ is the number of distinct part sizes in $\lambda$. This is equal to the number of cells that can be removed from the Ferrers diagram of $\lambda$ while leaving behind a valid diagram. Such cells are called *outer corners*. Thus, the coefficient of the penultimate term for a rectangle is 1, while the corresponding term for a non-rectangular partition is at least two.

From the previous paragraph, we see that the $q$-binomials are in some sense all at the "bottom" of $P_n$; they cannot be larger than any non-rectangular shape. Unsurprisingly, rectangles never yield maxima. For example, the rectangle $(b^a)$ (where we can assume $a > 1$) has a strictly smaller generating function than the partition $(b+1, b^{a-2}, b-1)$. Indeed, we have

$$G_{(b+1, b^{a-2}, b-1)} - G_{(b^a)} = q^{b+1}G_{(b^{a-2}, b-1)} - q^{ab}.$$

Bergeron's conjecture asserts that these form a chain in $P_n$ that progresses towards

$q$-binomials arising from more square-like rectangles. This chain (if it exists) is saturated.

In addition to the $q$-binomials near the bottom of the poset, other small generating functions are those that arise from partitions having a single hook or exactly 2 parts. As we saw above, partitions with single hooks will not produce maxima (for $n \geq 6$). The First Balancing Theorem 4.3.2, which we prove in the next section, claims that balancing a partition improves the generating function. Furthermore since balancing any partition with exactly 2 parts (which is not a single hook) always yields a partition with exactly one more outer corner (as long as $n \geq 8$), this inequality is strict in the penultimate degree. So partitions with exactly 2 parts can never yield maxima for $n \geq 8$.

Note that the small generating functions discussed above come from partitions with few outer corners. In general, *staircases* (partitions of the form $(k, k-1, \ldots, 3, 2, 1)$, which have size $\binom{k+1}{2}$) yield the largest possible number of outer corners. Thus, a partition of $n$ can have as many as $\left\lfloor \frac{\sqrt{8n+1}-1}{2} \right\rfloor$ outer corners. Heuristically, it appears that generating functions corresponding to partitions with many outer corners tend to be larger in the poset than those corresponding to partitions with fewer outer corners.

To give a more intuitive sense of these observations, we illustrate the Hasse diagram of the poset $P_{16}$ in Fig. 4.3 below. The nodes are labeled by Ferrers shapes of partitions

having the associated generating function of the element in the poset. These labels are not necessarily unique. For example, the node labeled by (6, 6, 4) could also be labeled by (3, 3, 2, 2, 2, 2), the conjugate of (6, 6, 4). The labels in the diagram below were chosen by first appearance in reverse lexicographical order.

Note that neither the partition (8, 5, 3) nor its conjugate appear as labels in the diagram below. This is because $G_{(8,5,3)} = G_{(9,5,2)}$, so both (8, 5, 3) and (9, 5, 2) correspond to the same node, which is labeled below by the latter partition. This concept of non-conjugate partitions of $n$ yielding the same generating function (and thus corresponding to the same element in $P_n$) will be discussed in more depth in Section 4.4.

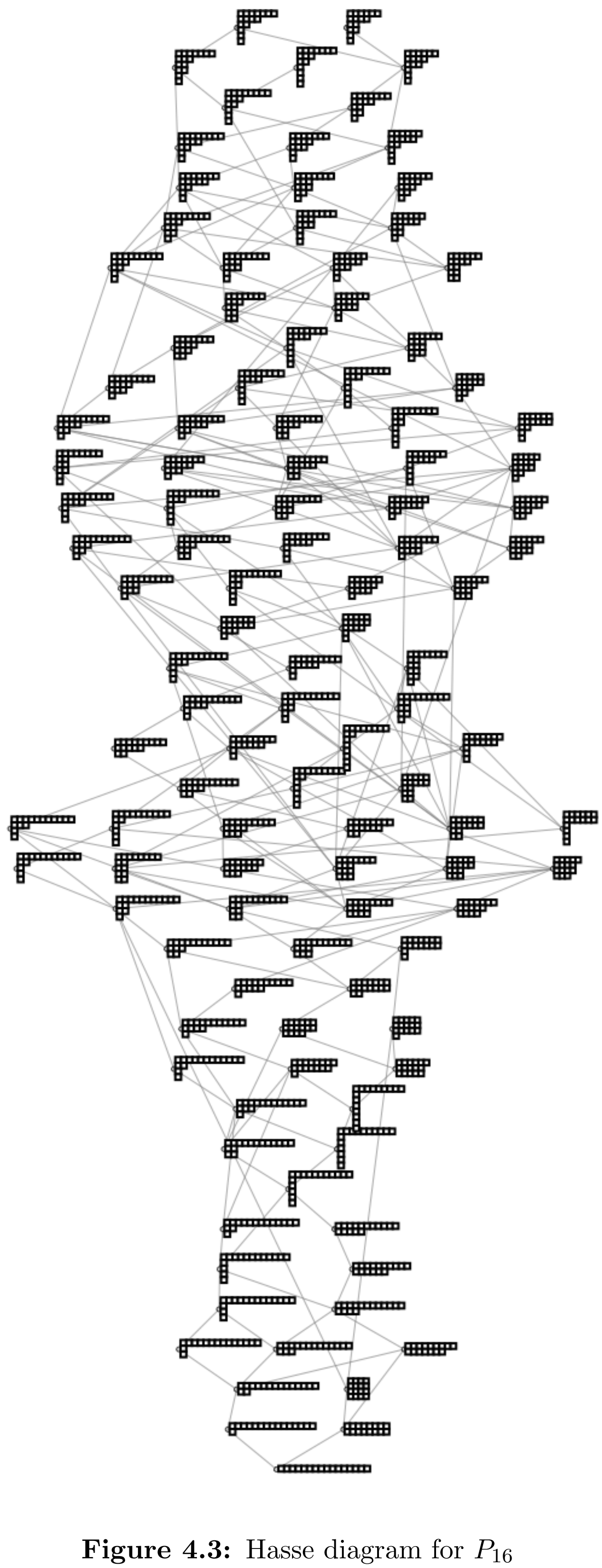


**Figure 4.3:** Hasse diagram for $P_{16}$

## 4.3 The Balancing Theorems

The purpose of this section is to prove the two Balancing Theorems, which were mentioned in the previous section.

### 4.3.1 The First Balancing Theorem

Recall the balancing operation $\lambda \to \tilde{\lambda}$ defined in the previous section. Given a partition $\lambda = \left(\begin{smallmatrix} A_k & A_{k-1} & \cdots & A_1 \\ B_k & B_{k-1} & \cdots & B_1 \end{smallmatrix}\right)$, we define

$$\tilde{\lambda} = \begin{pmatrix} \tilde{A}_k & \tilde{A}_{k-1} & \cdots & \tilde{A}_1 \\ \tilde{B}_k & \tilde{B}_{k-1} & \cdots & \tilde{B}_1 \end{pmatrix}, \text{ where } \tilde{A}_i + \tilde{B}_i = A_i + B_i, \text{ and } \tilde{A}_i = \begin{cases} A_i - 1 & \text{if } A_i > B_i + 1 \\ A_i + 1 & \text{if } A_i + 1 < B_i \\ A_i & \text{otherwise.} \end{cases}$$

Note that the number and sizes of principal hooks are preserved from $\lambda$ to $\tilde{\lambda}$. For a partition $\mu = \left(\begin{smallmatrix} a_\ell & a_{\ell-1} & \cdots & a_1 \\ b_\ell & b_{\ell-1} & \cdots & b_1 \end{smallmatrix}\right)$, call $w_i = a_i + b_i$ the *weight* of the $i^{\text{th}}$ principal hook ($w_i$ is simply hook length minus 1), and define the *weight vector* of $\mu$ by $\mathrm{wt}(\mu) = (a_\ell + b_\ell, \ldots, a_1 + b_1)$. We will show the following:

**Lemma 4.3.1.** *For any integer vector* $\mathbf{w}$*, we have*

$$\#\{\mu \le \lambda \,|\, \mathrm{wt}(\mu) = \mathbf{w}\} \le \#\{\tilde{\mu} \le \tilde{\lambda} \,|\, \mathrm{wt}(\tilde{\mu}) = \mathbf{w}\}.$$

A few remarks on this lemma are in order. First, partitions of weight $\mathbf{w} \in \mathbb{P}^k$ exist in $\lambda$ (or $\tilde{\lambda}$) if and only if $w_i \le A_i + B_i$ and $w_i > w_{i-1} + 1$ for all $i$. In this case we say that $\mathbf{w}$ is a *valid* weight vector for $\lambda$. Also, without loss of generality we may assume that the length of $\mathbf{w}$ is equal to the number of principal hooks of $\lambda$. For if not (say, if $\lambda$ had $k$ hooks and the length of $\mathbf{w}$ were $\ell < k$), then any partition with weight $\mathbf{w}$ in $\lambda$ (or $\tilde{\lambda}$) would occupy only the $\ell$ outermost hooks of $\lambda$. Thus we could discard the $k - \ell$ excess (inner) hooks of $\lambda$ to obtain a partition with $\ell$ principal hooks, and the resulting counts would be the same as those in the inequality above.

**Theorem 4.3.2** (The First Balancing Theorem)**.**

$$G_\lambda \le G_{\tilde{\lambda}}.$$

*Remark* 4.3.3*.* This theorem is easily verified in the single hook case (i.e., $\lambda = \binom{A}{B}$) by a simple injection. Without loss of generality we may assume $A > B + 1$, so $\tilde{\lambda} = \binom{A-1}{B+1}$. Map $\mu = \binom{a}{b} \le \lambda$ to $\tilde{\mu} \le \tilde{\lambda}$ as follows: If $a < A$, set $\tilde{\mu} = \mu$. Otherwise (if $a = A$), set $\tilde{\mu} = \binom{a+b-B-1}{B+1}$.

We will prove Lemma 4.3.1 by means of two rather technical lemmas. We proceed to

provide the background necessary to state and prove these lemmas.

Given a partition $\lambda = \left(\begin{smallmatrix} A_k & A_{k-1} & \cdots & A_1 \\ B_k & B_{k-1} & \cdots & B_1 \end{smallmatrix}\right)$ and valid weight vector $\mathbf{w} = (w_k, \ldots, w_1)$, we can represent all partitions $\mu \leq \lambda$ of weight $\mathrm{wt}(\mu) = \mathbf{w}$ in the following way:

On the coordinate axes, consider the diagonal $x + y = w_i$, and place points $(x, y)$ (with nonnegative integer-valued coordinates) on this diagonal whenever $x \leq A_i$ and $y \leq B_i$. Such a point $(x, y)$ corresponds to an $i^{\text{th}}$ principal hook of $\mu$ with arm length $x$ and leg length $y$. By construction this hook fits inside the $i^{\text{th}}$ principal hook of $\lambda$ and has weight $w_i$.

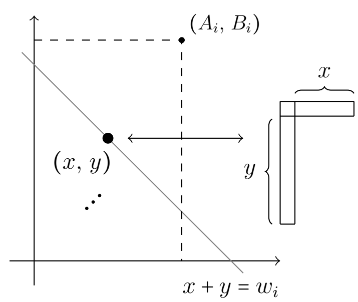



**Figure 4.4:** Correspondence between points and hooks

After placing such points on each diagonal line $x + y = w_1, \ldots, x + y = w_k$, join any point $(x, y)$ on the $i^{\text{th}}$ diagonal to a point $(X, Y)$ on the $(i+1)^{\text{st}}$ diagonal if and only if $x < X$ and $y < Y$. This connection corresponds to an $i^{\text{th}}$ principal hook properly "nesting under" an $(i+1)^{\text{st}}$ principal hook to form a valid portion of a Ferrers diagram, as shown in Fig. 4.5.

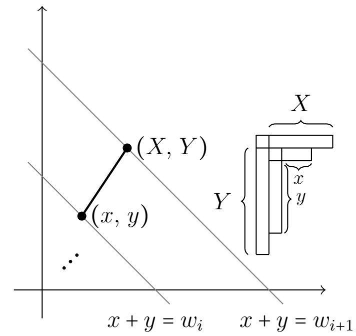



**Figure 4.5:** Correspondence between diagonals and consecutive hooks

Next, beginning at the second diagonal ($x + y = w_2$) and proceeding to subsequent ones, discard any points that do not connect to the lower adjacent diagonal. This produces precisely the structure whose points on the line $x + y = w_i$ are inside the $A_i \times B_i$ rectangle for each $i$, with strictly northeast paths. We call this structure the *trellis* for $\lambda$ determined by $\mathbf{w}$, denoted by $\mathcal{T}_{\lambda,\mathbf{w}}$.

*Example* 4.3.4. Let $\lambda = \left(\begin{smallmatrix}10&8&6&4\\9&6&5&2\end{smallmatrix}\right)$, with $\mathbf{w} = (16, 10, 6, 1)$. We have $\tilde{\lambda} = \left(\begin{smallmatrix}10&7&6&3\\9&7&5&3\end{smallmatrix}\right)$. The trellises $\mathcal{T}_{\lambda,\mathbf{w}}$ and $\mathcal{T}_{\tilde{\lambda},\mathbf{w}}$ are given below:

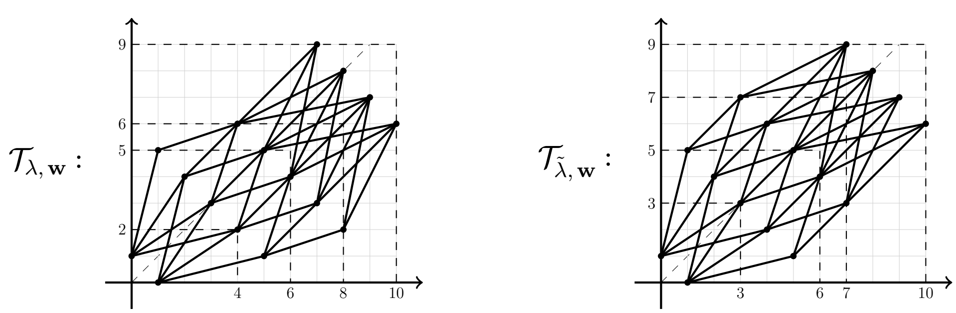



**Figure 4.6:** Trellises for $\left(\begin{smallmatrix}10&8&6&4\\9&6&5&2\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}10&7&6&3\\9&7&5&3\end{smallmatrix}\right)$ with $\mathbf{w} = (16, 10, 6, 1)$

Consider points $Q_1, \ldots, Q_k$ on the diagonals $x + y = w_1, \ldots, x + y = w_k$, respectively, where each $Q_i$ is connected to $Q_{i+1}$. The path formed by traversing $Q_1 - Q_2 - \cdots - Q_k$ corresponds to a partition with weight $\mathbf{w}$ whose $i^{\text{th}}$ principal hook is given by $Q_i = (x_i, y_i) \leftrightarrow \left(\begin{smallmatrix} \cdots & x_i & \cdots \\ \cdots & y_i & \cdots \end{smallmatrix}\right)$.

*Example* 4.3.5. The path $(0,1) - (4,2) - (5,5) - (10,6)$ in the trellis $\mathcal{T}_{\lambda,\mathbf{w}}$ from Fig. 4.6 corresponds to $\left(\begin{smallmatrix} 10 & 5 & 4 & 0 \\ 6 & 5 & 2 & 1 \end{smallmatrix}\right) \leq \lambda$ having weight $(16, 10, 6, 1)$, as illustrated below:

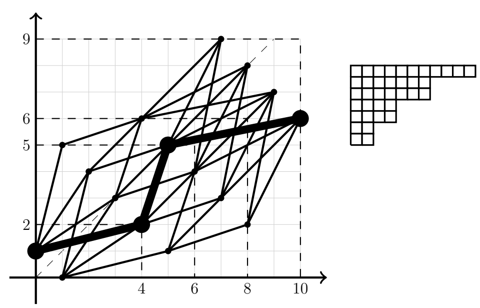



**Figure 4.7:** Correspondence between trellis paths and partitions

We make the following important observation: the total number of partitions $\mu$ of weight $\mathbf{w}$ in $\lambda$ corresponds to the total number of paths $Q_1 - Q_2 - \cdots - Q_k$ through the trellis $\mathcal{T}_{\lambda,\mathbf{w}}$.

*Example* 4.3.6. Consider the construction of the trellis for $\lambda = \left(\begin{smallmatrix} 14 & 8 & 7 & 4 \\ 12 & 10 & 5 & 2 \end{smallmatrix}\right)$ with $\mathbf{w} = (19, 14, 8, 4)$:

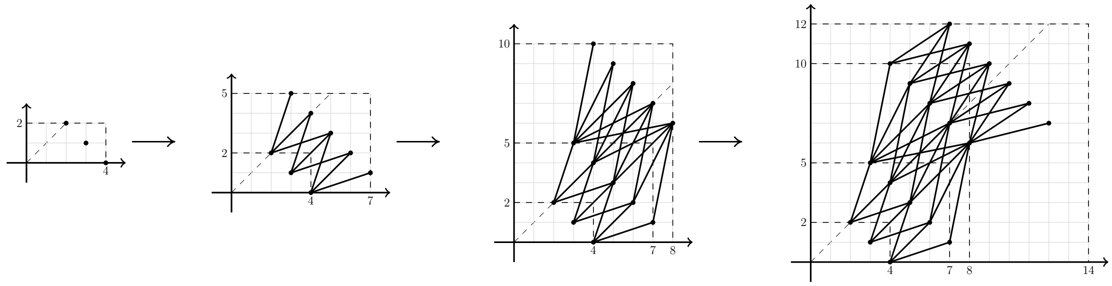


**Figure 4.8:** Construction of the trellis for $\left(\begin{smallmatrix}14 & 8 & 7 & 4\\ 12 & 10 & 5 & 2\end{smallmatrix}\right)$ with $\mathbf{w} = (19, 14, 8, 4)$

For comparison with the trellis diagrams given above, consider the progression of the more "balanced" trellis diagrams corresponding to $\tilde{\lambda} = \left(\begin{smallmatrix}13 & 9 & 6 & 3\\ 13 & 9 & 6 & 3\end{smallmatrix}\right)$ with the same weight vector:

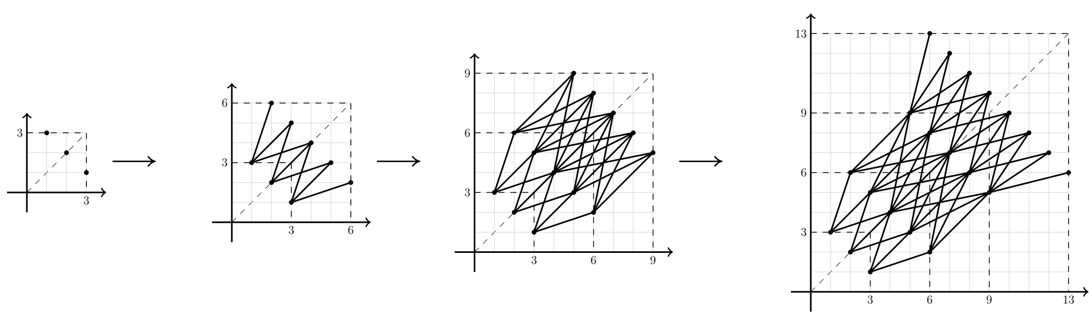


**Figure 4.9:** Construction of the trellis for $\left(\begin{smallmatrix}13 & 9 & 6 & 3\\ 13 & 9 & 6 & 3\end{smallmatrix}\right)$ with $\mathbf{w} = (19, 14, 8, 4)$

Denote the points on the $J^{\text{th}}$ diagonal of $\mathcal{T}_{\lambda,\mathbf{w}}$ by $\mathbf{P}^J = \{(x_1, y_1), \ldots, (x_m, y_m)\}$, where $x_{i+1} = x_i - 1$. Define $C(\mathbf{P}^J) = x_1 - y_m$. (The function $C$ measures the "centeredness" of the points of $\mathbf{P}^J$ along the line $y = x$.)

Comparing Fig. 4.8 to Fig. 4.9 leads us to the following lemma:

**Lemma 4.3.7.** *Let $\lambda$ be any partition and let $\mathbf{w}$ be a weight vector that is valid for $\lambda$. Define $\mathbf{P}^J$ as above, and similarly let $\tilde{\mathbf{P}}^J$ be the points on the $J^{th}$ diagonal of $\mathcal{T}_{\tilde{\lambda},\mathbf{w}}$. Then the following hold:*

- *If $C(\mathbf{P}^J) < -1$, then $\{(x_1+1, y_1-1), \ldots, (x_{m-1}, y_{m-1})\} \subseteq \tilde{\mathbf{P}}^J$.*
- *If $|C(\mathbf{P}^J)| \leq 1$, then $\{(x_1, y_1), \ldots, (x_m, y_m)\} \subseteq \tilde{\mathbf{P}}^J$.*
- *If $C(\mathbf{P}^J) > 1$, then $\{(x_2, y_2), \ldots, (x_m - 1, y_m + 1)\} \subseteq \tilde{\mathbf{P}}^J$.*

*In particular, for all cases we have $\#\tilde{\mathbf{P}}^J \geq \#\mathbf{P}^J$.*

*Remark* 4.3.8. Visually, Lemma 4.3.7 claims that the number of points on a diagonal never decreases after balancing $\lambda$ to $\tilde{\lambda}$, and the points on a diagonal are never less balanced around $y = x$. Later, we will without loss of generality assume $\#\tilde{\mathbf{P}}^J = \#\mathbf{P}^J$.

*Proof of Lemma 4.3.7.* We proceed by induction on the number of principal hooks of $\lambda$. Suppose $\lambda$ is a partition with 1 principal hook, $\lambda = \binom{A}{B}$, and let $\mathbf{w} = (w)$ (where $w \leq A + B$). Without loss of generality we may assume $A \geq B$ (if not, conjugate $\lambda$). If $A = B$, then $\tilde{\lambda} = \lambda$ and $C(\mathbf{P}^1) = 0$ so the statement is true. Likewise if $A = B + 1$, $0 \leq C(\mathbf{P}^1) \leq 1$ and the statement is again easily verified since $\tilde{\lambda} = \lambda$. If $A > B + 1$, then $\tilde{\lambda} = \binom{A-1}{B+1}$ and we have the following possibilities:

- If $w \leq B$, then $\mathbf{P}^1 = \{(w, 0), \ldots, (0, w)\} = \tilde{\mathbf{P}}^1$.

- If $B < w < A$, then $\mathbf{P}^1 = \{(w, 0), \ldots, (w - B, B)\}$ (so $C(\mathbf{P}^1) \geq 1$) and $\tilde{\mathbf{P}}^1 = \{(w, 0), \ldots, (w - B - 1, B + 1)\}$.

- If $w \geq A$, then $\mathbf{P}^1 = \{(A, w - A), \ldots, (w - B, B)\}$ (so $C(\mathbf{P}^1) > 1$) and $\tilde{\mathbf{P}}^1 = \{(A - 1, w - A + 1), \ldots, (w - B - 1, B + 1)\}$.

In each instance we see that the statement of the lemma is true, thus verifying the single hook case. (Note that this also follows from Remark 4.3.3.)

For some specific $k \geq 2$, we assume now that the property holds for any partition with $k - 1$ hooks. Let $\lambda = \left(\begin{smallmatrix} A_k & A_{k-1} & \cdots & A_1 \\ B_k & B_{k-1} & \cdots & B_1 \end{smallmatrix}\right)$ be any partition with $k$ hooks, and let $\mathbf{w} = (w_k, \ldots, w_1)$ be any valid weight vector for $\lambda$. Represent the points of $\mathbf{P}^{k-1}$ and $\mathbf{P}^k$ as $\mathbf{P}^{k-1} = \{(x_1, y_1), \ldots, (x_m, y_m)\}$ and $\mathbf{P}^k = \{(X_1, Y_1), \ldots, (X_n, Y_n)\}$ for $\mathcal{T}_{\lambda, \mathbf{w}}$. Analogously let $\tilde{\mathbf{P}}^{k-1} = \{(\tilde{x}_1, \tilde{y}_1), \ldots, (\tilde{x}_{\tilde{m}}, \tilde{y}_{\tilde{m}})\}$ and $\tilde{\mathbf{P}}^k = \{(\tilde{X}_1, \tilde{Y}_1), \ldots, (\tilde{X}_{\tilde{n}}, \tilde{Y}_{\tilde{n}})\}$ in $\mathcal{T}_{\tilde{\lambda}, \mathbf{w}}$. Furthermore, set $d = C(\mathbf{P}^{k-1})$ and $e = C(\mathbf{P}^k)$.

Observe that the inductive hypothesis can be restated as follows:

- If $d < -1$, then $\tilde{x}_1 \geq x_1 + 1$ and $\tilde{x}_{\tilde{m}} \leq x_m + 1$.

