Approximation of the Generalized Singular Value Expansion

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APPROXIMATION OF THE GENERALIZED SINGULAR VALUE EXPANSION

By
Matthew J. Roberts

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Abstract

Let $X$, $Y$, and $Z$ be real separable Hilbert spaces, let $T : X \to Y$ be a compact operator, and let $L : D(L) \to Z$ be a closed and densely defined linear operator. Then the generalized singular value expansion (GSVE) is an expansion that expresses $T$ and $L$ in terms of a common orthonormal basis. Under certain hypotheses on discretization, the GSVE of an approximate operator pair $(T_j, L_j)$, where $T_j : X_j \to Y_j$ and $L_j : X_j \to Z_j$, converges to the GSVE of $(T, L)$. Error estimates establish a rate of convergence that is consistent with numerical experiments in the case of discretization using piecewise linear finite elements. Further numerical testing suggests that a higher rate of convergence is attained by using higher order elements. However, the theory does not cover this case.
Chapter 1

Introduction

A linear inverse problem is a problem of the form $Tx = y$, where $T : X \rightarrow Y$ is a linear operator and we wish to estimate the exact solution $x^*$ from a noisy measurement $y$ of the exact data $y^*$ (see [1], [2], [3], and [4]). We consider only linear inverse problems defined on real separable Hilbert space. A particular class of inverse problems that is of interest consists of those in which the operator $T$ is an integral operator defined by

$$ (Tf)(s) = \int_{\Omega_1} k(s,t) f(t) d\Omega_1(t), \quad (1.1) $$

where $\Omega_1$ and $\Omega_2$ are either bounded intervals in $\mathbb{R}$, or are bounded two-dimensional domains. The corresponding separable Hilbert spaces are $X = L^2(\Omega_1)$ and $Y = L^2(\Omega_2)$ (see [5]).

We call $Tx = y$ an inverse problem only when the problem is unstable (that is, $x$ does not depend continuously on $y$); for this reason, it is necessary to use regularization of some sort to produce an acceptable approximation of the true solution $x$.

Let us consider the following model inverse problem. Let $X = L^2(0,1)$ and define $T : X \rightarrow X$ by the integral operator

$$ (Tx)(s) = \int_0^1 k(s,t) x(t) dt, $$

where $k(s,t) = \frac{1}{2} (s + t - |s - t|) - st$. It is a quick exercise to verify that $T$ is the solution operator to the following two-point boundary value problem.

$$ -y''(t) = x(t) \text{ in } (0,1) $$

$$ y(0) = 0 $$

$$ y(1) = 0. \quad (1.2) $$
More precisely, given the right hand side \(x(t)\) to the above two-point boundary value problem (1.2), the operator \(T\) gives back the solution \(y(t)\). Therefore,

\[
(Tx)(s) = \int_0^1 k(s,t)x(t)\,dt = y(s).
\]

In this example, the two-point boundary value problem (1.2) is the forward problem and the equation

\[
Tx = y
\]

defines the corresponding inverse problem.

To see that (1.3) actually defines an inverse problem, consider the case where \(x(t) = t\) is the exact solution. This produces the exact data \(y(s) = -\frac{s^3}{6} + \frac{s}{6}\). Let \(\beta \in \mathbb{R}^n\) be a measurement of \(y\) at \(n\) equally spaced points on \([0, 1]\), subject to uniformly distributed random noise, scaled to \(1\%\) in the Euclidean norm.

![Figure 1.1: Exact solution (left) and exact and noisy data (right) for the model inverse problem.](image)

We will now discretize the interval \([0, 1]\) in the following way. Let \(h = \frac{1}{n}\) and consider the mesh

\[
M = \left\{[0,h], [h,2h], \cdots, [(n-1)h,1]\right\}.
\]

We can discretize the space \(X\) by the finite dimensional subspace \(X_n = \text{span}\{x_1, x_2, \cdots, x_n\}\), where \(\{x_1, x_2, \cdots, x_n\}\) is the standard nodal basis for the space of continuous piecewise constant functions relative to the mesh \(M\). We discretize the operator \(T\) by the matrix \(A \in \mathbb{R}^{n \times n}\) using the Galerkin method (see [1], Section 3.2).
By replacing the equation \( Tx = y \) with the discretized equation \( A\alpha = \beta \), we then have the vector \( \hat{y} = \sum_{k=0}^{n-1} \beta_k x_k \) as our approximation (measurement) of \( y \), and the resulting vector \( \hat{x} = \sum_{k=0}^{n-1} \alpha_k x_k \) (where \( \alpha \) is the solution of \( A\alpha = \beta \)) as our approximation of \( x \). Figure 1.2 gives the plot of the approximated solution \( \hat{x} \).

As we can see, the approximate solution \( \hat{x} \) is not close to the actual solution \( x \). This illustrates the fact that the solution \( x \) to equation (1.3) does not continuously depend on the data \( y \). Hence, the equation \( Tx = y \) defines a linear inverse problem, and we will need to use regularization of some sort in order to solve it. We begin with considering classical Tikhonov regularization (see [6] and [5]).
In Tikhonov regularization, we consider the solution to the regularized problem

$$\min_{x \in X} \|Tx - y\|^2 + \lambda \|x\|^2,$$  \hfill (1.4)

where $\lambda > 0$ is a constant. In this case, $k \in L^2((0,1) \times (0,1))$ and $T$ is a compact operator, so the singular value expansion (SVE) of $T$ is invaluable for analyzing the solution to this regularized problem. The SVE of $T$ can be given as follows:

$$T = \sum_{k=1}^{\infty} \sigma_k \psi_k \otimes \phi_k.$$  \hfill (1.5)

Here, $\{\phi_k\}$ is a complete orthonormal sequence for the space $\mathcal{N}(T)^\perp$, $\{\psi_k\}$ is an orthonormal sequence in $X$, and $\{\sigma_k\}$ is the sequence of singular values of $T$, a sequence of positive numbers monotonically decreasing to 0 (see [7] Section 2.8 or see [4]). It is easy to show that the unique solution of (1.4) lies in $R(T^*)$ (see [4]).

Using the SVE of $T$, we have that for any $x = \sum_{k=1}^{\infty} \alpha_k \phi_k \in \mathcal{N}(T)^\perp$,

$$\|Tx - y\|^2 + \lambda \|x\|^2 = \left[ \sum_{k=0}^{\infty} \left( \sigma_k \langle x, \phi_k \rangle_X - \langle y, \psi_k \rangle_Y \right)^2 + \lambda \langle x, \phi_k \rangle_X^2 \right] + \|\hat{y}\|^2$$

$$= \sum_{k=0}^{\infty} \left( \sigma_k^2 + \lambda \left( \frac{\sigma_k^2}{\sigma_k^2 + \lambda} \langle y, \psi_k \rangle_Y \right)^2 + \left( 1 - \frac{\sigma_k^2}{\sigma_k^2 + \lambda} \right) \langle y, \psi_k \rangle_Y \right)$$

$$+ \|\hat{y}\|^2,$$

where $\hat{y}$ is the orthogonal projection of $y$ onto $R(T^\perp)$. Therefore, the unique solution to problem (1.4) is given by

$$x_{\lambda,y} = \sum_{k=0}^{\infty} \frac{\sigma_k}{\sigma_k^2 + \lambda} \langle y, \psi_k \rangle_Y \phi_k.$$  \hfill (1.6)

Using an appropriate choice for $\lambda$, the plots for the exact solution $x$ and the regularized solution $x_{\lambda,y}$ can be seen in Figure 1.3.

As we can see from figure 1.3, the regularized solution $x_{\lambda,y}$ inherits the Dirichlet boundary conditions of the forward problem (1.2). To understand why this is so, it is easy to show that $x_{\lambda,y} \in R(T^*)$. In this particular example, the operator $T$ is self-adjoint. This follows from the fact that the kernel $k(s,t)$ is symmetric about the line $s = t$ (i.e. $k(s,t) = k(t,s)$). Since $T$ is the solution operator to the forward problem (1.2) with Dirichlet boundary conditions, every element in the range of $T$ has Dirichlet boundary conditions. Thus, the regularized solution $x_{\lambda,y} \in R(T^*) = R(T)$ inherits
Dirichlet boundary conditions. For this reason, classical Tikhonov regularization does not work well for this problem.

We can generalize classical Tikhonov regularization by considering the unique solution to the problem

$$\min_{x \in D(L)} \|Tx - y\|^2 + \lambda \|Lx\|^2; \quad (1.7)$$

where again $\lambda > 0$ is a constant. Here, $L$ is a closed operator with domain $D(L)$ and is densely defined in $X$. It is chosen to have the property that $\|Lx\|$ is small for reasonable solutions $x$ and large for those $x$ with some undesirable feature. Such regularization is called Tikhonov regularization with seminorms since $\|x\|_L = \|Lx\|$ defines a seminorm on $D(L)$ (see [6] or [4]). In the case of classical Tikhonov regularization, $L$ is the identity operator, and the undesirable feature of $x$ is being large in magnitude.

In many cases, the true solution $x$ has a certain level of smoothness or regularity. In other words, the undesirable approximations for $x$ in the problem $Tx = y$ are those approximations in which the derivative is undefined or large in magnitude. In the case that $X = L^2(a,b)$, the derivative operator $L : D(L) \to Z$ is linear, closed, and densely defined in $X$. Thus, a method of regularization appropriate for problems of this kind will be Tikhonov regularization with seminorms defined by the derivative operator. Similarly, if $\Omega \subseteq \mathbb{R}^d$ ($d = 2$ or $d \geq 3$), the gradient operator is a suitable regularization operator for many problems.

Let problem (1.7) be discretized to produce the following linear algebra problem:

$$\min_{\bar{x} \in \mathbb{R}^n} \|A\bar{x} - \bar{y}\|^2 + \lambda \|B\bar{x}\|^2 \quad (1.8)$$
Here, \( A \in \mathbb{R}^{m \times n} \), \( B \in \mathbb{R}^{p \times n} \), \( \vec{x} \in \mathbb{R}^{n} \), and \( \vec{y} \in \mathbb{R}^{m} \), and we assume that \( \mathcal{N}(A) \cap \mathcal{N}(B) = \{0\} \), where \( \mathcal{N}(A) \) represents the null space of the matrix \( A \). In order to solve this problem, it is beneficial to simultaneously diagonalize the matrices \( A \) and \( B \). This is done by computing the generalized singular value decomposition (GSVD) of the matrix pair \((A, B)\). We present one version of the GSVD that is relevant to our discussion (see [8], Theorem 22.2).

**Theorem 1.** Let \( A \in \mathbb{R}^{m \times n} \) and \( B \in \mathbb{R}^{p \times n} \) be matrices such that \( m \geq n \) and \( \mathcal{N}(A) \cap \mathcal{N}(B) = \{0\} \). Then there exist a nonsingular matrix \( W \in \mathbb{R}^{n \times n} \), matrices \( U \in \mathbb{R}^{m \times n} \) and \( V \in \mathbb{R}^{p \times p} \) with orthonormal columns, and diagonal matrices \( S \in \mathbb{R}^{n \times n} \) and \( M \in \mathbb{R}^{p \times n} \) such that

\[
A = U S W^{-1}, \quad B = V M W^{-1}.
\]

Moreover, the diagonal entries \( s_i \) of \( S \) and \( m_1 \) of \( M \) are nonnegative, and satisfy

\[
s_i^2 + m_i^2 = 1 \text{ for } i = 1, 2, \ldots, p,
\]

\[
s_i = 1 \text{ for } i = p + 1, p + 2, \ldots, n
\]

(assuming for convenience that \( n \geq p \)). In matrix form,

\[
S^T S + M^T M = I.
\]

Let \( A = U S W^{-1} \) and \( B = V M W^{-1} \) be the GSVD of \((A, B)\) as given in the theorem. Then

\[
\|A \vec{x} - \vec{y}\|^2 + \lambda \|B \vec{x}\|^2 = \|USW^{-1} \vec{x} - \vec{y}\|^2 + \lambda \|VMW^{-1}\vec{x}\|^2
\]

\[
= \sum_{k=1}^{n} (s_k w_k - \langle u_k, \vec{y} \rangle)^2 + \sum_{k=1}^{p} \lambda (m_k w_k)^2
\]

\[
= \sum_{k=1}^{p} [(s_k w_k - \langle u_k, \vec{y} \rangle)^2 + \lambda (m_k w_k)^2] + \sum_{k=p+1}^{n} (w_k - \langle u_k, \vec{y} \rangle)^2.
\]

By regrouping the terms of these sums, we have

\[
\sum_{k=1}^{p} [(s_k^2 + \lambda m_k^2) (w_k - \frac{s_k \langle u_k, \vec{y} \rangle}{s_k^2 + \lambda m_k^2})^2 + \left(1 - \frac{s_k^2}{s_k^2 + \lambda m_k^2}\right) \langle u_k, \vec{y} \rangle]
\]
\[ + \sum_{k=p+1}^{n} (w_k - \langle u_k, \bar{y} \rangle)^2 \]

Therefore, the solution to the discretized problem is \( \bar{x} = W\bar{w} \) where
\[
 w_k = \frac{s_k}{s_k^2 + \lambda m_k^2} (u_k \cdot \bar{y}).
\]

Figure 1.4 shows the plots of the true solution \( x \) with the regularized solution \( x_{\lambda, y} \) coming from seminorm regularization. (A good value of \( \lambda \) was chosen by trial and error.)

![Figure 1.4: The exact solution of the model inverse problem, together with a solution produced by seminorm regularization.](image)

The generalized singular value expansion (GSVE) of an operator pair \((T, L)\), introduced in [4], allows the two operators \( T \) and \( L \) to be simultaneously diagonalized in
the same way that the GSVD of a matrix pair \((A, B)\) simultaneously diagonalizes the matrices \(A\) and \(B\). Therefore, the GSVE of the operator pair \((T, L)\) makes the analysis of problem \((1.7)\) relatively transparent in the same way the GSVD of the matrix pair \((A, B)\) does for the discretized problem \((1.8)\). To describe the GSVE, we establish the following conditions on the operators \(A, B\) and \(T, L\) (see [4]). The following theorem asserts the existence of the generalized singular value expansion of the operator pair \((T, L)\) (see [9], Theorem 4.2).

**Theorem 2.** Let \(X, Y,\) and \(Z\) be real separable Hilbert spaces. Assume that \(T : X \rightarrow Y\) is a compact linear operator and \(L : D(L) \rightarrow Z\) is a closed densely defined linear operator. Assume that there exists \(\gamma > 0\) such that \((1.9)\) holds. Then there exists a complete orthonormal set \(\{\phi_k : k \in I\}\) for \(D(L)\) where \(I\) is a countable index set, a partition \(M_0 \cup M_a \cup M_b\) of \(I\), orthonormal sets \(\{\psi_k : k \in M_0 \cup M_b\}\) in \(Y\), \(\{\theta_k : k \in M_0 \cup M_a\}\) in \(Z\), and subsets \(\{a_k : k \in I\}, \{b_k : k \in I\}\) of the nonnegative real numbers such that

\[
T = \sum_{k \in M_0 \cup M_b} a_k \psi_k \otimes_* \phi_k, \quad L = \sum_{k \in M_0 \cup M_a} b_k \theta_k \otimes_* \phi_k, \tag{1.11}
\]

and \(0 \leq a_k, b_k \leq 1, a_k^2 + b_k^2 = 1\) for every \(k \in I\). Here, \(\otimes_*\) refers to the outer product with respect to the \(*\)-norm. (i.e. \((\psi_k \otimes_* \phi_k)x = \langle \phi_k, x \rangle \psi_k\) for any \(x \in D(L)\)).

Using the GSVE of the operator pair \((T, L)\) given by \((1.11)\), we have, for any \(x \in D(L)\),

\[
\|Tx - y\|^2_X + \lambda \|Lx\|^2_Z = \|\hat{y}\|^2 + \sum_{k \in M_0 \cup M_b} (a_k \langle x, \phi_k \rangle_* - \langle y, \psi_k \rangle_Y)^2 + \sum_{k \in M_0 \cup M_a} \lambda b_k^2 \langle x, \phi_k \rangle_*^2 \\
= \|\hat{y}\|^2 + \sum_{k \in M_b} (a_k \langle x, \phi_k \rangle_* - \langle y, \psi_k \rangle_Y)^2 + \sum_{k \in M_a} \left[(a_k \langle x, \phi_k \rangle_* - \langle y, \psi_k \rangle_Y)^2 + \lambda b_k^2 \langle x, \phi_k \rangle_*^2\right] \\
+ \sum_{k \in M_a} \lambda b_k^2 \langle x, \phi_k \rangle_*^2
\]

\[8\]
\[ \sum_{k \in M_0} \left[ \left( a_k^2 + \lambda b_k^2 \right) \left( \langle x, \phi_k \rangle_* - \frac{a_k}{a_k^2 + \lambda b_k^2} \langle y, \psi_k \rangle_Y \right)^2 + \left( 1 - \frac{a_k^2}{a_k^2 + \lambda b_k^2} \right) \langle y, \psi_k \rangle_Y \right] \\
+ \sum_{k \in M_a} \lambda b_k^2 \langle x, \phi_k \rangle_*^2 + \sum_{k \in M_b} \left( a_k \langle x, \phi_k \rangle_* - \langle y, \psi_k \rangle_Y \right)^2 + \sum_{k \in M_a} \lambda b_k^2 \langle x, \phi_k \rangle_*^2 + \| \hat{y} \|^2. \]

Here, \( \hat{y} \) is the orthogonal projection of \( y \) onto \( R(T)^\perp \). Therefore, the solution of problem (1.7) is given by

\[ x_{\lambda,y} = \sum_{k \in M_0} \frac{a_k}{a_k^2 + \lambda b_k^2} \langle y, \psi_k \rangle_Y \phi_k + \sum_{k \in M_b} \langle y, \psi_k \rangle_Y \phi_k. \quad (1.12) \]

In practice, the GSVE of the operator pair \((T, L)\) is a useful tool for analyzing methods such as Tikhonov regularization with seminorms. The GSVE of \((T, L)\) can also be used directly to make computations, as can be seen in the above derivations for the regularized problem (1.7). In the next section, we provide an algorithm for computing the GSVE of \((T_j, L_j)\), where the operator pair \((T_j, L_j)\) is a finite dimensional discretization of the operator pair \((T, L)\). In order to compute the GSVE of the discretized operator pair \((T_j, L_j)\), we compute the GSVD of an associated pair of matrices.
Chapter 2

The approximate GSVE of \((T, L)\)

The purpose of this thesis is to propose and analyze a general approach to estimating the GSVE of an operator pair \((T, L)\). Two approaches were presented in [9]. The first is based on recognizing that the pairs \((a_k^2, \phi_k)\) for \(k \in I\) (with \(a_k = 0\) for \(k \in M_a\)) are the eigenpairs of the compact self-adjoint operator \(T^\#T\). These eigenpairs can be estimated using the general theory for symmetric, variationally posed eigenvalue problems, as presented in [10]. However, this approach has two shortcomings. We must choose a finite-dimensional subspace \(X_j\) of \(D(L)\) with basis \(\{x_1, x_2, \cdots, x_n\}\) and solve the generalized (matrix) eigenvalue problem

\[
G\alpha = \lambda M\alpha,
\]

where \(G \in \mathbb{R}^{n \times n}\) and \(M \in \mathbb{R}^{n \times n}\) are defined by

\[
G_{ij} = \langle x_j, x_i \rangle_Y, \quad M_{ij} = \langle Tx_i, Tx_j \rangle_Y.
\]

The first issue with this approach is the need to compute the matrix \(M\); generally, this matrix is expensive to compute. (For example, if \(T\) is a Fredholm integral operator, then each \(M_{ij}\) is defined by a triple integral.) The second difficulty is that, in the typical application (\(R(T)\) infinite-dimensional and not closed), \(M_0\) has infinite cardinality and \(a_k \to 0\) as \(k \to \infty\). It follows that by using an algorithm that computes \(a_k^2\) (instead of computing \(a_k\) directly), we artificially restrict the ability to compute small singular values; roughly speaking, at best we can compute values of \(a_k\) down to \(\sqrt{u}\) (where \(u\) is the unit round), rather than down to \(u\) itself.

It should be noted that the approach described in the previous paragraph, which is described fully in [9], does have the advantage that its convergence follows directly from the theory of symmetric, variationally posed eigenvalue problems.

