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PARTITIONING THE BLOCKS OF A STEINER TRIPLE SYSTEM INTO PARTIAL PARALLEL CLASSES

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**PARTITIONING THE BLOCKS OF A STEINER TRIPLE SYSTEM INTO
PARTIAL PARALLEL CLASSES**

**By
Jezerca Hodaj**

A REPORT

Submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCES

in Mathematical Sciences

**MICHIGAN TECHNOLOGICAL UNIVERSITY
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This report has been approved in partial fulfillment of the requirements for the Degree of MASTER OF SCIENCE in Mathematical Sciences.

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To the people I love the most in this world, my parents,
who never stopped being on my side even though they were miles away.

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Abstract

Does there exist a Steiner Triple System on v points, whose blocks can be partitioned into partial parallel classes of size m , where $m \leq \lfloor \frac{v}{3} \rfloor$, $m \mid b$ and b is the number of blocks of the $STS(v)$? We give the answer for $9 \leq v \leq 43$. We also show that whenever $2 \mid b$, $v \equiv 3 \pmod{6}$ we can find an $STS(v)$ whose blocks can be partitioned into partial parallel classes of size 2, and whenever $4 \mid b$, $v \equiv 3 \pmod{6}$, there exists an $STS(v)$ whose blocks can be partitioned into partial parallel classes of size 4.

Chapter 1

About Coloring triples in every way

1.1 Introduction

A *Steiner triple system* on v points, denoted as $STS(v)$ is a pair (V, \mathcal{B}) , where V denotes a set of v elements and \mathcal{B} denotes a set of b triples, called blocks, from V , such that $|\{B \in \mathcal{B} : \{x, y\} \subset B\}| = 1$, for each distinct pair $\{x, y\} \subset V$. Each point of the Steiner triple system is contained in exactly $r = \frac{v-1}{2}$ blocks. Kirkman [7], showed the existence of such a system whenever the necessary conditions are met, in 1847 .

Theorem 1 [7] *A Steiner triple system of order v exists if and only if $v \equiv 1, 3 \pmod{6}$.*

The smallest Steiner triple system is $STS(7)$, and is commonly known as the Fano Plane. The six lines and one circle drawn in Figure 1.1 gives a pictorial representation of the Steiner triple system with blocks;

$\mathcal{B} = \{\{1, 4, 3\}, \{3, 5, 7\}, \{1, 5, 2\}, \{1, 6, 7\}, \{2, 3, 6\}, \{2, 4, 7\}, \{4, 5, 6\}\}$. A *parallel class* in a Steiner triple system is a collection of disjoint blocks that partition its point set V . Because each point is in r blocks we must have r parallel classes, each consisting of $\frac{v}{3}$ blocks.

A *Kirkman triple system* on v points, denoted as $KTS(v)$ is an $STS(v)$ whose blocks can be partitioned into parallel classes.

Example 1.1.1 *A Kirkman triple system of order 9*

1 2 3	1 4 7	1 5 9	1 6 8
4 5 6	2 5 8	2 6 7	2 4 9
7 8 9	3 6 9	3 4 8	3 5 7

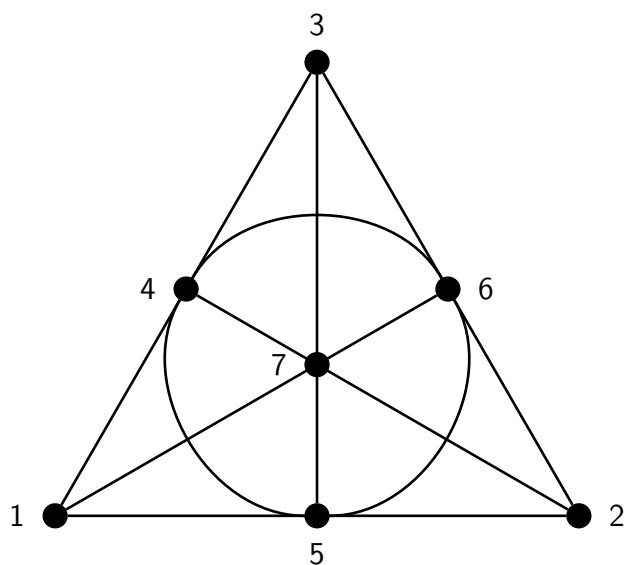


Figure 1.1: STS(7), The Fano Plane

Example 1.1.1 gives the 12 blocks of a $KTS(9)$. The blocks are partitioned into four parallel classes, and each parallel class partitions the set of nine points.