- If $|d| \leq 1$, then $\tilde{x}_1 \geq x_1$ and $\tilde{x}_{\tilde{m}} \leq x_m$.

- If $d > 1$, then $\tilde{x}_1 \geq x_1 - 1$ and $\tilde{x}_{\tilde{m}} \leq x_m - 1$.

Similarly, the conditions we must prove for $\tilde{\mathbf{P}}^k$ relative to $\mathbf{P}^k$ can be restated as follows:

- If $e < -1$, show that $\tilde{X}_1 \geq X_1 + 1$ and $\tilde{X}_{\tilde{n}} \leq X_n + 1$.
- If $|e| \leq 1$ show that $\tilde{X}_1 \geq X_1$ and $\tilde{X}_{\tilde{n}} \leq X_n$.
- If $e > 1$ show that $\tilde{X}_1 \geq X_1 - 1$ and $\tilde{X}_{\tilde{n}} \leq X_n - 1$.

To prove these inequalities, we need to determine constraints on the values of $X_1$, $X_n$, $\tilde{X}_1$ and $\tilde{X}_{\tilde{n}}$. Notice that adding a $k^{\text{th}}$ principal outer hook (together with a valid weight to the weight vector) does not alter the original points and connections of the trellis for the partition with $k-1$ hooks (refer to Figs. 4.8 and 4.9 for an illustration of this). With this in mind, we have bounds on the values of $X_1$ and $X_n$ as discussed below.

To guarantee connectivity between the $(k-1)^{\text{st}}$ and $k^{\text{th}}$ diagonals, we see that $Y_1 > y_1$ (equivalently, $w_k - X_1 > w_{k-1} - x_1$) and $X_n > x_m$. Furthermore, since the points on the $k^{\text{th}}$ diagonal correspond to hooks fitting in the outermost hook of $\lambda$, we have that $X_1 \leq A_k$ and $Y_n \leq B_k$ (equivalently, $w_k - X_n \leq B_k$).

Thus $X_1 = \min(w_k - w_{k-1} + x_1 - 1,\, A_k)$ and $X_n = \max(w_k - B_k,\, x_m + 1)$. Similarly for $\mathcal{T}_{\tilde{\lambda},\mathbf{w}}$ we have $\tilde{X}_1 = \min(w_k - w_{k-1} + \tilde{x}_1 - 1,\, \tilde{A}_k)$ and $\tilde{X}_{\tilde{n}} = \max(w_k - \tilde{B}_k,\, \tilde{x}_{\tilde{m}} + 1)$.

Clearly the values of $X_1$, $X_n$, $\tilde{X}_1$ and $\tilde{X}_{\tilde{n}}$ break apart into many cases based on the interdependencies of the other parameters. We now consider what the cases for those other parameters may be. Recall that $\tilde{A}_k + \tilde{B}_k = A_k + B_k$, and the values of $\tilde{A}_k$ and $\tilde{B}_k$ are determined as follows:

$$(\tilde{A}_k, \tilde{B}_k) = \begin{cases} (A_k - 1, B_k + 1) & \text{if } A_k > B_k + 1 \\ (A_k + 1, B_k - 1) & \text{if } A_k + 1 < B_k \\ (A_k, B_k) & \text{otherwise.} \end{cases}$$

Next, notice that the conclusion given by the inductive hypothesis depends on the value of $d$, and similarly the property that we wish to show for $\tilde{\mathbf{P}}^k$ with respect to $\mathbf{P}^k$ is dependent on $e$.

To summarize, we have two cases for determining each of the values $X_1$, $X_n$, $\tilde{X}_1$, $\tilde{X}_{\tilde{n}}$; three cases for the values of $\tilde{A}_k$ (also determining $\tilde{B}_k$); three cases for $d$ and three cases for $e$. Thus, there are altogether 432 different possible cases that must be considered. We relied on symbolic logic computations in Mathematica to verify the inequalities in question for each of these cases (in fact, it turned out that 331 of these cases yield contradictory assumptions and can therefore be ignored, while for each of the valid 101 cases the desired result holds). The full code used for this verification is provided in Appendix C.2. $\square$

*Example* 4.3.9. It is beneficial to illustrate the verification of one of the cases in the previous proof by hand. Consider the case where:

1. $d > 1$ (so $\tilde{x}_1 \geq x_1 - 1$ and $\tilde{x}_{\tilde{m}} \leq x_m - 1$ by the inductive hypothesis)

2. $|e| \leq 1$ (so we wish to show that $\tilde{X}_1 \geq X_1$ and $\tilde{X}_{\tilde{n}} \leq X_n$)

3. $A_k + 1 < B_k$ (implying $\tilde{A}_k = A_k + 1$ and $\tilde{B}_k = B_k - 1$)

4. $A_k \leq w_k - w_{k-1} + x_1 - 1$ (implying $X_1 = A_k$)

5. $w_k - B_k \leq x_m + 1$ (implying $X_n = x_m + 1$)

6. $\tilde{A}_k \leq w_k - w_{k-1} + \tilde{x}_1 - 1$ (implying $\tilde{X}_1 = \tilde{A}_k = A_k + 1$)

7. $\tilde{x}_{\tilde{m}} + 1 \leq w_k - \tilde{B}_k$ (implying $\tilde{X}_{\tilde{n}} = w_k - \tilde{B}_k = w_k - B_k + 1$)

From the cases above, we easily see that $\tilde{X}_1 = A_k + 1 > A_k = X_1$.

From (5) and (7) above, we have $\tilde{X}_{\tilde{n}} = w_k - B_k + 1 \leq x_m + 2 = X_n + 1$. Note that $e = X_1 + X_n - w_k$, by definition. If $\tilde{X}_{\tilde{n}} = X_n + 1$, then $X_n = w_k - B_k$ and $e = A_k - B_k$, which contradicts the assumption that $|e| \leq 1$. So we have $\tilde{X}_{\tilde{n}} \leq X_n$, as desired.

Mathematica symbolically performed these types of comparisons for this and the other 100 viable cases, and verified that the desired inequalities for each case were true.

*Remark* 4.3.10. From now on, we choose to discard any (potential) excess points in $\tilde{\mathbf{P}}^J$ that are (in general) not guaranteed to exist by Lemma 4.3.7. In other words, we assume the following.

- If $C(\mathbf{P}^J) < -1$, then $\tilde{\mathbf{P}}^J = \{(x_1 + 1, y_1 - 1), \ldots, (x_{m-1}, y_{m-1})\}$.
- If $|C(\mathbf{P}^J)| \leq 1$ then $\tilde{\mathbf{P}}^J = \{(x_1, y_1), \ldots, (x_m, y_m)\}$.
- If $C(\mathbf{P}^J) > 1$, then $\tilde{\mathbf{P}}^J = \{(x_2, y_2), \ldots, (x_m - 1, y_m + 1)\}$.

We move on to define more concepts and tools that will be used in the next lemma. Let $\mathbf{p}^J = (p_1, \ldots, p_m)$, where $p_i$ is the number of paths from any point on the first diagonal of the trellis $\mathcal{T}_{\lambda,\mathbf{w}}$ to the point $(x_i, y_i) \in \mathbf{P}^J$. In other words, $p_i$ is the number of partitions with outer hook $\binom{x_i}{y_i}$ fitting inside the innermost $J$ hooks of $\lambda$ and having weight $(w_1, \ldots, w_J)$. We call $\mathbf{p}^J$ the *path-counting* vector to the $J^{\text{th}}$ diagonal, and we define $\tilde{\mathbf{p}}^J$ for $\tilde{\mathbf{P}}^J$ similarly.

*Remark* 4.3.11. Note that $\sum_{i=1}^{m} p_i = \mathbf{p}^J \cdot \mathbf{1}$ is the total number of paths $Q_1 - Q_2 - \cdots - Q_J$ through $\mathcal{T}_{\lambda,\mathbf{w}}$ from the first to the $J^{\text{th}}$ diagonal. (Here "$\cdot$" denotes the ordinary dot product and $\mathbf{1}$ is the all one vector of appropriate size.) Because of Remark 4.3.10, $\mathbf{p}^J$ and $\tilde{\mathbf{p}}^J$ have the same length. Note that Lemma 4.3.1 can be expressed as

$$\mathbf{p}^k \cdot \mathbf{1} \leq \tilde{\mathbf{p}}^k \cdot \mathbf{1}$$

(recall that $k$ is the number of principal hooks of $\lambda$).

*Example* 4.3.12. From Example 4.3.4, $\mathbf{p}^4 = (15, 20, 19, 16)$ and $\tilde{\mathbf{p}}^4 = (14, 19, 22, 19)$. Thus $\mathbf{p}^4 \cdot \mathbf{1} = 70 < 74 = \tilde{\mathbf{p}}^4 \cdot \mathbf{1}$.

*Remark* 4.3.13. For the remainder of this section, we will be working in a sufficient level of generality that the connection of these trellis structures to partitions may in a sense be ignored. We may simply regard a given trellis as a collection of vertices and edges, with the property that points balance along $y = x$ as given by Remark 4.3.10. Much of what follows will be dealing with relationships between certain 0-1 matrices. These more general results may be of some independent interest.

We now consider the correlation between these path-counting vectors in successive diagonals. As before, $\mathbf{P}^{J-1} = \{(x_1, y_1), \ldots, (x_m, y_m)\}$ and $\mathbf{P}^J = \{(X_1, Y_1), \ldots, (X_n, Y_n)\}$ are the points on successive diagonals of $\mathcal{T}_{\lambda, \mathbf{w}}$. Also, define $\sigma = X_1 - x_1$ and $\tau = Y_1 - y_1$. (We similarly define $\tilde{\mathbf{P}}^{J-1}$ and $\tilde{\mathbf{P}}^J$, together with $\tilde{\sigma} = \tilde{X}_1 - \tilde{x}_1$ and $\tilde{\tau} = \tilde{Y}_1 - \tilde{y}_1$.)

Viewing $\mathbf{P}^{J-1}$ and $\mathbf{P}^J$ as the vertices of a bipartite graph (together with edge set corresponding to the connections between $\mathbf{P}^{J-1}$ and $\mathbf{P}^J$), we consider the $m \times n$ biadjacency matrix, $M$. Its rows are labeled (in order) by the points of $\mathbf{P}^{J-1}$ and columns by the points of $\mathbf{P}^J$. In particular, $M_{i,j} = 1$ if $x_i < X_j$ and $y_i < Y_j$, and 0 otherwise.

*Remark* 4.3.14. Note that $M$ (of dimensions $m \times n$) has the following properties:

1. Entries in any parallel of the main diagonal of $M$ are the same, i.e., $M_{i,j} =$

$M_{i+\ell,\,j+\ell}$ whenever $1 \le i+\ell \le m$ and $1 \le j+\ell \le n$.

2. $M$ has $s = \min(\sigma, n)$ ones in the first row (in columns 1 through $s$). This is because we have an edge from $(x_1, y_1)$ to $(X_j, Y_j) \in \mathbf{P}^J$ whenever $x_1 < X_j \le X_s$ (note that $y_1 < Y_1 \le Y_j$ for all $j$).

3. Similarly to (2), $M$ has $t = \min(\tau, m)$ ones in the first column (in rows 1 through $t$).

4. It follows from (1)-(3) that $M$ is completely determined by $s$ and $t$. Visually, $s$ is the number of edges from $(x_1, y_1)$ to the points of $\mathbf{P}^J$, and $t$ is the number of edges from $(X_1, Y_1)$ to points of $\mathbf{P}^{J-1}$. This is illustrated in Fig. 4.10.

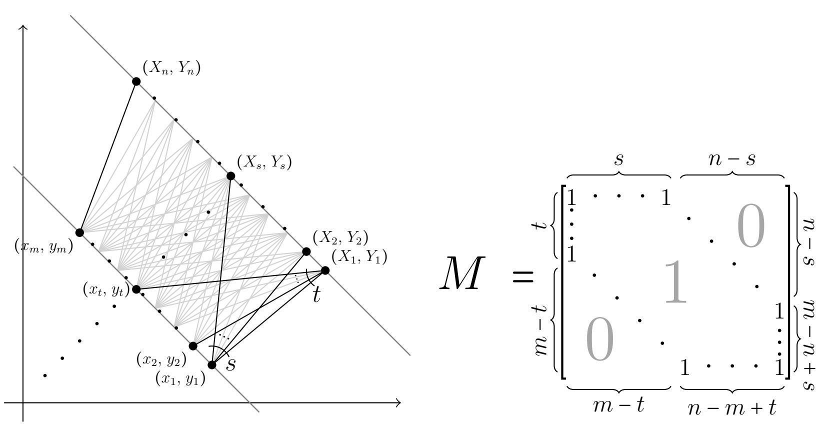



**Figure 4.10:** Correspondence between diagonals and biadjacency matrix

*Remark* 4.3.15. We derive a simple identity to be used later. Recall that $\sigma = X_1 - x_1$ and $\tau = Y_1 - y_1$. Furthermore, $Y_n = Y_1 + n - 1$ and $y_m = y_1 + m - 1$. Since $d = x_1 - y_m$ and

$e = X_1 - Y_n$, we have that $e + n + \tau = X_1 - Y_n + n + Y_1 - y_1 = x_1 + \sigma + 1 - y_m + m - 1 = d + m + \sigma$. Thus we have

$$e + n + \tau = d + m + \sigma. \tag{4.1}$$

The biadjacency matrix $M$ provides the correlation between the path-counting vectors $\mathbf{p}^{J-1}$ and $\mathbf{p}^{J}$ as follows: Consider the matrix product $\mathbf{p}^{J-1}M$. This yields a vector with $n$ components whose $j^{\text{th}}$ entry is $\mathbf{p}^{J-1} \cdot M_j$ (where $M_j$ is the $j^{\text{th}}$ column of $M$). $M_j$ has a 1 in the $i^{\text{th}}$ entry if and only if $(x_i, y_i) \in \mathbf{P}^{J-1}$ is connected to $(X_j, Y_j) \in \mathbf{P}^{J}$. In other words, the $j^{\text{th}}$ entry of $\mathbf{p}^{J-1}M$ is the number of trellis paths starting from the first diagonal and ending at $(X_j, Y_j)$. Thus we have the important identity

$$\mathbf{p}^{J-1}M = \mathbf{p}^{J}. \tag{4.2}$$

We analogously define $\tilde{M}$ for $\tilde{\mathbf{P}}^{J-1}$ and $\tilde{\mathbf{P}}^{J}$ in the more balanced trellis. Because of Remark 4.3.10, $\tilde{M}$ and $M$ have the same dimensions. The $m \times n$ matrix $\tilde{M}$ is completely determined by $\tilde{s} = \min(\tilde{\sigma}, n)$ and $\tilde{t} = \min(\tilde{\tau}, m)$.

*Remark* 4.3.16. Because of Remark 4.3.10, $\tilde{\sigma}$ and $\tilde{\tau}$ are completely determined by – and can be expressed in terms of – the parameters of the pre-balanced trellis. These expressions for the various cases (and the corresponding expressions for $\tilde{s}$ and $\tilde{t}$) are included in Appendix A for completeness sake. The conclusions given there follow from Lemma 4.3.7.

Let $\mathbf{x}$ be a vector of the form $(\overbrace{0,\ldots,0}^{a\geq 0},\ \overbrace{1,\ldots,1}^{b\geq 1},\ \overbrace{0,\ldots,0}^{c\geq 0})$. We will call such vectors *admissible*. Define $D(\mathbf{x}) = a - c$, the *centeredness* of $\mathbf{x}$. Define $B(\mathbf{x})$, the *centering* of $\mathbf{x}$, to be the admissible vector $\tilde{\mathbf{x}}$ with numbers $\tilde{a}$, $\tilde{b} = b$, $\tilde{c}$ such that

$$(\tilde{a}, \tilde{c}) = \begin{cases} (a-1,\, c+1) & \text{if } a > c \\ (a+1,\, c-1) & \text{if } a < c \\ (a,\, c) & \text{if } a = c. \end{cases}$$

Note that $\mathbf{x}$ and $B(\mathbf{x})$ have the same number of 1's.

*Example* 4.3.17. For $a = 4, b = 3, c = 1$, we have $\mathbf{x} = (0,0,0,0,1,1,1,0)$ and $D(\mathbf{x}) = 3$. Then $B(\mathbf{x}) = (0,0,0,1,1,1,0,0)$, $B^2(\mathbf{x}) = (0,0,1,1,1,0,0,0)$ and $B^3(\mathbf{x}) = (0,0,0,1,1,1,0,0)$, where $B^i$ is the function $B$ applied $i$ times.

We now have the language necessary to state a general lemma that immediately implies Lemma 4.3.1.

**Lemma 4.3.18.** *Define $\mathbf{p}^J$ and $\tilde{\mathbf{p}}^J$ as above, and let $d = C(\mathbf{P}^J)$. Let $\mathbf{u}$ be an admissible vector with $a + b + c = \#\mathbf{P}^J$. Then $\mathbf{p}^J \cdot \mathbf{u} \leq \tilde{\mathbf{p}}^J \cdot \tilde{\mathbf{u}}$, where $\tilde{\mathbf{u}}$ is defined by the following cases:*

- *If $d < -1$, then:*
  - *If $D(\mathbf{u}) > d + 1$, then $\tilde{\mathbf{u}} = \mathbf{u}$.*

– *If* $d - 1 \le D(\mathbf{u}) \le d + 1$, *then* $\tilde{\mathbf{u}} = B(\mathbf{u})$.

– *If* $D(\mathbf{u}) < d - 1$, *then* $\tilde{\mathbf{u}} = B^2(\mathbf{u})$.

- *If* $|d| \le 1$, *then:*

  – *If* $d - 1 \le D(\mathbf{u}) \le d + 1$, *then* $\tilde{\mathbf{u}} = \mathbf{u}$.

  – *If* $D(\mathbf{u}) < d - 1$ *or* $D(\mathbf{u}) > d + 1$, *then* $\tilde{\mathbf{u}} = B(\mathbf{u})$.

- *If* $d > 1$, *then:*

  – *If* $D(\mathbf{u}) < d - 1$, *then* $\tilde{\mathbf{u}} = \mathbf{u}$.

  – *If* $d - 1 \le D(\mathbf{u}) \le d + 1$, *then* $\tilde{\mathbf{u}} = B(\mathbf{u})$.

  – *If* $D(\mathbf{u}) > d + 1$, *then* $\tilde{\mathbf{u}} = B^2(\mathbf{u})$.

We will prove Lemma 4.3.18 by induction on the number of principal hooks of $\lambda$. However, before giving the proof, we first show how it implies Lemma 4.3.1 and in turn how Lemma 4.3.1 implies the First Balancing Theorem 4.3.2.

*Proof of Lemma 4.3.1.* In each of the three cases of Lemma 4.3.18, notice that if $D(\mathbf{u}) = 0$, then $\tilde{\mathbf{u}} = \mathbf{u}$. Thus since $D(\mathbf{1}) = 0$, we have in every case that $\mathbf{p}^k \cdot \mathbf{1} \le \tilde{\mathbf{p}}^k \cdot \mathbf{1}$. As noted in Remark 4.3.11, this is a restatement of the inequality in Lemma 4.3.1. □

*Remark* 4.3.19. Note that the proof of Lemma 4.3.1 only required the assertion of Lemma 4.3.18 specifically for $\mathbf{u} = \mathbf{1}$. However, the full generality of Lemma 4.3.18

(allowing for *any* admissible vector $\mathbf{u}$) seems to be necessary in order for the proof by induction to be possible. Determining the appropriate conditions for induction to work in the proof of Lemma 4.3.18 was a significant part of discovering the proof.

*Proof of Theorem 4.3.2 (The First Balancing Theorem).* All $\mu \leq \lambda$, with $\mu$ of given weight $\mathbf{w}$, partition the same number. Thus they contribute to the coefficient of the same degree in $G_\lambda$ (similarly for $\tilde{\mu} \leq \tilde{\lambda}$, with $\mathrm{wt}(\tilde{\mu}) = \mathbf{w}$). Thus the result follows from Lemma 4.3.1. $\square$

The remainder of this section will be devoted to proving Lemma 4.3.18.

*Proof of Lemma 4.3.18.* We proceed by induction on the number of principal hooks of $\lambda$.

Note that there are no "paths" leading to the points of $\mathbf{P}^1$ in the single hook case. Because of the correspondence to partitions, the $i^{\text{th}}$ entry of $\mathbf{p}^1$ is the number of partitions in $\lambda$ having outer hook $(x_i, y_i)$. This number is simply 1. So $\mathbf{p}^1 = \tilde{\mathbf{p}}^1 = \mathbf{1}$ and the inequalities in the lemma are trivially true (since $\mathbf{u}$ and $B(\mathbf{u})$ always have the same number of 1's).

We assume now that Lemma 4.3.18 is true for any partition having some specific number, $k-1$, of principal hooks.

Let $\lambda$ be any partition with $k$ hooks, and let $\mathbf{w}$ be a valid weight vector for $\lambda$. As before, $\mathbf{P}^{k-1}$ and $\mathbf{P}^k$ are the points on the $(k-1)^{\text{st}}$ and $k^{\text{th}}$ diagonals of $\mathcal{T}_{\lambda,\mathbf{w}}$, respectively, and $\mathbf{p}^{k-1}$ and $\mathbf{p}^k$ are the corresponding path-counting vectors. Similarly, we have $\tilde{\mathbf{P}}^{k-1}$, $\tilde{\mathbf{P}}^k$, $\tilde{\mathbf{p}}^{k-1}$ and $\tilde{\mathbf{p}}^k$ for $\mathcal{T}_{\tilde{\lambda},\mathbf{w}}$. Let $d = C(\mathbf{P}^{k-1})$ and $e = C(\mathbf{P}^k)$.

For any admissible $n$-vector $\mathbf{v}$ (with $a+b+c=n$), we wish to show that $\mathbf{p}^k \cdot \mathbf{v} \leq \tilde{\mathbf{p}}^k \cdot \tilde{\mathbf{v}}$. Define $V$ to be the $n \times b$ matrix whose $i^{\text{th}}$ column is all zero except for a 1 in the $(a+i)^{\text{th}}$ position (note that this corresponds to a position of a 1 in $\mathbf{v}$). We define $\tilde{V}$ based on $\tilde{\mathbf{v}}$ similarly. Then $MV$ is the $m \times b$ matrix consisting of the columns of $M$ whose indices correspond to 1's in $\mathbf{v}$, as illustrated below:

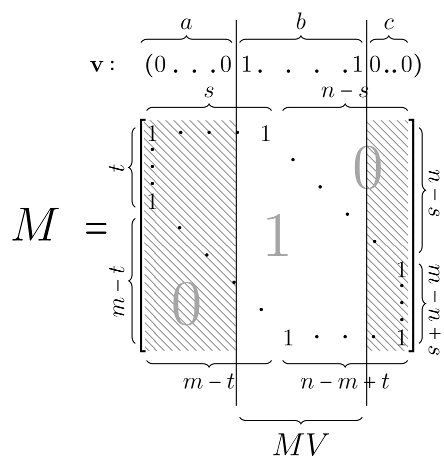



**Figure 4.11:** Structure of $MV$ from $M$ and $\mathbf{v}$

Note that since $\mathbf{p}^k = \mathbf{p}^{k-1}M$ (Equation (4.2)), the sum of the entries of $\mathbf{p}^{k-1}MV$ is

$\mathbf{p}^k \cdot \mathbf{v}$. That is,

$$\mathbf{p}^k \cdot \mathbf{v} = \mathbf{p}^{k-1} MV \cdot \mathbf{1}. \tag{4.3}$$

Observe that any permutation of the entries within a row of $MV$ does not affect the expression on the right-hand side of (4.3). With this in mind, we define $\overline{MV}$ to be the matrix obtained from $MV$ by "justifying" the 1's in each row of $MV$ to the left (the number of 1's in each row remains the same). So we have

$$\mathbf{p}^k \cdot \mathbf{v} = \mathbf{p}^{k-1} \overline{MV} \cdot \mathbf{1}. \tag{4.4}$$

Let $B\left(\overline{MV}\right)$ be the matrix obtained by replacing each nonzero column $\mathbf{u}$ of $\overline{MV}$ by $\tilde{\mathbf{u}}$, as dictated by the lemma. By the inductive hypothesis, $\mathbf{p}^{k-1} \cdot \mathbf{u} \leq \tilde{\mathbf{p}}^{k-1} \cdot \tilde{\mathbf{u}}$. Together with (4.4), this implies that

$$\mathbf{p}^k \cdot \mathbf{v} \leq \tilde{\mathbf{p}}^{k-1} B\left(\overline{MV}\right) \cdot \mathbf{1}. \tag{4.5}$$

From (4.2), we have $\tilde{\mathbf{p}}^k = \tilde{\mathbf{p}}^{k-1} \tilde{M}$. We also consider $\overline{\tilde{M}\tilde{V}}$ (the matrix consisting of the columns of $\tilde{M}$ corresponding to 1's in $\tilde{\mathbf{v}}$ with the 1's in each row then justified left). Analogously to (4.4), we have

$$\tilde{\mathbf{p}}^k \cdot \tilde{\mathbf{v}} = \tilde{\mathbf{p}}^{k-1} \overline{\tilde{M}\tilde{V}} \cdot \mathbf{1}. \tag{4.6}$$

Thus, in order to prove $\mathbf{p}^k \cdot \mathbf{v} \leq \tilde{\mathbf{p}}^k \cdot \tilde{\mathbf{v}}$, it suffices to show that $\tilde{\mathbf{p}}^{k-1} B\left(\overline{MV}\right) \cdot \mathbf{1} \leq$

$\tilde{\mathbf{p}}^{k-1}\overline{\tilde{M}\tilde{V}}\cdot\mathbf{1}$. We will in fact demonstrate that $\overline{\tilde{M}\tilde{V}} - B\left(\overline{MV}\right)$ has nonnegative entries, which produces the desired inequality since $\overline{\tilde{M}\tilde{V}}$ and $B\left(\overline{MV}\right)$ are both 0-1 matrices. We represent this claim simply as $\overline{\tilde{M}\tilde{V}} - B\left(\overline{MV}\right) \geq \mathbf{0}$ (where $\mathbf{0}$ denotes the all-zero matrix of appropriate size). We will continue the proof of Lemma 4.3.18 after the next remark and an example.