The second approach, as described in this chapter, is based on reducing the computation to that of a (matrix) generalized singular value decomposition. The GSVE of a
pair of operators \((T_j, L_j)\) is related to the GSVD of a pair of related matrices, where \(T_j\) and \(L_j\) are finite dimensional operators that approximate \(T\) and \(L\) respectively in some sense. We now elaborate on this.

Let \(\{X_j\}, \{Y_j\}, \text{ and } \{Z_j\}\) be sequences of finite dimensional spaces contained in \(D(L)\), \(Y\), and \(Z\), respectively, such that for each \(j \in \mathbb{Z}^+\),

\[
X_j = \text{span}\{x_1^{(j)}, x_2^{(j)}, \ldots, x_{n_j}^{(j)}\},
\]

\[
Y_j = \text{span}\{y_1^{(j)}, y_2^{(j)}, \ldots, y_{m_j}^{(j)}\},
\]

\[
Z_j = \text{span}\{z_1^{(j)}, z_2^{(j)}, \ldots, z_{p_j}^{(j)}\}.
\]

Suppose that the sequence of spaces \(\{X_j\}\) approximate the space \(D(L)\) in that for any \(x \in D(L)\),

\[
\|\Pi_{X_j} x - x\|_* \to 0 \text{ as } j \to \infty.
\]

Here, \(\Pi_{X_j} : D(L) \to X_j\) is the orthogonal projection of \(D(L)\) onto \(X_j\) with respect to the \(*\)-norm as defined in equation (1.10). Similarly, suppose that \(\{Y_j\}\) and \(\{Z_j\}\) approximate the spaces \(Y\) and \(Z\), respectively, such that for any \(y \in Y\) and for any \(z \in Z\),

\[
\|P_{Y_j} y - y\|_Y \to 0 \text{ as } j \to \infty,
\]

\[
\|P_{Z_j} z - z\|_Z \to 0 \text{ as } j \to \infty.
\]

Here, \(P_{Y_j} : Y \to Y_j\) and \(P_{Z_j} : Z \to Z_j\) are the orthogonal projections of \(Y\) onto \(Y_j\) and \(Z\) onto \(Z_j\) respectively. For each \(j \in \mathbb{Z}^+\), let \(T_j : X_j \to Y_j\) and \(L_j : X_j \to Z_j\) be linear operators that approximate \(T\) and \(L\) in some sense. The conditions under which \(T_j\) and \(L_j\) should approximate \(T\) and \(L\) are made clear in the next chapter. For each \(j \in \mathbb{Z}^+\), we define the \(*_j\)-inner product on the space \(X_j\) by

\[
\langle x, y \rangle_{*_j} = \langle T_j x, T_j y \rangle_Y + \langle L_j x, L_j y \rangle_Z \text{ for all } x, y \in X_j
\]

In general, \(\langle \cdot, \cdot \rangle_{*_j}\) need not be positive definite on \(X_j\). To ensure that \(\langle \cdot, \cdot \rangle_{*_j}\) defines an inner product, we will assume that

\[
\mathcal{N}(T_j) \cap \mathcal{N}(L_j) = 0 \text{ for every } j \in \mathbb{Z}^+.
\]

Under the assumptions placed on \((T_j, L_j)\) in the next chapter, this must hold for all \(j \in \mathbb{Z}^+\) sufficiently large. Define the matrices \(A_j \in \mathbb{R}^{m_j \times n_j}\) and \(B_j \in \mathbb{R}^{p_j \times n_j}\) by

\[
(A_j)_{k\ell} = \langle y_k^{(j)}, T_j x_\ell^{(j)} \rangle_Y,
\]

\[
(B_j)_{k\ell} = \langle z_k^{(j)}, L_j x_\ell^{(j)} \rangle_Z.
\]
Let $H_j \in \mathbb{R}^{m_j \times m_j}$ and $J_j \in \mathbb{R}^{p_j \times p_j}$ be the Gram matrices for $\text{span}\{y_1^{(j)}, y_2^{(j)}, \ldots, y_{m_j}^{(j)}\}$ and $\text{span}\{z_1^{(j)}, z_2^{(j)}, \ldots, z_{p_j}^{(j)}\}$, respectively, which are defined by \\
\[
(H_j)_{k \ell} = \langle y_k^{(j)}, y_\ell^{(j)} \rangle_Y,\\
(J_j)_{k \ell} = \langle z_k^{(j)}, z_\ell^{(j)} \rangle_Z.
\]

The next theorem shows how to compute the GSVE of $(T_j, L_j)$ using the GSVD of the matrix pair $(H_j^{-1/2}A, J_j^{-1/2}B)$.

**Theorem 3.** Let $A_j, B_j, H_j,$ and $J_j$ be as defined above, and let \\
\[
H_j^{-1/2}A_j = US_jW_j^{-1}, \quad J_j^{-1/2}B_j = VM_jW_j^{-1/2}
\]
be the GSVD of the matrix pair $(H_j^{-1/2}A_j, J_j^{-1/2}B_j)$. Define the matrices $U_j = H_j^{-1/2}U$ and $V_j = J_j^{-1/2}V$. Then the GSVE of the operator pair $(T_j, L_j)$ is given by \\
\[
T_j = \sum_{k=1}^{\min\{m_j, n_j\}} a_k^{(j)} \psi_k^{(j)} \otimes_x \phi_k^{(j)},
\]
\[
L_j = \sum_{k=1}^{\min\{p_j, n_j\}} b_k^{(j)} \theta_k^{(j)} \otimes_v \phi_k^{(j)}.
\]

The values $a_1^{(j)}, a_2^{(j)}, \ldots, a_{\min\{m_j, n_j\}}^{(j)}$ are the diagonal entries of $S_j$, the values $b_1^{(j)}, b_2^{(j)}, \ldots, b_{\min\{p_j, n_j\}}^{(j)}$ are the diagonal entries of $M_j$, and \\
\[
\phi_k^{(j)} = \sum_{i=1}^{n_j} (W_j)_{ik} x_k,
\]
\[
\psi_k^{(j)} = \sum_{i=1}^{m_j} (U_j)_{ik} y_k,
\]
\[
\theta_k^{(j)} = \sum_{i=1}^{p_j} (V_j)_{ik} z_k.
\]

The sets $\{\phi_1^{(j)}, \phi_2^{(j)}, \ldots, \phi_{n_j}^{(j)}\}$, $\{\psi_1^{(j)}, \psi_2^{(j)}, \ldots, \psi_{m_j}^{(j)}\}$, and $\{\theta_1^{(j)}, \theta_2^{(j)}, \ldots, \theta_{n_j}^{(j)}\}$ are orthonormal in $X_j, Y,$ and $Z$, respectively, where the $*_j$-inner product is used on $X_j$.

The proof the Theorem is similar to that of Theorem 4.4 of [9]. In that paper, the special case of $T_j = P_{Y_j}T|_{X_j}$ and $L_j = P_{Z_j}L|_{X_j}$ is considered. However, the derivation
of the GSVE of an arbitrary pair of discretized operators \((T_j, L_j)\) is similar to the special case covered in [9]. For this reason, the proof of Theorem 3 is omitted.

The GSVE of the operator pair \((T_j, L_j)\) can be seen to be directly related to the GSVD of the pair of matrices from the last theorem. One advantage for computing this GSVE is that the approximate generalized singular vectors computed are orthogonal with respect to the spaces \(X_j, Y\), and \(Z_j\) with respect to the \(*j\)-norm, \(Y\)-norm, and \(Z\)-norm respectively. It is this orthogonality that makes analysis and computations transparent in much of applied mathematics.

As noted at the beginning of the chapter, our main goal is to analyze the convergence of the GSVE of \((T_j, L_j)\) to the GSVE of \((T, L)\). The next example demonstrates that a seemingly natural discretization need not lead to convergence of the GSVE.

**Example 1.** Let \(X = D(L) = H^1(0, 1)\) and \(Y = Z = L^2(0, 1)\). Define operators \(T : X \rightarrow Y\) and \(L : D(L) \rightarrow Z\) by \(T x = x\) and \(L x = x'\), respectively. By Rellich’s lemma, \(T\) (the identity operator) is compact. In this example, the \(*\)-norm is precisely the \(H^1(0, 1)\)-norm.

We can easily derive the GSVE of \((T, L)\) using Fourier analysis; the result is

\[
T = \sum_{k=0}^{\infty} a_k \psi_k \otimes \phi_k, \\
L = \sum_{k=1}^{\infty} b_k \theta_k \otimes \phi_k,
\]

where, for \(k \geq 1,
\[
\phi_k(t) = \sqrt{\frac{2}{k^2 \pi^2 + 1}} \cos(k\pi t),
\psi_k = \sqrt{2} \cos(k\pi t),
\theta_k(t) = -\sqrt{2} \sin(k\pi t),
\]

\[
a_k = \frac{1}{\sqrt{k^2 \pi^2 + 1}},
b_k = \frac{k\pi}{\sqrt{k^2 \pi^2 + 1}},
\]

and \(\phi_0(t) = 1, \psi_0(t) = 1, a_0 = 1,\) and \(b_0 = 0\). It can be verified that \(\{\phi_k\}_{k=1}^{\infty}, \{\psi_k\}_{k=1}^{\infty},\) and \(\{\theta_k\}_{k=1}^{\infty}\) are orthonormal in the \(*, Y,\) and \(Z\) inner products, respectively. Also

\[
a_k^2 + b_k^2 = 1,\ T \phi_k = a_k \psi_k, \text{ and } L \phi_k = b_k \theta_k \text{ for all } k \in \mathbb{Z}^+.
\]

In the notation of Theorem 1.11, we have \(M_0 = \mathbb{Z}^+, M_a = \emptyset,\) and \(M_b = \{0\}.

We discretize \((T, L)\) by defining \(X_j = Y_j = Z_j\) to be the space of continuous piecewise linear functions on a uniform mesh of \([0, 1]\) with \(j\) elements. Let \(\{x_0, x_1, \ldots, x_j\}\) be the standard nodal basis. Define \(T_j = P_j T|_{X_j}\) and \(L_j = P_j L|_{X_j}\). We compute the GSVE of \((T_{100}, L_{100})\) as described in Theorem 3 and graph \(\phi_k^{(100)}, \psi_k^{(100)}\), and \(\theta_k^{(100)}\) for \(k = 1, 2, 3\) (see Figures 1-3). We see that \(\phi_1^{(100)}, \psi_1^{(100)}, \theta_1^{(100)}\) and \(\phi_2^{(100)}, \psi_2^{(100)}, \theta_2^{(100)}\)
are accurate approximations of the corresponding exact functions, but \( \phi_{3}^{(100)} \), \( \psi_{3}^{(100)} \), and \( \theta_{3}^{(100)} \) are completely wrong. The behavior seen in Figure 3 is consistent with the type of "spurious modes" observed in the numerical solution of variationally posed eigenvalue problems (see [10]). The spurious mode persists as the mesh is refined.

Although we do not show any more results here, in fact every triple \((\phi_{k}^{(100)}, \psi_{k}^{(100)}, \theta_{k}^{(100)})\) for \( k > 3 \) is far from the exact generalized singular functions \((\phi_{k}, \psi_{k}, \theta_{k})\). Moreover, this behavior is not eliminated by refining the mesh. Every fourth generalized singular mode is spurious.

In the next chapter, we analyze the convergence of the GSVE of \((T_{j}, L_{j})\) to that of \((T, L)\), presenting a condition on the convergence of \((T_{j}, L_{j})\) to \((T, L)\) that guarantees that the corresponding GSVEs converge. We will see that the condition fails for the discretization in Example 1 and also see how to modify the discretization to obtain convergence.
Figure 2.2: The computed functions $\phi_2^{(100)}$ (top), $\psi_2^{(100)}$ (middle), and $\theta_2^{(100)}$ (bottom) for Example 1, together with the corresponding exact functions $\phi_2$, $\psi_2$, and $\theta_2$. In each graph, the approximate function is the solid curve and the exact function is the dashed curve. The approximate and exact curves are indistinguishable at this scale.

Figure 2.3: The computed functions $\phi_3^{(100)}$ (top), $\psi_3^{(100)}$ (middle), and $\theta_3^{(100)}$ (bottom) for Example 1, together with the corresponding exact functions $\phi_3$, $\psi_3$, and $\theta_3$. In each graph, the approximate function is the solid curve and the exact function is the dashed curve.
Chapter 3

Convergence

Let $X$, $Y$, and $Z$ be separable Hilbert spaces, let $T : X \to Y$ be a compact linear operator, and let $L : D(L) \to Z$ be a closed linear operator, where $D(L)$ is a dense subspace of $X$. We assume that there exists $\gamma > 0$ such that

$$\langle Tx, Tx \rangle_Y + \langle Lx, Lx \rangle_Z \geq \gamma \|x\|_X^2 \quad \forall x \in D(L).$$

We define the bilinear form $\langle \cdot, \cdot \rangle_* : D(L) \times D(L) \to \mathbb{R}$ by

$$\langle x, y \rangle_* = \langle Tx, Ty \rangle_Y + \langle Lx, Ly \rangle_Z.$$

Condition (1.9) from Chapter 1, which has been restated above, guarantees $\langle \cdot, \cdot \rangle_*$ defines an inner product on $D(L)$. Then, by Theorem 2, the GSVE of $(T, L)$ is given by

$$T = \sum_{k \in M_0 \cup M_b} a_k \psi_k \otimes_\ast \phi_k,$$

$$L = \sum_{k \in M_0 \cup M_a} b_k \theta_k \otimes_\ast \phi_k.$$ 

Here, $\{(a_k, b_k)\}$ are the generalized singular values of $(T, L)$, and the sets $\{\phi_k\}$, $\{\psi_k\}$, and $\{\theta_k\}$ are the generalized singular vectors of $(T, L)$.

In the last section, we provided an algorithm for computing the approximate GSVE of $(T_j, L_j)$, which is given by

$$\phi_k^{(j)} = \sum_{i=1}^{n_j} (W_j)_{ik} x_k, \quad \psi_k^{(j)} = \sum_{i=1}^{m_j} (U_j)_{ik} y_k, \quad \theta_k^{(j)} = \sum_{i=1}^{p_j} (V_j)_{ik} z_k.$$

We next provide sufficient conditions under which the GSVE of $(T_j, L_j)$ is guaranteed.
to converge to the GSVE of \((T, L)\). Informally, this means that the approximate generalized singular values \((a^{(j)}_k, b^{(j)}_k)\) converge to the exact generalized singular values \((a_k, b_k)\), and the approximate generalized singular vectors \(\{\phi^{(j)}_k\}, \{\psi^{(j)}_k\}, \{\theta^{(j)}_k\}\) converge to the exact generalized singular vectors \(\{\phi_k\}, \{\psi_k\}, \{\theta_k\}\). Convergence of the generalized singular vectors poses a complicated matter since the sets of generalized singular vectors correspond to subspaces of the Hilbert spaces \(D(L), Y, Z\). The issues are comparable to those faced in approximating the eigenvalues and eigenvectors of a linear operator \(A : X \to X\) by the eigenvalues and eigenvectors of an approximation \(A_j\) of \(A\). We refer the reader to Boffi’s survey article [10] for a detailed discussion. In the case of eigenvalues and eigenvectors, we can expect that the eigenvalues of \(A_j\) to converge to the corresponding eigenvalues of \(A\) in the expected manner. However, since a given eigenspace does not have a unique basis, there is no reason that the computed basis of the corresponding eigenspace of \(A_j\) to converge directly to a given basis of an eigenspace of \(A\). Therefore, we have to refer to convergence of a sequence of subspaces to a given subspace, not the convergence of individual eigenvectors. Moreover, if \(\lambda\) is an eigenvalue of \(A\) of multiplicity \(k\), then there are probably \(k\) simple eigenvalues of \(A_j\) that converge to \(\lambda\) as \(j \to \infty\).

When discussing the convergence of the GSVE of \((T_j, L_j)\) to the GSVE of \((T, L)\), we have an additional complication, namely that both \(T\) and \(L\) can have a nontrivial null space. It is straightforward to show that \(\mathcal{N}(L)\) must be finite-dimensional (otherwise, the inequality (1.9) is incompatible with the compactness of \(T\)). However, \(\mathcal{N}(T)\) could be infinite-dimensional. We will assume throughout our discussion that \(\mathcal{R}(T)\) is infinite-dimensional, since this is the interesting case in applications.

In terms of the GSVE of the operator pair \((T, L)\),

\[
T = \sum_{k \in M_0 \cup M_b} a_k \psi_k \otimes_\ast \phi_k,
\]

\[
L = \sum_{k \in M_0 \cup M_a} b_k \theta_k \otimes_\ast \phi_k,
\]

the generalized singular values of \((T, L)\) have the following properties:

\[
k \in M_b \implies a_k = 1 \text{ and } b_k = 0,
\]

\[
k \in M_0 \implies 0 < a_k, b_k < 1,
\]

\[
k \in M_a \implies a_k = 0 \text{ and } b_k = 1.
\]

To compare the singular values of \((T_j, L_j)\) with those of \((T, L)\), we have to order the generalized singular values of \((T, L)\) consistently. Since \(\mathcal{N}(L)\) is finite-dimensional, we will assume that \(\dim(\mathcal{N}(L)) = \ell\) and that \(M_b = \{1, 2, \cdots, \ell\}\). Since \(\mathcal{R}(T)\) is infinite-dimensional by assumption, we will define the index set \(M_0\) by \(M_0 = \{\ell + 1, \ell + 2, \cdots\}\) and assume that \(a_{\ell+1} \geq a_{\ell+2} \geq \ldots\). Since \(a_k^2 + b_k^2 = 1\), this implies that \(b_{\ell+1} \leq b_{\ell+2} \leq \ldots\).
With these definitions for \( M_b \) and \( M_a \), we see that \( \{a_k : k \in \mathbb{Z}^+\} \) is a nonincreasing sequence of positive real numbers, and \( \{b_k : k \in \mathbb{Z}^+\} \) is a nondecreasing sequence of nonnegative real numbers. However, if \( M_a \) is nonempty (that is, if \( T \) has a nontrivial null space), then there is no natural definition for \( M_a \) that maintains the monotonicity of the sequences \( \{a_k\} \) and \( \{b_k\} \). Therefore, we will continue to denote \( M_a \) as a (countable) abstract index set. We can now write the GSVE of \((T, L)\) as follows:

\[
T = \sum_{k=1}^{\infty} a_k \psi_k \otimes_s \phi_k, 
\]

\[
L = \sum_{k=1}^{\infty} b_k \theta_k \otimes_s \phi_k + \sum_{k \in M_a} b_k \theta_k \otimes_s \phi_k. 
\]  

Recall that the GSVE of \((T_j, L_j)\) is given by

\[
T_j = \min\{m, n_j\} \sum_{k=1}^{m} a_k^{(j)} \psi_k^{(j)} \otimes_{s_j} \phi_k^{(j)}, 
\]

\[
L_j = \min\{p_j, n_j\} \sum_{k=1}^{p_j} b_k^{(j)} \theta_k^{(j)} \otimes_{s_j} \phi_k^{(j)}. 
\]

Here, we assume that \( a_1^{(j)} \geq a_2^{(j)} \geq \cdots \geq a_{n_j}^{(j)} \) and \( b_1^{(j)} \leq b_2^{(j)} \leq \cdots \leq b_{n_j}^{(j)} \).

To describe the convergence of the singular vectors of \((T_j, L_j)\) to those of \((T, L)\), we will use the concept of the gap between two subspaces (see [10]).