Theorem 2 [4, 5, 7] *A Kirkman triple system of order v exists if and only if $v \equiv 3 \pmod{6}$.*

We are providing now more formal definitions.

A *partial Steiner triple system* on v points, $PSTS(v)$, is a pair (V, \mathcal{B}) , where V denotes a set of v elements and \mathcal{B} denotes a set of b triples from V , such that $|B \cap B'| \leq 1$, when $B, B' \in \mathcal{B}$ and $B \neq B'$.

When $\mathcal{R} \subseteq \mathcal{B}$ and $|B \cap B'| = 0$ when $B, B' \in \mathcal{R}$ and $B \neq B'$, then \mathcal{R} is called a *partial parallel class (PPC)* of (V, \mathcal{B}) .

A partial parallel class containing all but one point of the points of a Steiner triple system, is called an *almost parallel class*.

A *block coloring* of a $PSTS(v)(V, \mathcal{B})$ in c colors is a mapping $\chi : \mathcal{B} \rightarrow \{1, \dots, c\}$, so that every color class $\chi^{-1}(i), i \in \{1, \dots, c\}$ is a partial parallel class.

The *color type* of a block coloring of (V, \mathcal{B}) in c colors is the sequence (m_1, \dots, m_c)

where $m_i = |\{B \in \mathcal{B} : \chi(B) = i\}|$.

Often the sequence (m_1, \dots, m_c) is written in exponential form; color type $w_1^{u_1} \dots w_s^{u_s}$ indicates that u_i of the color classes have size w_i for $1 \leq i \leq s$.

Using partial parallel classes we can give the definition of a Hanani triple system.

A *Hanani triple system*, (HATS) on $v = 6t + 1$ points is a Steiner triple system having $\frac{v-1}{2} = 3t$ pairwise disjoint almost parallel classes of size $2t = \frac{v-1}{3}$ (set of pairwise disjoint triples that span $v-1$ elements) and the remaining triples form a partial parallel class of size $t = \frac{v-1}{6}$. We give the number of partial parallel classes and their sizes with respect to t , as a Hanani triple system is equivalent to a partial Steiner triple system that admits a color type $(2t)^{3t}t^1$.

Theorem 3 [6] *A HATS(v) exists if and only if $v \equiv 1 \pmod{6}$.*

Steiner, Kirkman, and Hanani triple systems have been studied for a long time now, [7, 4, 5, 6] and they are useful for other constructions as we will see in this report.

Another type of configuration that aids in the construction of certain designs is the Group Divisible Design (GDD). A *3-GDD* is a collection of 3-elements subsets of a v -set V , called blocks, which satisfies the following properties;

- each point v appears in r of the b blocks;
- the $v = nl$ elements of V are partitioned into l subsets, which are called groups, each of size n ;
- no pair of points from the same group occur together in any block;
- any two points not in the same group appear together in λ blocks.

We use exponential notation to describe the type of the GDD. So a $GDD(g^u)$ denotes a GDD of type g^u , which has u groups of size g . Staying in the same topic, a *Resolvable Group Divisible Design* is a GDD whose blocks can be partitioned into parallel classes. An example of an RGDD on 15 points is given in Example 1.1.2.

Example 1.1.2 *An RGDD(5^3) with groups, $G_1 = \{0, 1, 2, 3, 4\}$, $G_2 = \{5, 6, 7, 8, 9\}$, $G_3 = \{10, 11, 12, 13, 14\}$.*

0 5 10	0 6 12	0 7 14	0 8 11	0 9 13
1 6 11	1 7 13	1 8 10	1 9 12	1 5 14
2 7 12	2 8 14	2 9 11	2 5 13	2 6 10
3 8 13	3 9 10	3 5 12	3 6 14	3 7 11
4 9 14	4 5 11	4 6 13	4 7 10	4 8 12

A *frame parallel class* (FPC) for group G is a collection of blocks that partition the points $V \setminus G$, hence it is a partial parallel class that misses every vertex in G but contains every other vertex.

Let V be a finite set, and let \mathcal{G} be a partition of V . A *3-frame* of type $g_1^{u_1} g_2^{u_2} \dots g_t^{u_t}$ is a $3 - GDD$ of type $g_1^{u_1} g_2^{u_2} \dots g_t^{u_t}$ together with a partition of its blocks into frame parallel classes. For every group $G \in \mathcal{G}$ there are exactly $\frac{|G|}{2}$ frame parallel classes.