*Remark* 4.3.20. The preceding discussion allows us to no longer deal with path-counting vectors and instead simply work with comparing 0-1 matrices that arise from connections between the last two diagonals of a trellis diagram.

*Example* 4.3.21. Before continuing with the proof, we provide an example to give some evidence and intuition to this claim.

Suppose in some trellis structure we have $m = 8$ points in $\mathbf{P}^{k-1}$ and $n = 10$ points in $\mathbf{P}^k$. Moreover, suppose $C(\mathbf{P}^{k-1}) = d = -2$, $C(\mathbf{P}^k) = e = 3$ and $M = \begin{pmatrix} 1&1&1&1&1&1&1&1&0&0 \\ 0&1&1&1&1&1&1&1&1&0 \\ 0&0&1&1&1&1&1&1&1&1 \\ 0&0&0&1&1&1&1&1&1&1 \\ 0&0&0&0&1&1&1&1&1&1 \\ 0&0&0&0&0&1&1&1&1&1 \\ 0&0&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&1&1&1 \end{pmatrix}$.

This implies $\tilde{M} = \begin{pmatrix} 1&1&1&1&1&1&0&0&0&0 \\ 1&1&1&1&1&1&1&0&0&0 \\ 1&1&1&1&1&1&1&1&0&0 \\ 0&1&1&1&1&1&1&1&1&0 \\ 0&0&1&1&1&1&1&1&1&1 \\ 0&0&0&1&1&1&1&1&1&1 \\ 0&0&0&0&1&1&1&1&1&1 \\ 0&0&0&0&0&1&1&1&1&1 \end{pmatrix}$. (An example with these specific values can be seen in the trellis $\mathcal{T}_{\left(\begin{smallmatrix} 16 & 8 & 7 \\ 13 & 10 & 8 \end{smallmatrix}\right),(20,11,8)}$ in the 2$^{\text{nd}}$ and 3$^{\text{rd}}$ diagonals.)

It can be verified for these specific matrices that, for any $\mathbf{v}$, we have $\overline{\tilde{M}\tilde{V}} - B\left(\overline{MV}\right) \geq \mathbf{0}$. We illustrate the case $\mathbf{v} = (0,0,1,1,1,1,1,1,0,0)$. We have $MV = \begin{pmatrix} 1&1&1&1&1&1 \\ 1&1&1&1&1&1 \\ 1&1&1&1&1&1 \\ 0&1&1&1&1&1 \\ 0&0&1&1&1&1 \\ 0&0&0&1&1&1 \\ 0&0&0&0&1&1 \\ 0&0&0&0&0&1 \end{pmatrix}$, so

$\overline{MV} = \begin{pmatrix} 1&1&1&1&1&1 \\ 1&1&1&1&1&1 \\ 1&1&1&1&1&1 \\ 1&1&1&1&1&0 \\ 1&1&1&1&0&0 \\ 1&1&1&0&0&0 \\ 1&1&0&0&0&0 \\ 1&0&0&0&0&0 \end{pmatrix}$. Because of the inductive hypothesis, any column $\mathbf{u}$ with $D(\mathbf{u}) < d - 1 = -3$ "balances" twice ($\tilde{\mathbf{u}} = B^2(\mathbf{u})$), which applies to the last two columns. Any

column with $-3 \le D(\mathbf{u}) \le -1$ balances once (applied to all other columns except the first).

So we have $B\left(\overline{MV}\right) = \begin{pmatrix} 1&0&0&0&0&0\\ 1&1&1&1&0&0\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&0\\ 1&1&1&0&0&0\\ 1&1&0&0&0&0 \end{pmatrix}$.

Since $D(\mathbf{v}) = 0 < e-1$, we have $\tilde{\mathbf{v}} = \mathbf{v}$, so $\tilde{M}\tilde{V} = \begin{pmatrix} 1&1&1&1&0&0\\ 1&1&1&1&1&0\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 0&1&1&1&1&1\\ 0&0&1&1&1&1\\ 0&0&0&1&1&1 \end{pmatrix}$ and $\overline{\tilde{M}\tilde{V}} = \begin{pmatrix} 1&1&1&1&0&0\\ 1&1&1&1&1&0\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&0\\ 1&1&1&1&0&0\\ 1&1&1&0&0&0 \end{pmatrix}$.

Therefore $\overline{\tilde{M}\tilde{V}} - B\left(\overline{MV}\right) = \begin{pmatrix} 0&1&1&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0 \end{pmatrix} \ge \mathbf{0}$, as desired.

*Proof of Lemma 4.3.18, continued.* It remains to prove in general the claim that $\overline{\tilde{M}\tilde{V}} - B\left(\overline{MV}\right) \ge \mathbf{0}$. We begin by developing a simple method for representing and comparing matrices $\overline{\tilde{M}\tilde{V}}$ and $B\left(\overline{MV}\right)$. Note that by construction, the row sums of $\tilde{M}\tilde{V}$ (and therefore of $\overline{\tilde{M}\tilde{V}}$) form a unimodal sequence where consecutive terms differ by at most 1. Moreover this sequence is strictly unimodal except for possibly admitting a peak with multiplicity greater than 1. Also, because of the left-justification of rows in $\overline{\tilde{M}\tilde{V}}$, the column sums form a weakly decreasing sequence where consecutive terms first repeat (if the sequence starts with $m$), then decrease by exactly 1 (if the first and last rows of $\overline{\tilde{M}\tilde{V}}$ do not have the same number of zeros), then decrease by exactly 2 for the remaining positive terms. So we conclude that if the $(i, j)$ entry of $\overline{\tilde{M}\tilde{V}}$ is a 1, then the $(i-1, j-1)$, the $(i, j-1)$ and the $(i+1, j-1)$ entries (if they exist) are also 1's. We remark that these claims can readily be visualized in the illustrations in Appendix B.2.

Thus $\overline{\tilde{M}\tilde{V}}$ of fixed dimensions $m \times b$ can be described by the three numbers $\alpha$, $\beta$, $\gamma$, where $\alpha$ (with $1 \leq \alpha \leq b$) indicates the index position of the last nonzero column, $\beta$ (with $1 \leq \beta \leq m$) indicates the row of the first 1 in column $\alpha$, and $\gamma$ (with $\beta \leq \gamma \leq m$) indicates the row of the last 1 in column $\alpha$. Since $\overline{\tilde{M}\tilde{V}}$ is uniquely determined by these three numbers, we write $\overline{\tilde{M}\tilde{V}} = (\alpha, \beta, \gamma)$. Note that this 3-tuple representation is completely determined by the last nonzero column of the matrix.

*Example* 4.3.22. The $8 \times 6$ matrix $\begin{pmatrix} 1&1&1&1&0&0\\ 1&1&1&1&1&0\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&1\\ 1&1&1&1&1&0\\ 1&1&1&1&0&0\\ 1&1&1&0&0&0 \end{pmatrix}$ from Example 4.3.21 can be represented as $(6, 3, 5)$.

*Proof of Lemma 4.3.18, continued.* Analogously, we see that $\overline{MV}$ (of fixed dimensions $m \times b$) can also be uniquely expressed by a 3-tuple.

For the matrix $B\left(\overline{MV}\right)$, we similarly let $\rho$ be the index position of the last nonzero column, and let $\eta$ and $\zeta$ indicate the row of the first and last 1, respectively, in column $\rho$. Unfortunately $B\left(\overline{MV}\right)$ is not in general equal to the matrix determined (as above) by $(\rho, \eta, \zeta)$. An instance of this can be seen in Example 4.3.21. However, note that columns, $\mathbf{u}$, of $\overline{MV}$ either all have nonnegative centeredness, or else they all have nonpositive centeredness (i.e., either $D(\mathbf{u}) \geq 0$ or else $D(\mathbf{u}) \leq 0$ for all $\mathbf{u}$). Furthermore, the sequence of centeredness numbers of the columns weakly increases in absolute value, from left to right, by at most 1. Under the assumptions of the inductive hypothesis, namely the result of the lemma applied to columns of $B(\overline{MV})$, we infer

from these observations that a column centered by two under $B$ cannot be adjacent to a (nonzero) column that is fixed under $B$. Also, if a column of $\overline{MV}$ is centered under $B$, then nonzero columns to the right of it are also centered. Furthermore, notice that consecutive columns with 0's in the first and last entries always have the same centeredness. (We remark that these observations are clearly illustrated in Appendix B.2.) From this discussion, we conclude that, although $B\left(\overline{MV}\right) \neq (\rho, \eta, \zeta)$, we do have $(\rho, \eta, \zeta) - B\left(\overline{MV}\right) \geq \mathbf{0}$.

So to show that $\overline{\tilde{M}\tilde{V}} - B\left(\overline{MV}\right) \geq \mathbf{0}$, it suffices to show that $(\alpha, \beta, \gamma) - (\rho, \eta, \zeta) \geq \mathbf{0}$. Based on the structure of these matrices, we have that $(\alpha, \beta, \gamma) - (\rho, \eta, \zeta) \geq \mathbf{0}$ if and only if $\rho \leq \alpha$ and

- If $\rho = \alpha$, then $\eta \geq \beta$ and $\zeta \leq \gamma$, or else
- If $\rho < \alpha$, then $\beta - \eta \leq \alpha - \rho$ and $\zeta - \gamma \leq \alpha - \rho$.

In order to determine $(\alpha, \beta, \gamma)$ and $(\rho, \eta, \zeta)$, it is necessary to examine a rather large number of cases based on the comparative values of the parameters $n, m, s, t, a, b, c, d, e$.

We first examine various cases for precisely determining the structure of $MV$. From these we will obtain the form of $\overline{MV}$ (also $\overline{\tilde{M}\tilde{V}}$ by extension), and subsequently determine the various possibilities for $B\left(\overline{MV}\right)$. We refer the reader to Fig. 4.11

for understanding why the following cases are relevant, and why the corresponding implications hold.

Case 1. $c \geq n - s$ (implying the first row of $MV$ is all 1's)

Case 2. $a < s$ and $c < n - s$ (first row of $MV$ contains 0's and 1's)

Case 3. $a \geq s$ (first row of $MV$ is all 0's)

For each of Cases 1.-3., we must also consider the following:

Case A. $a \geq m - t$ (implying the last row of $MV$ is all 1's)

Case B. $a < m - t$ and $c < n - m + t$ (last row of $MV$ contains 0's and 1's)

Case C. $c \geq n - m + t$ (last row of $MV$ is all 0's)

Illustrations of the 9 possible forms of $MV$ are given in Appendix B.1.

From these, we can describe the structure of $\overline{MV}$. The form of $\overline{MV}$ in Case 1. or Case A. can be determined immediately. However, in order to determine the form of $\overline{MV}$ in Cases 2.B., 3.B., 2.C., or 3.C., the following subcases must also be considered:

Case i. $s + t \leq b$ (implying the final column of $\overline{MV}$ is all 0's)

Case ii. $s + t > b$ (implying the final column of $\overline{MV}$ contains 1's)

Illustrations of the 13 possible forms of $\overline{MV}$ are given in Appendix B.2.

Since $\tilde{M}$ and $\tilde{V}$ are matrices having the same structural properties as $M$ and $V$, note that replacing all parameter names (other than $n$, $m$ and $b$, which are fixed under balancing) with their respective "tilde" names (i.e., $m - t - a \to m - \tilde{t} - \tilde{a}$) produces all possible cases for $\overline{\tilde{M}\tilde{V}}$. Thus (noting some symmetries in the various cases) we have from Appendix B.2 the following possible values for the matrix $\overline{\tilde{M}\tilde{V}} = (\alpha, \beta, \gamma)$.

**Table 4.1**
Possible cases for $\overline{\tilde{M}\tilde{V}} = (\alpha, \beta, \gamma)$

| Case | $\overline{\tilde{M}\tilde{V}}$ |
|---|---|
| $\tilde{1}.\tilde{A}.$ | $(b, 1, m)$ |
| $\tilde{1}.\tilde{B}.$, $\tilde{1}.\tilde{C}.$ | $(b, 1, \tilde{t} + \tilde{a})$ |
| $\tilde{2}.\tilde{A}.$, $\tilde{3}.\tilde{A}.$ | $(b, n - \tilde{s} - \tilde{c} + 1, m)$ |
| $\tilde{2}.\tilde{B}.\tilde{i}.$, $\tilde{2}.\tilde{C}.\tilde{i}.$, $\tilde{3}.\tilde{B}.\tilde{i}.$, $\tilde{3}.\tilde{C}.\tilde{i}.$ | $(\tilde{s} + \tilde{t} - 1, \tilde{t} + \tilde{a}, n - \tilde{s} - \tilde{c} + 1)$ |
| $\tilde{2}.\tilde{B}.\tilde{ii}.$, $\tilde{2}.\tilde{C}.\tilde{ii}.$, $\tilde{3}.\tilde{B}.\tilde{ii}.$, $\tilde{3}.\tilde{C}.\tilde{ii}.$ | $(b, n - \tilde{s} - \tilde{c} + 1, \tilde{t} + \tilde{a})$ |

We now turn our attention to determining $(\rho, \eta, \zeta)$ for the various cases. Note that we need only consider how the function $B$ affects the last nonzero column of $\overline{MV}$. Once again, we have a further breakdown of the cases, dependent on the centeredness

of this last nonzero column compared with $C(\mathbf{P}^{k-1}) = d$ as directed by the inductive hypothesis. In each case, we use $\delta$ to represent the centeredness of this final nonzero column of $\overline{MV}$.

We are again able to take advantage of some symmetries (noticeable in Appendix B.2) between the various cases. We have the following:

- For Case 1.A., $B(\overline{MV}) \leq (b, 1, m)$.
- For Cases 1.B. and 1.C., we have $\delta = -(m - t - a)$.
  - If $d < -1$, then:
    - If $\delta > d + 1$, then: $B(\overline{MV}) \leq (b, 1, t + a)$
    - If $d - 1 \leq \delta \leq d + 1$, then: $B(\overline{MV}) \leq (b, 2, t + a + 1)$
    - If $\delta < d - 1$, then: $B(\overline{MV}) \leq (b, 3, t + a + 2)$
  - If $|d| \leq 1$, then:
    - If $\delta \geq d - 1$, then: $B(\overline{MV}) \leq (b, 1, t + a)$
    - If $\delta < d - 1$, then: $B(\overline{MV}) \leq (b, 2, t + a + 1)$
  - If $d > 1$, then: $B(\overline{MV}) \leq (b, 1, t + a)$
- For Cases 2.A. and 3.A., we have $\delta = n - s - c$.
  - If $d < -1$, then: $B(\overline{MV}) \leq (b, n - s - c + 1, m)$
  - If $|d| \leq 1$, then:

* If $\delta \leq d+1$, then: $$B(\overline{MV}) \leq (b,\, n-s-c+1,\, m)$$
* If $\delta > d+1$, then: $$B(\overline{MV}) \leq (b,\, n-s-c,\, m-1)$$

– If $d > 1$, then:

* If $\delta < d-1$, then: $$B(\overline{MV}) \leq (b,\, n-s-c+1,\, m)$$
* If $d-1 \leq \delta \leq d+1$, then: $$B(\overline{MV}) \leq (b,\, n-s-c,\, m-1)$$
* If $\delta > d+1$, then: $$B(\overline{MV}) \leq (b,\, n-s-c-1,\, m-2)$$

Before continuing to describe $B(\overline{MV})$ for Cases 2.B., 2.C., 3.B. and 3.C., we note a few simplifications. In any of these cases we have $\delta = (t+a-1)-(m-n+s+c-1)$. Furthermore, since $s=\sigma$ and $t=\tau$ in these cases, the simple identity (4.1) we noted in Remark 4.3.15 is equivalent to $e+n+t = d+m+s$. It follows that $\delta = a-c+d-e$. Consequently, note that the cases $(\delta < d-1,\ d-1 \leq \delta \leq d+1,\ \delta > d+1)$ are equivalent to $(a-c < e-1,\ e-1 \leq a-c \leq e+1,\ a-c > e+1)$. We proceed now with listing the representations of $B(\overline{MV})$ for these remaining cases.

- For Cases 2.B.i., 2.C.i., 3.B.i. and 3.C.i. [and for 2.B.ii., 2.C.ii., 3.B.ii. and 3.C.ii., as given below in brackets], we have $\delta = a-c+d-e$.

  – If $\delta = 0$, then: $$B(\overline{MV}) \leq (s+t-1,\, t+a,\, n-s-c+1)$$ $$[(b,\, n-s-c+1,\, a+t)]$$

  – If $\delta \leq -1$, then:

  * If $d < -1$, then:

· If $\delta < d-1$, then: $B(\overline{MV}) \leq (s+t-1,\, t+a+2,\, n-s-c+3)$

$[(b,\, n-s-c+3,\, a+t+2)]$

· If $d-1 \leq \delta \leq d+1$, then: $B(\overline{MV}) \leq (s+t-1,\, t+a+1,\, n-s-c+2)$

$[(b,\, n-s-c+2,\, a+t+1)]$

· If $\delta > d+1$, then: $B(\overline{MV}) \leq (s+t-1,\, t+a,\, n-s-c+1)$

$[(b,\, n-s-c+1,\, a+t)]$

∗ If $|d| \leq 1$, then:

· If $\delta < d-1$, then: $B(\overline{MV}) \leq (s+t-1,\, t+a+1,\, n-s-c+2)$

$[(b,\, n-s-c+2,\, a+t+1)]$

· If $\delta \geq d-1$, then: $B(\overline{MV}) \leq (s+t-1,\, t+a,\, n-s-c+1)$

$[(b,\, n-s-c+1,\, a+t)]$

∗ If $d > 1$, then: $B(\overline{MV}) \leq (s+t-1,\, t+a,\, n-s-c+1)$

$[(b,\, n-s-c+1,\, a+t)]$

– If $\delta \geq 1$, then:

∗ If $d < -1$, then: $B(\overline{MV}) \leq (s+t-1,\, t+a,\, n-s-c+1)$

$[(b,\, n-s-c+1,\, a+t)]$

∗ If $|d| \leq 1$, then:

· If $\delta \leq d+1$, then: $B(\overline{MV}) \leq (s+t-1,\, t+a,\, n-s-c+1)$

$[(b,\, n-s-c+1,\, a+t)]$

· If $\delta > d+1$, then: $B(\overline{MV}) \leq (s+t-1,\, t+a-1,\, n-s-c)$

$[(b,\, n-s-c,\, a+t-1)]$

∗ If $d > 1$, then:

· If $\delta < d - 1$, then: $$B(\overline{MV}) \leq (s + t - 1, t + a, n - s - c + 1)$$ $$[(b, n - s - c + 1, a + t)]$$

· If $d - 1 \leq \delta \leq d + 1$, then: $$B(\overline{MV}) \leq (s + t - 1, t + a - 1, n - s - c)$$ $$[(b, n - s - c, a + t - 1)]$$

· If $\delta > d + 1$, then: $$B(\overline{MV}) \leq (s + t - 1, t + a - 2, n - s - c - 1)$$ $$[(b, n - s - c - 1, a + t - 2)]$$

We again turned to Mathematica to symbolically compare $(\rho, \eta, \zeta)$ and $(\alpha, \beta, \gamma)$ for the many possible valid cases (there are 1990 of them). In each case, Mathematica verified that $(\alpha, \beta, \gamma) - (\rho, \eta, \zeta) \geq \mathbf{0}$, thus completing the proof. The full code used for this verification is provided in Appendix C.3. □

*Example* 4.3.23. We illustrate the proof of Lemma 4.3.18 for two cases.

- **Example 4.3.23.a.**

  Suppose we are in the following case for the various parameters (assuming as always some arbitrary but fixed positive integer values of $m$ and $n$):

  – $|d| \leq 1$ and $e > 1$

  – $\sigma = n - 1$ and $\tau < m - 1$ (so $s = n - 1$ and $t = \tau$)

  – Case 2.B.i. holds

Recall that in any case, $b \leq n$ and $t \geq 1$. Since we are assuming that $s = n - 1$ and $s+t \leq b$ (Case i.), it follows that $s+t = b$, so $t = 1$ and $b = n$ (this also implies $a = c = 0$). Since $e > 1$ we have $a-c < e-1$, which we noted is equivalent to $\delta < d-1$ in Cases 2.B., 2.C., 3.B., or 3.C.. Thus $B(\overline{MV}) \leq (s+t-1, t+a+1, n-s-c+2) = (b-1, 2, 3)$.

We now determine $\overline{\tilde{M}\tilde{V}}$ for the current case. Since $|d| \leq 1$ and $e > 1$, we have (from the cases given in Appendix A) that $(\tilde{\sigma}, \tilde{\tau}) = (\sigma-1, \tau+1)$ and also $\tilde{s} = n-2$ and $\tilde{t} = t+1 = 2$. Also, since $D(\mathbf{v}) = a - c = 0$, we have that $\tilde{\mathbf{v}} = \mathbf{v}$ so $\tilde{a} = a = 0$ and $\tilde{c} = c = 0$. So we are in Case $\tilde{2}.\tilde{\mathrm{B}}$. Also, since $\tilde{s} + \tilde{t} = s + t \leq b$, Case i. still holds. Thus we have $\overline{\tilde{M}\tilde{V}} = (\tilde{s} + \tilde{t} - 1, \tilde{t} + \tilde{a}, n - \tilde{s} - \tilde{c} + 1) = (b - 1, 2, 3)$. So we see that $\overline{\tilde{M}\tilde{V}} = B(\overline{MV})$ and we have the expected result.

- **Example 4.3.23.b.**

  Suppose we have the following case:

  - $d > 1$ and $e < -1$
  - $D(\mathbf{v}) > e + 1$ (i.e., $a - c > e + 1$)
  - $\sigma < n - 1$ and $\tau = m + 1$ (so $s = \sigma$ and $t = m$)
  - Case 2.A. holds and $d - 1 \leq n - s - c \leq d + 1$

  We immediately see that $B(\overline{MV}) \leq (b, n - s - c, m - 1)$.

  From Appendix A, we determine that $\tilde{s} = s + 2$ and $\tilde{t} = m - 1$. Also, since $e < -1$ and $D(\mathbf{v}) > e + 1$ we have $\tilde{\mathbf{v}} = \mathbf{v}$, so $\tilde{a} = a$ and $\tilde{c} = c$.

The case that holds true for the "tilde" parameters is not completely determined by the assumptions we made about the "non-tilde" ones, but we can rule out some possibilities.

Note that since $a < s$ (Case 2.), we have that $\tilde{a} < \tilde{s}$, so Case $\tilde{3}$. cannot hold.

Also, Case $\tilde{\text{C}}$. cannot hold, for if it did then we would have $\tilde{c} \geq n - m + \tilde{t}$, which is equivalent to $c \geq n - 1$. But this contradicts the fact that $s \geq 1$ and $c < n - s$ (from Case 2.).