**Definition 4.** Let \( H \) be a Hilbert space, and let \( U \) and \( V \) be closed subspaces of \( H \). The gap between \( U \) and \( V \) is defined to be \( \hat{\delta}(U,V) \), where

\[
\delta(U,V) = \sup_{u \in U} \inf_{v \in V} \|u - v\|, 
\]

\[
\hat{\delta}(U,V) = \max\{\delta(U,V), \delta(V,U)\}. 
\]

We now introduce the concept of angle between subspaces \( U \) and \( V \) of a Hilbert space \( H \). We define the asymmetric angle \( \theta(U,V) \), denoted more simply by \( \theta \), by

\[
\cos(\theta(U,V)) = \cos(\theta) = \inf_{u \in U} \sup_{v \in V} \frac{\langle u, v \rangle}{\|u\|=1 \|v\|=1}. 
\]

Notice that the above quantity is bounded between 0 and 1, so \( \theta \in [0, \pi/2] \) is well defined. We next derive the following properties about the asymmetric angle and asymmetric gap.
Theorem 5. Let $U$ and $V$ be closed subspaces of a Hilbert space $H$. Then the asymmetric angle $\theta = \theta(U, V)$ has the following property:

$$\cos(\theta(U, V)) = \cos(\theta) = \inf_{u \in U} \|P_V u\|.$$ 

Proof. Let $u \in U$ such that $\|u\| = 1$. If $\|P_V u\| = 0$, then $u \in V^{\perp}$ and hence

$$\langle u, v \rangle = \langle P_V u, v \rangle = 0.$$ 

Thus, $\sup_{v \in V} \langle u, v \rangle_H = \|P_V u\|_H$ in this case. Suppose that $\|P_V u\| \neq 0$. Then,

$$\|P_V u\| = \sup_{v \in V} \left\langle P_V u, \frac{P_V u}{\|P_V u\|} \right\rangle \leq \sup_{v \in V} \langle P_V u, v \rangle \left\| \frac{P_V u}{\|P_V u\|} \right\| \leq \sup_{v \in V} \|P_V u\| \|v\| = \|P_V u\|.$$ 

Since $\langle u, v \rangle = \langle P_V u, v \rangle$ for every $v \in V$, we have

$$\sup_{v \in V} \langle u, v \rangle_H = \sup_{\|v\|_H = 1} \inf_{\|u\| = 1} \|P_V u - v\|^2 = \sup_{\|u\| = 1} \|P_V u\|^2.$$ 

Thus we have shown that $\sup_{\|v\|_H = 1} \langle u, v \rangle_H = \|P_V u\|_H$ in every case. We then have

$$\inf_{u \in U} \|P_V u\| = \inf_{u \in U} \sup_{\|v\| = 1} \langle u, v \rangle = \cos(\theta).$$

Theorem 6. Let $U, V$ be closed subspaces of a Hilbert space $H$. Then

$$\delta(U, V) = \sin(\theta(U, V)).$$

Proof. This follows directly from the definition of asymmetric gap. Writing $\theta = \theta(U, V)$,

$$\delta(U, V)^2 = \left( \sup_{u \in U} \inf_{v \in V} \|u - v\| \right)^2 \leq \sup_{\|u\| = 1} \|u - P_V u\|^2$$
\[ \delta(U, V) = \sin(\theta). \]

**Theorem 7.** If the asymmetric angles \( \theta(U, V) \) and \( \theta(V, U) \) are strictly less than \( \pi/2 \), then \( \theta(U, V) = \theta(V, U) \) and

\[ \delta(U, V) = \delta(V, U). \]

**Proof.** Let \( \theta = \theta(U, V) \) and let \( \omega = \theta(V, U) \). Suppose that \( \theta, \omega < \pi/2 \). Then \( \cos(\theta), \cos(\omega) > 0 \), and

\[ \inf_{u \in U} \|P_V u\| > 0, \quad \inf_{v \in V} \|P_U v\| > 0. \]

Therefore, the projections \( P_U \) and \( P_V \) are bounded below when restricted to \( V \) and \( U \) respectively. Hence, \( P_U : V \to U \) and \( P_V : U \to V \) each have closed range in \( U \) and \( V \), respectively, and are both injective. Also, we have for any \( u \in U \) and for any \( v \in V \),

\[ \langle P_V u, v \rangle = \langle u, v \rangle = \langle u, P_U v \rangle. \]

Thus, as operators between the spaces \( U \) and \( V \), \( P_V^* = P_U \) and \( P_U^* = P_V \). Then

\[ \mathcal{R}(P_U) = \mathcal{N}(P_V)^\perp = U, \]
\[ \mathcal{R}(P_V) = \mathcal{N}(P_U)^\perp = V, \]

which shows that \( P_U \) and \( P_V \) are bijections between \( U \) and \( V \). Therefore, for any unit vector \( u \in U \), there exists a unit vector \( v \in V \) such that \( u = \frac{P_U v}{\|P_U v\|} \). It follows that

\[ \|P_V u\| = \sup_{x \in V} \langle P_V u, x \rangle \]
\[ = \sup_{x \in V} \langle u, P_U x \rangle \]
\[ = \sup_{x \in V} \langle \frac{P_U v}{\|P_U v\|}, P_U x \rangle \geq \langle \frac{P_U v}{\|P_U v\|}, P_U v \rangle = \|P_U v\|. \]
Therefore we have
\[ \inf_{v \in V} \|P_Uv\| \leq \|P_Vu\| \quad \forall u \in U, \|u\| = 1 \]
\[ \implies \inf_{v \in V} \|P_Uv\| \leq \inf_{u \in U} \|P_Vu\|. \]

By symmetry of $U$ and $V$ in the above formulation, we then have
\[ \inf_{v \in V} \|P_Uv\| \geq \inf_{u \in U} \|P_Vu\|. \]
Thus, $\cos(\theta(U,V)) = \cos(\theta(V,U))$, and we have
\[ \delta(U,V) = \sin(\theta(U,V)) = \sin(\theta(V,U)) = \delta(V,U). \]

\[ \square \]

**Corollary 8.** If $\delta(U,V)$ and $\delta(V,U)$ are strictly less than 1, then
\[ \delta(U,V) = \delta(V,U). \]

*Proof.* Suppose $\delta(U,V), \delta(V,U) < 1$. Then $\sin(\theta(U,V)), \sin(\theta(V,U)) < 1$ and therefore the angles $\theta(U,V)$ and $\theta(V,U)$ are strictly greater than 0, and the previous theorem then follows. \[ \square \]

Given the sequences $\{a_k\}$ and $\{b_k\}$ of singular values and the sequences $\{\phi_k\}, \{\psi_k\},$ and $\{\theta_k\}$ of singular vectors for $(T,L)$, we define the corresponding singular spaces by
\[ S_k(\phi) = \text{span}\{\phi_i : a_i = a_k\} \]
\[ S_k(\psi) = \text{span}\{\psi_i : a_i = a_k\} \]
\[ S_k(\theta) = \text{span}\{\theta_i : a_i = a_k\}. \]

Typically, if $a_k$ is a multiple singular value (that is, $\dim(S_k(\phi)) > 1$), then each approximate singular value $a^{(j)}_k$ converging to $a_k$ will be a simple singular value of $(T_j, L_j)$, meaning that
\[ \dim \{\text{span}\{\phi^{(j)}_i : a^{(j)}_i = a^{(j)}_k\}\} = 1. \]
For this reason, we define the approximate singular spaces of $(T_j, L_j)$ by
\[ S^{(j)}_k(\phi) = \text{span}\{\phi^{(j)}_i : a^{(j)}_i \to a_k \text{ as } \ell \to \infty\} \]
\[
S_k^{(j)}(\psi) = \text{span}\{\psi_i^{(j)} : a_i^{(\ell)} \to a_k \text{ as } \ell \to \infty\}
\]
\[
S_k^{(j)}(\theta) = \text{span}\{\phi_i^{(j)} : a_i^{(\ell)} \to a_k \text{ as } \ell \to \infty\}.
\]

Note that because \(a_k^2 + b_k^2 = 1\) for every \(k \in \mathbb{Z}^+\), it follows that
\[
\{i \in \mathbb{Z}^+ : a_i = a_k\} = \{i \in \mathbb{Z}^+ : b_i = b_k\}
\]

Therefore, we could have defined the above subspaces with reference to \(\{b_k\}\) instead of \(\{a_k\}\).

We can now define what it means for the GSVE of \((T_j, L_j)\) to converge to the GSVE of \((T, L)\) (see [11], Definition 5).

**Definition 9.** We say that the GSVE of \((T_j, L_j)\), \(j \in \mathbb{Z}^+\), converges to the GSVE of \((T, L)\) if, for all \(N \in \mathbb{Z}^+\) and all \(\varepsilon > 0\), there exists an integer \(j_0\) such that for all integers \(j \geq j_0\),

\[
\left| a_k^{(j)} - a_k \right| < \varepsilon \text{ for every } k = 1, 2, \ldots, N,
\]
\[
\left| b_k^{(j)} - b_k \right| < \varepsilon \text{ for every } k = 1, 2, \ldots, N,
\]
\[
\hat{\delta}\left(S_k^{(j)}(\phi), S_k(\phi)\right) < \varepsilon \text{ for every } k = 1, 2, \ldots, N,
\]
\[
\hat{\delta}\left(S_k^{(j)}(\psi), S_k(\psi)\right) < \varepsilon \text{ for every } k = 1, 2, \ldots, N,
\]
\[
\hat{\delta}\left(S_k^{(j)}(\theta), S_k(\theta)\right) < \varepsilon \text{ for every } k = 1, 2, \ldots, N.
\]

In computing the gaps, we use the \(\ast\), \(Y\), and \(Z\) norms for

\[
\hat{\delta}\left(S_k^{(j)}(\phi), S_k(\phi)\right), \hat{\delta}\left(S_k^{(j)}(\psi), S_k(\psi)\right), \hat{\delta}\left(S_k^{(j)}(\theta), S_k(\theta)\right)
\]

respectively.

Notice that Definition 9 does not refer to \(\{\phi_k : k \in M_a\}\) or \(\{\theta_k : k \in M_a\}\). Our theory with show that, in the representation

\[
T = \sum_{k=1}^{\infty} a_k \psi_k \otimes_\ast \phi_k,
\]
\[
L = \sum_{k=1}^{\infty} b_k \theta_k \otimes_\ast \phi_k + \sum_{k \in M_a} b_k \theta_k \otimes_\ast \phi_k,
\]

the series for \(T\) and the first series in the representation of \(L\) are approximated. It is not guaranteed that we can approximate the second series in the representation of \(L\).
For each \( j \in \mathbb{Z}^+ \), we refer to three different inner products on the space \( X_j \), namely the \( * \)-inner product, the \( *_j \)-inner product, and the \( X \)-inner product. Therefore, there are three different adjoint operators for the operator \( T_j \). The adjoint of \( T \) with respect to the \( * \)-inner product is denoted by \( T^* \), the adjoint of \( T_j \) with respect to the \( *_j \)-inner product by \( T_j^* \), and the adjoint of \( T \) with respect to the \( X \)-inner product by \( T^* \). To study the convergence of the GSVE of \((T_j, L_j)\) to that of \((T, L)\), we consider the operators \( T_j^# \), \( T_j P_{X_j} \) and \( T^# T \). Using the expansion for \( T \) as in equation (3.1), we see that the eigenpairs of \( T^# T \) are \((a_k^2, \phi_k)\), \( k = 1, 2, \ldots \). Similarly, the eigenpairs for the operator \( T_j^# \), \( T_j \) are \( \left((a_k^{(j)})^2, \phi_k^{(j)}\right) \), \( k = 1, 2, \ldots, n_j \). Our goal is to show that the eigensystem of \( T_j^# \) converges to that of \( T^# T \); we can then show that the GSVE of \((T_j, L_j)\) converges to the GSVE of \((T, L)\).

The theory of Babuska and Osborn ([12]; see also [10], Sections 6 and 9) shows that if a sequence \( \{A_j\} \) of compact operators \( A_j : X \to X \) converges in the operator norm to the compact operator \( A : X \to X \), then eigensystems of \( A_j \) converge to the eigensystem of \( A \) as \( j \to \infty \), provided we exclude the zero eigenvalues of \( A \) from consideration. Specifically, we have the following theorem ([10], Theorem 9.1) (in which \( \rho(A) \) denotes the resolvent set of \( A \)).

\textbf{Theorem 10.} Let \( A : X \to X \) be a compact linear operator, and let \( \{A_j\} \) be a sequence of compact linear operators from \( X \) to \( X \) such that

\[ \|A - A_j\|_{\mathcal{L}(X, X)} \to 0 \text{ as } j \to \infty. \]

Then for any compact set \( K \subseteq \rho(A) \), there exists \( j_0 \in \mathbb{Z}^+ \) such that for every \( j \geq j_0 \), we have \( K \subseteq \rho(A_j) \). If \( \lambda \) is a non-zero eigenvalue of \( A \) with multiplicity \( m \), then there are \( m \) eigenvalues \( \lambda_1^{(j)}, \lambda_2^{(j)}, \ldots, \lambda_m^{(j)} \) of \( A_j \), repeated according to their algebraic multiplicities, such that each \( \lambda_i^{(j)} \) converges to \( \lambda \) as \( j \to \infty \). Moreover, if we define \( E_j(\lambda) \) to be the direct sum of the eigenspaces corresponding to the eigenvalues of \( \lambda_1^{(j)}, \lambda_2^{(j)}, \ldots, \lambda_m^{(j)} \), then the gap between \( E_j(\lambda) \) and the eigenspace \( E(\lambda) \) corresponding to the eigenvalue \( \lambda \) tends to 0 as \( j \to \infty \).

By the above discussion, if we show that \( A_j = T_j^# \), \( T_j P_{X_j} \) converges to \( A = T^# T \) in the operator norm, then it will follow that the eigensystem of \( T_j^# \), \( T_j P_{X_j} \) converges to the eigensystem of \( T^# T \). We will use the following fundamental result (See [13]).
Theorem 11. Let $U$, $V$, and $W$ be Hilbert spaces. Let $M : V \to W$ be a bounded linear operator, let $T : U \to V$ be a compact linear operator, and let $M_j : V \to W$ be a bounded linear operator for each $j \in \mathbb{Z}^+$. Suppose that $M_j \to M$ pointwise on $V$. Then

$$\|(M_j - M)T\|_{L(U,W)} \to 0 \text{ as } j \to \infty.$$ 

Example 1 shows that the GSVE of $(T_j, L_j)$ need not converge to the GSVE of $(T, L)$. We now describe the fundamental assumption on the sequences $\{T_j\}$ and $\{L_j\}$ that will allow us to prove convergence of the GSVE. For each $j \in \mathbb{Z}^+$, we define

$$t_{j,1} = \max_{x \in X_j, x \neq 0} \frac{\|(T - T_j)x\|_Y}{\|x\|_X},$$

$$t_{j,2} = \max_{x \in X_j, x \neq 0} \frac{\|(T - T_j)x\|_Y}{\|x\|_*},$$

$$t_j = \max\{t_{j,1}, t_{j,2}\},$$

$$\ell_j = \max_{x \in X_j, x \neq 0} \frac{\|(L - L - j)x\|_Z}{\|x\|_*},$$

$$c_j = \sqrt{t_j^2 + \ell_j^2}.$$

Henceforth, we will assume that $c_j \to 0$ as $j \to \infty$. We will see that this is enough to imply that the GSVE of $(T_j, L_j)$ converges to the GSVE of $(T, L)$.

By (3.5), we have

$$\|(T_j - T)x\|_Y \leq t_j \|x\|_X \text{ and } \|(T_j - T)x\|_Y \leq t_j \|x\|_* \text{ for all } x \in X_j,$$

and, by (3.6)

$$\|(L_j - L)x\|_Z \leq \ell_j \|x\|_* \text{ for all } x \in X_j.$$

Therefore, for all $x \in X_j$,

$$\|T_jx\|_Y^2 = \langle T_jx, T_jx \rangle_Y = \langle (T_j - T)x, T_jx \rangle_Y + \langle Tx, T_jx \rangle_Y \leq \|(T_j - T)x\|_Y \|T_jx\|_Y + \|Tx\|_Y \|T_jx\|_Y \leq t_j \|x\|_* \|T_jx\|_Y + \|Tx\|_Y \|T_jx\|_Y.$$

Hence,

$$\|T_jx\|_Y \leq t_j \|x\|_* + \|Tx\|_Y \leq (1 + t_j) \|x\|_* \text{ for all } x \in X_j \quad (3.8)$$

Henceforth, we will assume that $c_j \to 0$ as $j \to \infty$. We will see that this is enough to imply that the GSVE of $(T_j, L_j)$ converges to the GSVE of $(T, L)$. By (3.5), we have

$$\|(T_j - T)x\|_Y \leq t_j \|x\|_X \text{ and } \|(T_j - T)x\|_Y \leq t_j \|x\|_* \text{ for all } x \in X_j,$$

and, by (3.6)

$$\|(L_j - L)x\|_Z \leq \ell_j \|x\|_* \text{ for all } x \in X_j.$$
(since obviously $\|Tx\|_Y \leq \|x\|_*$ for all $x \in D(L)$). Similarly,
\begin{equation}
\|L_jx\|_Z \leq (1 + \ell_j)\|x\|_* \text{ for all } x \in X_j.
\end{equation}

In our analysis, it will useful to define the quantity $\eta_j$ by
\begin{equation}
\eta_j = c_j^2 + 2(t_j + \ell_j).
\end{equation}

We will need the following bound.

**Lemma 12.** For every $j \in \mathbb{Z}^+$, and for every $x, y \in X_j$,
\begin{equation}
|\langle x, y \rangle_* - \langle x, y \rangle_{*j}| \leq \eta_j \|x\|_*\|y\|_*.
\end{equation}

**Proof.** Let $j \in \mathbb{Z}^+$, and let $x, y \in X_j$. Then
\begin{align*}
|\langle x, y \rangle_* - \langle x, y \rangle_{*j}| &= |\langle Tx, Ty \rangle_Y + \langle Lx, Ly \rangle_Z - \langle T_jx, T_jy \rangle_Y - \langle L_jx, L_jy \rangle_Z| \\
&= |\langle Tx, (T - T_j)y \rangle_Y + \langle Lx, (L - L_j)y \rangle_Z + \langle (T - T_j)x, T_jy \rangle_Y \\
&\quad + \langle (L - L_j)x, L_jy \rangle_Z| \\
&\leq \|Tx\|_Y \|(T - T_j)y\|_Y + \|Lx\|_Z \|(L - L_j)y\|_Z \\
&\quad + \|(T - T_j)x\|_Y \|T_jy\|_Y + \|(L - L_j)x\|_Z \|L_jy\|_Z \\
&\leq t_j \|Tx\|_Y \|y\|_* + \ell_j \|Lx\|_Z \|y\|_* + t_j \|x\|_* \|T_jy\|_Y + \ell_j \|x\|_* \|L_jy\|_Z \\
&\leq (t_j^2 + \ell_j^2) \|x\|_* \|y\|_* + t_j (1 + t_j) \|x\|_* \|y\|_* + \ell_j (1 + \ell_j) \|x\|_* \|y\|_* \\
&= (t_j^2 + \ell_j^2 + 2(t_j + \ell_j)) \|x\|_* \|y\|_* \\
&= \eta_j \|x\|_* \|y\|_*.
\end{align*}

From this, we have the following Corollary.

**Corollary 13.** For any $j \in \mathbb{Z}^+$ and for any $x \in X_j$,
\begin{align}
(1 - \eta_j)\|x\|_*^2 &\leq \|x\|_{*j}^2 \leq (1 + \eta_j)\|x\|_*^2, \\
\frac{1}{1 + \eta_j} \|x\|_{*j} &\leq \|x\|_*^2 \leq \frac{1}{1 - \eta_j} \|x\|_{*j}^2.
\end{align}

Next, we define $M_j : X_j \to X_j$ and $M : D(L^*L) \to X$ by
\begin{align*}
M &= T^*T + L^*L \\
M_j &= T_j^*T_j + L_j^*L_j.
\end{align*}

Let $x \in D(L)$ and $y \in D(L^*L)$. Notice that
\begin{equation}
\langle x, y \rangle_* = \langle Tx, Ty \rangle_Y + \langle Lx, Ly \rangle_Z = \langle x, (T^*T + L^*L)y \rangle_X.
\end{equation}
Therefore, $M$ has the following property:
\[
\langle x, y \rangle_* = \langle x, My \rangle_X \quad \forall x \in D(L) \quad \forall y \in D(L^*L).
\] (3.13)

Similarly, the operator $M_j$ has the following property:
\[
\langle x, y \rangle_{*j} = \langle x, M_jy \rangle_X \quad \forall x, y \in X_j.
\] (3.14)

These operators will be central to our analysis; the following three results come from these two properties of $M$ and $M_j$.