Example 1.1.3 A 3-frame of type 2^4 , with groups, $G_1 = \{1, 2\}$, $G_2 = \{3, 4\}$, $G_3 = \{5, 6\}$, $G_4 = \{7, 8\}$. The blocks of this 3-frame are as follows:

FPC1	FPC2	FPC3	FPC4
3 6 7	1 6 8	2 3 8	1 3 5
4 5 8	2 5 7	1 4 7	2 4 6

In this report we study the following problem: Does there exist an $STS(v)$ whose triples can be partitioned into partial parallel classes of size m , where $m_i \leq \lfloor \frac{v}{3} \rfloor$, $m_i \mid b$ and b is the number of triples in an $STS(v)$? This question was initially asked by Alex Rosa. Colbourn, Horsley and Wang studied a more general version of this problem in [1]. In this paper they studied the decomposition of partial Steiner Triple Systems, (PSTS) into partial parallel classes of mixed sizes. More specifically they ask, given v , t and m_1, m_2, \dots, m_s , with $t = \sum_{i=1}^s m_i$, does there exist a partial Steiner triple system of order v , whose triples can be partitioned into partial parallel classes of sizes m_1, m_2, \dots, m_s ? Elementary necessary conditions are that there exist a partial Steiner triple system of order v having the number of triples equal to the sum of the triples in each partial parallel class. Thus $t = \sum_{i=1}^s m_i$. Also each partial parallel class can have at most $\lfloor \frac{v}{3} \rfloor$ triples in it, so $m_i \leq \lfloor \frac{v}{3} \rfloor$ for $1 \leq i \leq s$. These necessary conditions are not sufficient when $v \in \{6, 7, 9, 11, 12, 13\}$. In the paper they conjecture that these conditions are sufficient for $v \geq 14$. They also show that they are sufficient for $14 \leq v \leq 32$.

Let $\mu(v)$ denote the number of triples in a partial Steiner triple system of order v . This number cannot exceed, $\mu(v) = \lfloor \frac{v}{3} \lfloor \frac{v-1}{2} \rfloor \rfloor - \varepsilon$, where $\varepsilon = 1$ if $v \equiv 5 \pmod{6}$ and $\varepsilon = 0$ otherwise. Then

$$\mu(v) = \begin{cases} 6t^2 - 2t & \text{when } v = 6t \\ 6t^2 + t & \text{when } v = 6t + 1 \\ 6t^2 + 2t & \text{when } v = 6t + 2 \\ 6t^2 + 5t + 1 & \text{when } v = 6t + 3 \\ 6t^2 + 6t + 1 & \text{when } v = 6t + 4 \\ 6t^2 + 9t + 2 & \text{when } v = 6t + 5 \end{cases}$$

Schönheim [8] showed that a (PSTS) with $\mu(v)$ triples exists for all $v \geq 0$. Such a system is a *maximum partial Steiner triple system*, $\text{MPT}(v)$.

The basic necessary conditions for a $\text{PSTS}(v)$ to admit a color type (m_1, \dots, m_c) are as follows.

1. For each $1 \leq i \leq c$, there must be a partial parallel class containing m_i triples, so then $m_i \leq \lfloor \frac{v}{3} \rfloor$.
2. There must exist a $\text{PSTS}(v)$ having $\sum_{i=1}^c m_i$ triples, so $\sum_{i=1}^c m_i \leq \mu(v)$.

The interest of the authors in [1] is to determine the possible color types of block colorings, (the possible decomposition of the blocks) of a $\text{PSTS}(v)$. They conjecture the following.

Conjecture 4 *Let $v \geq 14$. Let (m_1, \dots, m_c) satisfy $\sum_{i=1}^c m_i \leq \mu(v)$ and $m_i \leq \lfloor \frac{v}{3} \rfloor$ for $1 \leq i \leq c$. Then there exists a partial Steiner triple system that admits a block coloring of color type (m_1, \dots, m_c) .*

In the paper they divide the coloring of the triples in two cases. They deal first with orders up to 13. They next consider the orders $14 \leq v \leq 32$, where they prove their conjecture all such v .

We will call a decomposition in which each color class has the same size, *uniform*. Thus in the context of their paper a Kirkman triple system $\text{KTS}(6t + 3)$, is equivalent to a $\text{PSTS}(6t + 3)$ that admits a uniform color type $(2t + 1)^{3t+1}$.