Case $\tilde{1}.\tilde{\text{B}}$. cannot hold. For if it did, then Case $\tilde{\text{B}}$. gives $\tilde{a} < m - \tilde{t}$ which implies $a < 1$, so $a = 0$. Thus the general equality $n = a + b + c$ gives $n = b + c$, and the identity from Remark 4.3.15 is equivalent in this case to $d = b - s + e + c + 1$. Furthermore, since we are assuming $a - c > e + 1$, we have $e + c + 1 < 0$. Assuming Case $\tilde{1}$. gives $\tilde{c} \geq n - \tilde{s}$, which is equivalent to $b - s \leq 2$. Combining the (in)equalites from the end of the last three sentences gives $d = b - s + e + c + 1 < 2$, which contradicts our assumption at the beginning of this example that $d > 1$.

So one of Cases $\tilde{1}.\tilde{\text{A}}$., $\tilde{2}.\tilde{\text{A}}$., or $\tilde{2}.\tilde{\text{B}}$. holds. Note that $\tilde{s} + \tilde{t} = s + m + 1 > b$, so we do know that Case $\tilde{\text{ii}}$. holds. We consider each of the possibilities:

- If Case $\tilde{1}.\tilde{\text{A}}$. holds, then $\overline{\tilde{M}\tilde{V}} = (b, 1, m)$
- If Case $\tilde{2}.\tilde{\text{A}}$. holds, then $\overline{\tilde{M}\tilde{V}} = (b, n - \tilde{s} - \tilde{c} + 1, m) = (b, n - s - c - 1, m)$
- If Case $\tilde{2}.\tilde{\text{B}}.\tilde{\text{ii}}$. holds, then $\overline{\tilde{M}\tilde{V}} = (b, n - \tilde{s} - \tilde{c} + 1, \tilde{t} + \tilde{a}) = (b, n - s - c - 1, m + a - 1)$

In each case, we have $\overline{\tilde{M}\tilde{V}} - B(\overline{MV}) \geq 0$, as expected.

### 4.3.2 The Second Balancing Theorem

We now consider the second balancing operation on partitions, as introduced in Section 4.2. As we proceed to prove, this balancing operation also improves the generating function of a partition.

For a partition $\lambda = \left(\begin{smallmatrix} A_k & A_{k-1} & \cdots & A_1 \\ B_k & B_{k-1} & \cdots & B_1 \end{smallmatrix}\right)$, define $\overline{\lambda} = \left(\begin{smallmatrix} \overline{A}_k & \overline{A}_{k-1} & \cdots & \overline{A}_1 \\ \overline{B}_k & \overline{B}_{k-1} & \cdots & \overline{B}_1 \end{smallmatrix}\right)$, where $\overline{A}_i = \max(A_i, B_i)$ and $\overline{B}_i = \min(A_i, B_i)$.

Note that the number and sizes of principal hooks are preserved from $\lambda$ to $\overline{\lambda}$.

**Lemma 4.3.24.** *For any integer vector* $\mathbf{w}$*, we have*

$$\#\{\mu \leq \lambda \,|\, \mathrm{wt}(\mu) = \mathbf{w}\} \leq \#\{\overline{\mu} \leq \overline{\lambda} \,|\, \mathrm{wt}(\overline{\mu}) = \mathbf{w}\}.$$

The theorem given below follows from Lemma 4.3.24 in exactly the same way that Theorem 4.3.2 follows from Lemma 4.3.1.

**Theorem 4.3.25** (The Second Balancing Theorem)**.**

$$G_\lambda \leq G_{\overline{\lambda}}.$$

*Remark* 4.3.26. The two Balancing Theorems (4.3.2 and 4.3.25) produce sequences of

partitions with successively improved generating functions. This is illustrated below. The first inequality is from the Second Balancing Theorem; the second and third follow from the First Balancing Theorem 4.3.2.

$$G_{\begin{pmatrix} 12 & 4 & 3 & 0 \\ 8 & 7 & 2 & 1 \end{pmatrix}} \le G_{\begin{pmatrix} 12 & 7 & 3 & 1 \\ 8 & 4 & 2 & 0 \end{pmatrix}} \le G_{\begin{pmatrix} 11 & 6 & 3 & 1 \\ 9 & 5 & 2 & 0 \end{pmatrix}} \le G_{\begin{pmatrix} 10 & 6 & 3 & 1 \\ 10 & 5 & 2 & 0 \end{pmatrix}}$$

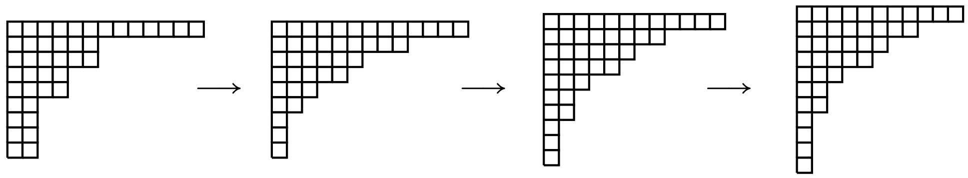


**Figure 4.12:** Successively improved generating functions

*Proof of Lemma 4.3.24.* We prove the desired inequality with an injective argument. For any partition $\mu \le \lambda$, let $h_i = \binom{a_i}{b_i}$ represent the $i^{\text{th}}$ hook of $\mu$. Since $\mu \le \lambda$, we have that $a_i \le A_i$ and $b_i \le B_i$.

We define a map from the set of partitions fitting inside $\lambda$ to those fitting inside $\overline{\lambda}$ as follows. Take any partition $\mu$ where $\mu \le \lambda$. Any hook of $\mu$ that fits in the corresponding hook of $\overline{\lambda}$ is (initially) unchanged. Note that a hook $h_i = \binom{a_i}{b_i}$ will not fit in $\overline{\lambda}$ if and only if $\overline{B_i} = A_i$ and $b_i > \overline{B}_i$. All such hooks are "flipped", that is,

$$\begin{pmatrix} a_i \\ b_i \end{pmatrix} \to \begin{pmatrix} b_i \\ a_i \end{pmatrix}.$$

Clearly, all the hooks produced by this "flip/no flip" procedure will fit inside $\overline{\lambda}$. However, it is possible that the rows of the resulting array may not strictly decrease,

meaning the array does not give a partition. For instance, if the $i^{\text{th}}$ hook is flipped and the $(i-1)^{\text{st}}$ is not, then there could be hooks $\left(\begin{smallmatrix} b_i & a_{i-1} \\ a_i & b_{i-1} \end{smallmatrix}\right)$, where $a_i \leq b_{i-1}$. To correct this, we simply flip the $(i-1)^{\text{st}}$ hook to obtain $\left(\begin{smallmatrix} b_i & b_{i-1} \\ a_i & a_{i-1} \end{smallmatrix}\right)$. This clearly fixes the non-decreasing issue between the $i^{\text{th}}$ and $(i-1)^{\text{st}}$ hooks. Furthermore, because $a_{i-1} < a_i \leq b_{i-1} \leq \overline{B}_{i-1} \leq \overline{A}_{i-1}$, we see that the flipped $(i-1)^{\text{st}}$ hook fits inside $\overline{\lambda}$. We continue flipping subsequent hooks ($h_{i-2}$, etc) until we reach a hook where the rows of the array are strictly decreasing.

This issue of non-decreasing rows could also have happened between the $(i+1)^{\text{st}}$ and $i^{\text{th}}$ hooks. We correct this in an analogous way by flipping hooks $h_{i+1}$, $h_{i+2}$, etc. The argument for this case is essentially the same as that given in the paragraph above.

When this procedure is complete, we will be left with a partition $\overline{\mu}$ in $\overline{\lambda}$. It is not hard to see that this map is indeed an injection. Suppose that $\overline{\mu} \leq \overline{\lambda}$ is the image of some partition $\mu \leq \lambda$ as constructed by the map described above. To recover $\mu$, first flip all hooks in $\overline{\mu}$ that do not fit inside $\lambda$. Once again we will, in general, be left with an invalid array. To fix this, we incrementally flip any hooks before or after any particular flipped hook that caused an issue until the array gives a valid partition. The hooks that are flipped in this reversal are exactly those that were flipped from $\mu$ to obtain $\overline{\mu}$. $\square$

*Example* 4.3.27. To illustrate the map from the proof, consider

$$\lambda = \left( \begin{smallmatrix} 28, & 26, & 25, & 17, & 14, & 12, & 4, & 3, & 0 \\ 31, & 23, & 22, & 19, & 16, & 10, & 7, & 3, & 1 \end{smallmatrix} \right),$$

which gives

$$\overline{\lambda} = \left( \begin{smallmatrix} 31, & 26, & 25, & 19, & 16, & 12, & 7, & 3, & 1 \\ 28, & 23, & 22, & 17, & 14, & 10, & 4, & 3, & 0 \end{smallmatrix} \right).$$

Consider the following partition $\mu \prec \lambda$:

$$\mu = \left( \begin{smallmatrix} 22, & 20, & 19, & 17, & 11, & 7, & 4, & 2, & 0 \\ 29, & 22, & 21, & 17, & 13, & 9, & 7, & 2, & 1 \end{smallmatrix} \right).$$

We show in detail how to obtain $\overline{\mu}$, emphasizing the flipped hooks for clarity. We first flip the hooks of $\mu$ that do not fit inside the corresponding hooks of $\overline{\lambda}$:

$$\left( \begin{smallmatrix} \mathbf{29}, & 20, & 19, & 17, & 11, & 7, & \mathbf{7}, & 2, & \mathbf{1} \\ \mathbf{22}, & 22, & 21, & 17, & 13, & 9, & \mathbf{4}, & 2, & \mathbf{0} \end{smallmatrix} \right).$$

Next we iteratively flip adjacent hooks until the result is a valid partition:

$$\left( \begin{smallmatrix} 29, & \mathbf{22}, & 19, & 17, & 11, & \mathbf{9}, & 7, & 2, & 1 \\ 22, & \mathbf{20}, & 21, & 17, & 13, & \mathbf{7}, & 4, & 2, & 0 \end{smallmatrix} \right),$$

$$\left( \begin{smallmatrix} 29, & 22, & \mathbf{21}, & 17, & 11, & 9, & 7, & 2, & 1 \\ 22, & 20, & \mathbf{19}, & 17, & 13, & 7, & 4, & 2, & 0 \end{smallmatrix} \right),$$

so we conclude that

$$\overline{\mu} = \left( \begin{smallmatrix} 29, & 22, & 21, & 17, & 11, & 9, & 7, & 2, & 1 \\ 22, & 20, & 19, & 17, & 13, & 7, & 4, & 2, & 0 \end{smallmatrix} \right).$$

For completeness' sake, we show how to go backwards to recover $\mu$. We first flip the hooks of $\overline{\mu}$ that do not fit in $\lambda$ and proceed from there:

$$\left(\begin{smallmatrix} \mathbf{22}, & 22, & 21, & 17, & 11, & 9, & \mathbf{4}, & 2, & \mathbf{0} \\ \mathbf{29}, & 20, & 19, & 17, & 13, & 7, & \mathbf{7}, & 2, & \mathbf{1} \end{smallmatrix}\right),$$

$$\left(\begin{smallmatrix} 22, & \mathbf{20}, & 21, & 17, & 11, & \mathbf{7}, & 4, & 2, & 0 \\ 29, & \mathbf{22}, & 19, & 17, & 13, & \mathbf{9}, & 7, & 2, & 1 \end{smallmatrix}\right),$$

$$\left(\begin{smallmatrix} 22, & 20, & \mathbf{19}, & 17, & 11, & 7, & 4, & 2, & 0 \\ 29, & 22, & \mathbf{21}, & 17, & 13, & 9, & 7, & 2, & 1 \end{smallmatrix}\right),$$

which is $\mu$.

As a final point of interest, note that the injection described here is not generally a bijection. For example, the partition $\left(\begin{smallmatrix} 31, & 25 \\ 24 & 23 \end{smallmatrix}\right)$ fits inside $\overline{\lambda}$, but attempting to find a preimage by reversing the map from the proof gives $\left(\begin{smallmatrix} 24, & 23 \\ 31, & 25 \end{smallmatrix}\right)$, which is not in $\lambda$.

We now consider a certain refinement (which we call the *principal hook refinement*) of partition generating functions. For a partition $\lambda = \left(\begin{smallmatrix} A_k & A_{k-1} & \cdots & A_1 \\ B_k & B_{k-1} & \cdots & B_1 \end{smallmatrix}\right)$, we can write the generating function $G_\lambda$ as

$$1+qG_{(A_k-1)}G_{(B_k-1)}+q^4G_{(A_k-2,\,A_{k-1}-1)}G_{(B_k-2,\,B_{k-1}-1)}+\ldots+q^{k^2}G_{(A_k-k,\ldots,A_1-1)}G_{(B_k-k,\ldots,B_1-1)}.$$

The polynomial associated with the term $q^{i^2}$ in the expansion above is

$$q^{i^2}G_{(A_k-i,\,A_{k-1}-(i-1)\ldots,A_{k-i-1}-1)}G_{(B_k-i,\,B_{k-1}-(i-1)\ldots,B_{k-i-1}-1)}.$$

This term counts the partitions inside $\lambda$ having Durfee square of size $i$ (that is, having exactly $i$ principal hooks).

*Example* 4.3.28. The generating function for $\left(\begin{smallmatrix}10 & 8 & 6 & 4\\ 9 & 6 & 5 & 2\end{smallmatrix}\right)$ can be written as

$$G_{\left(\begin{smallmatrix}10 & 8 & 6 & 4\\ 9 & 6 & 5 & 2\end{smallmatrix}\right)} = 1 + qG_{(9)}G_{(8)} + q^4G_{(8,7)}G_{(7,5)} + q^9G_{(7,6,5)}G_{(6,4,4)} + q^{16}G_{(6,5,4,3)}G_{(5,3,3,1)}.$$

**Proposition 4.3.29.** *For any partitions* $\mu = (\mu_1, \ldots, \mu_{\ell_1})$ *and* $\eta = (\eta_1, \ldots, \eta_{\ell_2})$ *(where without loss of generality* $\ell_1 \geq \ell_2$*), let* $\tilde{\mu}_j$ *and* $\tilde{\eta}_j$ *be as follows:*

$$(\tilde{\mu}_j, \tilde{\eta}_j) = \begin{cases} (\mu_j - 1,\, \eta_j + 1) & \text{if } \mu_j > \eta_j + 1 \\ (\mu_j + 1,\, \eta_j - 1) & \text{if } \mu_j < \eta_j + 1 \\ (\mu_j,\, \eta_j) & \text{otherwise.} \end{cases}$$

*(If* $\ell_1 > \ell_2$*, we define* $\eta_{\ell_2+1} = \eta_{\ell_2+2} = \cdots = \eta_{\ell_1} = 0$*.) Then*

$$G_\mu G_\eta \leq G_{(\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_{\ell_1})}G_{(\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_{\ell_1})}.$$

*Proof.* Let $\lambda$ be any partition whose $q^{i^2}$ term in the principal hook refinement is

$q^{i^2}G_\mu G_\eta$. Note that the $q^{i^2}$ term in the refinement of $\tilde{\lambda}$ will be

$$q^{i^2}G_{(\tilde{\mu}_1,\tilde{\mu}_2,\ldots,\tilde{\mu}_{\ell_1})}G_{(\tilde{\eta}_1,\tilde{\eta}_2,\ldots,\tilde{\eta}_{\ell_1})}.$$

These polynomials are the generating functions for partitions inside $\lambda$ and $\tilde{\lambda}$, respectively having $i$ principal hooks.

Recall that the First Balancing Theorem 4.3.2 implies that there is an injection from partitions $\mu$ in $\lambda$ to partitions $\tilde{\mu}$ in $\tilde{\lambda}$ such that $|\mu| = |\tilde{\mu}|$. This followed from Lemma 4.3.1, which more strongly guarantees that this injection also preserves the number and sizes of principal hooks. Thus we have the desired inequality. $\square$

**Proposition 4.3.30.** *For any partitions $\mu = (\mu_1, \ldots, \mu_{\ell_1})$ and $\eta = (\eta_1, \ldots, \eta_{\ell_2})$ (where without loss of generality $\ell_1 \geq \ell_2$), let $\overline{\mu}_i = \max(\mu_i, \eta_i)$ and $\overline{\eta}_i = \min(\mu_i, \eta_i)$. Then*

$$G_\mu G_\eta \leq G_{(\overline{\mu}_1,\overline{\mu}_2,\ldots,\overline{\mu}_{\ell_1})}G_{(\overline{\eta}_1,\overline{\eta}_2,\ldots,\overline{\eta}_{\ell_1})}.$$

The proof of Proposition 4.3.30 (using Lemma 4.3.24) is completely analogous to the proof of Proposition 4.3.29, and is therefore omitted.

## 4.4 Non-conjugate Partitions With the Same Generating Function

In this section, we consider the size of $P_n$: the number of distinct generating functions of partitions of $n$. As we have seen before, any partition and its conjugate have the same generating function. Representing the number of self-conjugate partitions of $n$ by $\mathrm{sc}(n)$, we have that

$$|P_n| \leq \frac{p(n) + \mathrm{sc}(n)}{2}.$$

As we will see, it is also possible for two *non-conjugate* partitions $\lambda$ and $\eta$ to share the same generating function. Despite the existence of such partitions, computational evidence suggests that the number of such examples is negligible with respect to the total number of partitions of $n$. In other words, we expect that $|P_n| \sim \frac{p(n)+\mathrm{sc}(n)}{2}$. (We say that two arbitrary functions $f(n)$ and $g(n)$ are *asymptotic* and write $f(n) \sim g(n)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$.)

The asymptotic formula for $p(n)$, due to G. Hardy and S. Ramanujan [3, 14], is a famous result from partition theory:

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}.$$

M. Wildon proved in [40] that

$$\mathrm{sc}(n) \sim \frac{e^{\pi\sqrt{n/6}}}{2^{7/4}3^{1/4}n^{3/4}}.$$

Since $\sqrt{n/6} < \sqrt{2n/3}$, this easily implies the asymptotic

$$p(n) + \mathrm{sc}(n) \sim p(n).$$

This leads us to conjecture the following:

**Conjecture 4.4.1.**

$$|P_n| \sim \frac{p(n)}{2}$$

We now proceed to give an infinite family of pairs of non-conjugate partitions having the same generating function.

**Proposition 4.4.2.** *Let $m$, $k$, $i \in \mathbb{P}$ with $i \le m$, and let $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ be a binary vector. Then setting*

$$\lambda = (km-1,\, (k-1)m-1, \ldots, 2m-1,\, m-1,\, i-1) + i(x_1,\, x_2, \ldots, x_{k-1},\, x_k,\, 0)$$

*and*

$$\eta = (km-1,\, (k-1)m-1, \ldots, 2m-1,\, m-1,\, i-1) + i(x_k,\, x_{k-1}, \ldots, x_2,\, x_1,\, 0),$$

*we have* $G_\lambda = G_\eta$.

In the case when $i = 1$, we simply ignore the part of size 0 at the end of the partition. Note that the condition $i \leq m$ guarantees that $\lambda$ and $\eta$ are indeed partitions. Before proving Proposition 4.4.2, we give a quick example:

*Example* 4.4.3. Let $k = 5$, $m = 3$, $i = 2$ and $(x_1, x_2, x_3, x_4, x_5, 0) = (1, 0, 1, 1, 0, 0)$. Then we have

$$\lambda = (14, 11, 8, 5, 2, 1) + 2(1, 0, 1, 1, 0, 0) = (16, 11, 10, 7, 2, 1)$$

and

$$\eta = (14, 11, 8, 5, 2, 1) + 2(0, 1, 1, 0, 1, 0) = (14, 13, 10, 5, 4, 1)$$

The associated Ferrers shapes are:

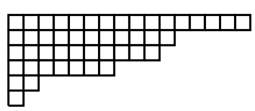
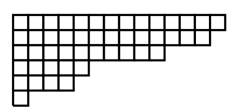
$\lambda :$  $\eta :$  .

**Figure 4.13:** Ferrers diagrams for $(16, 11, 10, 7, 2, 1)$ and $(14, 13, 10, 5, 4, 1)$

It can be easily verified by a computer that

$$G_{(16,11,10,7,2,1)} = G_{(14,13,10,5,4,1)}.$$

*Proof of Proposition 4.4.2.* For simplicity, let $\mu = (km-1,\, (k-1)m-1, \ldots, 2m-1,\, m-1,\, i-1)$ denote the "base" partition to which we are adding the vector of 0's and $i$'s.

If $\mathbf{x} = \mathbf{0}$, then $\lambda = \eta$ and the result is trivial. So we may assume that $\ell > 0$ entries of $\mathbf{x}$ are nonzero: $x_{j_1}, x_{j_2}, \ldots, x_{j_\ell}$. For any subset $\{j'_1, j'_2, \ldots, j'_s\}$ of $\{j_1, j_2, \ldots, j_\ell\}$, there are partitions in $\lambda$ (and $\eta$) that "extend beyond" $\mu$ in exactly rows $j'_1, j'_2, \ldots, j'_s$. We will demonstrate that $\lambda$ and $\eta$ both have an equal number of such partitions. Thus we can expand the generating functions for $\lambda$ and $\eta$ into $2^\ell$ matching components, which gives $G_\lambda = G_\eta$.

Consider the partitions fitting inside $\lambda$ that extend beyond $\mu$ in exactly $s$ parts: $j_1, j_2, \ldots, j_s$ (that is, precisely the entries $x_{j_1}, x_{j_2}, \ldots, x_{j_s}$ in $\mathbf{x}$ are nonzero). Similarly, consider the partitions in $\eta$ that extend beyond $\mu$ in parts $j_1, j_2, \ldots, j_s$ (appearing in reverse order in $\eta$). It is not difficult to see (by the symmetry of the part sizes) that the contribution from these two classes of partitions to $G_\lambda$ and $G_\eta$, respectively, is the same.

Indeed, the associated partitions in $\lambda$ contribute to $G_\lambda$ the following amount:

$$q^M \prod_{t=0}^{s} G_{((|j'_{t+1}-j'_t|-1)m-1,\, (|j'_{t+1}-j'_t|-2)m-1, \ldots, 2m-1,\, m-1,\, i-1)}, \tag{4.7}$$

where

$$M = m(j_1'(k - j_1' + 1) + (j_2' - j_1')(k - j_2' + 1) + \ldots + (j_s' - j_{s-1}')(k - j_s' + 1))$$

and we define $j_0' = 0$ and $j_{s+1}' = k + 1$.

By construction, replacing each $j_t'$ above with $k - j_t' + 1$ produces the associated partitions in $\eta$ (since these indices label rows of $\eta$ in reverse order). Careful inspection shows that the corresponding portion of the generating function contributing to $G_\eta$ is the same as (4.7).

For example, consider $k = 12$, $m = 5$, $i = 4$ together with the binary vector

$$(1, 1, 0, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0).$$

This yields the partitions

$$\lambda = (63, 58, 49, 44, 43, 38, 33, 24, 23, 18, 13, 8, 3)$$

and

$$\eta = (63, 58, 53, 48, 39, 38, 33, 28, 19, 14, 13, 8, 3).$$

The partitions in $\lambda$ having part sizes that extend beyond $\mu$ in exactly rows 2, 6, 11

and 12 are counted by

$$q^{305} G_{(4,3)} G_{(14,9,4,3)} G_{(19,14,9,4,3)} G_{(3)} G_{(3)}, \tag{4.8}$$

as illustrated in the Ferrers shape below. (For clarity, the base partition $\mu$ is highlighted in light gray. All dark gray cells within the bold border must contribute to the partitions being counted, while none of the cells with an "×" can contribute.)