**Theorem 14.** The operator $M$ is a bijection with bounded inverse, and

\[
\|M^{-1}\|_{\mathcal{L}(X,D(L))} < \frac{1}{\sqrt{\gamma}},
\]

that is,

\[
\|M^{-1}x\|_* \leq \frac{\|x\|_X}{\sqrt{\gamma}} \quad \forall x \in X.
\]

**Proof.** See [4], Theorem 5.25. \qed

**Theorem 15.** For each $j \in \mathbb{Z}^+$, the operator $M_j$ is invertible and

\[
\|M_j\|_{\mathcal{L}(X,D(L))} \leq \frac{1}{(1 - \eta_j)\sqrt{\gamma}}.
\]

That is,

\[
\|M_j^{-1}x\|_* \leq \frac{1}{(1 - \eta_j)\sqrt{\gamma}}\|x\|_X \quad \forall x \in X_j.
\]

**Proof.** Let $j \in \mathbb{Z}^+$ and let $x \in X_j$. By Corollary [13], we have
\[
\|M_j^{-1}x\|^2 \leq \frac{1}{1 - \eta_j} \|M_j^{-1}x\|^2_{*j} = \frac{1}{1 - \eta_j} \langle M_j^{-1}x, M_j^{-1}x \rangle_{*j} = \frac{1}{1 - \eta_j} \langle M_j^{-1}x, x \rangle_x \\
\leq \frac{1}{1 - \eta_j} \|M_j^{-1}x\|_X \|x\|_X \\
\leq \frac{1}{(1 - \eta_j)\sqrt{\gamma}}\|M_j^{-1}x\|_* \|x\|_X.
\]
The desired result follows from dividing both sides of the inequality by \( \|M_j^{-1}x\|_* \).

Next, recall that \( \Pi_{X_j} : D(L) \to X_j \) denotes the orthogonal projection with respect to the \( * \)-inner product onto the subspace \( X_j \). The following result allows us to compare \( M^{-1} \) and \( M_j^{-1} \).

**Theorem 16.** For every \( x \in X \),

\[
\|\Pi_{X_j}M^{-1}x - M_j^{-1}P_{X_j}x\|_* \leq \frac{\eta_j}{(1 - \eta_j)\sqrt{\gamma}}\|x\|_X.
\]

**Proof.** Let \( x \in X \). Then

\[
\|\Pi_{X_j}M^{-1}x - M_j^{-1}P_{X_j}x\|_*^2 = \langle (\Pi_{X_j}M^{-1} - M_j^{-1}P_{X_j})x, (\Pi_{X_j}M^{-1} - M_j^{-1}P_{X_j})x \rangle_*
\]

\[
= \langle M^{-1}x, (\Pi_{X_j}M^{-1} - M_j^{-1}P_{X_j})x \rangle_* - \langle M_j^{-1}P_{X_j}x, (\Pi_{X_j}M^{-1} - M_j^{-1}P_{X_j})x \rangle_*
\]

Notice that

\[
\langle M^{-1}x, (\Pi_{X_j}M^{-1} - M_j^{-1}P_{X_j})x \rangle_* = \langle x, (\Pi_{X_j}M^{-1} - M_j^{-1}P_{X_j})x \rangle_X
\]

\[
= \langle P_{X_j}x, (\Pi_{X_j}M^{-1} - M_j^{-1}P_{X_j})x \rangle_X
\]

\[
= \langle M_j^{-1}P_{X_j}x, (\Pi_{X_j}M^{-1} - M_j^{-1}P_{X_j})x \rangle_{*j}.
\]

Therefore we have

\[
\|\Pi_{X_j}M^{-1}x - M_j^{-1}P_{X_j}x\|_*^2
\]

\[
= \langle M_j^{-1}P_{X_j}x, (\Pi_{X_j}M^{-1} - M_j^{-1}P_{X_j})x \rangle_{*j} - \langle M_j^{-1}P_{X_j}x, (\Pi_{X_j}M^{-1} - M_j^{-1}P_{X_j})x \rangle_*
\]

\[
\leq \eta_j\|M_j^{-1}P_{X_j}x\|_*(\Pi_{X_j}M^{-1} - M_j^{-1}P_{X_j})x\|_*^2,
\]

where we have applied Lemma [12] for the last inequality. Hence,

\[
\|(\Pi_{X_j}M^{-1} - M_j^{-1}P_{X_j})x\|_* \leq \eta_j\|M_j^{-1}P_{X_j}x\|_*.
\]

Applying Theorem [15] (and the fact that \( \|P_{X_j}x\|_X \leq \|x\|_X \)), we obtain

\[
\|(\Pi_{X_j}M^{-1} - M_j^{-1}P_{X_j})x\|_* \leq \frac{\eta_j}{(1 - \eta_j)\sqrt{\gamma}}\|x\|_X,
\]

as desired. 

We can now prove that \( M_jP_{X_j} \) converges pointwise to \( M^{-1} \) on \( X \).
**Theorem 17.** For every $x \in X$, 
\[ \|M^{-1}x - M_j^{-1}P_{X_j}x\|_* \to 0 \text{ as } j \to \infty. \]

**Proof.** Let $x \in X$. Then 
\[ \|(M^{-1} - M_j^{-1}P_{X_j})x\|_* \leq \|M^{-1}x - \Pi_{X_j}M^{-1}x\|_* + \|(\Pi_{X_j}M^{-1} - M_j^{-1}P_{X_j})x\|_* \]
\[ \leq \|(I - \Pi_{X_j})M^{-1}x\|_* + \frac{\eta_j}{1 - \eta_j} \|x\|_X. \]

By assumption, $\Pi_{X_j}$ converges pointwise to the identity operator on $D(L)$, and $\eta_j \to 0$ as $j \to \infty$. The desired result then follows. 

For every $y \in Y_j$ and for every $x \in X_j$, we have 
\[ \langle T_jx, y \rangle_Y = \langle x, T_j^#y \rangle_{X_j} = \langle T_jx, T_jT_j^#y \rangle_Y + \langle L_jx, L_jT_j^#y \rangle_{Z_j} \]
\[ = \langle x, (T_j^#T_j + L_j^*L_j)T_j^#y \rangle_{X_j}. \]

Also, 
\[ \langle T_jx, y \rangle_Y = \langle x, T_j^#y \rangle_X. \]

Because this is true for every $x \in X_j$ and for every $y \in Y_j$, we see that 
\[ T_j^* = (T_j^*T_j + L_j^*L_j)T_j^# = M_jT_j^#. \]

Similarly, 
\[ T^* = (T^*T + L^*L)T^# = MT^#. \]

We define $S_j : X_j \to Y$ by $S_j = T_j - T|_{X_j}$. By definition, we have that $t_j = \|S_j\|_{\mathcal{L}(X_j, Y)}$ and hence, by assumption, $\|S_j\|_{\mathcal{L}(X_j, Y)} \to 0$ as $j \to \infty$. We now compute the adjoint of $S_j$. To do this, let $x \in X_j$ and let $y \in Y$. Then 
\[ \langle S_jx, y \rangle_Y = \langle (T_j - T)x, y \rangle_Y = \langle T_jx, y \rangle_Y - \langle Tx, y \rangle_Y \]
\[ = \langle T_jx, P_{Y_j}y \rangle_Y - \langle x, T_j^#y \rangle_X \]
\[ = \langle x, T_j^#P_{Y_j}y \rangle_X - \langle x, P_{X_j}T^*x \rangle_X \]
\[ = \langle x, (T_j^#P_{Y_j} - P_{X_j}T^*)y \rangle_X. \]

Therefore, $S_j^* = T_j^*P_{Y_j} - P_{X_j}T^*$, and since $\|S_j^*\|_{\mathcal{L}(Y, X_j)} = \|S_j\|_{\mathcal{L}(X_j, Y)}$, we see that 
\[ \|P_{X_j}T^* - T_j^*P_{Y_j}\|_{\mathcal{L}(Y, X_j)} \to 0 \text{ as } j \to \infty. \] (3.15)
The following theorem will be used to show that $T_j^* T_j P_{X_j} \rightarrow T^* T$ uniformly.

**Theorem 18.** $T_j^* P_{Y_j} T \rightarrow T^* T$ in the $\mathcal{L}(D(L), D(L))$ norm.

**Proof.** We have shown that $T^* = M^{-1} T^*$ and $T_j^* = M_j^{-1} T_j^*$. From this, it follows that

$$T_j^* P_{Y_j} T - T^* T = (M_j^{-1} T_j^* P_{Y_j} - M^{-1} T^*) T.$$  

By Theorem 11, it suffices to prove that $M_j^{-1} T_j^* P_{Y_j} \rightarrow M^{-1} T^*$ pointwise on $Y$ as $j \rightarrow \infty$. Let $y \in Y$. Then

$$
\|M^{-1} T^* y - M_j^{-1} T_j^* P_{Y_j} y\|_*
\leq \|(M^{-1} - M_j^{-1} P_{X_j} ) T^* y\|_* + \|M_j^{-1} P_{X_j} T^* y - M_j^{-1} T_j^* P_{Y_j} y\|_*
= \|(M^{-1} - M_j^{-1} P_{X_j} ) T^* y\|_* + \|M_j^{-1} (P_{X_j} T^* y - T_j^* P_{Y_j} y)\|_*
\leq \|(M^{-1} - M_j^{-1} P_{X_j} ) T^* y\|_* + \frac{1}{(1 - \eta_j)^2} \|(P_{X_j} T^* - T_j^* P_{Y_j}) y\|_x.
$$

It now follows from Theorem 17 and (3.15) that $\|M^{-1} T^* y - M_j^{-1} T_j^* P_{Y_j} y\|_* \rightarrow 0$ as $j \rightarrow \infty$. \qed

We need two more results.

**Lemma 19.** If $\{v_j\} \subseteq X$ and $v_j \rightarrow v$ weakly as $j \rightarrow \infty$, then $P_{X_j} v_j \rightarrow v$ weakly.

**Proof.** For any $x \in X$, we have

$$
\langle P_{X_j} v_j, x \rangle_x = \langle v_j, x \rangle_x + \langle v_j, (P_{X_j} - I) x \rangle_x \rightarrow \langle v, x \rangle_x
$$

(notice that $\{v_j\}$ is a bounded sequence in $X$, and that $(P_{X_j} - I) x \rightarrow 0$ in norm). This shows that $P_{X_j} v_j \rightarrow v$ weakly as $j \rightarrow \infty$. \qed

**Theorem 20.** $T_j P_{X_j} \rightarrow T$ in the $\mathcal{L}(X, Y)$ norm.

**Proof.** We argue by contradiction and assume that there exist $\varepsilon_0 > 0$ and a subsequence $\{j_k\}$ of $\mathbb{Z}^+$ such that for every $k \in \mathbb{Z}^+$, there exists $v_{j_k} \in X$ satisfying

$$
\|v_{j_k}\|_X = 1 \text{ and } \|T_{j_k} P_{X_{j_k}} v_{j_k} - T v_{j_k}\|_Y \geq \varepsilon_0.
$$  

(3.16)

Since $T$ is compact and $X$ is separable, without loss of generality, we can assume that there exists $v \in X$ and $y \in Y$ such that $v_{j_k} \rightarrow v$ weakly in $X$ and $Tv_{j_k} \rightarrow y$ in $Y$. We then have

$$T_j P_{X_j} v_{j_k} = T P_{X_{j_k}} v_{j_k} + (T_{j_k} - T) P_{X_{j_k}} v_{j_k} \rightarrow T v + 0 = y
$$

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\[(\| (T_{jk} - T) P_{X_{jk} v_{jk}} \|_Y \leq t_j \| P_{X_{jk} v_{jk}} \|_X) \to 0, \text{ and } T P_{X_{jk}} \to Tv \text{ because } P_{X_{jk} v_{jk}} \to v \text{ weakly from Lemma 19 and } T \text{ is compact). But then we have}
\]
\[
T_{jk} P_{X_{jk} v_{jk}} - T v_{jk} \to y - y = 0,
\]
contradicting (3.16). The contradiction completes the proof. □

We have been working towards the following result.

**Theorem 21.** $T_j^\# T_j P_{X_j} \to T^\# T$ in the $\mathcal{L}(D(L), D(L))$ norm.

**Proof.** We have
\[
\| T_j^\# T_j P_{X_j} - T^\# T \|_{\mathcal{L}(D(L), D(L))}
\]
\[
\leq \| T_j^\# T_j P_{X_j} - T^\# P_{T_j X_j T} \|_{\mathcal{L}(D(L), D(L))} + \| T_j^\# P_{T_j X_j T} - T^\# T \|_{\mathcal{L}(D(L), D(L))}
\]
\[
= \| M_j^{-1} T_j^* P_{T_j X_j T} \|_{\mathcal{L}(D(L), D(L))} + \| T_j^\# P_{T_j X_j T} - T^\# T \|_{\mathcal{L}(D(L), D(L))}.
\]

The second term to the right of the equals sign goes to 0 by Theorem 18. Therefore, it suffices to show that the first term goes to 0. Applying Theorem 15, we have
\[
\| M_j^{-1} T_j^* P_{T_j X_j T} \|_{\mathcal{L}(D(L), D(L))}
\]
\[
\leq \frac{1}{(1 - \eta_j) \sqrt{\gamma}} \| T_j^* P_{T_j X_j} - T \|_{\mathcal{L}(D(L), X)}
\]
\[
\leq \frac{\| T_j \|_{\mathcal{L}(Y_j, X_j)}}{(1 - \eta_j) \sqrt{\gamma}} \| T_j P_{X_j} - T \|_{\mathcal{L}(D(L), Y)}
\]
\[
= \frac{\| T_j \|_{\mathcal{L}(X_j, Y_j)}}{(1 - \eta_j) \sqrt{\gamma}} \| T_j P_{X_j} - T \|_{\mathcal{L}(D(L), Y)}
\]
\[
\leq \frac{t_j + \| T \|_{\mathcal{L}(X, Y)}}{(1 - \eta_j) \sqrt{\gamma}} \| T_j P_{X_j} - T \|_{\mathcal{L}(D(L), Y)}
\]
\[
\leq \frac{t_j + \| T \|_{\mathcal{L}(X, Y)}}{(1 - \eta_j) \gamma} \| T_j P_{X_j} - T \|_{\mathcal{L}(X, Y)}.
\]

Since $t_j \to 0$ as $j \to \infty$ and $T$ is a bounded operator from $X$ to $Y$, it follows from Theorem 20 that $\| M_j^{-1} T_j^* P_{T_j X_j T} \|_{\mathcal{L}(D(L), D(L))} \to 0$ as $j \to \infty$. This completes the proof. □

Theorem 3 and Theorem 10 show that the eigensystem of $T_j^\# T_j$, which is the same as the eigensystem of $T_j^\# T_j P_{X_j}$, converges to the eigensystem of $T^\# T$. We can now prove the following theorem.

**Theorem 22.** Assuming that $c_j \to 0$ as $j \to \infty$ (where $c_j$ is defined by (3.7)), the GSVE of $(T_j, L_j)$ converges to the GSVE of $(T, L)$ in the sense of Definition 9.
Proof. Since \( ((a_k^{(j)})^2, \phi_k^{(j)}) \), \( k = 1, 2, \ldots, n_j \), are the eigenpairs of \( T_j^*T_j P_{X_j} \), \( (a_k^2, \phi_k) \) are the eigenpairs of \( T^*T \), and \( T_j^*T_j P_{X_j} \rightarrow T^*T \) in the operator norm, it follows from Theorem 10 that the set of approximate generalized singular values \( \{a_k^{(j)}\} \) converges to the set of true generalized singular values \( \{a_k\} \), and the set of approximate generalized singular functions \( \{\phi_k^{(j)}\} \) converges to the set of true generalized singular functions \( \{\phi_k\} \) in the manner described by Definition 9. Moreover, since \( (a_k^{(j)})^2 + (b_k^{(j)})^2 = 1 \) for every \( k = 1, 2, \ldots, n_j \), and \( a_k^2 + b_k^2 = 1 \) for every \( k \in \mathbb{Z}^+ \), it follows that the set of approximate generalized singular values \( \{b_k^{(j)}\} \) also converges to the true set of generalized singular values \( \{b_k\} \) in the manner described in Definition 9.

It now remains only to show that \( \{\psi_k^{(j)}\} \) converges to \( \{\psi_k\} \) and \( \{\theta_k^{(j)}\} \) converges to \( \{\theta_k\} \) as \( j \rightarrow \infty \) in the sense of Definition 9. To show this, let \( k \) be an arbitrary positive integer and let \( \varepsilon > 0 \) be given. We must show that there exists \( j_0 \in \mathbb{Z}^+ \) such that

\[
j \geq j_0 \implies \max \{ \delta(S_k(\psi), S_k^{(j)}(\psi)), \delta(S_k^{(j)}(\psi), S_k(\psi)) \} < \varepsilon.
\]

First, we show that \( j_0 \in \mathbb{Z}^+ \) can be chosen such that \( \delta(S_k(\phi), S_k^{(j)}(\phi)) < \varepsilon \) for every \( j \geq j_0 \). That is, we show that \( j_0 \) can be chosen so that

\[
j \geq j_0 \implies \sup_{y \in S_k(\psi)} \inf_{v \in S_k^{(j)}(\phi)} \|y - v\|_Y < \varepsilon. \quad (3.17)
\]

We know that there exists \( j_0 \in \mathbb{Z}^+ \) such that

\[
j \geq j_0 \implies \sup_{x \in S_k(\phi)} \inf_{v \in S_k^{(j)}(\phi)} \|x - v\|_* < \frac{a_k \varepsilon}{4} \text{ and } t_j < \min \left\{ \frac{a_k \varepsilon}{2}, 1 \right\}.
\]

We will show that this value of \( j_0 \) satisfies (3.17). It suffices to show that for any \( j \geq j_0 \) and for any \( y \in S_k(\psi) \) satisfying \( \|y\|_Y = 1 \), there exists \( v \in S_k^{(j)}(\psi) \) such that \( \|y - v\|_Y < \varepsilon \). Suppose

\[
S_k(\psi) = \text{span}\{\psi_{k1}, \psi_{k2}, \ldots, \psi_{kq}\}.
\]

Then there exists real numbers \( \alpha_1, \alpha_2, \ldots, \alpha_q \) such that

\[
y = a_k^{-1} \sum_{i=1}^q \alpha_i a_k \psi_{ki} = a_k^{-1} \sum_{i=1}^q \alpha_i T \phi_{ki} = a_k^{-1} T x,
\]

where \( x = \sum_{i=1}^q \alpha_i \phi_{ki} \). Moreover, since \( \{\phi_{k1}, \phi_{k2}, \ldots, \phi_{kq}\} \) is orthonormal in \( D(L) \) with respect to the \( * \)-inner product, we see that \( \|x\|_* = \|y\|_Y = 1 \). Hence, there exists
\[ u \in S^{(j)}_k(\psi) \] such that
\[ \|x - u\|_* < \frac{a_k \varepsilon}{4}. \]

By construction, \( Tx = a_k y \), and the vector \( v \) defined by \( v = a_k^{-1} T_j u \) lies in \( S^{(j)}_k(\psi) \). Moreover,
\[ \|y - v\|_Y = a_k^{-1} \|Tx - T_j u\|_Y \leq \frac{\|(T - T_j)x\|_Y + \|T_j x - T_j v\|_Y}{a_k} \]
\[ \leq \frac{t_j + \|T_j \|_{\mathcal{L}(D(L),Y)} \|x - u\|_*}{a_k} \]
\[ \leq \frac{t_j + (1 + t_j) \|x - u\|_*}{a_k} \]
\[ \leq \frac{1}{a_k} \left( \frac{a_k \varepsilon}{2} + \frac{2a_k \varepsilon}{4} \right) = \varepsilon \]

Thus, \( \delta(S_k(\psi),S^{(j)}_k(\psi)) < \varepsilon \) for every \( j \geq j_0 \). The proof that \( j_0 \) can be chosen such that \( \delta(S^{(j)}_k(\theta),S_k(\theta)) < \varepsilon \) for every \( j \geq j_0 \) is similar. Thus, we have shown that \( \{\psi^{(j)}_k\} \) converges to \( \{\psi_k\} \) in the sense of Definition 9.

The proof that \( \{\theta^{(j)}_k\} \) converges to \( \{\theta_k\} \) in the sense of Definition 9 is exactly the same, and the proof is complete.

In Example 1, it appeared that the GSVE of \((T_j, L_j)\) did not converge to the GSVE of \((T, L)\). Thus, the sequence of discretized operator pairs \((T_j, L_j)\) must fail to satisfy the hypotheses of Theorem 22.

**Example 2.** In this example, we analyze the discretization of Example 1. In Example 1, \( T_j = P_j T|_{X_j} \), and since \( T \) is the identity operator, it follows that \( T_j = T|_{X_j} \) (\( X_j = Y_j \) for each \( j \in \mathbb{Z}^+ \)). Therefore, \( t_j = 0 \) for every \( j \in \mathbb{Z}^+ \). However (recalling that \( x_i \) is the \( i \)th standard nodal basis function), a direct calculation shows that
\[ \ell_j = \sup_{x \in X_j, x \neq 0} \frac{\|L_j - L\|_{L^2(0,1)}}{\|x\|_{H^1(0,1)}} \geq \frac{\|L_j - L\|_{L^2(0,1)}}{\|x_j\|_{H^1(0,1)}} \geq \frac{1}{2\sqrt{2}\sqrt{1 + h^2/6}} \]
(\( h = 1/j \)) and hence \( \ell_j \) is bounded away from 0. Therefore, Theorem 22 does not apply to this example.

We now present a discretization of the operators of Example 1 that satisfies the hypotheses of Theorem 22 and hence leads to convergence of the GSVE.

**Example 3.** Let \( T, L, X_j, \) and \( Y_j \) be defined as in Example 1, but now define \( Z_j \) to be the space of piecewise constant functions on the uniform mesh with \( j \) elements.
As before, \( T_j \) is to be \( P_{Y_j} T|_{X_j} = T|_{X_j} \), and we define \( L_j = P_{Z_j} L \). Since \( L \) maps \( X_j \) into \( Z_j \), it follows that \( L_j = L|_{X_j} \). Therefore, for this discretization, we have that \( t_j = \ell_j = 0 \) for all \( j \in \mathbb{Z}^+ \), and hence Theorem 22 guarantees that the GSVE of \( (T_j, L_j) \) converges to the GSVE of \( (T, L) \) in the sense of Definition 7 as \( j \to \infty \).

Figures 3.1-3.3 show the approximate and exact singular functions for \( k = 1, 2, 3 \) (analogous to Figures 2.1-2.3 from Example 1). As in Example 1, we use \( j = 100 \) to obtain these numerical results. In contrast to Example 1, now all three of the examined singular modes are well approximated.

Each of the generalized singular spaces is one-dimensional and, therefore, we can compare the generalized singular functions directly rather than referring to the gap between subspaces (we just have to normalize the vectors and multiply by -1 when necessary so that the angle between each singular vector and its estimate is close to 0 rather than close to \( \pi \)). We observe

\[
\left\| \phi^{(j)}_k - \phi_k \right\|_{L^2(0,1)} = O(h^2) \quad \text{as} \quad j \to \infty,
\]

\[
\left\| \psi^{(j)}_k - \psi_k \right\|_{L^2(0,1)} = O(h^2) \quad \text{as} \quad j \to \infty,
\]

Extensive numerical testing suggests that

\[
\left| a^{(j)}_k - a_k \right| = O(h^2) \quad \text{as} \quad j \to \infty,
\]

\[
\left| b^{(j)}_k - b_k \right| = O(h^2) \quad \text{as} \quad j \to \infty.
\]
In each case, the rate of convergence is optimal for the given discretization.
Chapter 4

Rates of convergence

In Example 3 of the previous chapter, we compared the generalized singular values and vectors of the operator pairs \((T, L)\) and \((T_j, L_j)\). In that example, we observed the following rates of convergence for the generalized singular values:

\[
\left| a^{(j)}_k - a_k \right| = O(h^2) \text{ as } j \to \infty,
\]
\[
\left| b^{(j)}_k - b_k \right| = O(h^2) \text{ as } j \to \infty.
\]

We also observed the following rates of convergence for the generalized singular functions:

\[
\left\| \phi^{(j)}_k - \phi_k \right\|_{L^2(0,1)} = O(h^2) \text{ as } j \to \infty,
\]
\[
\left\| \psi^{(j)}_k - \psi_k \right\|_{L^2(0,1)} = O(h^2) \text{ as } j \to \infty,
\]
\[
\left\| \theta^{(j)}_k - \theta_k \right\|_{L^2(0,1)} = O(h) \text{ as } j \to \infty.
\]

Here, \( h = 1/j \) was the width of each interval of continuous piecewise linear finite elements in the discretization \( X_j \) of \( X \). In this chapter, we analyze the rate of convergence of the generalized singular values and vectors of \((T_j, L_j)\) to those of \((T, L)\). We next consider a less trivial example that demonstrates the same rates of convergence observed in Example 3.

Example 4. Let \( X = L^2(0,1) \) and \( Y = Z = L^2(0,1) \). Define operators \( T : X \to Y \) and \( L : D(L) \to Z \) by \( Lx = x' \), and \( Tx = \int_0^1 se^{xt}x(t) \, dt \). We discretize \((T, L)\) by defining \( X_j = Y_j \) to be the space of continuous piecewise linear functions on a uniform mesh with \( j \) elements, and \( Z_j \) to be the set of piecewise constant functions defined on each subinterval of the mesh. Let \( \{x_0, x_1, \ldots, x_j\} \) be the standard nodal basis. Define \( T_j = P_{Y_j}T|_{X_j} \) and \( L_j = P_{Z_j}L|_{X_j} \). Using the method of computation for
the GSVE from Chapter 2, we are able to compute the singular values and vectors for \((T_j, L_j)\). The GSVE of \((T, L)\) is unknown in this example, so convergence rates are approximated using Richardson extrapolation. In our discretization, we will have a total of 7 refinements of our finite element space. At each stage of refinement, the previous discretization is also interpolated into the new refinement in order to compare the functions from each refinement. Using Richardson extrapolation, we can use any 3 consecutive refinements of our discretization to produce a rate of approximation for each of the singular values and singular vectors.

We begin with a discretization using 40 elements, and at each refinement of the discretization, we double the number of elements. Tables 4.1-4.4 give the rates of convergence for the first 5 generalized singular vectors and generalized singular values of \(T\). The numbers in Table 4.1 are the estimates for \(p\) using Richardson extrapolation such that

\[
|a_{jk}^{(j)} - a_k| = Ch^p.
\]

The numbers in Table 4.2 (and similarly in Tables 4.3 and 4.4) are estimates of \(p\) using Richardson extrapolation such that

\[
\|\phi_{jk}^{(j)} - \phi_k\|_{L^2(0,1)} = Ch^p.
\]

The results of Table 4.1 suggest the following rates of convergence of the generalized singular values of \((T_j, L_j)\) to those of \((T, L)\).

\[
|a_{jk}^{(j)} - a_k| = O(h^2) \text{ as } j \to \infty,
\]

\[
|b_{jk}^{(j)} - b_k| = O(h^2) \text{ as } j \to \infty.
\]

The rate of convergence of \(b_{jk}^{(j)}\) to \(b_k\) follows immediately from the equations

\[
a_k^2 + b_k^2 = 1,
\]

\[
\left(a_{jk}^{(j)}\right)^2 + \left(b_{jk}^{(j)}\right)^2 = 1.
\]

Tables 4.2-4.4 suggest the following rates of convergence of the generalized singular vectors of \((T_j, L_j)\) to those of \((T, L)\).

\[
\|\phi_{jk}^{(j)} - \phi_k\|_{L^2(0,1)} = O(h^2) \text{ as } j \to \infty,
\]

\[
\|\psi_{jk}^{(j)} - \psi_k\|_{L^2(0,1)} = O(h^2) \text{ as } j \to \infty,
\]

\[
\|\theta_{jk}^{(j)} - \theta_k\|_{L^2(0,1)} = O(h) \text{ as } j \to \infty.
\]
Table 4.1
Rate of convergence of $a_k^{(j)}$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$j = 160$</th>
<th>$j = 320$</th>
<th>$j = 640$</th>
<th>$j = 1240$</th>
<th>$j = 2480$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>NaN</td>
<td>NaN</td>
<td>NaN</td>
<td>NaN</td>
<td>NaN</td>
</tr>
<tr>
<td>2</td>
<td>2.00065</td>
<td>2.00016</td>
<td>2.00004</td>
<td>2.00001</td>
<td>2.00000</td>
</tr>
<tr>
<td>3</td>
<td>2.00121</td>
<td>2.00031</td>
<td>2.00008</td>
<td>2.00002</td>
<td>2.00000</td>
</tr>
<tr>
<td>4</td>
<td>2.00339</td>
<td>2.00087</td>
<td>2.00022</td>
<td>2.00005</td>
<td>2.00001</td>
</tr>
<tr>
<td>5</td>
<td>2.00816</td>
<td>2.00214</td>
<td>2.00054</td>
<td>2.00013</td>
<td>2.00008</td>
</tr>
</tbody>
</table>

Table 4.2
Rate of convergence of $\phi_k^{(j)}$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$j = 160$</th>
<th>$j = 320$</th>
<th>$j = 640$</th>
<th>$j = 1240$</th>
<th>$j = 2480$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.99998</td>
<td>2.00000</td>
<td>2.00000</td>
<td>2.00000</td>
<td>2.00000</td>
</tr>
<tr>
<td>2</td>
<td>2.00013</td>
<td>2.00003</td>
<td>2.00001</td>
<td>2.00000</td>
<td>2.00000</td>
</tr>
<tr>
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<td>2.00005</td>
<td>2.00001</td>
<td>2.00000</td>
<td>2.00000</td>
</tr>
<tr>
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<td>1.99984</td>
<td>1.99996</td>
<td>1.99999</td>
<td>2.00000</td>
</tr>
<tr>
<td>5</td>
<td>1.99644</td>
<td>1.99912</td>
<td>1.99978</td>
<td>1.99995</td>
<td>2.00000</td>
</tr>
</tbody>
</table>

In both examples, the same rates of convergence of the generalized singular values and vectors were observed when using continuous piecewise linear elements to discretize the space $D(L)$ and when using piecewise constant elements to discretize the space $Y$. In this chapter, we provide a theory and analysis to prove these rates of convergence. In our analysis, it will be important to consider the space $D(L^*L) \subseteq X$.

To be consistent with the notation in Chapter 5, we will denote $D(L^*L)$ by $S_2$. We define the bilinear form $\langle \cdot, \cdot \rangle_{S_2} : S_2 \times S_2 \to \mathbb{R}$ by
\[
\langle x, y \rangle_{S_2} = \langle Mx, My \rangle_X, \quad \forall x, y \in S_2,
\]
where $M : S_2 \to X$ was defined by $M = T^*T + L^*L$ in Chapter 3.

By Theorem 14, $M$ has bounded inverse and is, therefore, injective. Hence, for any $x \in S_2$,
\[
\langle x, x \rangle_{S_2} = 0 \iff x = 0.
\]

Therefore, $\langle \cdot, \cdot \rangle_{S_2}$ defines an inner product on $S_2$. The next two lemmas show that $S_2$ is a dense subspace of $D(L)$ and that $S_2$ is a Hilbert space with norm $\| \cdot \|_{S_2}$ defined.
by

\[ \|x\|_{S_2} = \|Mx\|_X, \ \forall x \in X. \]

**Lemma 23.** $S_2$ is dense in $D(L)$.

*Proof.* It will suffice to prove that $S_2^\perp_* = \{0\}$ where $\perp_*$ denotes the orthogonal complement in $D(L)$ of $S_2$ with respect to the $*$-inner product. Let $w \in S_2^\perp*$, and define $u = M^{-1}w$. Then $u \in D(L^*L)$, so by definition of $w$, $\langle w, u \rangle_* = 0$. Therefore,

\[ \|w\|_{X}^2 = \langle w,w \rangle_X = \langle w,M^{-1}w \rangle_* = \langle w,u \rangle_* = 0. \]

Thus, $w = 0$ and the proof is complete. \qed

**Lemma 24.** $\| \cdot \|_{S_2}$ is a stronger norm than $\| \cdot \|_*$. In particular, for every $x \in S_2$,

\[ \|x\|_* \leq (\gamma)^{-1/2} \|x\|_{S_2}. \]

*Proof.* Let $x \in S_2$. Then it follows that

\[ \|x\|_*^2 = \langle x,x \rangle_* \]
\[
= \langle Tx, Tx \rangle_Y + \langle Lx, Lx \rangle_Z \\
= \langle x, (T^* T + L^* L)x \rangle_X \\
= \langle x, Mx \rangle_X \\
\leq \|x\|_X \|Mx\|_X \\
\leq (\gamma)^{-1/2} \|x\|_* \|Mx\|_X.
\]

After dividing each side of the inequality by \(\|x\|_*\), we have
\[
\|x\|_* \leq (\gamma)^{-1/2} \|Mx\|_X = (\gamma)^{-1/2} \|x\|_{S_2},
\]
and this completes the proof.

We will now assume the following property of the discretization: There exists a sequence of positive real numbers \(\{\varepsilon_j\}\) such that \(\varepsilon_j \to 0\) as \(j \to \infty\) and
\[
\|\Pi_{X_j} x - x\|_* \leq \varepsilon_j \|x\|_{S_2}, \ \forall x \in S_2.
\] (4.1)

This assumption is consistent with finite element approximation results. For instance if \(X_j\) is the space of continuous piecewise linear finite elements in \(L^2(0,1)\), and \(L : H^1(0,1) \to L^2(0,1)\) is the derivative operator, then the \(*\)-norm is equivalent to the \(H^1(0,1)\) norm, and the \(S_2\)-norm is related to the \(H^2\)-seminorm. In this case, we can take \(\varepsilon_j = Ch_j\) for some constant \(C > 0\), where \(h_j\) is the mesh size of the \(j\)-th mesh (see [14] Theorem 4.4.20).

**Theorem 25.** For every \(x \in S_2\),
\[
\|\Pi_{X_j} x - x\|_X \leq \varepsilon_j^2 \|x\|_{S_2}.
\] (4.2)

**Proof.** Let \(x \in S_2\), and define \(\hat{x} = \Pi_{X_j} x\) and \(w = M^{-1}(x - \hat{x})\). Then
\[
\|x - \hat{x}\|_X^2 = \langle x - \hat{x}, x - \hat{x} \rangle_X \\
= \langle M^{-1}(x - \hat{x}), x - \hat{x} \rangle_* \\
= \langle w, x - \hat{x} \rangle_*.
\]

Since \(x - \hat{x}\) is orthogonal to the space \(X_j\) with respect to the \(*\)-inner product, it follows that \(\langle \Pi_{X_j} w, x - \hat{x} \rangle_* = 0\). Therefore,
\[
\langle w, x - \hat{x} \rangle_* = \langle w - \Pi_{X_j} w, x - \hat{x} \rangle_*.
\]

Putting these results together, we have
\[
\|x - \hat{x}\|_X^2 = \langle w, x - \hat{x} \rangle_* = \langle w - \Pi_{X_j} w, x - \hat{x} \rangle_* \\
\leq \|w - \Pi_{X_j} w\|_* \|x - \hat{x}\|.
\]
Both $w$ and $x$ are in $S_2$, so by (4.1), it follows that
\[
\|w - \Pi_{X_j}w\|_* \leq \varepsilon_j \|w\|_{S_2} \\
\|x - \hat{x}\|_* \leq \varepsilon_j \|x\|_{S_2}.
\]
From this we have
\[
\|x - \hat{x}\|_X^2 \leq \|w - \Pi_{X_j}\|_* \|x - \hat{x}\|_* \leq (\varepsilon_j \|w\|_{S_2})(\varepsilon_j \|x\|_{S_2}) = \varepsilon_j^2 \|w\|_{S_2} \|x\|_{S_2}.
\]
Therefore,
\[
\|x - \hat{x}\|_X^2 \leq \varepsilon_j^2 \|Mw\|_X \|x\|_{S_2} \leq \varepsilon_j^2 \|M(M^{-1}(x - \hat{x}))\|_X \|x\|_{S_2} = \varepsilon_j^2 \|x - \hat{x}\|_X \|x\|_{S_2}.
\]
After eliminating a factor of $\|x - \hat{x}\|_X$ from each side of the inequality, we have
\[
\|x - \hat{x}\|_X \leq \varepsilon_j^2 \|x\|_{S_2}.
\]
It should be noted that a similar argument yields that
\[
\|x - \Pi_{X_j}x\|_X \leq \varepsilon_j \|x\|_*, \forall x \in D(L). \tag{4.3}
\]
It then follows that
\[
\|\Pi_{X_j}x\|_X \leq \|x\|_X + \varepsilon_j \|x\|_* \leq (\gamma^{-1/2} + \varepsilon_j)\|x\|_* \forall x \in D(L). \tag{4.4}
\]
The following lemma will be used in the analysis of the rate of convergence of the generalized singular values.

**Lemma 26.** There exists a constant $C > 0$ such that
\[
\|T_j \Pi_{X_j}x - Tx\|_Y \leq C(t_j \|x\|_* + \varepsilon_j^2 \|x\|_{S_2}) \forall x \in S_2. \tag{4.5}
\]

**Proof.** For every $x \in S_2$, we have
\[
\|T_j \Pi_{X_j}x - Tx\|_Y \leq \|T_j \Pi_{X_j}x - T\Pi_{X_j}x\|_Y + \|T\Pi_{X_j}x - Tx\|_Y \\
\leq \|(T - T_j)\Pi_{X_j}\|_Y + \|T(\Pi_{X_j}x - x)\|_Y \\
\leq t_j \|\Pi_{X_j}x\|_X + \|T\|_{L(X,Y)} \|\Pi_{X_j}x - x\|_X \\
\leq t_j \gamma^{-1/2} \|x\|_* + \|T\|_{L(X,Y)} \varepsilon_j^2 \|x\|_{S_2} \\
\leq C(t_j \|x\|_* + \varepsilon_j^2 \|x\|_{S_2}),
\]
where $C = \max\{\gamma^{-1/2}, \|T\|_{\mathcal{L}(X,Y)}\}$.

Recall that the GSVE of $(T, L)$ can be expressed as

$$T = \sum_{k=1}^{\infty} a_k \psi_k \otimes_\ast \phi_k,$$

$$L = \sum_{k=1}^{\infty} b_k \theta_k \otimes_\ast \phi_k + \sum_{k \in M_a} b_k \theta_k \otimes_\ast \phi_k,$$

and similarly, the GSVE of $(T_j, L_j)$ is given by

$$T_j = \min\{m_j, n_j\} \sum_{k=1}^{m_j} a_k^{(j)} \psi_k \otimes_\ast \phi_k^{(j)},$$

$$L_j = \min\{p_j, n_j\} \sum_{k=1}^{p_j} b_k^{(j)} \theta_k \otimes_\ast \phi_k^{(j)}.$$

Here, we order the generalized singular terms in the GSVE of $(T, L)$ such that the generalized singular values of $T$ are nonincreasing and the generalized singular values of $L$ are nondecreasing, as was done in Chapter 3. To be precise, we assume that the index sets $M_b$ and $M_0$, defined in Definition 1.11 of Chapter 3, are given by

$$M_b = \{1, 2, \cdots, N_b\}, \quad M_0 = \{N_b + 1, N_b + 2, \cdots\},$$

where $N_b = \dim(\mathcal{N}(L))$. Since $T$ is a compact operator, the null space of $L$ must be finite-dimensional, hence making such an indexing possible. Similarly, the GSVE for $(T_j, L_j)$, as given above, is such that the sequence of generalized singular values $\{a_k^{(j)}\}_{k=1}^{\infty}$ is nonincreasing and the sequence of generalized singular values $\{b_k^{(j)}\}_{k=1}^{\infty}$ is nondecreasing for every $j \in \mathbb{Z}^+$.

The generalized singular values $a_k$ of $(T, L)$ can be characterized as

$$a_k = \max_{S \subseteq D(L)} \min_{\substack{x \in S \quad x \neq 0 \quad \dim(S) = k \quad \|x\|_*}} \frac{\|T x\|_Y}{\|x\|_*},$$

where this maximum is attained for each $k \in \mathbb{Z}^+$ by the space

$$S = \Phi_k = \text{span}\{\phi_1, \phi_2, \cdots, \phi_k\}.$$
The corresponding singular values of $T_j$ can be characterized as

$$a_k^{(j)} = \max_{S \subseteq X_j} \min_{\dim(S) = k, x \neq 0} \frac{\|T_j x\|_Y}{\|x\|_*}.$$

We will often have to compare the norms $\| \cdot \|_*$ and $\| \cdot \|_{*j}$ on the subspace $X_j$ when analyzing the rates of convergence of the generalized singular vectors. Recall that $\eta_j = c_j^2 + 2(t_j + \ell_j)$. Then $\eta_j = O(c_j) \to 0$ as $j \to \infty$, and hence $0 \leq \eta_j < 1$ for all $j \in \mathbb{Z}^+$ that are sufficiently large. We will need the following fact about the generalized singular vectors of the operator pair $(T, L)$.