$\lambda$ :

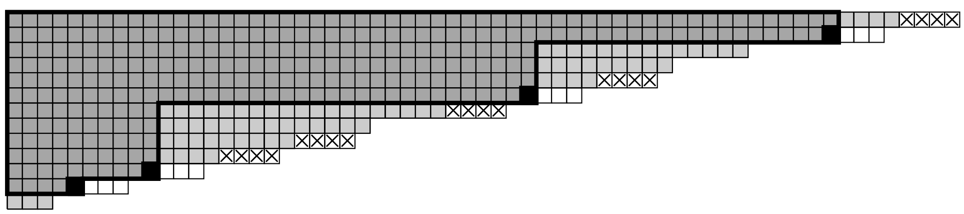


**Figure 4.14:** Counting $q^{305} G_{(4,3)} G_{(14,9,4,3)} G_{(19,14,9,4,3)} G_{(3)} G_{(3)}$ in $\lambda$

Similarly, the partitions in $\eta$ having part sizes that extend beyond $\mu$ in the "symmetric" rows 1, 2, 7 and 11 are also counted by (4.8), as illustrated below:

$\eta$ :

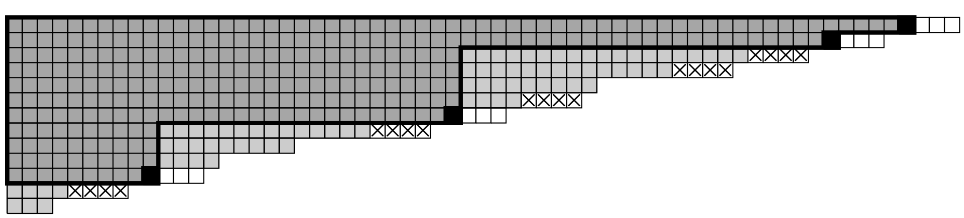


**Figure 4.15:** Counting $q^{305} G_{(4,3)} G_{(14,9,4,3)} G_{(19,14,9,4,3)} G_{(3)} G_{(3)}$ in $\eta$

As an example of the complete expansion, the generating functions for $\lambda$ and $\eta$ from

Example 4.4.3 can be expressed as

$$G_\lambda = G_{(14,11,8,5,2,1)} + q^{15}G_{(1)}G_{(11,8,5,2,1)} + q^{27}G_{(5,2,1)}G_{(5,2,1)} + q^{24}G_{(8,5,2,1)}G_{(2,1)}$$

$$+q^{33}G_{(1)}G_{(2,1)}G_{(5,2,1)} + q^{33}G_{(1)}G_{(5,2,1)}G_{(2,1)} + q^{33}G_{(5,2,1)}G_{(1)}G_{(2,1)}$$

$$+q^{39}G_{(1)}G_{(2,1)}G_{(1)}G_{(2,1)}$$

and

$$G_\eta = G_{(14,11,8,5,2,1)} + q^{24}G_{(2,1)}G_{(8,5,2,1)} + q^{27}G_{(5,2,1)}G_{(5,2,1)} + q^{15}G_{(11,8,5,2,1)}G_{(1)}$$

$$+q^{33}G_{(2,1)}G_{(1)}G_{(5,2,1)} + q^{33}G_{(2,1)}G_{(5,2,1)}G_{(1)} + q^{33}G_{(5,2,1)}G_{(2,1)}G_{(1)}$$

$$+q^{39}G_{(2,1)}G_{(1)}G_{(2,1)}G_{(2,1)}.$$

□

We only remark here that it can easily be shown that the family given above does not affect the asymptotics of $|P_n|$ as speculated in Conjecture 4.4.1. However, this family does not cover all instances when two non-conjugate partitions have the same generating function. For instance, computational evidence suggests that

$$G_{\left((2m+2j+i)^j,\,(m+j)^{j-1},\,(m+j-1)^{i+1}\right)} = G_{\left((2m+2j+i)^{j-1},\,2m+2j,\,(m+j)^{j+i}\right)}$$

for any $i$, $j$, $m \in \mathbb{P}$.

An exhaustive classification of all such cases of distinct partitions $\lambda$ and $\eta$ where $G_\lambda = G_\eta$ (and $\lambda' \neq \eta$) seems quite difficult. Despite this, computational evidence suggests that such cases are negligible with respect to the asymptotics of $|P_n|$, as given in Conjecture 4.4.1.

## 4.5 Maxima in $P_n$

In this section, we study the number of maxima in $P_n$, which we denote by $M_n$. From the Balancing Theorems (4.3.2 and 4.3.25), we have a necessary condition on the structure of partitions that yield maxima in $P_n$. This restriction on possible maxima gives an upper bound on $M_n$.

Later in this section, we will describe a certain infinite family of partitions of $n$ that gives maximal generating functions in $P_n$. In particular, this will show that $M_n \gg n^{1/4}$, which implies $\lim_{n\to\infty} M_n = \infty$ (if $f$ and $g$ are positive value functions, $f(n) \gg g(n)$ means $f(n) \geq Cg(n)$ for some positive constant $C$ and all $n$ large).

### 4.5.1 An Upper Bound On $M_n$

For any positive integer $n$, let $\mathrm{sd}(n)$ be the number of "super-distinct" partitions of $n$, that is, partitions of $n$ with consecutive parts differing by at least 2 (see [4, p. 23]). Also, let $\mathrm{Bal}_n$ be the set of completely balanced (and all offset in the same direction) partitions of $n$. In other words,

$$\mathrm{Bal}_n = \{ \left( \begin{smallmatrix} A_k & A_{k-1} & \cdots & A_1 \\ B_k & B_{k-1} & \cdots & B_1 \end{smallmatrix} \right) \vdash n \,|\, A_i = B_i \text{ or } A_i = B_i + 1 \text{ for all } i \}.$$

By the Balancing Theorems (4.3.2 and 4.3.25), any maximum in $P_n$ must be the generating function of some partition in $\mathrm{Bal}_n$. In other words, $M_n \leq |\mathrm{Bal}_n|$. We will use this to prove the following proposition.

**Proposition 4.5.1.**

$$M_n \leq \mathrm{sd}(n).$$

*Proof.* It suffices to show that $|\mathrm{Bal}_n| = \mathrm{sd}(n)$. This follows from the following bijective correspondence between partitions counted by $\mathrm{sd}(n)$ and partitions in $\mathrm{Bal}_n$, respectively:

$$(\eta_1, \ldots, \eta_\kappa) \longleftrightarrow \begin{pmatrix} \left\lceil \frac{\eta_1 - 1}{2} \right\rceil & \left\lceil \frac{\eta_2 - 1}{2} \right\rceil & \cdots & \left\lceil \frac{\eta_\kappa - 1}{2} \right\rceil \\ \left\lfloor \frac{\eta_1 - 1}{2} \right\rfloor & \left\lfloor \frac{\eta_2 - 1}{2} \right\rfloor & \cdots & \left\lfloor \frac{\eta_\kappa - 1}{2} \right\rfloor \end{pmatrix}.$$

□

*Example* 4.5.2. We illustrate this bijection for clarity sake. Consider $(18, 14, 11, 9, 4, 1)$, a partition in $\mathrm{sd}(57)$. Subtract one from each part size and divide by two to form the columns of the Frobenius notation for the completely balanced partition in $\mathrm{Bal}_{57}$:

$$(18, 14, 11, 9, 4, 1) \qquad \longmapsto \qquad \left(\begin{smallmatrix} 9 & 7 & 5 & 4 & 2 & 0 \\ 8 & 6 & 5 & 4 & 1 & 0 \end{smallmatrix}\right)$$

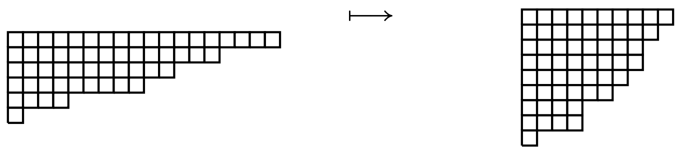


**Figure 4.16:** Correspondence between $(18, 14, 11, 9, 4, 1)$ and $\left(\begin{smallmatrix} 9 & 7 & 5 & 4 & 2 & 0 \\ 8 & 6 & 5 & 4 & 1 & 0 \end{smallmatrix}\right)$

To illustrate the other direction, consider $\left(\begin{smallmatrix} 13 & 10 & 5 & 1 & 0 \\ 12 & 9 & 4 & 1 & 0 \end{smallmatrix}\right)$ in $\mathrm{Bal}_{60}$. Sum the columns and add one to each result to get the parts of the corresponding partition in $\mathrm{sd}(60)$:

$$\left(\begin{smallmatrix} 13 & 10 & 5 & 1 & 0 \\ 12 & 9 & 4 & 1 & 0 \end{smallmatrix}\right) \qquad \longmapsto \qquad (26, 20, 10, 3, 1)$$

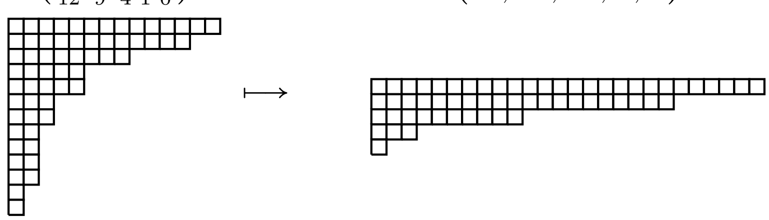


**Figure 4.17:** Correspondence between $\left(\begin{smallmatrix} 13 & 10 & 5 & 1 & 0 \\ 12 & 9 & 4 & 1 & 0 \end{smallmatrix}\right)$ and $(26, 20, 10, 3, 1)$

The first Rogers-Ramanujan identity (see for example [3]) asserts that $\mathrm{sd}(n)$ is equal to the number of partitions of $n$ with parts equal to 1 or 4 (mod 5). So the generating

function for $\mathrm{sd}(n)$ is

$$\sum_{n\geq 0} \mathrm{sd}(n) q^n = \prod_{i=0}^{\infty} \frac{1}{(1-q^{5i+1})(1-q^{5i+4})}.$$

M. Lugo showed in [21] that

$$\mathrm{sd}(n) \sim \frac{\csc(\pi/5)}{4\pi \cdot 15^{1/4}} n^{-3/4} e^{\pi\sqrt{4n/15}}.$$

Recall the well-known asymptotics for $p(n)$, the partition function [3, 14]:

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}.$$

Thus if we assume Conjecture 4.4.1 from the previous section, then $M_n$ is negligible with respect to $|P_n|$. In other words, we have

$$\frac{M_n}{|P_n|} \leq \frac{\mathrm{sd}(n)}{|P_n|} \ll \frac{\mathrm{sd}(n)}{p(n)} \ll \frac{n^{1/4} e^{\pi\sqrt{4n/15}}}{e^{\pi\sqrt{2n/3}}}.$$

Then since $\sqrt{4n/15} < \sqrt{2n/3}$, this discussion proves the following fact:

**Theorem 4.5.3.** *If Conjecture 4.4.1 is true, then*

$$\lim_{n\to\infty} \frac{M_n}{|P_n|} = 0.$$

### 4.5.2 A Lower Bound On $M_n$

We now show that the number of maxima in $P_n$ tends to infinity as $n$ grows. We begin by showing how one particular partition yields a maximum in $P_n$, and then discuss how to generate more maxima from this starting point.

For any positive integer $n$, it is easy to see that there exist unique positive integers $k$ and $m$ such that

$$\binom{k+2}{2} - 3 \leq n \leq \binom{k+3}{2} - 4,$$

where $n$ can be written in the form

$$n = \binom{k+1}{2} + m = \binom{k+1}{2} + \left\lfloor \frac{m}{2} \right\rfloor + \left\lceil \frac{m}{2} \right\rceil.$$

Indeed, from the inequality above we can determine that $k = \left\lfloor \frac{\sqrt{8n+25}-3}{2} \right\rfloor$, and we also have that $k - 2 \leq m \leq 2k - 1$.

**Lemma 4.5.4.** *For $n \in \mathbb{P}$, let $k$ and $m$ be as defined above. Then the partition*

$$\Lambda = \left(k + \left\lceil \frac{m}{2} \right\rceil, k-1, k-2, \ldots, 3, 2, 1^{\lfloor \frac{m}{2} \rfloor + 1}\right)$$

*yields a maximum in $P_n$, where $1^{\lfloor \frac{m}{2} \rfloor + 1}$ denotes the part of size 1 repeated $\left\lfloor \frac{m}{2} \right\rfloor + 1$ times.*

*Proof.* Note that the shape of $\Lambda$ is that of one large outer hook enclosing a "staircase" of size $k-2$. We observe two facts about $G_\Lambda$:

- Since $\Lambda$ has $k$ outer corners, the penultimate term of $G_\Lambda$ is $kq^{n-1}$.
- Since $m \le 2k-1$, it is not hard to see that the coefficients of $G_\Lambda$ match the partition function up to and including degree $k + \left\lfloor \frac{m}{2} \right\rfloor$ (the number of parts of $\Lambda$). Furthermore, in the case when $m$ is odd, the coefficient of $q^{k+\lceil \frac{m}{2} \rceil}$ in $G_\Lambda$ is $p\left(k + \left\lceil \frac{m}{2} \right\rceil\right) - 1$.

We claim that $\Lambda$ is the only partition of $n$ – up to conjugation – that satisfies these two conditions. Indeed, in order to satisfy the second condition, the arm of the outer principal hook must be at least $k + \left\lceil \frac{m}{2} \right\rceil - 1$, and leg at least $k + \left\lfloor \frac{m}{2} \right\rfloor - 1$ (recall that the corner of the hook is not counted). Sharply meeting these bounds leaves us with $\binom{k-1}{2}$ more cells for the portion of the partition within the outer hook. In order to satisfy the first condition, the shape within the outer hook must have $k-2$ distinct part sizes. This can only be obtained by constructing the $k-2$ staircase within the outer hook, so $\Lambda$ is uniquely determined. It is clear from this discussion that no other partition of $n$ can have a coefficient at least $k$ in the penultimate term and simultaneously satisfy the second condition. Thus $\Lambda$ yields a maximum in $P_n$.

$\square$

Our next goal is to show how more maxima in $P_n$ can be generated starting from this partition $\Lambda$. To do this, we require the concept of *double outer corners* of a partition. A double outer corner is a pair of two adjacent cells in a partition such that, if both are removed, a valid Ferrers diagram will be left. Therefore, a double outer corner can arise either from a part that is at least 2 larger than the next part, or from a repeated part size.

This concept is useful because it determines how many partitions of $n-2$ fit inside a given partition of $n$. In particular, if some partition has $k$ outer corners and $m$ double outer corners, then the coefficient of the *antepenultimate* (third to last) degree in the generating function is $\binom{k}{2}+m$.

*Example* 4.5.5. The partition of $n = 217$ whose Ferrers diagram is shown below has $k = 10$ outer corners and $m = 6 + 5 = 11$ double outer corners.

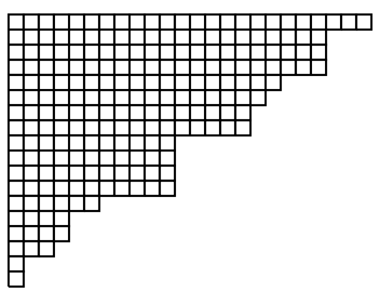


**Figure 4.18:** A partition with 10 outer corners and 11 double outer corners

Hence the number of partitions of size $n - 2 = 215$ that fit inside this partition (i.e., the coefficient of the $q^{215}$ term in the generating function) is $\binom{10}{2} + 11 = 56$.

**Lemma 4.5.6.** *For nonnegative integers $k$ and $t$ with $2t < k$, the smallest partition having $k$ outer corners and $2t$ double outer corners is*

$$\left(\begin{smallmatrix} k+t-1, & k+t-4, & k+t-7, & \cdots & ,k-2t-1, & k-2t-3, & k-2t-5, & \cdots \\ k+t-1, & k+t-4, & k+t-7, & \cdots & ,k-2t-1, & k-2t-3, & k-2t-5, & \cdots \end{smallmatrix}\right).$$

The last hook of this partition is either $\binom{1}{1}$ or $\binom{0}{0}$, depending on the parity of $k$. Before proving Lemma 4.5.6, we give a small example.

*Example* 4.5.7. The smallest partition having $k = 13$ outer corners and $2t = 8$ double outer corners is $\left(\begin{smallmatrix} 16 & 13 & 10 & 7 & 4 & 2 & 0 \\ 16 & 13 & 10 & 7 & 4 & 2 & 0 \end{smallmatrix}\right)$, of size 111.

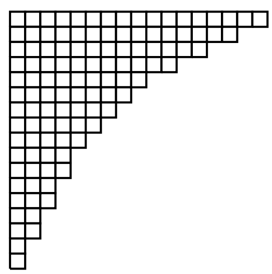


**Figure 4.19:** The smallest partition with 13 outer corners and 8 double outer corners

In particular, no other partition of size less than or equal to 111 produces a generating function that simultaneously has a penultimate term of $k = 13$ and an antepenultimate term of $\binom{k}{2} + 2t = 86$.

*Proof of Lemma 4.5.6.* Let $\eta$ be a partition of smallest size having $k$ outer corners

and $2t$ double outer corners. We do not assume a priori that $\eta$ is unique.

By minimality, we may assume that no part of $\eta$ is repeated three or more times, since removing all but two of a repeated part size does not alter the number of outer (or double outer) corners. Analogously, no part is more than two larger than the subsequent part.

Furthermore, if some part $\eta_i$ is repeated, then all parts smaller than $\eta_i$ are also repeated. Otherwise, we could reduce the size of the partition by eliminating one of the parts of size $\eta_i$ and instead repeating the smaller part. (This reduction clearly preserves the number of outer and double outer corners.) Analogously, if some part $\eta_j$ is two more than the subsequent part, then each of the parts greater than $\eta_j$ is also two more than the subsequent parts.

By the preceding paragraphs and under the assumption that $2t < k$, the first part of the partition is distinct and the last part is 1. Thus for some $r, s \geq 0$ where $r + s = 2t$, each of the first $r$ parts of $\eta$ are exactly two larger than the next part, and the last $2s$ parts of $\eta$ are $(s, s, s-1, s-1, \ldots, 2, 2, 1, 1)$. Since $\eta$ has exactly $k$ distinct part sizes, it follows from this discussion that $\eta$ is of the form

$$(\underbrace{k+r,\, k+r-2, \ldots, k-r+2}_{r},\, \underbrace{k-r,\, k-r-1, \ldots, s+1}_{k-2t},\, \underbrace{s,\, s,\, s-1,\, s-1, \ldots, 2,\, 2,\, 1,\, 1}_{2s}).$$

Hence the total size of $\eta$ is

$$|\eta| = 2\binom{s+1}{2} + s(k-2t) + \binom{k-2t+1}{2} + r(k-r) + 2\binom{r+1}{2}.$$

Replacing $s$ with $2t-r$ and simplifying, we obtain

$$|\eta| = r^2 - (2t)r + 2t(t+1) + \binom{k+1}{2}.$$

Regarding $k$ and $t$ as fixed and treating this expression as a quadratic in $r$, it is easy to see that $|\eta|$ is minimized precisely when $r = t$. This implies that there is a *unique* partition of smallest size having $k$ outer corners and $2t$ double outer corners. It is easily checked that setting $s = r = t$ gives the partition in Lemma 4.5.6. $\square$

We are now in a position to state the following:

**Proposition 4.5.8.** *The poset $P_n$ has at least $\left\lfloor \frac{\sqrt{8\lfloor \frac{k-2}{2} \rfloor + 1} + 1}{2} \right\rfloor - 1$ maxima, where $k = \left\lfloor \frac{\sqrt{8n+25}-3}{2} \right\rfloor$.*

Before proving Proposition 4.5.8, we give an important corollary that follows immediately from it.

**Corollary 4.5.9.** *The number of maxima in $P_n$ goes to infinity as $n$ increases. That is,*

$$\lim_{n\to\infty} M_n = \infty.$$

*Proof.* From Proposition 4.5.8, there are at least asymptotically $(2n)^{1/4}$ maxima in $P_n$. $\square$

*Proof of Proposition 4.5.8.* Let

$$\Lambda = \left(k + \left\lceil \frac{m}{2} \right\rceil,\ k-1,\ k-2, \ldots, 3,\ 2,\ 1^{\lfloor \frac{m}{2} \rfloor + 1}\right),$$

as in Lemma 4.5.4. We construct $\ell = \left\lfloor \frac{\sqrt{8\lfloor \frac{k-2}{2} \rfloor + 1} + 1}{2} \right\rfloor - 1$ partitions: $\Lambda_0, \ldots, \Lambda_{\ell-1}$, that yield maxima in $P_n$. This is done by an iterative process on $\Lambda$ of removing cells from the outermost (principal) hook and distributing these cells to the interior $k-2$ staircase.

We first take $\Lambda_0 = \Lambda$. Then form $\Lambda_1$ by removing one cell from both the arm and leg of the outer hook of $\Lambda_0$ and adding them to the second part and to the first part of size 1 in $\Lambda_0$.

Next, form $\Lambda_2$ by removing two cells from both the arm and leg of the outer hook of $\Lambda_1$ and adding one to the second and third parts, and to the first parts of sizes 2 and 1 in $\Lambda_1$.

In general, the partition $\Lambda_j$ is formed by removing $j$ cells from both the arm and leg of the outer hook of $\Lambda_{j-1}$. Each of the first $j$ parts of the inner staircase is increased by one, as is each of the first occurrences of parts of size $j, j-1, \ldots, 1$ in $\Lambda_{j-1}$.

The generating function of $\Lambda_j$ satisfies the following:

- $G_{\Lambda_j}$ matches the partition function up to degree $k + \lfloor \frac{m}{2} \rfloor - \binom{j+1}{2}$, and is one less than the partition function in degree $k + \lceil \frac{m}{2} \rceil - \binom{j+1}{2}$.

- $G_{\Lambda_j}$ has coefficient $k$ in the penultimate term.

- $G_{\Lambda_j}$ has coefficient $\binom{k}{2} + 2j + 2$ in the antepenultimate term.

No other partition of $n$ (up to conjugation) satisfies these three conditions. To see this, note that the first condition above dictates precisely what the outer principal hook of the partition is. In order to satisfy the last two conditions, the portion of the partition within the outer hook must have $k - 2$ outer corners and $2j$ double outer corners. This interior shape as constructed above is precisely the modified staircase shape demanded by Lemma 4.5.6. Thus no other partition of $n$ (up to conjugation) satisfies these three conditions. □

*Example* 4.5.10. Applying the argument given above to $P_{87}$ produces four partitions corresponding to maxima, whose Ferrers diagrams are illustrated below.

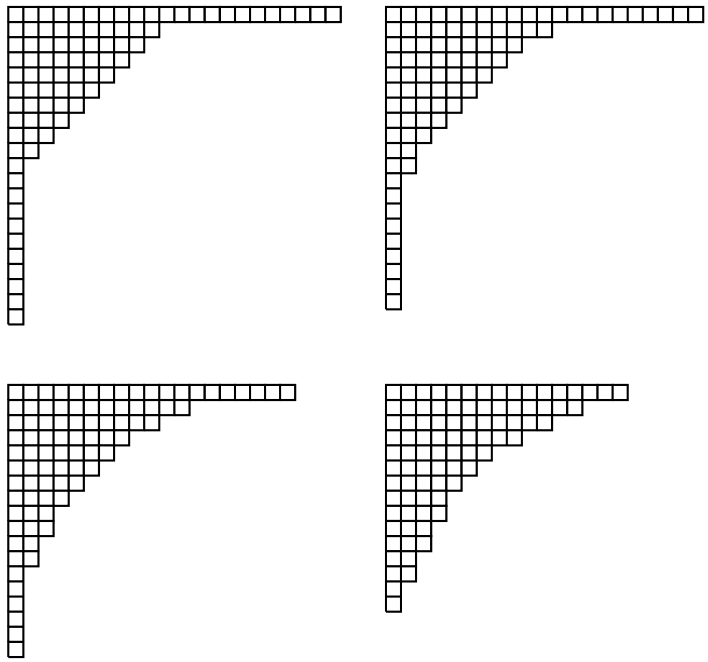


**Figure 4.20:** Some partitions yielding maxima in $P_{87}$

# Chapter 5

# Conclusion

In this brief chapter, we summarize the main results of this dissertation and propose questions for future work. Rather than redefine the terms and notation in the following discussions, we refer the reader to the appropriate chapter for the context.

## 5.1 Results and Questions for Chapter 2

### 5.1.1 Summary of Results

In Section 2.2, we presented the Full Rank Lemma 2.2.1, which is a linear-algebraic generalization of Pouzet's Multicolor Theorem 2.3.4.

In Section 2.3, we demonstrated how the Multicolor Theorem can be applied to prove the flawlessness of several significant combinatorial sequences. In particular, we leveraged the Multicolor Theorem in the case of the order ideal generated by a single monomial. We then extended this to multiple generators to re-prove flawlessness of arbitrary pure $O$-sequences. Our next application was to the generating function of compositions embedded inside a larger composition. We then discussed how the Multicolor Theorem implies the flawlessness of $G_\lambda$, which had originally been observed by Pouzet. Finally, we discussed an analogy of multiple generators (as in the case of pure $O$-sequences) that carries over to other settings as well, including partitions.

### 5.1.2 Questions

In each case where we proved flawlessness, the full rank matrix $M$ we started with was the adjacency matrix from the boolean algebra.

It seems certain that further interesting flawlessness results can be obtained using the Full Rank Lemma 2.2.1 without relying on the adjacency matrix from the boolean algebra as the Multicolor Theorem 2.3.4 does. Broadly, we ask:

*Question* 5.1.1. What meaningful applications can be drawn from the Full Rank Lemma that do not follow from the Multicolor Theorem?

Our second question deals with a conjecture of Stanley and Zanello (Conjecture 3.9

in [35]).

*Question* 5.1.2. Can the Multicolor Theorem or the Full Rank Lemma be used to prove that the generating function associated with a partition *into distinct parts* is flawless?