**Lemma 27.** For all $k \in \mathbb{Z}^+ = M_b \cup M_0$, $\phi_k \in S_2$.

**Proof.** Let $k \in M_0 \cup M_b$. Then we have

$$T \phi_k = a_k \psi_k, \quad T^\# \psi_k = a_k \phi_k,$$

where $a_k > 0$. Putting both of these things together, we have

$$T^\# T \phi_k = a_k^2 \phi_k.$$

Since $T^\# = M^{-1} T^*$ (see [9], Theorem 5.27), it follows that $\phi_k \in D(M) = S_2$. \qed

Notice that when $a_k \in M_a$, the same argument does not hold since $T^\# T \phi_k = 0$. We will need a few more preliminary results in order to prove a particular rate of convergence.

**Lemma 28.** For every $n \in \mathbb{Z}^+$,

$$\max_{S \subseteq D(L)} \min_{\dim(S) = k} \frac{\|T x\|_Y}{\|x\|_*} = \max_{S \subseteq S_2} \min_{\dim(S) = k} \frac{\|T x\|_Y}{\|x\|_*}.$$

**Proof.** This follows immediately from the fact that $S_2$ is dense in $D(L)$. \qed

**Lemma 29.** For every $k \in \mathbb{Z}^+$ and for every sufficiently large positive integer $j$,

$$\max_{S \subseteq D(L)} \min_{\dim(S) = k} \frac{\|T_j \Pi_{X_j} x\|_Y}{\|x\|_*} = \max_{S \subseteq X_j} \min_{\dim(S) = k} \frac{\|T_j x\|_Y}{\|x\|_*}. \quad (4.6)$$

**Proof.** We assume that $j$ is sufficiently large such that $\dim(X_j) \geq k$. Clearly, the left-hand side of (4.6) is at least as big as the right-hand side. Therefore, we must prove that

$$\max_{S \subseteq D(L)} \min_{\dim(S) = k} \frac{\|T_j \Pi_{X_j} x\|_Y}{\|x\|_*} \leq \max_{S \subseteq X_j} \min_{\dim(S) = k} \frac{\|T_j x\|_Y}{\|x\|_*}. \quad (4.7)$$

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Let $S$ be a $k$-dimensional subspace of $X$, and let $\hat{S} = \Pi_{x_j}S$. We consider two cases. If $S \cap X_j^\perp$ is nontrivial, where $X_j^\perp$ denotes the orthogonal complement of $X_j$ in $D(L)$ with respect to the $\ast$-norm, then there exists $x \in S$ such that $x \neq 0$ and $T_j \Pi_{x_j} x = 0$. It then follows that

$$\min_{x \in S, x \neq 0} \frac{\| T_j \Pi_{x_j} x \|_Y}{\| x \|_*} = 0 \implies \min_{x \in S, x \neq 0} \frac{\| T_j \Pi_{x_j} x \|_Y}{\| x \|_*} \leq \max_{K \subseteq X_j, \dim(K) = k} \min_{x \in K, x \neq 0} \frac{\| T_j x \|_Y}{\| x \|_*}.$$ 

The second case is that $S \cap X_j^\perp$ is trivial. In this case, $\dim(\hat{S}) = k$, and there is a one-to-one correspondence between $x \in S$ and $\bar{x} \in \hat{S}$ ($\bar{x} = \Pi_{x_j} x$). For each such $x$ and $\bar{x}$, we have $\bar{x} - x \in X_j^\perp$, and hence

$$T_j \bar{x} = T_j \Pi_{x_j} \bar{x} = T_j \Pi_{x_j} x.$$ 

Also, we have that $\| \bar{x} \|_* = \| \Pi_{x_j} x \|_* \leq \| x \|_*$. Therefore,

$$\frac{\| T_j \Pi_{x_j} x \|_Y}{\| x \|_*} \leq \frac{\| T_j \bar{x} \|_Y}{\| \bar{x} \|_*}.$$ 

From this, it follows that

$$\min_{x \in S, x \neq 0} \frac{\| T_j \Pi_{x_j} x \|_Y}{\| x \|_*} \leq \min_{\bar{x} \in \hat{S}, \bar{x} \neq 0} \frac{\| T_j \bar{x} \|_Y}{\| \bar{x} \|_*} \leq \max_{S \subseteq X_j, \dim(S) = k} \min_{x \in S, x \neq 0} \frac{\| T_j x \|_Y}{\| x \|_*},$$

and, thus, we have shown

$$\max_{S \subseteq D(L), \dim(S) = k} \min_{x \neq 0} \frac{\| T_j \Pi_{x_j} x \|_Y}{\| x \|_*} \leq \max_{S \subseteq X_j, \dim(S) = k} \min_{x \neq 0} \frac{\| T_j x \|_Y}{\| x \|_*},$$

which completes the proof.

We can now prove the desired result.

**Theorem 30.** For each $k \in \mathbb{Z}^+$, there exists constants $C_1$ and $C_2$ such that for all sufficiently large positive integers $j$,

$$a_k - C_1 (c_j + \varepsilon_j^2) \leq a_k^{(j)} \leq a_k + C_2 c_j. \quad (4.8)$$
Proof. By Corollary 9 of [11],

$$\frac{\|x\|_*}{\|x\|_{s_j}} \leq \frac{1}{\sqrt{1 - \eta_j}} \forall x \in X_j,$$

where $\eta_j = c_j^2 + 2(t_j + c_j)$. Using this result, we have

$$a_k^{(j)} = \max_{S \subseteq X_j} \min_{x \in S \text{ s.t. dim}(S) = k} \frac{\|T_j x\|_Y}{\|x\|_{s_j}} = \max_{S \subseteq X_j} \min_{x \in S \text{ s.t. dim}(S) = k} \frac{\|T_j x\|_Y}{\|x\|_*} \frac{\|x\|_*}{\|x\|_{s_j}}$$

$$\leq \frac{1}{\sqrt{1 - \eta_j}} \max_{S \subseteq X_j} \min_{x \in S \text{ s.t. dim}(S) = k} \frac{\|T_j x\|_Y}{\|x\|_*}$$

$$\leq \frac{1}{\sqrt{1 - \eta_j}} \max_{S \subseteq X_j} \min_{x \in S \text{ s.t. dim}(S) = k} \frac{\|T x\|_Y + \|T - T_j x\|_Y}{\|x\|_*}$$

$$\leq \frac{1}{\sqrt{1 - \eta_j}} \left( \max_{S \subseteq X_j} \min_{x \in S \text{ s.t. dim}(S) = k} \frac{\|T x\|_Y}{\|x\|_*} + t_j \right)$$

$$= \frac{1}{\sqrt{1 - \eta_j}} (a_k + t_j).$$

Since

$$\frac{1}{\sqrt{1 - \eta_j}} = 1 + \delta(\eta_j) \eta_j \text{ where } 0 < \delta(\eta_j) < 1,$$

for all $\eta_j$ sufficiently small ($0 \leq \eta_j < 3/8$ suffices), and since $t_j = O(c_j)$ and $\eta_j = O(c_j)$, this establishes the upper bound in (4.8). To prove the lower bound, we define the subspace $\Phi_k = \text{span}\{\phi_1, \phi_2, \cdots, \phi_k\}$ and apply Lemma 26:

$$a_k = \max_{S \subseteq D(L)} \min_{x \in S \text{ s.t. dim}(S) = k} \frac{\|T x\|_Y}{\|x\|_*}$$

$$= \min_{x \in \Phi_k \text{ s.t. } x \neq 0} \frac{\|T x\|_Y}{\|x\|_*}$$

$$\leq \min_{x \in \Phi_k \text{ s.t. } x \neq 0} \frac{\|T_j I x\|_Y + C(t_j \|x\|_* + \varepsilon_j^2 \|x\|_{S_2})}{\|x\|_*}.$$
Since $\Phi_k$ is fixed and finite-dimensional, there exists $C_k > 0$ such that

$$\frac{\|x\|_{S_2}}{\|x\|_*} \leq C_k \forall x \in \Phi_k.$$

If we define $C' = \max\{C, CC_k\}$, we obtain

$$a_k \leq \min_{x \in \Phi_k, x \neq 0} \left( \frac{\|T_j \Pi X_j x\|_Y}{\|x\|_*} + C'(t_j + \varepsilon_j^2) \right)$$

$$= \min_{x \in \Phi_k, x \neq 0} \frac{\|T_j \Pi X_j x\|_Y}{\|x\|_*} + C'(t_j + \varepsilon_j^2)$$

$$\leq \max_{S \subseteq X_j} \min_{x \in S, x \neq 0} \frac{\|T_j x\|_Y}{\|x\|_*} + C'(t_j + \varepsilon_j^2).$$

By Corollary 9 of [11],

$$\frac{\|x\|_*}{\|x\|_*} \leq \sqrt{1 + \eta_j} \forall x \in X_j.$$

Therefore,

$$a_k \leq \sqrt{1 + \eta_j} \left( \max_{S \subseteq X_j} \min_{x \in S, x \neq 0} \frac{\|T_j x\|_Y}{\|x\|_*} \right) + C'(t_j + \varepsilon_j^2)$$

$$= \sqrt{1 + \eta_j} \left( a_k^{(j)} \right) + C'(t_j + \varepsilon_j^2)$$

$$\implies \frac{1}{\sqrt{1 + \eta_j}}(a_k - C'(t_j + \varepsilon_j^2)) \leq a_k^{(j)}.$$

Since

$$\frac{1}{\sqrt{1 + \eta_j}} = 1 - \varsigma(\eta_j)\eta_j, \text{ where } 0 \leq \varsigma(\eta_j) \leq 1/2,$$

the lower bound in (4.8) follows (again using the fact that both $t_j$ and $\eta_j$ are $O(c_j)$), and the proof is complete.

We now wish to analyze the convergence of the generalized singular vectors of $(T_j, L_j)$ to the generalized singular vectors of $(T, L)$. Recall that for each $k$, we define the spaces $E_k$ and $E_k^{(j)}$ by

$$E_k = \text{span}\{\phi_i : a_i = a_k\}.$$
and
\[ E_{k}^{(j)} = \text{span}\{\phi_{i}^{(j)} : a_{i}^{(\ell)} \to a_{k} \text{ as } \ell \to \infty\}. \]

We will need to prove that the subspace \( E_{k}^{(j)} \) converges to the subspace \( E_{k} \) as \( j \to \infty \) in the sense that the gap between \( E_{k} \) and \( E_{k}^{(j)} \) converges to 0 as \( j \to \infty \). Recall that the gap between subspaces \( U \) and \( V \) in \( H \), denoted by \( \delta(U, V) \) is defined by
\[
\delta(U, V) = \max\{\delta(U, V), \delta(V, U)\},
\]
\[
\delta(U, V) = \sup_{u \in U} \inf_{v \in V} \|u - v\|_{H}.
\]

We wish to derive estimates for \( \hat{\delta}(E_{k}, E_{k}^{(j)}) \), where gap is defined by either the \( X \)-norm or the \( * \)-norm. We will, therefore, write \( \hat{\delta}_{*} \) and \( \hat{\delta}_{X} \) for the gap defined by the \( * \)-norm and \( \hat{\delta}_{X} \) and \( \hat{\delta}_{X} \) for the gap defined by the \( X \)-norm.

It was shown in [11] that, under the assumptions made here, \( \hat{\delta}_{*}(E_{k}, E_{k}^{(j)}) \to 0 \) as \( j \to \infty \). We will need the following results to conclude that \( \hat{\delta}_{X}(E_{k}, E_{k}^{(j)}) = \hat{\delta}_{X}(E_{k}^{(j)}, E_{k}) \) for all positive integers \( j \) sufficiently large.

**Theorem 31.** If \( U \) and \( V \) are \( k \)-dimensional subspaces of a Hilbert space \( H \), where \( k \) is a positive integer, then \( \delta(U, V) = \delta(V, U) \).

**Proof.** Without loss of generality, let us assume that \( \delta(U, V) \leq \delta(V, U) \). By Theorem 7, the result holds if \( \delta(U, V) \) and \( \delta(V, U) \) are both strictly less than 1. It suffices, therefore, to show that the assumption \( \delta(U, V) < \delta(V, U) = 1 \) produces a contradiction. Since \( V \) is finite dimensional,
\[
\delta(V, U) = \max_{\|v\|=1} \|P_{V}v - v\|.
\]

Therefore, the assumption that \( \delta(V, U) = 1 \) implies that there exists \( \hat{v} \in V \) such that \( \|\hat{v}\| = 1 \) and \( \|P_{U}\hat{v} - \hat{v}\| = 1 \). This is possible only if \( P_{U}\hat{v} = 0 \). That is, if \( \hat{v} \in U^\perp \). On the other hand, the assumption that \( \delta(U, V) < 1 \) implies that \( \|P_{V}u - u\| < 1 \) for all \( u \in U \) and hence that the null space of \( P = P_{V}|_{U} \) (\( P_{V} \) restricted to \( U \)) is trivial. Since \( \text{dim}(U) = \text{dim}(V) = k \), the fundamental theorem of linear algebra implies that \( P \) maps \( U \) onto \( V \); thus, there exists \( \hat{u} \in U \) such that \( P\hat{u} = \hat{v} \). But then
\[
\left\| P\left( \frac{\hat{u}}{\|\hat{u}\|} \right) - \frac{\hat{u}}{\|\hat{u}\|} \right\| < 1 \implies \|P\hat{u} - \hat{u}\| < \|\hat{u}\|
\]
\[
\implies \|\hat{v} - \hat{u}\| < \|\hat{u}\|
\]
\[
\implies \|\hat{v}\|^{2} + \|\hat{u}\|^{2} < \|\hat{u}\|^{2}
\]

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(where we used \( \delta(U, V) < 1 \) in the first step and \( \hat{v} \in U^\perp \) in the last step). Since \( \|\hat{v}\| = 1 \), the last inequality is impossible, and the proof is complete.

The previous theorem implies that \( \delta_X(E_k, E_k^{(j)}) = \delta_X(E_k^{(j)}, E_k) \). It follows, therefore, that \( \hat{\delta}_X(E_k, E_k^{(j)}) = \delta(E_k, E_k^{(j)}) \), and it, therefore, suffices to analyze the convergence of \( \delta_X(E_k, E_k^{(j)}) \) to zero. The same comments apply to \( \hat{\delta}_* : \hat{\delta}_*(E_k, E_k^{(j)}) = \delta_*(E_k, E_k^{(j)}) \). Specifically, we will show that

\[
\delta_X(E_k, E_k^{(j)}) = O(c_j + \varepsilon_j^2),
\]
\[
\delta_*(E_k, E_k^{(j)}) = O(c_j + \varepsilon_j).
\]

By definition of \( \delta_X(E_k, E_k^{(j)}) \), we must show that there exists \( C = C_k > 0 \) such that, for all \( v \in E_k \) with \( \|v\|_X = 1 \), there exists \( w \in E_k^{(j)} \) such that

\[
\|v - w\|_X \leq C(c_j + \varepsilon_j^2).
\]

We will show that the same vector \( w \) also satisfies

\[
\|v - w\|_* \leq C(c_j + \varepsilon_j)
\]

(albeit with a different constant for \( C \)).

We now proceed to show that there exists \( C > 0 \) such that given \( v \in E_k \) with \( \|v\|_* = 1 \), one can define \( w \in E_k^{(j)} \) such that inequality (4.10) holds. To do this, we will need some more notation and several preliminary results. Recall that the dimension of \( X_j \) is \( n_j \); let \( r_j \) be the rank of \( T_j \). Then there exist \( r_j \) generalized singular value/singular vector pairs \( (a_i^{(j)}, \phi_i^{(j)}) \) of \( T_j \). If \( r_j < n_j \), extend the set \( \{\phi_i^{(j)} : i = 1, 2, \ldots, r_j\} \) to an orthonormal basis \( \{\phi_i^{(j)} : i = 1, 2, \ldots, n_j\} \) for \( X_j \), and define \( a_i^{(j)} = 0 \) for \( i = r_j + 1, r_j + 2, \ldots, n_j \). It should be noted that \( \{\phi_i^{(j)} : i = 1, 2, \ldots, n_j\} \) is orthonormal with respect to the \( *_j \)-inner product.

We will write

\[
I_k = \{i \in \mathbb{Z}^+ : a_i = a_k\};
\]

then

\[
E_k = \text{span}\{\phi_i : i \in I_k\},
\]
\[
E_k^{(j)} = \text{span}\{\phi_i^{(j)} : i \in I_k\}.
\]

We will also write \( J_j = \{1, 2, \ldots, n_j\} \).
Recall that we have defined the operators $M : D(L^*L) \to X$ and $M_j : X_j \to X_j$ by

$$M = T^*T + L^*L,$$
$$M_j = T_j^*T_j + L_j^*L_j.$$

Both such operators are bijections and each has bounded inverse (see Chapter 3). For each $j \in \mathbb{Z}^+$, we define the operator $\Lambda_j : D(L^*L) \to X_j$ by

$$\Lambda_j = M_j^{-1}P_{X_j}M.$$

Here, again, $P_{X_j}$ denotes the orthogonal projection onto the subspace $X_j$ of $X$ with respect to the $X$-norm. Notice that, for any $u, v \in X_j$,

$$\langle u, v \rangle_{X_j} = \langle (T_j^*T_j + L_j^*L_j)u, v \rangle_X = \langle M_ju, v \rangle_X.$$

Similarly, for any $u, v \in D(L)$ such that $u \in S_2$,

$$\langle u, v \rangle_X = \langle Mu, v \rangle_X.$$

Therefore, for any $u \in S_2$ and for any $v \in X_j$, we have

$$\langle \Lambda_ju, v \rangle_{X_j} = \langle M_j^{-1}P_{X_j}Mu, v \rangle_{X_J} = \langle P_{X_j}Mu, v \rangle_X = \langle Mu, v \rangle_X = \langle u, v \rangle_X,$$

that is,

$$\langle \Lambda_ju, v \rangle_{X_j} = \langle u, v \rangle_X, \quad \forall u \in S_2, \quad \forall v \in X_j. \quad (4.11)$$

The operator $\Lambda_j$ approximates the operator $\Pi_{X_j}$ in the following sense.

**Lemma 32.** For every $j \in \mathbb{Z}^+$ and for every $u \in S_2$,

$$\|(\Pi_{X_j} - \Lambda_j)u\|_* \leq \frac{\eta_j}{(1 - \eta_j)\sqrt{\gamma}}\|u\|_{S_2}.$$  

**Proof.** By Theorem [16] for every $x \in X$,

$$\|(\Pi_{X_j}M^{-1} - M_j^{-1}P_{X_j})x\|_* \leq \frac{\eta_j}{(1 - \eta_j)\sqrt{\gamma}}\|x\|_X.$$

Since $M : S_2 \to X$, it follows that for every $u \in S_2$,

$$\|(\Pi_{X_j}M^{-1} - M_j^{-1}P_{X_j})Mu\|_* \leq \frac{\eta_j}{(1 - \eta_j)\sqrt{\gamma}}\|Mu\|_X.$$
By definition, $\|u\|_{S^2} = \|Mu\|_X$. Thus we have
\[
\| (\Pi_{X_j} - \Lambda_j) u \|_* = \| (\Pi_{X_j} - M_j^{-1}P_{X_j} M) u \|_* = \| (\Pi_{X_j} M^{-1} - M_j^{-1} P_{X_j} M) u \|_* \\
\leq \frac{\eta_j}{(1 - \eta_j)^{\frac{1}{2}}} \|u\|_{S^2}.
\]

Lemma 26 and Lemma 32 yield the following estimate.

**Lemma 33.** There exists a constant $C > 0$ such that for every $j \in \mathbb{Z}^+$,
\[
\| T - T_j \Lambda_j \|_{L(S^2,Y)} \leq C(c_j + \varepsilon_j^2).
\]

**Proof.** By Lemma 26, $\| \cdot \|_{S^2}$ is a stronger norm than $\| \cdot \|_*$ and there exists $C' > 0$ such that for every $j \in \mathbb{Z}^+$,
\[
\| T - T_j \Lambda_j \|_{L(S^2,Y)} \leq \| T - T_j \Pi_{X_j} \|_{L(S^2,Y)} + \| T_j (\Pi_{X_j} - \Lambda_j) \|_{L(S^2,Y)} \\
\leq C'(t_j + \varepsilon_j^2) + \| T_j \|_{L(D(L),Y)} \| \Pi_{X_j} - \Lambda_j \|_{L(S^2,D(L))}.
\]

Since $\| T_j \|_{L(D(L),Y)} \leq \| T \|_{L(D(L),Y)} + t_j$ and $t_j, \eta_j = O(c_j)$, it follows that
\[
\| T - T_j \Lambda_j \|_{L(S^2,Y)} \leq C'(t_j + \varepsilon_j^2) + \| T_j \|_{L(D(L),Y)} \| \Pi_{X_j} - \Lambda_j \|_{L(S^2,D(L))} \\
\leq C'(t_j + \varepsilon_j^2) + \left( \| T \|_{L(D(L),Y)} + t_j \right) \frac{\eta_j}{(1 - \eta_j)^{\frac{1}{2}}} = O(c_j + \varepsilon_j^2).
\]

This completes the proof.

Let $v \in E_k$ such that $\|v\|_X = 1$. We now define $w \in E^{(j)}_k$ by
\[
w = \Pi^{(j)}_{E^{(j)}_k} \Lambda_j v,
\]
where $\Pi^{(j)}_{E^{(j)}_k} : X_j \to E^{(j)}_k$ is defined to be the orthogonal projection of $X_j$ onto $E^{(j)}_k$ with respect to the $*_j$-inner product defined on $X_j$. By the triangle inequality,
\[
\| v - w \|_X \leq \| v - \Lambda_j v \|_X + \| \Lambda_j v - w \|_X. \tag{4.12}
\]

We first consider the first term to the right of inequality (4.12) above. We already know that
\[
\| v - \Pi_{X_j} v \|_X \leq \varepsilon_j^2 \|v\|_{S^2} \quad \forall v \in E_k \subseteq S_2
\]
and

\[ \| v - \Pi_{X_j} v \| \leq \varepsilon_j \| v \|_{S_2} \quad \forall v \in E_k \subseteq S_2. \]

Therefore, since \( E_k \) is a finite-dimensional subspace of \( S_2 \), there exists a constant \( C_{k}^{(1)} > 0 \) such that

\[ v \in E_k, \| v \|_X = 1 \quad \Rightarrow \quad \| v - \Pi_{X_j} v \|_X \leq C_{k}^{(1)} \varepsilon_j, \]

\[ v \in E_k, \| v \|_* = 1 \quad \Rightarrow \quad \| v - \Pi_{X_j} v \|_* \leq C_{k}^{(1)} \varepsilon_j. \]

Putting this together with Lemma 32, we have the following lemma.

**Lemma 34.** For every \( v \in E_k \) such that \( \| v \|_X = 1 \),

\[ \| v - \Lambda_j v \|_X = O(c_j + \varepsilon_j^2), \quad \| v - \Lambda_j v \|_* = O(c_j + \varepsilon_j). \]

**Proof.** By the triangle inequality,

\[ \| v - \Lambda_j v \|_X \leq \| v - \Pi_{X_j} v \|_X + \| (\Pi_{X_j} - \Lambda_j) v \|_X \]

From the above argument together with Lemma 32, we have

\[
\| v - \Pi_{X_j} v \|_X + \| (\Pi_{X_j} - \Lambda_j) v \|_X \leq \| v - \Pi_{X_j} v \|_X + \gamma^{-1/2} \| (\Pi_{X_j} - \Lambda_j) v \|_* \\
\leq C_{k}^{(1)} \varepsilon_j^2 + \frac{\eta_j}{(1 - \eta_j) \gamma} \| v \|_{S_2} \\
\leq C_{k}^{(1)} \varepsilon_j^2 + \frac{\eta_j}{(1 - \eta_j) \gamma} C_{k}^{(2)} \| v \|_X \\
= C_{k}^{(1)} \varepsilon_j^2 + C_{k}^{(2)} \frac{\eta_j}{(1 - \eta_j) \gamma},
\]

Here, such a constant \( C_{k}^{(2)} > 0 \) exists because \( X_j \) is a finite dimensional space and, therefore, all norms on \( X_j \) are equivalent. Since \( \eta_j = O(c_j) \), it follows that

\[ C_{k}^{(2)} \frac{\eta_j}{(1 - \eta_j) \gamma} = O(c_j). \]

Therefore, we have

\[ \| v - \Lambda_j v \|_X \leq C_{k}^{(1)} \varepsilon_j^2 + C_{k}^{(2)} \frac{\eta_j}{(1 - \eta_j) \gamma} \\
= O(c_j + \varepsilon_j^2). \]

Similarly, we have that

\[ \| v - \Lambda_j v \|_* \leq \| v - \Pi_{X_j} v \|_* + \| (\Pi_{X_j} - \Lambda_j) v \|_* \\
= O(c_j + \varepsilon_j). \]
\[ \leq C_k(1) \varepsilon_j + \frac{\eta_j}{(1 - \eta_j)^\gamma} \|v\|_{s_2}. \]

By exactly the same argument as above, it then follows that
\[ \|v - \Lambda_jv\|_* = O(c_j + \varepsilon_j). \]

We now consider the second term \( \|\Lambda_j v - w\|_X \) to the right of inequality (4.12). By our coercivity condition (1.9), we have that
\[ \|\Lambda_j v - w\|_X \leq \gamma^{-1/2} \|\Lambda_j v - w\|_*. \]

Therefore, it suffices to prove that
\[ \|\Lambda_j v - w\|_* = O(c_j + \varepsilon_j^2). \]

Notice that \( \Lambda_j v - w \in X_j \). Therefore, as a consequence of Corollary 13, we have
\[ \|\Lambda_j v - w\|_*^2 \leq \frac{1}{1 - \eta_j} \|\Lambda_j v - w\|_{*j}^2. \]

Before we prove that \( \|\Lambda_j v - w\|_{*j} = O(c_j + \varepsilon_j^2) \), we need the following result.

**Lemma 35.** For each \( k \in \mathbb{Z}^+ \), the quantity
\[ \rho_k^{(j)}(\cdot) = \max \left\{ \frac{1}{a_i^k - (a_i^j)^2} : i \in J_j \setminus I_k \right\} \]

is bounded by a constant \( C \) for all sufficiently large \( j \in \mathbb{Z}^+ \). In other words, there exists \( C = C_k > 0 \) such that for all \( j \in \mathbb{Z}^+ \) sufficiently large,
\[ \rho_k^{(j)} \leq C. \]

**Proof.** The proof is similar to that of Lemma 7 in [15].

The next result shows that the quantity \( \|\Lambda_j v - w\|_{*j} \) can be bounded by a constant times the quantity \( \|(\Lambda_j T^i T - T_j^{i^i} T_j T_j^i \Lambda_j)v\|_{*j}^2 \) for sufficiently large \( j \in \mathbb{Z}^+ \). Using previous results, we can then show that this upper bound converges to 0 at a rate at least as fast as \( O(\varepsilon_j^2 + c_j) \).

**Lemma 36.** For every \( j \in \mathbb{Z}^+ \),
\[ \|\Lambda_j v - w\|_{*j} \leq \rho_k^{(j)} \|(\Lambda_j T^i T - T_j^{i^i} T_j T_j^i \Lambda_j)v\|_{*j}. \]
Proof. By definition, \( w = \Pi_{E_k}^{(j)} \Lambda_j v \), where \( \Pi_{E_k}^{(j)} \) is the orthogonal projection of \( X_j \) onto \( E_k \) with respect to the \( *_j \)-inner product defined on \( X_j \). Since \( E_k = \text{span}\{\phi_i^{(j)}: i \in I_k\} \), we can express these vectors as

\[
\Lambda_j v = \sum_{i \in J} \langle \phi_i^{(j)}, \Lambda_j v \rangle_{*_j} \phi_i^{(j)}, \quad w = \sum_{i \in I_k} \langle \phi_i^{(j)}, \Lambda_j v \rangle_{*_j} \phi_i^{(j)}.
\]

Taking the difference then gives

\[
\|\Lambda_j v - w\|^2_{*_j} = \sum_{i \in J \setminus I_k} \langle \phi_i^{(j)}, \Lambda_j v \rangle^2_{*_j}.
\]

For \( i \in J \setminus I_k \), we have

\[
a_k^2 \langle \phi_i^{(j)}, T_j^{\#} T_j \Lambda_j v \rangle_{*_j} = a_k^2 \left( a_i^{(j)} \right)^2 \langle \phi_i^{(j)}, \Lambda_j v \rangle_{*_j} = a_k^2 \left( a_i^{(j)} \right)^2 \langle \phi_i^{(j)}, v \rangle_{*_j}
\]

\[
= \left( a_i^{(j)} \right)^2 \langle \phi_i^{(j)}, T^\# T v \rangle_{*_j}
\]

\[
= \left( a_i^{(j)} \right)^2 \langle \phi_i^{(j)}, \Lambda_j T^\# T v \rangle_{*_j}.
\]

Subtracting \( \left( a_i^{(j)} \right)^2 \langle \phi_i^{(j)}, T_j^{\#} T_j \Lambda_j v \rangle_X \) from each side of the equation above, we have

\[
\left( a_k^2 - (a_i^{(j)})^2 \right) \langle \phi_i^{(j)}, T_j^{\#} T_j \Lambda_j v \rangle_{*_j} = \left( a_i^{(j)} \right)^2 \langle \phi_i^{(j)}, (\Lambda_j T^\# T - T_j^{\#} T_j \Lambda_j) v \rangle_{*_j}.
\]

Therefore, for any \( i \in J \setminus I_k \),

\[
\frac{1}{\left( a_i^{(j)} \right)^2} \langle \phi_i^{(j)}, T_j^{\#} T_j \Lambda_j v \rangle_{*_j} = \frac{1}{a_k^2 - (a_i^{(j)})^2} \langle \phi_i^{(j)}, (\Lambda_j T^\# T - T_j^{\#} T_j \Lambda_j) v \rangle_{*_j}.
\]

Therefore, we have

\[
\|\Lambda_j v - w\|^2_{*_j} = \sum_{i \in J \setminus I_k} \langle \phi_i^{(j)}, \Lambda_j v \rangle^2_{*_j}
\]

\[
= \sum_{i \in J \setminus I_k} \left( \frac{1}{\left( a_i^{(j)} \right)^2} \langle \phi_i^{(j)}, T_j^{\#} T_j \Lambda_j v \rangle_{*_j} \right)^2
\]

\[
= \sum_{i \in J \setminus I_k} \left( \frac{1}{a_k^2 - (a_i^{(j)})^2} \langle \phi_i^{(j)}, (\Lambda_j T^\# T - T_j^{\#} T_j \Lambda_j) v \rangle_{*_j} \right)^2
\]

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\[
\leq \sum_{i \in J_k \setminus I_k} \left( \rho_k^{(j)} \langle \phi_i^{(j)}, (\Lambda_j T^\# T - T_j^\# T_j \Lambda_j) v \rangle_{s_j} \right)^2 \\
\leq \left( \rho_k^{(j)} \right)^2 \| (\Lambda_j T^\# T - T_j^\# T_j \Lambda_j) v \|_{s_j}.
\]
This completes the proof. \qed

We want to show that \(\|(\Lambda_j T^\# T - T_j^\# T_j \Lambda_j) v\|_{s_j} = O(c_j + \varepsilon_j^2)\). Let \(y \in E_k\) such that \(\|y\|_X = 1\), and let \(x \in X_j\). Then

\[
\langle x, (\Lambda_j T^\# T - T_j^\# T_j \Lambda_j) y \rangle_{s_j} \\
= \langle x, \Lambda_j T^\# T y \rangle_{s_j} - \langle x, T_j^\# T_j \Lambda_j y \rangle_{s_j} \\
= \langle T x, T y \rangle_Y - \langle T_j x, T_j \Lambda_j y \rangle_Y = \langle (T - T_j) x, T y \rangle_Y + \langle T_j x, (T - T_j \Lambda_j) y \rangle_Y.
\]

Therefore,

\[
\langle x, (\Lambda_j T^\# T - T_j^\# T_j \Lambda_j) y \rangle_{s_j} \\
= \langle (T - T_j) x, T y \rangle_Y + \langle T_j x, (T - T_j \Lambda_j) y \rangle_Y \\
\leq \|(T - T_j) x\|_Y \|T y\|_Y + \|T_j x\|_Y \|(T - T_j \Lambda_j) y\|_Y.
\]

As a consequence of Lemma 33, there exists a constant \(C' > 0\) such that for every \(j \in \mathbb{Z}^+\) and for every \(u \in S_2\),

\[
\|(T - T_j \Lambda_j) u\|_Y \leq C'(c_j + \varepsilon_j^2) \|u\|_{s_2}.
\]

From this, it then follows that

\[
\|(T - T_j) x\|_Y \|T y\|_Y + \|T_j x\|_Y \|(T - T_j \Lambda_j) y\|_Y \\
\leq t_j \|x\|_s \|T\|_{\mathcal{L}(X,Y)} \|y\|_X + (1 + t_j) \|x\|_s C'(c_j + \varepsilon_j^2) \|y\|_{s_2}.
\]

Since \(E_k\) is finite dimensional, all norms on \(E_k\) are equivalent. Thus, there exists \(C'' > 0\) such that for every \(u \in E_k\), \(C'' \|u\|_{s_2} \leq C'' \|u\|_X\). Therefore, we have

\[
t_j \|x\|_s \|T\|_{\mathcal{L}(X,Y)} \|y\|_X + C'(c_j + \varepsilon_j^2)(1 + t_j) \|x\|_s \|y\|_{s_2} \\
\leq t_j \|x\|_s \|T\|_{\mathcal{L}(X,Y)} \|y\|_X + (1 + t_j) C'' \|x\|_s \|y\|_X \\
\leq (t_j \|T\|_{\mathcal{L}(X,Y)} + (1 + t_j) C'' (c_j + \varepsilon_j^2)) \frac{1}{\sqrt{1 - \eta_j}} \|x\|_{s_j}.
\]
Hence,

\[ \langle x, (\Lambda_j^T T - T_j^\#_j T_j \Lambda_j)y \rangle_{\ast j} \leq \left( t_j \|T\|_{\mathcal{L}(X,Y)} + (1 + t_j)C''(c_j + \varepsilon_j^2) \right) \frac{1}{\sqrt{1 - \eta_j}} \|x\|_{\ast j}. \] (4.13)

Since \( t_j = O(c_j) \) and \( \eta_j = O(c_j) \) as \( j \to \infty \), it follows that

\[ \left( t_j \|T\|_{\mathcal{L}(X,Y)} + (1 + t_j)C''(c_j + \varepsilon_j^2) \right) \frac{1}{\sqrt{1 - \eta_j}} = O(c_j + \varepsilon_j^2). \]

Therefore, there exists \( C > 0 \) such that for all \( j \in \mathbb{Z}^+ \) sufficiently large,

\[ \left( t_j \|T\|_{\mathcal{L}(X,Y)} + (1 + t_j)C''(c_j + \varepsilon_j^2) \right) \frac{1}{\sqrt{1 - \eta_j}} \leq C(c_j + \varepsilon_j^2). \]

We then combine this with inequality (4.13) to give

\[ \langle x, (\Lambda_j^T T - T_j^\#_j T_j \Lambda_j)y \rangle_{\ast j} \leq C(c_j + \varepsilon_j^2) \|x\|_{\ast j}. \] (4.14)

Notice that the constant \( C \) does not depend on the choice of \( y \) or \( x \) from above. With inequality (4.14), we can prove the following theorem.

**Lemma 37.** Let \( k \in \mathbb{Z}^+ \). Then there exists \( C = C_k > 0 \) and \( j_k \in \mathbb{Z}^+ \) such that for every \( v \in E_k \) with \( \|v\|_X = 1 \) and for all \( j \in \mathbb{Z}^+ \) with \( j \geq j_k \),

\[ \| (\Lambda_j T^\# T - T_j^\#_j T_j \Lambda_j) v \|_{\ast j} \leq C(c_j + \varepsilon_j^2). \]

**Proof.** Let \( v \in E_k \) such that \( \|v\|_X = 1 \), and let \( z = (\Lambda_j T^\# T - T_j^\#_j T_j \Lambda_j) v \). Then

\[ \|z\|_{\ast j} = \sup_{\|x\|_{X_j} = 1} \langle x, z \rangle_{\ast j} = \sup_{\|x\|_{X_j} = 1} \langle x, (\Lambda_j T^\# T - T_j^\#_j T_j \Lambda_j) v \rangle_{\ast j}, \]

and the previous theorem gives us the desired bound for sufficiently large \( j \). \( \Box \)

From Lemmas 34, 36, and 37 it follows that there exists \( C_1, C_2 > 0 \) and \( j_k \in \mathbb{Z}^+ \) such that for all \( j \in \mathbb{Z}^+ \) with \( j \geq j_k \),

\[ \|v - w\|_X \leq \|v - \Lambda_j v\|_X + \|\Lambda_j v - w\|_X \]
≤ \|v - \Lambda_jv\|_X + \frac{1}{\sqrt{(1 - \eta_j)\gamma}} \|\Lambda_jv - w\|_s, \\
≤ C_1(\varepsilon_j^2 + c_j) + \frac{1}{\sqrt{(1 - \eta_j)\gamma}} \left( \mu_k^{(j)} \|\Lambda_jT^#T - T_j^# T_j\Lambda_j\|_s \right), \\
≤ C_1(\varepsilon_j^2 + c_j) + \frac{1}{(1 - \eta_j)\sqrt{\gamma}} \left( \mu_k^{(j)} C_2(c_j + \varepsilon_j^2) \right), \\
= \left( C_1 + \frac{\mu_k^{(j)} C_2}{\sqrt{(1 - \eta_j)\gamma}} \right) (\varepsilon_j^2 + c_j), \\
≤ C(c_j + \varepsilon_j^2),

where \( C > 0 \) is a constant depending on \( k \).

Since \( E_k \) is a finite-dimensional subspace, the preceding lemmas hold if we assume that \( \|v\|_s = 1 \) instead of \( \|v\|_X = 1 \). If the term \( \|v - w\|_X \) is replaced with \( \|v - w\|_s \), the same reasoning above using properties of norms and Lemma \( 34 \) shows that if we assume that \( v \) is chosen from \( E_k \) such that \( \|v\|_s = 1 \), then there exists a constant \( C_3 > 0 \) such that for all \( j \in \mathbb{Z}^+ \) sufficiently large,

\[ \|v - w\|_s \leq C_3(c_j + \varepsilon_j). \]

In this bound, we only have \( \varepsilon_j \), as oppose to \( \varepsilon_j^2 \). This is a consequence of Lemma \( 34 \) since the \( * \)-norm is stronger than \( X \)-norm from the coercivity condition (1.9). These results justify the following theorem about the rate of convergence of the gap between \( E_k \) and \( E_{k(j)} \) to 0 as \( j \to \infty \).

**Theorem 38.** Let \( k \in \mathbb{Z}^+ \) be given. Then there exists a constant \( C = C_k > 0 \) such that for any \( v \in E_k \) with \( \|v\|_X = 1 \) and for all \( j \in \mathbb{Z}^+ \) sufficiently large, there exists \( w \in E_{k(j)} \) satisfying

\[ \|v - w\|_X \leq C(c_j + \varepsilon_j^2), \]
\[ \|v - w\|_s \leq C(c_j + \varepsilon_j). \]

In terms of gap, it follows that for all \( j \in \mathbb{Z}^+ \) sufficiently large,

\[ \hat{\delta}_X(E_k, E_{k(j)}) \leq C(c_j + \varepsilon_j^2), \]
\[ \hat{\delta}_s(E_k, E_{k(j)}) \leq C(c_j + \varepsilon_j). \]

Using Theorem \( 38 \) we are able to derive the rates of convergence of the other generalized singular spaces. To do this, we define, for each \( k \in \mathbb{Z}^+ \), the following spaces:

\[ F_k = \text{span}\{\psi_i : i \in I_k\}, \]

where \( I_k \) is the index set for \( E_k \).
\[ F^{(j)}_k = \text{span}\{ \psi^{(j)}_i : i \in I_k \}, \]
\[ G_k = \text{span}\{ \theta_i : i \in I_k \}, \]
\[ G^{(j)}_k = \text{span}\{ \theta^{(j)}_i : i \in I_k \}. \]

Using the GSVE’s of \((T, L)\) and \((T_{j}, L_{j})\), it is clear that for any \(k \in \mathbb{Z}^+\),
\[ T(E_k) = F_k, \]
\[ L(E_k) = G_k, \]
\[ T_j \left( E^{(j)}_k \right) = F^{(j)}_k, \]
\[ L_j \left( E^{(j)}_k \right) = G^{(j)}_k. \]

The next theorem gives a rate of convergence of \(F^{(j)}_k\) to \(F_k\) and of \(G^{(j)}_k\) to \(G_k\) as \(j \to \infty\).