Based on our efforts, it seems likely that working within the context of the Multicolor Theorem is not the appropriate approach to answering this question. We believe that the Full Rank Lemma could be applied to show this result, but to do this appears to require some new ideas.

As mentioned in Section 2.4.3, our definition of composition "embedding" is not the natural inclusion of compositions as defined by Andrews [2] and Sagan [28]. In particular, the generating functions of the associated posets do not coincide.

*Question* 5.1.3. Are the generating functions corresponding to the poset of compositions (under the Andrews-Sagan inclusion) also flawless?

Again we believe that the answer is yes, but suspect that the result falls under the Full Rank Lemma and outside of the Multicolor Theorem.

Arguably the most significant open problem discussed here is the following:

*Question* 5.1.4. Can the "half-way" chain decomposition that we discussed in Section 2.4.4 be manipulated to create a full symmetric chain decomposition of $L(a, b)$?

Finally, we observe that the extension from order ideals generated by a single monomial to pure $O$-sequences is analogous to the extension from the poset generated by a single partition to pure $P$-sequences. Pure $O$-sequences enjoy a rich connection to such areas as algebraic combinatorics and combinatorial commutative algebra. Given the interest in the poset $P_\lambda$ generated by a single partition $\lambda$, we ask:

*Question* 5.1.5. What can be said about the general properties of pure $P$-sequences, and what connections do they share with other areas of interest?

## 5.2 Results and Questions for Chapter 3

### 5.2.1 Summary of Results

In Chapter 3, we extended a technique due to Stanton to provide some new infinite families of partitions $\lambda$ where $G_\lambda$ is unimodal. In particular, we proved that any partition in 5 or 6 parts is unimodal if the difference between the first and last part is at most one, with the single exception of (10, 9, 9, 9, 9, 9). We also proved that any partition in 5 or 6 parts with part sizes $b$ and $2b$ is unimodal for any given $b \geq 1$.

### 5.2.2 Questions

We expect that more results along the lines of these theorems could be obtained by adaptations and/or generalizations of the technique discussed in the proof of Theorem 3.2.2. In particular, we ask:

*Question* 5.2.1. Can this technique be generalized to show the unimodality of any partition in $\kappa = 5$, 6 or 7 parts where $\lambda_1 - \lambda_\kappa$ is bounded by a suitable constant (with finitely many exceptions when $\kappa = 6$)?

We anticipate that this is feasible with more computations, but it would require some additional analysis of the peaks. More ambitiously, we wonder:

*Question* 5.2.2. Can this technique be generalized to show that *all* partitions (with finitely many exceptions when $\kappa = 6$) in $\kappa = 5$, 6 or 7 parts are unimodal?

A proof of this by the method discussed previously would likely break down into a vast number of potential cases. It also appears unlikely in this situation that the peaks could be explicitly found, thus requiring significant modification to the technique.

Therefore, applying this method to more general cases (multiple parameters, partitions with many parts, etc.) appears intractable. A new approach seems necessary in order to obtain more general unimodality results.

## 5.3 Results and Questions for Chapter 4

### 5.3.1 Summary of Results

In Chapter 4, we introduced the study of the poset $P_n$, a natural context for Bergeron's conjecture. In Section 4.2, we demonstrated that $P_n$ is not graded and gave a general overview of some of its structural properties. We explained that partitions with single hooks or only two parts give "small" generating functions, while heuristically, partitions with more outer corners tend to be "higher" in the poset.

Our most significant results on the structure of $P_n$ were the two Balancing Theorems (4.3.2 and 4.3.25) from Section 4.3. These asserted that certain balancing operations on the hooks of a partition will improve the generating function. Proving the First Balancing Theorem 4.3.2 involved translating the problem into the language of trellis diagrams. Ultimately, we reduced the desired properties of these trellises to claims about certain 0-1 matrices. Finally, analyzing and proving every possible case involving these matrices required the help of Mathematica's symbolic comparison tools. The proof of the Second Balancing Theorem 4.3.25 was a much simpler injective argument.

In Section 4.4, we considered some concepts related to the size of $P_n$. Based on

computational evidence, we gave the conjecture that the size of $P_n$ is asymptotically $\frac{p(n)}{2}$ (Conjecture 4.4.1). We provided an example of an infinite family of pairs of non-conjugate partitions that have the same generating function (Proposition 4.4.2). Other examples outside this family exist, but giving a complete classification seems to be very difficult. As we observed, such examples do not appear to affect the asymptotics of $|P_n|$.

Finally, Section 4.5 dealt with maxima in $P_n$. We demonstrated a neat connection to the first Rogers-Ramanujan identity, which gave an upper bound on $M_n$ (Proposition 4.5.1). Conditional to our conjecture on $|P_n|$, we showed that asymptotically 100% of the elements in $P_n$ are not maxima (Theorem 4.5.3). On the other hand, we gave a construction showing that $M_n$ goes to infinity as $n$ grows by bounding $M_n$ from below (Proposition 4.5.8).

### 5.3.2 Questions

Recall that the motivation behind the poset $P_n$ was Bergeron's conjecture. The First Balancing Theorem 4.3.2 in particular appears to be of a similar flavor to that conjecture.

*Question* 5.3.1. Can a modification or clever application of the First Balancing Theorem be useful for making progress towards proving Bergeron's positivity conjecture?

Recall that Lemma 4.3.1 trivially implies the First Balancing Theorem. The proof of this lemma – though it made use of some novel and intrinsically interesting concepts – was long and complicated. We would like to see a simpler, more elegant proof. For instance, we ask:

*Question* 5.3.2. Is there a combinatorial proof of Lemma 4.3.1?

In particular, it would be interesting to find an explicit injection that maps partitions $\mu \leq \lambda$ to $\tilde{\mu} \leq \tilde{\lambda}$ in which every principal hook of $\tilde{\mu}$ is at least as balanced as the corresponding hook of $\mu$.

One of the more interesting open problems arising from this work deals with the size of $P_n$. In Conjecture 4.4.1, we guessed that $|P_n| \sim \frac{p(n)}{2}$. This appears to be a difficult question to answer, and such a solution would be of general interest. A more complete study of the topics in Section 4.4 may help shed light on this conjecture. A positive answer to the following question would be a significant step in this direction.

*Question* 5.3.3. Is it possible to classify all pairs $\lambda, \eta$ where $\lambda' \neq \eta$ and $G_\lambda = G_\eta$?

Recall that Proposition 4.4.2 gave infinitely many *pairs* of non-conjugate partitions having the same generating function. We also found instances of more than two (all pairwise non-conjugate) partitions having this property. The smallest such example occurs when $n = 18$: $(10, 5, 2, 1)$, $(8, 5, 4, 1)$ and $(7, 6, 3, 2)$ all have the same generating function. We also found examples of 4 such partitions for larger $n$. This leads

us to ask:

*Question* 5.3.4. For $n \gg 0$, are there arbitrarily many (non-conjugate) partitions of $n$ that have the same generating function?

Finally, while we gave a lower bound and an upper bound (assuming Conjecture 4.4.1) on $M_n$, we have no guess as to an asymptotic estimate.

*Question* 5.3.5. What is the size of $M_n$?

# References


[1] Levent Alpoge, *Proof of a conjecture of Stanley-Zanello*, J. Combin. Theory Ser. A **125** (2014), 166–176.

[2] George E. Andrews, *A theorem on reciprocal polynomials with applications to permutations and compositions*, Amer. Math. Monthly **82** (1975), no. 8, 830–833.

[3] ______, *The theory of partitions*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998, Reprint of the 1976 original.

[4] George E. Andrews and Kimmo Eriksson, *Integer partitions*, Cambridge University Press, Cambridge, 2004.

[5] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[6] François Bergeron, *The q-Foulkes conjecture*, From talk given at Bowdoin College, ME on September 24, 2016. Slides available at `http://bergeron.math.uqam.ca/wp-content/uploads/2016/10/Foulkes_AMS_Small.pdf`.

[7] Mats Boij, Juan C. Migliore, Rosa M. Miró-Roig, Uwe Nagel, and Fabrizio Zanello, *On the shape of a pure O-sequence*, Mem. Amer. Math. Soc. **218** (2012), no. 1024, viii+78.

[8] ———, *On the weak Lefschetz property for Artinian Gorenstein algebras of codimension three*, J. Algebra **403** (2014), 48–68.

[9] Francesco Brenti, *Log-concave and unimodal sequences in algebra, combinatorics, and geometry: an update*, Jerusalem combinatorics '93, Contemp. Math., vol. 178, Amer. Math. Soc., Providence, RI, 1994, pp. 71–89.

[10] Manoj K. Chari, *Two decompositions in topological combinatorics with applications to matroid complexes*, Trans. Amer. Math. Soc. **349** (1997), no. 10, 3925–3943.

[11] Leonard Eugene Dickson, *History of the theory of numbers. Vol. II: Diophantine analysis*, Chelsea Publishing Co., New York, 1966.

[12] Roland Fraïssé, *Theory of relations*, revised ed., Studies in Logic and the Foundations of Mathematics, vol. 145, North-Holland Publishing Co., Amsterdam, 2000, With an appendix by Norbert Sauer.

[13] Mark S. Gockenbach, *Finite-dimensional linear algebra*, Discrete Mathematics and its Applications (Boca Raton), CRC Press, Boca Raton, FL, 2010.

[14] G. H. Hardy and S. Ramanujan, *Asymptotic Formulaae in Combinatory Analysis*, Proc. London Math. Soc. (2) **17** (1918), 75–115.

[15] Tamás Hausel, *Quaternionic geometry of matroids*, Cent. Eur. J. Math. **3** (2005), no. 1, 26–38.

[16] Jürgen Herzog and Takayuki Hibi, *Monomial ideals*, Graduate Texts in Mathematics, vol. 260, Springer-Verlag London, Ltd., London, 2011.

[17] Takayuki Hibi, *What can be said about pure O-sequences?*, J. Combin. Theory Ser. A **50** (1989), no. 2, 319–322.

[18] ______, *Face number inequalities for matroid complexes and Cohen-Macaulay types of Stanley-Reisner rings of distributive lattices*, Pacific J. Math. **154** (1992), no. 2, 253–264.

[19] Martina Juhnke-Kubitzke and Dinh Van Le, *Flawlessness of h-vectors of broken circuit complexes*, Int. Math. Res. Not. IMRN (2018), no. 5, 1347–1367.

[20] Bernt Lindström, *A partition of* $L(3, n)$ *into saturated symmetric chains*, European J. Combin. **1** (1980), no. 1, 61–63.

[21] Michael T. Lugo, *Profiles of large combinatorial structures*, ProQuest LLC, Ann Arbor, MI, 2010, Thesis (Ph.D.)–University of Pennsylvania.

[22] Ezra Miller and Bernd Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005.

[23] Kathleen M. O'Hara, *Unimodality of Gaussian coefficients: a constructive proof*, J. Combin. Theory Ser. A **53** (1990), no. 1, 29–52.

[24] Igor Pak, *Partition bijections, a survey*, Ramanujan J. **12** (2006), no. 1, 5–75.

[25] Maurice Pouzet, *Application d'une propriété combinatoire des parties d'un ensemble aux groupes et aux relations*, Math. Z. **150** (1976), no. 2, 117–134.

[26] ———, private communication to D. Stanton (attached in Appendix D), 1990.

[27] Robert A. Proctor, *Solution of two difficult combinatorial problems with linear algebra*, Amer. Math. Monthly **89** (1982), no. 10, 721–734.

[28] Bruce E. Sagan, *Compositions inside a rectangle and unimodality*, J. Algebraic Combin. **29** (2009), no. 4, 405–411.

[29] Richard P. Stanley, *Cohen-Macaulay complexes*, Higher combinatorics (Proc. NATO Advanced Study Inst., Berlin, 1976), 1977, pp. 51–62. NATO Adv. Study Inst. Ser., Ser. C: Math. and Phys. Sci., 31.

[30] ———, *Weyl groups, the hard Lefschetz theorem, and the Sperner property*, SIAM J. Algebraic Discrete Methods **1** (1980), no. 2, 168–184.

[31] ———, *Log-concave and unimodal sequences in algebra, combinatorics, and geometry*, Graph theory and its applications: East and West (Jinan, 1986), Ann. New York Acad. Sci., vol. 576, New York Acad. Sci., New York, 1989, pp. 500–535.

[32] ———, *Combinatorics and commutative algebra*, second ed., Progress in Mathematics, vol. 41, Birkhäuser Boston, Inc., Boston, MA, 1996.

[33] ———, *Enumerative combinatorics. Volume 1*, second ed., Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012.

[34] ———, *Algebraic combinatorics*, Undergraduate Texts in Mathematics, Springer, New York, 2013.

[35] Richard P. Stanley and Fabrizio Zanello, *Unimodality of partitions with distinct parts inside Ferrers shapes*, European J. Combin. **49** (2015), 194–202.

[36] ———, *A generalization of a 1998 unimodality conjecture of Reiner and Stanton*, J. Comb. **11** (2020), no. 1, 111–126.

[37] Dennis Stanton, *Unimodality and Young's lattice*, J. Combin. Theory Ser. A **54** (1990), no. 1, 41–53.

[38] James J. Sylvester, *Proof of the hitherto undemonstrated fundamental theorem of invariants*, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science **5** (1878), no. 30, 178–188.

[39] Douglas B. West, *A symmetric chain decomposition of $L(4, n)$*, European J. Combin. **1** (1980), no. 4, 379–383.

[40] Mark Wildon, *Counting partitions on the abacus*, Ramanujan J. **17** (2008), no. 3, 355–367.

[41] Fabrizio Zanello, *On Bergeron's positivity problem for q-binomial coefficients*, Electron. J. Combin. **25** (2018), no. 2, Paper No. 2.17, 9.

[42] Samuel Zbarsky, *Unimodality of partitions in near-rectangular Ferrers diagrams*, Discrete Math. **338** (2015), no. 9, 1649–1658.

[43] Doron Zeilberger, *Kathy O'Hara's constructive proof of the unimodality of the Gaussian polynomials*, Amer. Math. Monthly **96** (1989), no. 7, 590–602.

# Appendix A

# Cases for $\tilde{\sigma}$ and $\tilde{\tau}$

The cases for expressing $\tilde{\sigma}$ and $\tilde{\tau}$ (as defined after Remark 4.3.15) in terms of the parameters of the unbalanced trellis are given (by Remark 4.3.10) as follows:

- If $d < -1$, then
  - If $e < -1$, then $(\tilde{\sigma}, \tilde{\tau}) = (\sigma, \tau)$.
  - If $|e| \leq 1$, then $(\tilde{\sigma}, \tilde{\tau}) = (\sigma - 1, \tau + 1)$.
  - If $e > 1$, then $(\tilde{\sigma}, \tilde{\tau}) = (\sigma - 2, \tau + 2)$.
- If $-1 \leq d \leq 1$, then
  - If $e < -1$, then $(\tilde{\sigma}, \tilde{\tau}) = (\sigma + 1, \tau - 1)$.
  - If $|e| \leq 1$, then $(\tilde{\sigma}, \tilde{\tau}) = (\sigma, \tau)$.

- If $e > 1$, then $(\tilde{\sigma}, \tilde{\tau}) = (\sigma - 1, \tau + 1)$.

- If $d > 1$, then

  - If $e < -1$, then $(\tilde{\sigma}, \tilde{\tau}) = (\sigma + 2, \tau - 2)$.

  - If $|e| \leq 1$, then $(\tilde{\sigma}, \tilde{\tau}) = (\sigma + 1, \tau - 1)$.

  - If $e > 1$, then $(\tilde{\sigma}, \tilde{\tau}) = (\sigma, \tau)$.

These give us the cases to consider for full determination of $\tilde{s}$:

- If $\sigma < n - 1$, then $s = \sigma$ and

  - If $\tilde{\sigma} = \sigma - 2$, then $\tilde{s} = s - 2$.

  - If $\tilde{\sigma} = \sigma - 1$, then $\tilde{s} = s - 1$.

  - If $\tilde{\sigma} = \sigma$, then $\tilde{s} = s$.

  - If $\tilde{\sigma} = \sigma + 1$, then $\tilde{s} = s + 1$.

  - If $\tilde{\sigma} = \sigma + 2$, then $\tilde{s} = s + 2 \leq n$.

- If $\sigma = n - 1$, then $s = n - 1$ and

  - If $\tilde{\sigma} = \sigma - 2$, then $\tilde{s} = n - 3$.

  - If $\tilde{\sigma} = \sigma - 1$, then $\tilde{s} = n - 2$.

  - If $\tilde{\sigma} = \sigma$, then $\tilde{s} = n - 1$.

  - If $\tilde{\sigma} = \sigma + 1, \sigma + 2$, then $\tilde{s} = n$.

- If $\sigma = n$, then $s = n$ and
  - If $\tilde{\sigma} = \sigma - 2$, then $\tilde{s} = n - 2$.
  - If $\tilde{\sigma} = \sigma - 1$, then $\tilde{s} = n - 1$.
  - If $\tilde{\sigma} = \sigma, \sigma + 1, \sigma + 2$, then $\tilde{s} = n$.
- If $\sigma = n + 1$, then $s = n$ and
  - If $\tilde{\sigma} = \sigma - 2$, then $\tilde{s} = n - 1$.
  - If $\tilde{\sigma} = \sigma - 1, \sigma, \sigma + 1, \sigma + 2$, then $\tilde{s} = n$.
- If $\sigma > n + 1$, then $s = n$ and $\tilde{s} = n$.

Similarly, the cases for $\tilde{t}$ are:

- If $\tau < m - 1$, then $t = \tau$ and
  - If $\tilde{\tau} = \tau - 2$, then $\tilde{t} = t - 2$.
  - If $\tilde{\tau} = \tau - 1$, then $\tilde{t} = t - 1$.
  - If $\tilde{\tau} = \tau$, then $\tilde{t} = t$.
  - If $\tilde{\tau} = \tau + 1$, then $\tilde{t} = t + 1$.
  - If $\tilde{\tau} = \tau + 2$, then $\tilde{t} = t + 2 \leq m$.
- If $\tau = m - 1$, then $t = m - 1$ and

  - If $\tilde{\tau} = \tau - 2$, then $\tilde{t} = m - 3$.
  - If $\tilde{\tau} = \tau - 1$, then $\tilde{t} = m - 2$.
  - If $\tilde{\tau} = \tau$, then $\tilde{t} = m - 1$.
  - If $\tilde{\tau} = \tau + 1$, $\tau + 2$, then $\tilde{t} = m$.

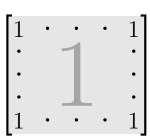
- If $\tau = m$, then $t = m$ and
  - If $\tilde{\tau} = \tau - 2$, then $\tilde{t} = m - 2$.
  - If $\tilde{\tau} = \tau - 1$, then $\tilde{t} = m - 1$.
  - If $\tilde{\tau} = \tau$, $\tau + 1$, $\tau + 2$, then $\tilde{t} = m$.
- If $\tau = m + 1$, then $t = m$ and
  - If $\tilde{\tau} = \tau - 2$, then $\tilde{t} = m - 1$.
  - If $\tilde{\tau} = \tau - 1$, $\tau$, $\tau + 1$, $\tau + 2$, then $\tilde{t} = m$.
- If $\tau > m + 1$, then $t = m$ and $\tilde{t} = m$.

# Appendix B

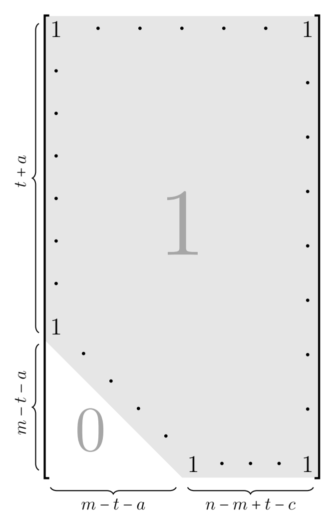
# Cases for $MV$ and $\overline{MV}$

## B.1 Cases for $MV$

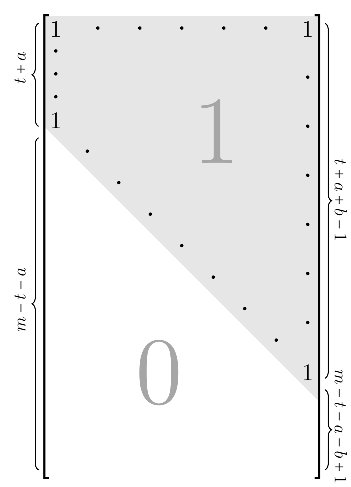
Case 1.A.



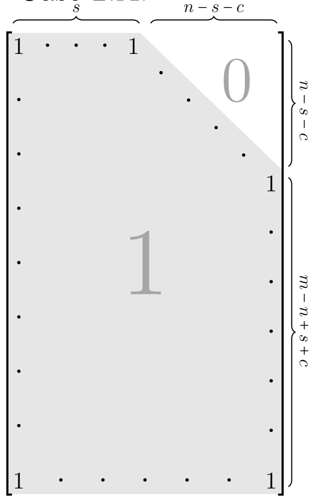
Case 1.B.



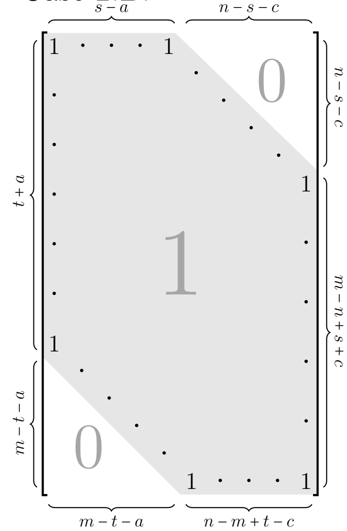

Case 1.C.



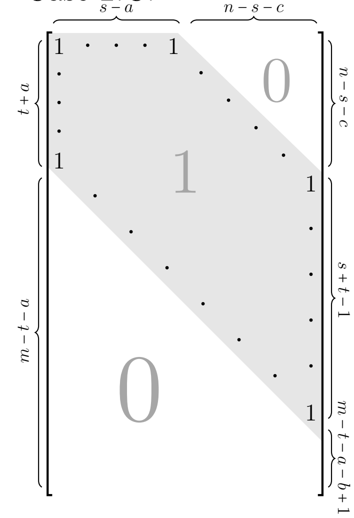
Case 2.A.



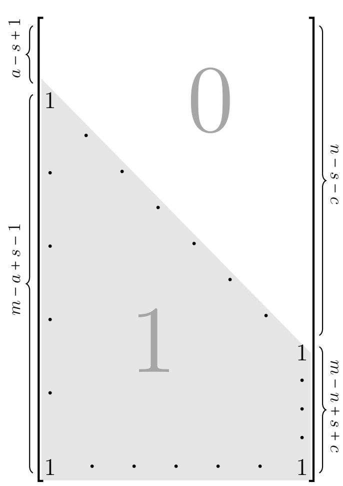

Case 2.B.



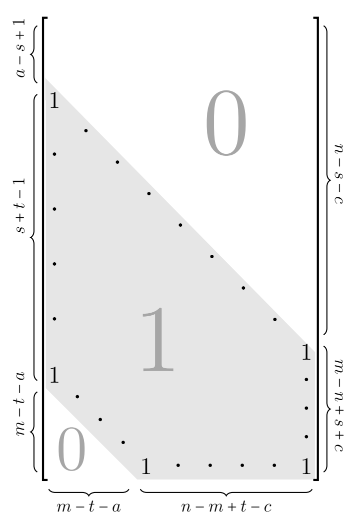

Case 2.C.



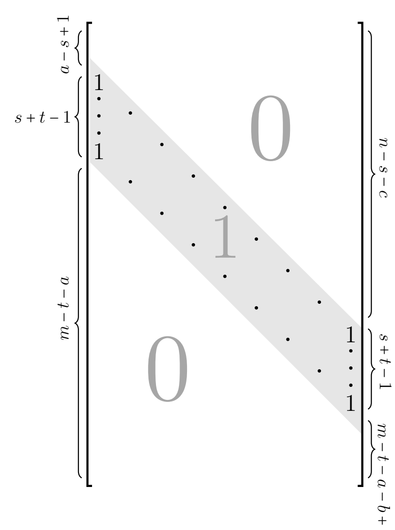

Case 3.A.




Case 3.B.

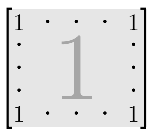



Case 3.C.