**Theorem 39.** Let \(k \in M_0\) be given. Then there exists a constant \(C = C_k > 0\) such that
\[ \hat{\delta}(F_k, F^{(j)}_k) \leq C(c_j + \varepsilon_j^2) \text{ and } \hat{\delta}(G_k, G^{(j)}_k) \leq C(c_j + \varepsilon_j). \]

**Proof.** Let \(k \in M_0\). Then \(T|_{E_k} : E_k \to F_k\) and \(L|_{E_k} : E_k \to G_k\) are bijections. In [11], it is proven that the space \(F^{(j)}_k\) converges to \(F_k\) and \(G^{(j)}_k\) converges to \(G_k\) as \(j \to \infty\) when \(k \in M_0\). Therefore, the respective gaps between these spaces is less than 1 for \(j \in \mathbb{Z}^+\) sufficiently large. Thus, it follows that for sufficiently large \(j \in \mathbb{Z}^+\),
\[ \hat{\delta}(F_k, F^{(j)}_k) = \hat{\delta}(F_k, F^{(j)}_k), \]
\[ \hat{\delta}(G_k, G^{(j)}_k) = \hat{\delta}(G_k, G^{(j)}_k). \]

By definition of the asymmetric gap,
\[ \hat{\delta}(F_k, F^{(j)}_k) = \max_{u, v \in F^{(j)}_k} \min_{\|u\|_Y = 1, \|v\|_Y = 1} \|u - v\|_Y. \]
When \(T : E_k \to F_k\) is understood to be restricted to \(E_k\), we will just write \(T\) instead of \(T|_{E_k}\). This map defines a bijection of finite dimensional spaces and therefore has a bounded inverse \(T^{-1} : F_k \to E_k\). Suppose that \(u \in E_k\) such that \(\|Tu\|_Y = 1\). Then,
\[ \|u\|_X = \|T^{-1}Tu\|_X \leq \|T^{-1}\|_{\mathcal{L}(F_k, E_k)} \|Tu\|_Y. \]
\[ ||T^{-1}||_{\mathcal{L}(F_k, E_k)}. \]

Then
\[ 1 \leq \frac{||T^{-1}||_{\mathcal{L}(F_k, E_k)}}{||u||_X}. \]

From this, it then follows that
\[ \max_{u \in E_k} \min_{v \in F_k^{(j)}} ||Tu - v||_Y \leq \max_{u \in E_k} \min_{v \in F_k^{(j)}} \frac{||Tu - v||_Y}{||u||_X} ||T^{-1}||_{\mathcal{L}(F_k, E_k)} \]
\[ = ||T^{-1}||_{\mathcal{L}(F_k, E_k)} \max_{u \in E_k} \min_{v \in F_k^{(j)}} ||Tu - v||_Y. \]

From Theorem 38, it follows that there exists a constant \( C' > 0 \) such that for any \( u \in E_k \) with \( ||u|| \leq ||T^{-1}||_{\mathcal{L}(F_k, E_k)} \), there exists \( w \in E_k^{(j)} \) such that for all \( j \in \mathbb{Z}^+ \) sufficiently large,
\[ ||u - w||_X \leq C'(c_j + \epsilon_j^2), \]
\[ ||u - w||_* \leq C'(c_j + \epsilon_j). \]

Since \( T_jw \in F_k^{(j)} \), we have
\[ ||T^{-1}||_{\mathcal{L}(F_k, E_k)} \max_{u \in E_k} \min_{v \in F_k^{(j)}} ||Tu - v||_Y \leq ||T^{-1}||_{\mathcal{L}(F_k, E_k)} \max_{u \in E_k} ||Tu - T_jw||_Y \]
\[ \leq ||T^{-1}||_{\mathcal{L}(F_k, E_k)} \max_{u \in E_k} \left( ||T||_{\mathcal{L}(X,Y)} ||u - w||_X + ||(T - T_j)w||_Y \right) \]
\[ \leq ||T^{-1}||_{\mathcal{L}(F_k, E_k)} \left( ||T||_{\mathcal{L}(X,Y)} C'(c_j + \epsilon_j^2) + \frac{t_j}{\sqrt{1 - \eta_j}} ||v||_* \right). \]

The term \( \frac{t_j}{\sqrt{1 - \eta_j}} ||v||_* \) at the end of this inequality follows from the fact
\[ ||(T - T_j)w||_Y \leq t_j ||w||_* \leq \frac{t_j}{\sqrt{1 - \eta_j}} ||w||_{s_j} \]
\[ = \frac{t_j}{\sqrt{1 - \eta_j}} ||\Pi_{E_k^{(j)}}^A v||_{s_j} \]
\[ \leq \frac{t_j}{\sqrt{1 - \eta_j}} ||A_j v||_{s_j} \]
\[ \leq \frac{t_j}{\sqrt{1 - \eta_j}} ||v||_* . \]
Since $E_k$ is a finite dimensional space, it follows that all norms are equivalent on $E_k$. Also, $t_j(1 - \eta_j)^{-1/2} = O(c_j)$ as $j \to \infty$. Hence, there exists a constant $C_1 > 0$ such that for any $v \in E_k$ with $\|v\|_X = 1$ and for all $j \in \mathbb{Z}^+$ sufficiently large,

$$\|T^{-1}\|_{\mathcal{L}(F_k, E_k)} \left( \|T\|_{\mathcal{L}(X,Y)} C'(c_j + \varepsilon_j^2) + \frac{t_j}{\sqrt{1 - \eta_j}} \|v\|_* \right) \leq C_1(c_j^2 + \varepsilon_j).$$

Thus, it follows that for all $j \in \mathbb{Z}^+$ sufficiently large,

$$\delta \left( F_k, F_k^{(j)} \right) = \max_{u \in F_k} \min_{v \in F_k^{(j)}} \|u - v\|_Y = \max_{u \in E_k} \min_{v \in F_k^{(j)}} \|Tu - v\|_Y$$

$$\leq \|T^{-1}\|_{\mathcal{L}(F_k, E_k)} \left( \|T\|_{\mathcal{L}(X,Y)} C'(c_j + \varepsilon_j^2) + \frac{t_j}{\sqrt{1 - \eta_j}} \|v\|_* \right)$$

$$\leq C_1(c_j + \varepsilon_j^2).$$

We now prove a rate of convergence of the spaces $G_k^{(j)}$ to $G_k$ as $j \to \infty$. Again considering only the asymmetric gap $\delta \left( G_k, G_k^{(j)} \right)$, the same argument above gives

$$\delta \left( G_k, G_k^{(j)} \right) = \max_{u \in G_k} \min_{v \in G_k^{(j)}} \|u - v\|_Z$$

$$= \max_{u \in E_k} \min_{v \in G_k^{(j)}} \|Gu - v\|_Z$$

$$\leq \|L^{-1}\|_{\mathcal{L}(G_k, E_k)} \max_{u \in E_k} \left( \|L\|_{\mathcal{L}(D(L), Y)} \|u - w\|_* + \|(L - L_j)w\|_Z \right),$$

where $w$ is the same as above. By the same argument as above together with Lemma 34, it follows that there exists a constant $C_2 > 0$ such that

$$\delta \left( G_k, G_k^{(j)} \right) \leq \|L^{-1}\|_{\mathcal{L}(G_k, E_k)} \max_{u \in E_k} \left( \|L\|_{\mathcal{L}(D(L), Y)} \|u - w\|_* + \|(L - L_j)w\|_Z \right)$$

$$\leq C_2(c_j + \varepsilon_j).$$

By letting $C = \max\{C_1, C_2\}$, the theorem is then proven.

In the previous theorem, the assumption that $k \in M_0$ is necessary for the spaces $G_k^{(j)}$ to converge to $G_k$ at the rate provided in the theorem. When $k \in M_b = \mathcal{N}(L)$, the space $G_k$ of left generalized singular vectors in $Z$ is trivial, and the gap between $G_k^{(j)}$ and $G_k$ is either 0 or 1, depending on whether $G_k^{(j)}$ is trivial or not. Therefore, a rate of convergence is not sensible when $k \in M_b$. Also, one should notice that the rate of convergence of $G_k^{(j)}$ to $G_k$ as $j \to \infty$ is one order of $\varepsilon_j$ worse than the convergence rate
of the other singular spaces. This follows from the possibility of $L$ being unbounded with respect to the weaker norm $\| \cdot \|_X$. Hence, in the derivation above, the quantity $\| L(v - w) \|_Y$ must be compared to $\| v - w \|_*$ instead of $\| v - w \|_X$, which yields a worse rate of convergence.
Chapter 5

Higher-order convergence and future work

The theory established in Chapter 4 proves the orders of convergence observed in Example 3 for the corresponding generalized singular values and vectors. In this example, $X_j$ was the discretization of the space $X = H^1(0, 1)$ using continuous piecewise linear finite elements on a mesh with elements of length $h = 1/j$. The rates of convergence for the generalized singular values were as follows:

$$
|a_k^{(j)} - a_k| = O(h^2) \text{ as } j \to \infty,
$$

$$
|b_k^{(j)} - b_k| = O(h^2) \text{ as } j \to \infty.
$$

Also, the following rates of convergence for the generalized singular functions were as follows:

$$
\left\| \phi_k^{(j)} - \phi_k \right\|_{L^2(0,1)} = O(h^2) \text{ as } j \to \infty,
$$

$$
\left\| \psi_k^{(j)} - \psi_k \right\|_{L^2(0,1)} = O(h^2) \text{ as } j \to \infty,
$$

$$
\left\| \theta_k^{(j)} - \theta_k \right\|_{L^2(0,1)} = O(h) \text{ as } j \to \infty.
$$

If we consider this same example, but with $X_j$ and $Y_j$ the space of continuous piecewise quadratic finite elements, and $Z_j$ the space of piecewise linear finite elements (not necessarily continuous), we then observe the following rates of convergence for the generalized singular values:

$$
|a_k^{(j)} - a_k| = O(h^3) \text{ as } j \to \infty,
$$

$$
|b_k^{(j)} - b_k| = O(h^3) \text{ as } j \to \infty.
$$
Also, the following rates of convergence of the generalized singular functions are observed to be

\[ \left\| \phi_k^{(j)} - \phi_k \right\|_{L^2(0,1)} = O(h^3) \text{ as } j \to \infty, \]
\[ \left\| \psi_k^{(j)} - \psi_k \right\|_{L^2(0,1)} = O(h^3) \text{ as } j \to \infty, \]
\[ \left\| \theta_k^{(j)} - \theta_k \right\|_{L^2(0,1)} = O(h^2) \text{ as } j \to \infty. \]

Similarly, if we let \( X_j \) and \( Y_j \) be the space of continuous piecewise cubic finite elements, and if we let \( Z_j \) be the space of piecewise quadratic finite elements (not necessarily continuous), we observe for the generalized singular values,

\[ \left| a_k^{(j)} - a_k \right| = O(h^4) \text{ as } j \to \infty, \]
\[ \left| b_k^{(j)} - b_k \right| = O(h^4) \text{ as } j \to \infty, \]

and for the generalized singular functions,

\[ \left\| \phi_k^{(j)} - \phi_k \right\|_{L^2(0,1)} = O(h^4) \text{ as } j \to \infty, \]
\[ \left\| \psi_k^{(j)} - \psi_k \right\|_{L^2(0,1)} = O(h^4) \text{ as } j \to \infty, \]
\[ \left\| \theta_k^{(j)} - \theta_k \right\|_{L^2(0,1)} = O(h^3) \text{ as } j \to \infty. \]

The theory of Chapter 4 is based on the assumption that there exists a sequence of positive real numbers \( \{\varepsilon_j\} \) such that \( \varepsilon_j \to 0 \) as \( j \to \infty \) and

\[ \left\| \Pi_{X_j} x - x \right\|_* \leq \varepsilon_j \left\| x \right\|_{S_2}, \forall x \in S_2. \]

A generalization of this assumption would make sense if we are using higher order finite elements. We begin by defining the sequence \( S_n \) of Hilbert spaces ordered by containment. Since \( M : D(L^*L) \to X \) is densely defined, self-adjoint, and strictly positive, it has a square root \( A = M^{1/2} \). We define

\[ \mathcal{M} = \bigcap_{k=0}^{\infty} D(A^k). \]

Then, by Lemma 8.17 of [6], \( \mathcal{M} \) is dense in \( X \), and we define

\[ \left\langle x, y \right\rangle_{S_k} = \left\langle A^k x, A^k y \right\rangle_X, \left\| x \right\|_{S_k} = \left\| A^k x \right\|_X, \forall x, y \in \mathcal{M}. \]
It is easy to see that $\langle \cdot , \cdot \rangle_{S_k}$ defines an inner product on $M$, and we define $S_k$ to be the completion of $M$ with respect to the norm $\| \cdot \|_{S_k}$. We say that the collection of spaces $\{ S_k : k \in \mathbb{Z}, k \geq 0 \}$ is the Hilbert scale defined by $A$. By definition, $S_0 = X$. We also have that $S_1 = D(L)$ and the definition of $S_2$ is consistent with that given in chapter 4. Further, the sequence of norms $\| \cdot \|_{S_k}$ is increasing in strength as $k \to \infty$. This is summarized in following theorem.

**Theorem 40.**

1.) $S_1 = D(L)$ and $\langle \cdot , \cdot \rangle_{S_1} = \langle \cdot , \cdot \rangle_{*}$.

2.) For every $n, m \in \mathbb{Z}$ such that $m > n \geq 0$, it follows that $\| \cdot \|_{S_m}$ is a stronger norm than $\| \cdot \|_{S_n}$, and $S_m$ is a dense subspace of $S_n$ with respect to the norm $\| \cdot \|_{S_n}$.

Proof. To prove (1), let $x, y \in D(L^*L)$. Since $M$ is self-adjoint with respect to the $X$-inner product, it follows that $A = M^{1/2}$ is also self-adjoint with respect to the $X$-inner product, and

$$\langle x, y \rangle_{S_1} = \langle Ax, Ay \rangle_X = \langle M^{1/2}x, M^{1/2}y \rangle_X = \langle x, My \rangle_X = \langle x, y \rangle_{*}.$$

By Lemma 23 from Chapter 4, $S_2$ is dense in $D(L)$ with respect to the norm $\| \cdot \|_{*}$. Thus, it follows that for every $x, y \in D(L)$,

$$\langle x, y \rangle_{S_1} = \langle x, y \rangle_{*}.$$

For a proof of (2), see [6], Proposition 8.19. □

We now make the following assumption about the discretizations $X_j$: There exists a sequence of positive real numbers $\{ \varepsilon_j \}$ and a positive integer $n$ such that $\varepsilon_j \to 0$ as $j \to \infty$ and

$$\| \Pi_{X_j} x - x \|_{*} \leq \varepsilon_j^k \| x \|_{S_{k+1}}, \ \forall x \in S_{k+1}, \ \forall k \in \mathbb{Z} \text{ such that } 0 \leq k \leq n. \quad (5.1)$$

**Theorem 41.** For every $x \in S_{k+1}$,

$$\| \Pi_{X_j} x - x \|_X \leq \varepsilon_j^{k+1} \| x \|_{S_{k+1}}, \ \forall k \in \mathbb{Z}^+ \text{ such that } 0 \leq k \leq n. \quad (5.2)$$

Proof. Let $x \in S_{k+1}$, and define $\hat{x} = \Pi_{X_j} x$ and $w = M^{-1}(x - \hat{x})$. Then $w \in S_2$ and

$$\| x - \hat{x} \|_X^2 = \langle x - \hat{x}, x - \hat{x} \rangle_X$$

$$= \langle M^{-1}(x - \hat{x}), x - \hat{x} \rangle_{*}$$

$$= \langle w, x - \hat{x} \rangle_{*}$$

$$= \langle w - \Pi_{X_j} w, x - \hat{x} \rangle_{*}$$

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Since \( w \in S_2 \) and \( x \in S_{k+1} \), it follows from (5.1), that
\[
\| w - \Pi_{X_j} w \|_* \leq \varepsilon_j \| w \|_{S_2} \text{, } \| x - \hat{x} \|_* \leq \varepsilon_j^k \| x \|_{S_{k+1}}.
\]
Therefore, we have
\[
\| x - \hat{x} \|_X \leq \| w - \Pi_{X_j} \|_* \| x - \hat{x} \|_* \\
\leq (\varepsilon_j \| w \|_{S_2}) (\varepsilon_j^k \| x \|_{S_{k+1}})
\]
\[
= \varepsilon_j^{k+1} \| w \|_{S_2} \| x \|_{S_{k+1}}
\]
\[
= \varepsilon_j^{k+1} \| M w \|_X \| x \|_{S_{k+1}}
\]
\[
= \varepsilon_j^{k+1} \| M (M^{-1}(x - \hat{x})) \|_X \| x \|_{S_{k+1}}
\]
\[
= \varepsilon_j^{k+1} \| x - \hat{x} \|_X \| x \|_{S_{k+1}}.
\]
After eliminating a factor of \( \| x - \hat{x} \|_X \) from each side of the inequality, we have
\[
\| x - \hat{x} \|_X \leq \varepsilon_j^{k+1} \| x \|_{S_{k+1}}.
\]

In the case that the generalized singular vectors of the operator pair \((T, L)\) are in the space \( S_{n+1} \), where \( n \) is defined in our discretization assumption above, the following theorems can be proven by following the analysis of Chapter 3.

**Theorem 42.** For each \( k \in \mathbb{Z}^+ \), there exists constants \( C_1 \) and \( C_2 \) such that for all sufficiently large positive integers \( j \),
\[
a_k - C_1(c_j + \varepsilon_j^{n+1}) \leq a_j^{(j)} \leq a_k + C_2 c_j.
\]

**Theorem 43.** Let \( k \in \mathbb{Z}^+ \) be given. Then there exists a constant \( C = C_k > 0 \) such that for any \( v \in E_k \) with \( \| v \|_X = 1 \) and for all \( j \in \mathbb{Z}^+ \) sufficiently large, there exists \( w \in E_k^{(j)} \) satisfying
\[
\| v - w \|_X \leq C(c_j + \varepsilon_j^{n+1}),
\]
\[
\| v - w \|_* \leq C(c_j + \varepsilon_j^n).
\]

In terms of gap, it follows that for all \( j \in \mathbb{Z}^+ \) sufficiently large,
\[
\delta_X(E_k, E_k^{(j)}) \leq C(c_j + \varepsilon_j^{n+1}),
\]
\[
\delta_*(E_k, E_k^{(j)}) \leq C(c_j + \varepsilon_j^n).
\]

**Theorem 44.** Let \( k \in M_0 \) be given. Then there exists a constant \( C = C_k > 0 \) such
that

\[ \hat{\delta}(F_k, F_k^{(j)}) \leq C(c_j + \varepsilon_j^{n+1}), \]
\[ \hat{\delta}(G_k, G_k^{(j)}) \leq C(c_j + \varepsilon_j^n). \]

The proofs for these theorems depend on the right generalized singular vectors \( \phi_k \) being in \( S_{n+1} \) for each \( k \in \mathbb{Z}^+ \). Let \( \phi_k \) be a right generalized singular vector of \( (T, L) \). Then

\[ T\phi_k = a_k \psi_k, \quad T^\# \psi_k = a_k \phi_k, \]

where \( a_k > 0 \). Putting both of these things together, we have

\[ T^\# T \phi_k = a_k^2 \phi_k. \]

Therefore, using the fact that \( T^\# T = M^{-1} T^* T \), it follows that

\[ \phi_k = \frac{1}{a_k^2} M^{-1} T^* T \phi_k. \]

It is clear that \( \phi_k \in S_2 \), but in order for \( \phi_k \in S_n \), it must follow that \( T^* T \phi_k \in S_{n-2} \). This assumption may be rather strong for a few reasons. One reason is that \( T^* T \) may not be smoothing with respect to the operator \( L^* L \). In other words, it may be that \( T^* T \) does not map \( S_i \) back into \( S_i \) for some integer \( i \). Another reason is that the functions in \( S_i \) may have to satisfy more and more boundary conditions as \( i \) becomes larger. Such boundary conditions arise, for example, when \( T \) is the solution operator to the Laplace equation.

The analysis for these observed rates of convergence is still incomplete. The above issues suggest that further assumptions may need to be made pertaining to the discretization. If this is so, then it should be possible to find a compact operator \( T \) such that even with higher order elements, convergence of the generalized singular values and vectors should be no better than \( O(h^p) \) for some a fixed integer \( p \), regardless the order of the finite elements used.
References