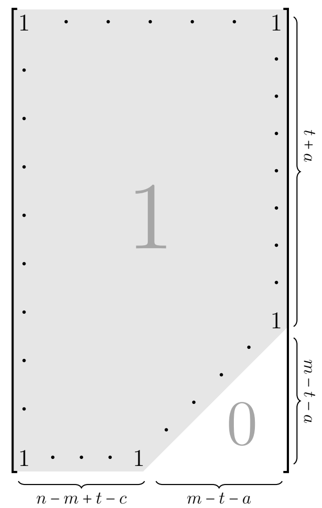


## B.2 Cases for $\overline{MV}$

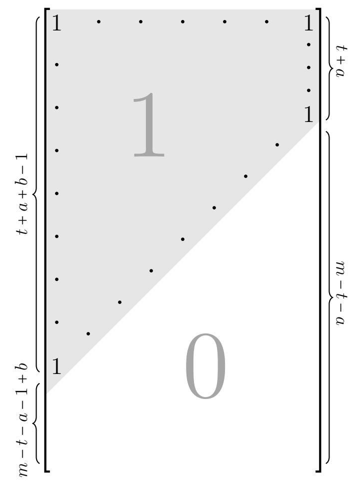
Case 1.A.



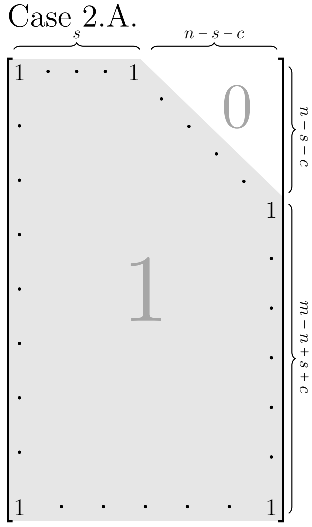
Case 1.B.



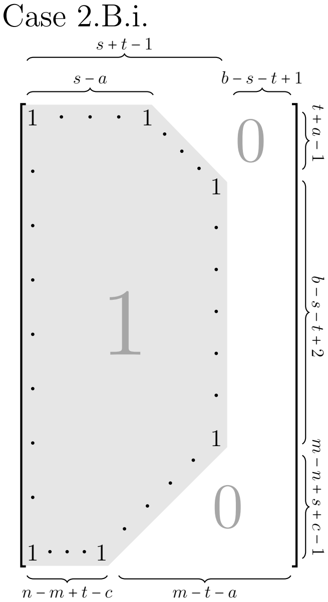

Case 1.C.



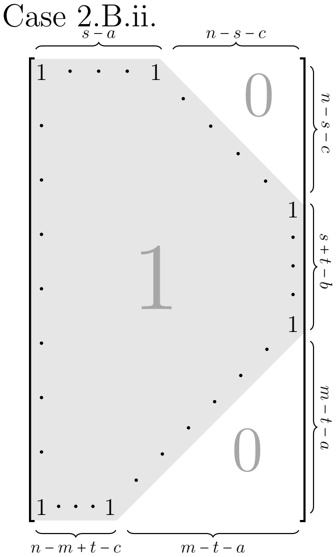

Case 2.A.



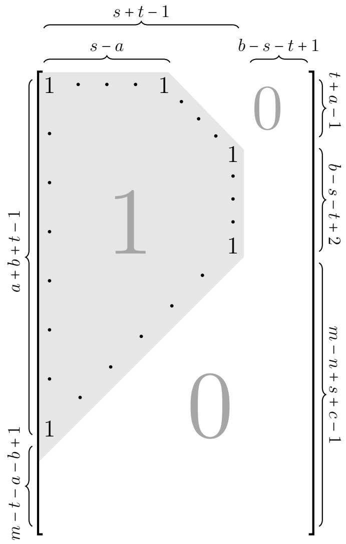

Case 2.B.i.



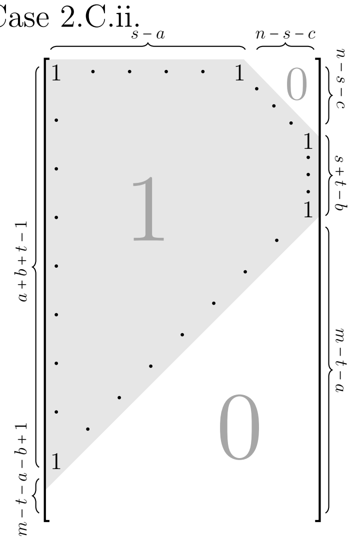

Case 2.B.ii.



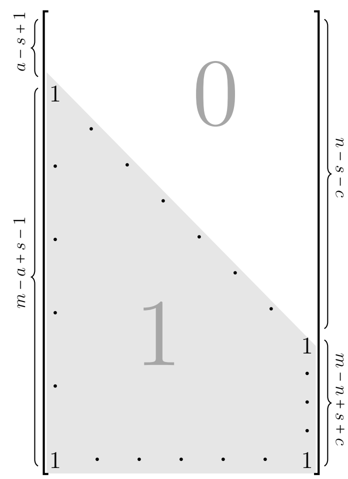
Case 2.C.i.



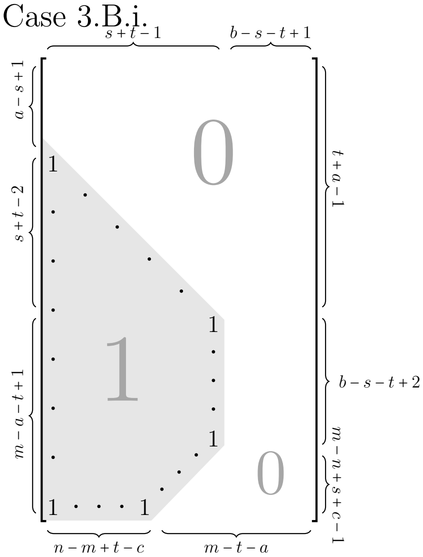

Case 2.C.ii.



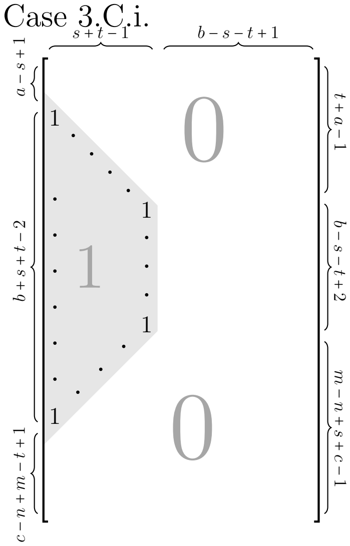

Case 3.A.



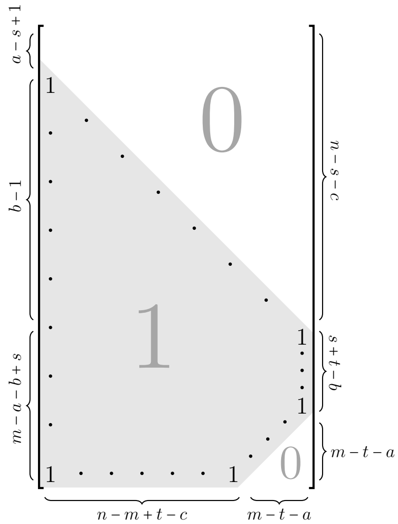

Case 3.B.i.



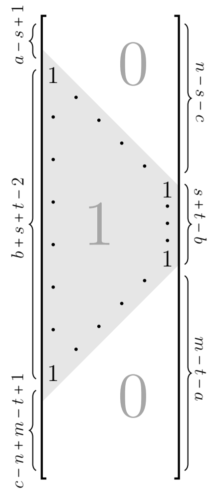

Case 3.C.i.



Case 3.B.ii.




Case 3.C.ii.



# Appendix C

# Mathematica Code

## C.1 Code for Proof of Theorem 3.2.2

```
IsUnimodal[sequence_] := Module[{result, pastPeak},
  Do[
   result = True;
   pastPeak = False;
   Do[
    If[pastPeak && sequence[[i - 1]] < sequence[[i]], result = False];
    If[sequence[[i - 1]] > sequence[[i]], pastPeak = True],
    {i, 2, Length[sequence]}];
   Return[result], 1]]
QB[a_, b_] := QBinomial[a, b, q];
QB[a_, b_] := Product[(1 - q^i), {i, a - b + 1, a}] / Product[(1 - q^i), {i, 1, b}]
Peaks[genfun_] := Flatten@
  Position[CoefficientList[genfun, q][[2 ;;]], Max@CoefficientList[genfun, q]]
```

The following cell is the generating function, $G(q)$, for $(b+1, b, b, b, b)$. We know that this is non-decreasing up to degree $\lfloor \frac{5\,b}{2} \rfloor$ and non-increasing from degree 3b+1 onward.

```
originalG = Simplify@FunctionExpand[QB[b + 5, 5] + q^(b + 1) * QB[b + 4, 4]]
```

```
gf[bb_] := Expand@Cancel[originalG /. b → bb];
gfStar[bb_] := Expand@Cancel[(1 - q) * originalG /. b → bb];
```

This simplifies to the fraction below:

$$G = \frac{(1-q^{1+b})\,(1-q^{2+b})\,(1-q^{3+b})\,(1-q^{4+b})\,(1+q^{1+b}-q^{5+b}-q^{6+b})}{(1-q^5)\,(1-q^4)\,(1-q^3)\,(1-q^2)\,(1-q)};$$

We can check equality with the line below:

```
Simplify[originalG == G, Assumptions → q ≠ 1]
```

The following cell makes a list (of size 60) that gives the $i^{\text{th}}$ coefficient of the denominator of $G(q)$ (depending on the value of $i \pmod{60}$). This is the series $f_5(q)$, the generating function for *all* partitions with 5 parts or fewer.

```
list = {};
Dynamic[prog1]
Do[
 AppendTo[list, Together[Expand[FullSimplify[SeriesCoefficient[(Denominator[G])^-1,
      {q, 0, n}, Assumptions → Mod[n, 60] == Mod[i, 60]],
     Assumptions → Mod[n, 60] == Mod[i, 60] && n ≥ 0]]]];
 prog1 = i, {i, 1, 60}]
```

The function `coeff[]` below gives the actual $i^{\text{th}}$ coefficient of $f_5(q)$ symbolically in terms of $i$, if the value of $i \pmod{60}$ is known. The first argument is the congruence class of the index (mod60) (in other words, telling us which value to pull from the list defined above). The second argument is the actual coefficient that we want. For example, if we want the $i^{\text{th}}$ coefficient of $f_5(q)$ where $i = 9 \pmod{60}$, then we compute $c(9, i)$.

```
coeff[ii_, i_] := list[[Mod[ii - 1, 60] + 1]] /. n → i;
```

The following cell contains the numerator of $G^*(q) := (1-q)\,G(q)$.

```
numerator = Expand[(1 - q) * Numerator[G]]
```

Note that a truncation of the **`numerator`** to some degree n (when multiplied by the terms from $f_5(q)$) will give the correct coefficients of $G^*(q)$ for all degrees $k$ where $n - 14 \le k \le N - 1$. Here, $N$ is the lowest power greater than $n$ appearing in **`numerator`**. The upper bound $N$ is obvious , for in the multiplication of **`numerator`** by $f_5(q)$, adding in the $q^N$ term from the numerator can only contribute to terms with degree greater than or equal to $N$. Thus all coefficients from the truncation must be correct up to degree $N - 1$.

The reason why the coefficients are correct even down to $n - 14$ is because the contribution to those terms by

the $k^{\text{th}}$ coefficient formula we found for $f_5(q)$ is 0. From the definition of multiplication of two series, for a term with degree less than n (say $n-i$), we would need to take expressions of the form **coeff[Mod[60-i,60],-i]** (before simplification) corresponding to coefficients of $f_5(q)$. Of course, these terms with negative degree do not truly exist in $f_5(q)$ when $i<0$, but calling for coefficients of the form **coeff[Mod[j,60],j]** (when we do not know a priori that $j$ is negative) may still give some nonzero value. However, it turns out that the first 14 of these values from **coeff** are indeed 0 (see the verification of this in the next cell). That is, **coeff[Mod[60-i,60],-i]= 0** for all $i$ where $-14 \leq i < 0$. Since these coefficients of 0 agree with the (non-existent) negative degree terms from $f_5(q)$, it follows that the truncation of **numerator** at degree $n$ yields correct coefficients of $G^*(q)$ for all degrees k where $n-14 \leq k < N$.

```
Table[coeff[-i, -i], {i, 1, 14}]
```

Based on the discussion from the previous paragraph, the following truncation of **numerator** (multiplied by $f_5(q)$) will yield the correct coefficients of $G^*(q)$ for all degrees $k$ where $2b-3 \leq k \leq 3b+3$.

(Note: we assume $b \geq 8$, or else we conflate the terms $-q^{11+2b} + q^{4+3b}$ from **numerator**).

```
truncatedNumerator =
  1 - q - q^(2+b) + q^(7+b) - q^(2+2 b) + q^(3+2 b) + q^(5+2 b) + q^(6+2 b) + q^(7+2 b) - q^(8+2 b) - q^(10+2 b) - q^(11+2 b);
```

The next cell finds the actual coefficients of $G^*(q)$, correct for all degrees $k$ where $2b-3 \leq k \leq 3b+3$. Note that the sum in the following code is precisely the definition of multiplication of two series.

```
lister = {};
Dynamic[prog2]
Do[
 prog2 = bb;
 Do[
  AppendTo[lister, Simplify@
    Sum[Coefficient[truncatedNumerator[[i]], q, Exponent[truncatedNumerator[[i]],
        q]] * coeff[(kk - (Exponent[truncatedNumerator[[i]], q] /. b → bb)),
       k - (Exponent[truncatedNumerator[[i]], q])],
     {i, 1, Length[truncatedNumerator]}]], {kk, 1, 60}], {bb, 1, 60}]
valuesModB[i_] := lister[[(Mod[i - 1, 60]) * 60 + 1 ;; 60 * (1 + Mod[i - 1, 60])]];
coefficient[bb_, kk_] := valuesModB[bb][[1 + Mod[kk - 1, 60]]] /. k → kk /. b → bb;
```

The cell below gives numeric illustration that the coefficients of $G^*(q)$ found above actually are correct for the coefficients of $q^k$ from $2b-3 \leq k \leq 3b+3$ (any value of $b \geq 8$ can be tested).

```
b = RandomInteger[{100, 200}]
CoefficientList[gfStar[b], q][[2 b - 2 ;; 3 b + 4]] ==
 Table[coefficient[b, k], {k, 2 b - 3, 3 b + 3}]
Clear[
 b]
```

We now split the coefficients of $G^*(q)$ found above into the categories for even and odd values of $b$ and eliminate duplicates:

```
evens = Catenate@Table[valuesModB[2 i], {i, 1, 30}];
odds = Catenate@Table[valuesModB[2 i - 1], {i, 1, 30}];
uniqueEvens = DeleteDuplicates[evens];
uniqueOdds = DeleteDuplicates[odds];
Length[uniqueEvens];
Length[uniqueOdds];
```

The following two cells check symbolically that the coefficients of $G^*(q)$ found above are nonnegative for $\lfloor \frac{5b-2}{2} \rfloor \leq k \leq \lfloor \frac{5b+5}{2} \rfloor$ and nonpositive for $\lfloor \frac{5b+7}{2} \rfloor \leq k \leq 3b+3$, for $b \geq 33$, thus finishing the proof. (Note that this handles the only interval we need to check: between degrees $\lfloor \frac{5b}{2} \rfloor$ and $3b+1$).

```
(*Next two lines shows that all coefficients of G*(q) are
  positive between degrees ⌊(5b-2)/2⌋ and ⌊(5b+5)/2⌋=(5b+4)/2 (for b even)*)
And @@ Table[Assuming[b ≥ 12 && (5 b - 2)/2 ≤ k ≤ (5 b + 4)/2, FullSimplify[coeff ≥ 0]],
  {coeff, uniqueEvens}]

(*Next line shows that all coefficients of G*(q) are
  negative between degrees ⌊(5b+7)/2⌋=(5b+6)/2 and 3b+3 (for b even)*)
And @@ Table[Assuming[b ≥ 8 && (5 b + 6)/2 ≤ k ≤ 3 b + 3, Simplify[coeff ≤ 0]],
  {coeff, uniqueEvens}]

(*Next line shows that all coefficients of G*(q) are
  positive between degrees ⌊(5b-2)/2⌋ and ⌊(5b+5)/2⌋=(5b+5)/2 (for b odd)*)
And @@ Table[Assuming[b ≥ 33 && (5 b - 3)/2 ≤ k ≤ (5 b + 5)/2, FullSimplify[coeff ≥ 0]],
  {coeff, uniqueOdds}]

(*Next line shows that all coefficients of G*(q) are
  negative between degrees ⌊(5b+7)/2⌋=(5b+7)/2 and 3b+3 (for b odd)*)
And @@ Table[Assuming[b ≥ 8 && (5 b + 7)/2 ≤ k ≤ 3 b + 3, Simplify[coeff ≤ 0]],
  {coeff, uniqueOdds}]
```

The following cell shows the partitions are unimodal for $b \leq 32$.

```
And @@ Table[IsUnimodal@CoefficientList[gf[b], q], {b, 1, 32}]
```

The next cell verifies that the peak is always at $\lfloor \frac{5b+5}{2} \rfloor$ for $b \geq 22$

```
And @@ Table[Peaks[gf[b]][[-1]] == Floor[(5 b + 5)/2], {b, 22, 32}]
```

## C.2 Code for Proof of Lemma 4.3.7

# Assumptions:

```
$Assumptions =
  (*Coordinates of points (and their weight, i.e., w) in (k-1)
    st diagonal are nonnegative integers (could be zero if k=2):*)
  (x1 | xm | x1p | xmp | w ) ∈ NonNegativeIntegers &&

   (*Coordinates of points
    (and their weight, i.e., W) in kth diagonal are positive integers:*)
   (X1 | Xn | X1p | Xnp | A | B | Ap | Bp | W) ∈ PositiveIntegers && W ≤ A + B &&

   (*Definitions of d and e:*)
   (d | e) ∈ Integers && e == X1 - (W - Xn) && d == x1 - (w - xm) &&

   (*Following conditions must hold
    to maintain connectivity of successive diagonals:*)
   x1 < X1 && xm < Xn && x1p < X1p && xmp < Xnp && w - x1 < W - X1 &&
   w - xm < W - Xn && w - x1p < W - X1p && w - xmp < W - Xnp;


(*These are assumptions that must hold in general,
but they are somehow not needed for proving the result:

  Nonnegativity of y-values
      w-x1≥0&&w-xm≥0&&w-x1p≥0&&w-xmp≥0&&

   Points in the diaganals are orientated this way with respect to each other
     x1≥xm&&x1p≥xmp&&X1≥Xn&&X1p≥Xnp&&


   A+B==Ap+Bp&&W≥w+2&&

*)
```

## Defining the cases and generating all possible combinations:

```
X1Cases = {W - A > w - x1 && X1 == A, W - A ≤ w - x1 && X1 == W - (w - x1 + 1)};
XnCases = {B < W - xm && Xn == W - B, B ≥ W - xm && Xn == xm + 1};
X1pCases = {W - Ap > w - x1p && X1p == Ap, W - Ap ≤ w - x1p && X1p == W - (w - x1p + 1)};
XnpCases = {Bp < W - xmp && Xnp == W - Bp, Bp ≥ W - xmp && Xnp == xmp + 1};
ApBpCases = {A < B - 1 && Ap == A + 1 && Bp == B - 1,
   B - 1 ≤ A ≤ B + 1 && Ap == A && Bp == B, A > B + 1 && Ap == A - 1 && Bp == B + 1};
dCases = {d < -1 && xmp ≤ xm + 1 && x1p ≥ x1 + 1, -1 ≤ d ≤ 1 && xmp ≤ xm && x1p ≥ x1,
   d > 1 && xmp ≤ xm - 1 && x1p ≥ x1 - 1};
eCases = {e < -1, -1 ≤ e ≤ 1, e > 1};
result :=
  Which[
   Simplify[eCases[[1]]],
   Xnp ≤ Xn + 1 && X1p ≥ X1 + 1,
   Simplify[eCases[[2]]],
   Xnp ≤ Xn && X1p ≥ X1,
   Simplify[eCases[[3]]],
   Xnp ≤ Xn - 1 && X1p ≥ X1 - 1
  ];

allCases = {};
Do[
 Do[
  Do[
   Do[
    Do[
     Do[
      Do[
       AppendTo[allCases, {i, j, k, l, m, n, o}],
       {o, 1, 3}],
      {n, 1, 3}],
     {m, 1, 3}],
    {l, 1, 2}],
   {k, 1, 2}],
  {j, 1, 2}],
 {i, 1, 2}]
```

## Eliminating Self-contradictory statements:

```
tempAllCases = allCases;
allCases = {};
prog = 0;
Dynamic[prog]
Do[
  prog++;
  aCase = tempAllCases[[i]];
  If[Simplify[X1Cases[[aCase[[1]]]] && XnCases[[aCase[[2]]]] &&
      X1pCases[[aCase[[3]]]] && XnpCases[[aCase[[4]]]] &&
      ApBpCases[[aCase[[5]]]] && dCases[[aCase[[6]]]] && eCases[[aCase[[7]]]]],
   Print["Well that's surprising"], Null, AppendTo[allCases, aCase]],
  {i, 1, Length[tempAllCases]}];
Print[StringForm["Original number of cases: ``", Length[tempAllCases]]]
Print[StringForm["Number of cases with non self-contradictory assumptions: ``",
  Length[allCases]]]
```

## Completion of the Proof:

```
ticker = 0;
Dynamic[ticker]
Do[
 ticker++;
 aCase = allCases[[i]];
 theCase = X1Cases[[aCase[[1]]]] && XnCases[[aCase[[2]]]] &&
   X1pCases[[aCase[[3]]]] && XnpCases[[aCase[[4]]]] &&
   ApBpCases[[aCase[[5]]]] && dCases[[aCase[[6]]]] && eCases[[aCase[[7]]]];
 If[Assuming[theCase, Simplify[result]] ≠ True, Print["Counterexample"],
  Null, Print["Can't decide with those assumptions"]],
 {i, 1, Length[allCases]}]
```

# C.3 Code for Proof of Lemma 4.3.18

## Introduction

This code completes the proof of Lemma 4.3.18. The cells should be evaluated in order (full runtime is likely to be anywhere from 15 minutes to 1 hour, depending on the computer).

The main purpose of the code is to compare two 3-tuples: `BMV={ρ,η,ξ}` and `MVTilde={α,β,γ}` (these uniquely determine two matrices). This comparison is performed by the function `Compare` (defined in Section 1a).

The values of `BMV` and `MVTilde` are determined by 9 parameters: `m, n, σ, τ, s, t, a, b, c, d, e`. The known assumptions that we can make about these parameters (e.g., that `m` and `n` are positive integers) are defined as `fixedAssumptions` in Section 1a.

There are a host of cases that must be considered for determining the exact form of `BMV` and `MVTilde` (based on bounds of, and relations among these parameters). The value for `BMV` (as given in Section 2a) can be determined directly from explicitly knowing which case holds for the 9 parameters. To determine `MVTilde`, we first must consider some other parameters ($\tilde{\sigma}$, $\tilde{\tau}$, $\tilde{s}$, $\tilde{t}$, $\tilde{a}$, and $\tilde{c}$). These “tilde” variables (denoted in the code as `sigmaTilde`, `tauTilde`, etc.,) are completely determined by the other 9 “non-tilde” parameters (as given in Section 2b). Once the values of these “tilde” parameters are decided, we can then determine `MVTilde` (although this determination sometimes requires considering further cases, which will be implicitly considered by `Simplify` in Section 3).

To shorten runtime, we explicitly tell the computer the various cases to assume for the bounds on, and relations among the parameters. These individual cases are represented as lists of mutually exclusive (in)equalities. For example, for the parameter **d** we want to consider separately the following cases: either `d<-1` or else `-1≤d≤1` or else `d>1`. So we set `dCases={d<-1,-1≤d≤1,d>1}`. To operate under the assumption `d<-1`, for example, we would add `dCases[[1]]` to `$Assumptions`. There are a total of 11 such lists of individual cases, as given in Section 1a. Certain of these cases should only be considered if a certain other case holds (or does not hold), as commented in the corresponding code. Section 1b creates all possible combinations of these cases. This is done by representing each combination of cases as a “case vector” (an integer-valued list). For example, `{3,1,2,...}` is interpreted as meaning that the 3$^{rd}$ expression in the first case list (`dCases`) is true, the 1$^{st}$ expression in the second case list (e`Cases`)is true, the 2$^{nd}$ expression in the third case list (v`Cases`) is true, etc. The cases are encoded in this way rather than created outright as logic statements (i.e., as `dCases[[3]]&&eCases[[1]]&&vCases[[2]]&&...` ) in order to speed up runtime and more easily deal with the fact that some cases should only be considered if a certain other case holds (or does not hold).

Section 1c converts the completed “case vectors” created in Section 1b to the corresponding logic

statements involving the 9 parameters (for example, `{3,1,2,...}` is converted to `dCases[[3]]&&eCases[[1]]&&vCases[[2]]&&...`). Some of these statements yield self-contradictory assumptions, which are also eliminated in Section 1c (this is in fact the most time-consuming code in the notebook).

Section 3 contains the actual verification of the lemma. For all the possible (non-self-contradictory) cases, the values of `BMV` and `MVTilde` are compared by `Compare`. Using `TrueQ` demands that a value of `True` explicitly be returned by `Compare` in order to verify the lemma for any particular case. (Thus `Compare` can never return that an expression is undecidable or `False` without us noticing it.) Running the code shows that the desired comparison always holds, thus the proof of the lemma is complete.

# Section 1: Setting the Assumptions and Creating all Cases

---

## 1a: Assignment of `fixedAssumptions`, definition of `Compare`, definition of the 11 lists of cases

```
(*These are the general assumptions
 that we may make about the various parameters*)
fixedAssumptions = (m | n | σ | τ | s | t | a | b | c | d | e) ∈ Integers &&
   m ≥ 1 && n ≥ 1 && e + n + τ == d + m + σ && s ≥ Max[n - m + 1, 1] && s == Min[σ, n] &&
   t ≥ Max[m - n + 1, 1] && t == Min[τ, m] && a ≥ 0 && b > 0 && c ≥ 0 && a + b + c == n;
$Assumptions = fixedAssumptions;


(*This function compares 3-tuples that represent 0-
  1 matrices and returns True if {α_,β_,γ_}-{ρ_,η_,ξ_}≥0 and False otherwise.*)
Compare[{ρ_, η_, ξ_}, {α_, β_, γ_}] :=
  Which[
   Simplify[ρ > α],
   Return[False],
   Simplify[ρ == α],
   If[Simplify[η ≥ β && ξ ≤ γ], Return[True], Return[False]],
   Simplify[ρ < α],
   If[Simplify[β - η ≤ α - ρ && ξ - γ ≤ α - ρ], Return[True], Return[False]]
  ];
```

```
(*These are the cases that should be considered:*)
dCases = {d < -1, -1 ≤ d ≤ 1, d > 1};
eCases = {e < -1, -1 ≤ e ≤ 1, e > 1};
vCases = {a - c < e - 1, e - 1 ≤ a - c ≤ e + 1, a - c > e + 1};
sigmaCases = {σ < n - 1, σ == n - 1, σ == n, σ == n + 1, σ > n + 1};
tauCases = {τ < m - 1, τ == m - 1, τ == m, τ == m + 1, τ > m + 1};

cases123 = {c ≥ n - s, a < s && c < n - s, a ≥ s};
(*[[2]] and [[3]] are only possible if σ<n (sigmaCases[[1 or 2]])*)

casesABC = {a ≥ m - t, a < m - t && c < n - m + t, c ≥ n - m + t};
(*[[2]] and [[3]] are only possible if τ<m (tauCases[[1 or 2]])*)

casesiAndii = {s + t ≤ b, s + t > b, True};
(*These cases do not need to be considered in
 Case 1. (cases123[[1]]) or Case A. (casesABC[[1]]).*)


colBalanceCases1 = {- (m - t - a) < d - 1, d - 1 ≤ - (m - t - a) ≤ d + 1, - (m - t - a) > d + 1, True};
(*These cases should only be considered in Case 1. (cases123[[1]])*)

colBalanceCases2 = {n - s - c < d - 1, d - 1 ≤ n - s - c ≤ d + 1, n - s - c > d + 1, True};
(*These cases should only be considered in Case A. (casesABC[[1]])*)

colBalanceCases3 = {a - c - e + d == 0, a - c - e + d ≤ -1, a - c - e + d ≥ 1, True};
(*These cases do not need to be considered in
 Case 1. (cases123[[1]]) or Case A. (casesABC[[1]]).*)
```

## 1b: All potential combinations of the cases

```
(*The following nested loops create all possible combinations of dCases,
eCases, vCases, sigmaCases, and tauCases *)
allCaseVectors = {};
Do[
  Do[
   Do[
    Do[
     Do[
      AppendTo[allCaseVectors, {h, i, j, k, l}],
```

```
      {l, 1, 5}],
     {k, 1, 5}],
    {j, 1, 3}],
   {i, 1, 3}],
  {h, 1, 3}];


(*Some of the lists created in the preceeding loop may give rise to self-
 contradictory statements. This next loop
  eliminates those (at least some of them). *)
tempAllCaseVectors = allCaseVectors;
allCaseVectors = {};
Do[
  aCase = tempAllCaseVectors[[i]];
  If[Simplify[dCases[[aCase[[1]]]] && eCases[[aCase[[2]]]] &&
     vCases[[aCase[[3]]]] && sigmaCases[[aCase[[4]]]] && tauCases[[aCase[[5]]]]],
   Print["This should not happen.  If it does, it means that
      <aCase> is trivially true, or that $Assumptions
      implies <aCase> (which should never be the case)"],
   Null, (*<aCase> gives self-contradictory assumptions,
   and will not be added to allCasesVectors*)
   AppendTo[allCaseVectors, aCase]
   (*<aCase> appears to not be self-contradictory*)
  ],
  {i, 1, Length[tempAllCaseVectors]}];

(*This next loop adds in all possibilities of cases123 (6th list)*)
tempAllCaseVectors = allCaseVectors;
allCaseVectors = {};
Do[
  If[tempAllCaseVectors[[i]][[4]] ≤ 2, (*check that σ<n*)
   Do[AppendTo[allCaseVectors, Append[tempAllCaseVectors[[i]], j]], {j, 1, 3}],
   AppendTo[allCaseVectors, Append[tempAllCaseVectors[[i]], 1]]],
  {i, 1, Length[tempAllCaseVectors]}];


(*This next loop adds in all possibilities of casesABC (7th list)*)
tempAllCaseVectors = allCaseVectors;
allCaseVectors = {};
Do[
  If[tempAllCaseVectors[[i]][[5]] ≤ 2, (*check that τ<m*)
   Do[AppendTo[allCaseVectors, Append[tempAllCaseVectors[[i]], j]], {j, 1, 3}],
   AppendTo[allCaseVectors, Append[tempAllCaseVectors[[i]], 1]]],
```

```
  {i, 1, Length[tempAllCaseVectors]}];


(*This next loop adds in all possibilities of casesiAndii (8th list)*)
tempAllCaseVectors = allCaseVectors;
allCaseVectors = {};
Do[
  If[tempAllCaseVectors[[i]][[6]] > 1 && tempAllCaseVectors[[i]][[7]] > 1,
   Do[AppendTo[allCaseVectors, Append[tempAllCaseVectors[[i]], j]], {j, 1, 2}],
   AppendTo[allCaseVectors, Append[tempAllCaseVectors[[i]], 3]]],
  {i, 1, Length[tempAllCaseVectors]}];


(*This next loop adds in all possibilities of colBalanceCases1 (9th list)*)
tempAllCaseVectors = allCaseVectors;
allCaseVectors = {};
Do[
  If[tempAllCaseVectors[[i]][[6]] == 1,
   Do[AppendTo[allCaseVectors, Append[tempAllCaseVectors[[i]], j]], {j, 1, 3}],
   AppendTo[allCaseVectors, Append[tempAllCaseVectors[[i]], 4]]],
  {i, 1, Length[tempAllCaseVectors]}];


(*This next loop adds in all possibilities of colBalanceCases2 (10th list)*)
tempAllCaseVectors = allCaseVectors;
allCaseVectors = {};
Do[
  If[tempAllCaseVectors[[i]][[7]] == 1,
   Do[AppendTo[allCaseVectors, Append[tempAllCaseVectors[[i]], j]], {j, 1, 3}],
   AppendTo[allCaseVectors, Append[tempAllCaseVectors[[i]], 4]]],
  {i, 1, Length[tempAllCaseVectors]}];


(*This next loop adds in all possibilities of colBalanceCases3(11th list)*)
tempAllCaseVectors = allCaseVectors;
allCaseVectors = {};
Do[
  If[tempAllCaseVectors[[i]][[6]] > 1 && tempAllCaseVectors[[i]][[7]] > 1,
   Do[AppendTo[allCaseVectors, Append[tempAllCaseVectors[[i]], j]], {j, 1, 3}],
   AppendTo[allCaseVectors, Append[tempAllCaseVectors[[i]], 4]]],
  {i, 1, Length[tempAllCaseVectors]}];
```

## 1c: Conversion to logic statements and elimination of self-contradictory statements

```
(*This converts the cases encoded by the lists in <allCaseVectors>
 to a list of actual logic statements involving the various parameters*)
allCases = Table[
   dCases[[allCaseVectors[[i]][[1]]]] &&
    eCases[[allCaseVectors[[i]][[2]]]] &&
    vCases[[allCaseVectors[[i]][[3]]]] &&
    sigmaCases[[allCaseVectors[[i]][[4]]]] &&
    tauCases[[allCaseVectors[[i]][[5]]]] &&
    cases123[[allCaseVectors[[i]][[6]]]] &&
    casesABC[[allCaseVectors[[i]][[7]]]] &&
    casesiAndii[[allCaseVectors[[i]][[8]]]] &&
    colBalanceCases1[[allCaseVectors[[i]][[9]]]] &&
    colBalanceCases2[[allCaseVectors[[i]][[10]]]] &&
    colBalanceCases3[[allCaseVectors[[i]][[11]]]],
   {i, 1, Length[allCaseVectors]}];

(*This next loop eliminates some self-contradictory statements in <allCases>*)
tempAllCases = allCases;
allCases = {};
prog1 = 0;
Dynamic[{prog1, Length[tempAllCases]}]
(*This simply keeps track of the progress of the loop*)
Do[
  prog1 = i;
  If[Simplify[tempAllCases[[i]]],
   Print["This should not happen.  If it does, it means that
      <tempAllCases[[i]]> is trivially true, or that $Assumptions
      implies <tempAllCases[[i]]> (which should never be the case)"];
   Break[],
   Null, (*<tempAllCases[[i]]> is self-contradictory,
   and will not be added to allCases*)
   AppendTo[allCases, tempAllCases[[i]]]
   (*<tempAllCases[[i]]> appears to not be self-contradictory*)
  ],
  {i, 1, Length[tempAllCases]}] // AbsoluteTiming
```

```
(*This next loop is exactly the same as the one in the previous cell,
except that it uses FullSimplify[] to eliminate any self-
 contradictory statements that Simplify[] missed in the previous cell.*)
tempAllCases = allCases;
allCases = {};
prog2 = 0;
Dynamic[{prog2, Length[tempAllCases]}]
(*This simply keeps track of the progress of the loop*)
Do[
  prog2 = i;
  If[FullSimplify[tempAllCases[[i]]],
   Print["This should not happen.  If it does, it means that
      <tempAllCases[[i]]> is trivially true, or that $Assumptions
      implies <tempAllCases[[i]]> (which should never be the case)"];
   Break[],
   Null, (*<tempAllCases[[i]]> is self-contradictory,
   and will not be added to allCases*)
   AppendTo[allCases, tempAllCases[[i]]]
   (*<tempAllCases[[i]]> appears to not be self-contradictory*)
  ],
  {i, 1, Length[tempAllCases]}] // AbsoluteTiming
```

# Section 2: Determinations of **BMV** and **MVTilde**

---

## 2a: Determination of BMV

```
(*The code below gives the value of BMV (i.e., B(MV)) *)
BMV :=
 Simplify@Which[
   cases123[[1]] && casesABC[[1]],
   {b, 1, m},
   cases123[[1]] && (casesABC[[2]] || casesABC[[3]]),
   Which[
    dCases[[1]],
    Which[
     colBalanceCases1[[1]],
     {b, 3, t + a + 2},
     colBalanceCases1[[2]],
     {b, 2, t + a + 1},
```

```
  colBalanceCases1[[3]],
  {b, 1, t + a}
 ],
 dCases[[2]],
 Which[
  colBalanceCases1[[1]],
  {b, 2, t + a + 1},
  colBalanceCases1[[2]] || colBalanceCases1[[3]],
  {b, 1, t + a}
 ],
 dCases[[3]],
 {b, 1, t + a}
],
casesABC[[1]] && (cases123[[2]] || cases123[[3]]),
Which[
 dCases[[1]],
 {b, n - s - c + 1, m},
 dCases[[2]],
 Which[
  colBalanceCases2[[1]] || colBalanceCases2[[2]],
  {b, n - s - c + 1, m},
  colBalanceCases2[[3]],
  {b, n - s - c, m - 1}
 ],
 dCases[[3]],
 Which[
  colBalanceCases2[[1]],
  {b, n - s - c + 1, m},
  colBalanceCases2[[2]],
  {b, n - s - c, m - 1},
  colBalanceCases2[[3]],
  {b, n - s - c - 1, m - 2}
 ]
],
(casesABC[[2]] || casesABC[[3]]) && (cases123[[2]] || cases123[[3]]),
Which[
 casesiAndii[[1]],
 Which[
  colBalanceCases3[[1]],
  {s + t - 1, t + a, n - s - c + 1},
  colBalanceCases3[[2]],
  Which[
   dCases[[1]],
```

```
   Which[
    vCases[[1]],
    {s + t - 1, t + a + 2, n - s - c + 3},
    vCases[[2]],
    {s + t - 1, t + a + 1, n - s - c + 2},
    vCases[[3]],
    {s + t - 1, t + a, n - s - c + 1}
   ],
   dCases[[2]],
   Which[
    vCases[[1]],
    {s + t - 1, t + a + 1, n - s - c + 2},
    vCases[[2]] || vCases[[3]],
    {s + t - 1, t + a, n - s - c + 1}
   ],
   dCases[[3]],
   {s + t - 1, t + a, n - s - c + 1}
  ],
  colBalanceCases3[[3]],
  Which[
   dCases[[1]],
   {s + t - 1, t + a, n - s - c + 1},
   dCases[[2]],
   Which[
    vCases[[1]] || vCases[[2]],
    {s + t - 1, t + a, n - s - c + 1},
    vCases[[3]],
    {s + t - 1, t + a - 1, n - s - c}
   ],
   dCases[[3]],
   Which[
    vCases[[1]],
    {s + t - 1, t + a, n - s - c + 1},
    vCases[[2]],
    {s + t - 1, t + a - 1, n - s - c},
    vCases[[3]],
    {s + t - 1, t + a - 2, n - s - c - 1}
   ]
  ]
 ],
 casesiAndii[[2]],
 Which[
  colBalanceCases3[[1]],
  {b, n - s - c + 1, a + t},
```

```
   colBalanceCases3[[2]],
   Which[
    dCases[[1]],
    Which[
     vCases[[1]],
     {b, n - s - c + 3, a + t + 2},
     vCases[[2]],
     {b, n - s - c + 2, a + t + 1},
     vCases[[3]],
     {b, n - s - c + 1, a + t}
    ],
    dCases[[2]],
    Which[
     vCases[[1]],
     {b, n - s - c + 2, a + t + 1},
     vCases[[2]] || vCases[[3]],
     {b, n - s - c + 1, a + t}
    ],
    dCases[[3]],
    {b, n - s - c + 1, a + t}
   ],
   colBalanceCases3[[3]],
   Which[
    dCases[[1]],
    {b, n - s - c + 1, a + t},
    dCases[[2]],
    Which[
     vCases[[1]] || vCases[[2]],
     {b, n - s - c + 1, a + t},
     vCases[[3]],
     {b, n - s - c, a + t - 1}
    ],
    dCases[[3]],
    Which[
     vCases[[1]],
     {b, n - s - c + 1, a + t},
     vCases[[2]],
     {b, n - s - c, a + t - 1},
     vCases[[3]],
     {b, n - s - c - 1, a + t - 2}
    ]
   ]
  ]
 ]
```

## 2b: Determination of $\tilde{\sigma}$ , $\tilde{\tau}$ , $\tilde{s}$ , $\tilde{t}$, $\tilde{a}$ and $\tilde{c}$

```
sigmaTilde := Simplify@
   Which[
    dCases[[1]],
    Which[
     eCases[[1]],
     σ,
     eCases[[2]],
     σ - 1,
     eCases[[3]],
     σ - 2
    ],
    dCases[[2]],
    Which[
     eCases[[1]],
     σ + 1,
     eCases[[2]],
     σ,
     eCases[[3]],
     σ - 1
    ],
    dCases[[3]],
    Which[
     eCases[[1]],
     σ + 2,
     eCases[[2]],
     σ + 1,
     eCases[[3]],
     σ
    ]];
tauTilde := Simplify@
   Which[
    dCases[[1]],
    Which[
     eCases[[1]],
     τ,
     eCases[[2]],
     τ + 1,
     eCases[[3]],
     τ + 2
    ],
```

```
     dCases[[2]],
     Which[
      eCases[[1]],
      τ - 1,
      eCases[[2]],
      τ,
      eCases[[3]],
      τ + 1
     ],
     dCases[[3]],
     Which[
      eCases[[1]],
      τ - 2,
      eCases[[2]],
      τ - 1,
      eCases[[3]],
      τ
     ]];

sTilde := Simplify@
    Which[
     sigmaCases[[1]],
     Which[
      sigmaTilde == σ - 2,
      s - 2,
      sigmaTilde == σ - 1,
      s - 1,
      sigmaTilde == σ,
      s,
      sigmaTilde == σ + 1,
      s + 1,
      sigmaTilde == σ + 2,
      s + 2
     ],
     sigmaCases[[2]],
     Which[
      sigmaTilde == σ - 2,
      n - 3,
      sigmaTilde == σ - 1,
      n - 2,
      sigmaTilde == σ,
      n - 1,
      sigmaTilde ≥ σ + 1,
      n
```

```
    ],
    sigmaCases[[3]],
    Which[
     sigmaTilde == σ - 2,
     n - 2,
     sigmaTilde == σ - 1,
     n - 1,
     sigmaTilde ≥ σ,
     n
    ],
    sigmaCases[[4]],
    Which[
     sigmaTilde == σ - 2,
     n - 1,
     sigmaTilde ≥ σ - 1,
     n
    ],
    sigmaCases[[5]],
    n
   ];

tTilde := Simplify@
   Which[
    tauCases[[1]],
    Which[
     tauTilde == τ - 2,
     t - 2,
     tauTilde == τ - 1,
     t - 1,
     tauTilde == τ,
     t,
     tauTilde == τ + 1,
     t + 1,
     tauTilde == τ + 2,
     t + 2
    ],
    tauCases[[2]],
    Which[
     tauTilde == τ - 2,
     m - 3,
     tauTilde == τ - 1,
     m - 2,
     tauTilde == τ,
     m - 1,
```

```
   tauTilde ≥ τ + 1,
   m
  ],
  tauCases[[3]],
  Which[
   tauTilde == τ - 2,
   m - 2,
   tauTilde == τ - 1,
   m - 1,
   tauTilde ≥ τ,
   m
  ],
  tauCases[[4]],
  Which[
   tauTilde == τ - 2,
   m - 1,
   tauTilde ≥ τ - 1,
   m
  ],
  tauCases[[5]],
  m
 ];

aTilde := Simplify@
 Which[
  eCases[[1]],
  Which[
   vCases[[1]],
   a + 2,
   vCases[[2]],
   a + 1,
   vCases[[3]],
   a
  ],
  eCases[[2]],
  Which[
   vCases[[1]],
   a + 1,
   vCases[[2]],
   a,
   vCases[[3]],
   a - 1
  ],
  eCases[[3]],
```

```
    Which[
     vCases[[1]],
     a,
     vCases[[2]],
     a - 1,
     vCases[[3]],
     a - 2
    ]
   ];
cTilde := Simplify@
   Which[
    eCases[[1]],
    Which[
     vCases[[1]],
     c - 2,
     vCases[[2]],
     c - 1,
     vCases[[3]],
     c
    ],
    eCases[[2]],
    Which[
     vCases[[1]],
     c - 1,
     vCases[[2]],
     c,
     vCases[[3]],
     c + 1
    ],
    eCases[[3]],
    Which[
     vCases[[1]],
     c,
     vCases[[2]],
     c + 1,
     vCases[[3]],
     c + 2
    ]
   ];
```

## 2c: Determination of MVTilde

```
cases123Tilde =
  {cTilde ≥ n - sTilde, aTilde < sTilde && cTilde < n - sTilde, aTilde ≥ sTilde};

casesABCTilde = {aTilde ≥ m - tTilde,
   aTilde < m - tTilde && cTilde < n - m + tTilde, cTilde ≥ n - m + tTilde};

casesiAndiiTilde = {sTilde + tTilde ≤ b, sTilde + tTilde > b};


(*The code below gives the value of
 MVTilde(i.e., M̃Ṽ)for specific cases of s̃, t̃, ã, c̃*)
MVTilde :=
 Simplify@Which[
   Simplify[cases123Tilde[[1]] && casesABCTilde[[1]]],
   {b, 1, m},
   Simplify[cases123Tilde[[1]] && casesABCTilde[[2]] ||
     cases123Tilde[[1]] && casesABCTilde[[3]]],
   {b, 1, tTilde + aTilde},
   Simplify[cases123Tilde[[2]] && casesABCTilde[[1]] ||
     cases123Tilde[[3]] && casesABCTilde[[1]]],
   {b, n - sTilde - cTilde + 1, m},
   Simplify[cases123Tilde[[2]] && casesABCTilde[[2]] ||
     cases123Tilde[[2]] && casesABCTilde[[3]] || cases123Tilde[[3]] &&
      casesABCTilde[[2]] || cases123Tilde[[3]] && casesABCTilde[[3]]],
   Which[
    Simplify[casesiAndiiTilde[[1]]],
    {sTilde + tTilde - 1, tTilde + aTilde, n - sTilde - cTilde + 1},
    Simplify[casesiAndiiTilde[[2]]],
    {b, n - sTilde - cTilde + 1, tTilde + aTilde}
   ]
  ]
```

# Section 3: Verification of the lemma using `Compare`

```
prog3 = 0;
Dynamic[prog3]
Do[
  prog3 = i;
  $Assumptions = True;
  $Assumptions = fixedAssumptions && allCases[[i]];
  If[! TrueQ[Simplify[Compare[BMV, MVTilde]]], Print["The verification failed"];
   Break[]],
  {i, 1, Length[allCases]}] // AbsoluteTiming
```

# Appendix D

# Letter from Pouzet to Stanton

The following letter was reproduced with written permission from D. Stanton.

Université Claude Bernard
Groupe **LOGIQUE, MATHEMATIQUES DISCRETES, INFORMATIQUE**

Bât. Doyen Jean Braconnier
43, bd du 11 novembre 1918
69622 Villeurbanne Cedex

Maurice Pouzet, Lyon, January 14, 1991

To Prof. D. Stanton,

Dear Colleague,

Thanks for the reprint of your paper "Unimodality and Young's lattice". Your results solve negatively a conjecture I had with Ivan Rival about the unimodality of intervals of the lattice of orders.

In your letter you mention that you conjectured

1. $a_0 \le a_1 \le \cdots \le a_{[n/2]}$
2. $a_i \le a_{n-i}$ $i \le [n/2]$

for the Witney numbers of intervals $[\phi, \lambda]$ of Young's lattice.

If I understand correctly the conjecture, then I must say that it is true and follows

Tél. : ~~78.89.81.24~~ ou ~~78.89.20.71~~
72.44.83.04 - 78 89 61 55

from a result I had in 1976 about
the profile of relational systems (in fact,
the key is a lemma from Bill Kantor, 1972)
The result is this: let R be a relational system
on a set E, let $a_i$ be the number of
non isomorphic subrelational systems induced
on subsets of size $i$ (eg if R is a graph,
$a_i$ is the number of non isomorphic induced subgraphs
on $i$ vertices). If $|E| = n$, then the $a_i$'s
satisfy 1.2. of my paper in MZ (III.5.1 p 129), enclosed
in a separate letter.
Now, to $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_k]$ with $\lambda_1 \geq \lambda_2, \dots \geq \lambda_k \geq 1$
associate an equivalence relation into k classes
$E_1, \dots E_k$ of size $\lambda_1, \dots \lambda_k$. Since to every finite
restriction of size $i$, we can associate an isomorphic
one consisting of $n_j$ elements in $E_j$ in such a way that
$n_1 \geq n_2 \geq \dots \geq n_k$, it follows that the number of
non-isomorphic one is the number of partitions of $i$.

I have not seen your conjecture mentioned
in your paper. Do you think it would be useful
to write a joint halfpage note about it for JCT?
I am interested by your conjecture about self dual
partitions - It is really very nice, and it attracts me
(the proof of Kantor results is based on the duality between
a set and its complement, here too there is some duality)

Sincerely yours [redacted]

e-mail: LMDI -at FRCPN11.BITNET