A partial Steiner triple system whose triples can be partitioned into s partial parallel classes, each of size m , is a signal set, denoted $SS(v, s, m)$. So a signal set $SS(v, s, m)$

is a PSTS(v) that admits a color type m^s . When $s = \lfloor \frac{\mu(v)}{m} \rfloor$ the $SS(v, s, m)$ is a Kirkman signal set, $KSS(v, m)$. Many of the uniform decompositions are covered by the next result.

Theorem 5 [9] *A $KSS(v, \lfloor \frac{v}{3} \rfloor)$ exists for each positive integer v such that $v \notin \{6, 7, 12\}$.*

A major construction technique for many of the non-uniform decompositions relies on the idea of dismantling closely related partial parallel classes that have been constructed previously on v points. We examine this technique more carefully in Chapter 2 of this report. Colbourn et. al. in [1], study the problem partitioning partial Steiner triple systems into partial parallel classes of mixed sizes. In this report, we take into consideration only Steiner triple systems, $|\{B \in \mathcal{B} : \{x, y\} \subset B\}| = 1$. Equivalent to the signal sets, we want to partition the b blocks of an $STS(v)$ into s partial parallel classes, all of size m , where, $sm = b = \frac{v(v-1)}{6}$ and $m \leq \frac{v}{3}$. Thus we focus on the uniform case of the problem.

1.2 Preliminary Results

We denote by a $STS_m(v)$ a $STS(v)$ whose blocks can be partitioned into partial parallel classes of size m . The next three results are straightforward, and can be found in [1].

Lemma 6 [1] *If $v \equiv 3 \pmod{6}$, then there exists a $STS_{\frac{v}{3}}(v)$.*

Proof If $v \equiv 3 \pmod{6}$ then the $STS(v)$ is a $KTS(v)$, and a Kirkman triple system is just a resolvable Steiner triple system. This means that we can partition its blocks into parallel classes of size $m = \frac{v}{3}$, and in total we would have $\frac{v-1}{2}$ parallel classes.

Lemma 7 [1] *Suppose $m_1 | m$ and there exists an $STS_m(v)$, then there exists an $STS_{m_1}(v)$.*

Proof The proof is obvious. We divide each partial parallel class of size m into partial parallel classes of size m_1 .

Recall a Hanani triple system, $HATS(6t + 1)$ is equivalent to an $STS(6t + 1)$ that admits a color type $(2t)^{3t}t^1$. It was shown in [6] that a Hanani triple system exists if and only if $v \equiv 1 \pmod{6}$ and $v \notin \{7, 13\}$. So we will have $3t$ partial parallel classes with $2t = \frac{v-1}{3}$ blocks in them and a partial parallel class with $t = \frac{v-1}{6}$ blocks. We partition each one of the $3t$ partial parallel classes of size $2t$ in two. We then have $6t + 1 = 6 \cdot \frac{v-1}{6} + 1 = v$ partial parallel classes of size $t = \frac{v-1}{6}$. Hence the following result.

Lemma 8 [1] *If $v = 6t + 1$, then there exists a $STS_t(v)$.*

An $l - PPC$ denotes a partial parallel class with l triples in it.

The following two theorems give results on decomposing an $STS(v)$ into partial parallel classes.

Theorem 9 [3] *Let $S = (V, \mathcal{B})$ be an $STS(v)$ with $v > 9l$. Then there exists a decomposition of \mathcal{B} into $l - PPC's$.*

Theorem 10 [3] *An $STS(v)$ can be decomposed into 2-PPC's if and only if $2|v$ and $v \neq 9$.*

From the previous theorem we know we are able to decompose an $STS(v)$ into partial parallel classes of size 2. In Chapter 2, we develop a method of decomposing the blocks of an $STS(v)$ into 2-PPC's, and we then extend this method to 4-PPC's. We also give a few special decompositions in which the techniques used could possibly be generalized.

Chapter 3 summarizes our results and gives a table of what is known on the problem for $v \leq 43$, and discusses some options for future research.

Chapter 2

New results

In this chapter we study one of the main techniques used in [1] to decompose the blocks of an $STS(v)$ into $l - PPC'$ s. We give the result, and illustrate the idea with the example $v = 25, m = 5$. Although the technique is used to obtain many of the non-uniform decompositions, it will also be useful for the uniform case.

Theorem 9 says we can always decompose an $STS(v)$, when $b = \frac{v(v-1)}{2}$ into partial parallel classes of size 2 except for the cases when $v = 9$. We develop a different technique to do so, that could possibly be generalized for partial parallel classes of larger size. We extend the idea behind this technique, to partial parallel classes of size 4. We also give the construction of a $5 - PPC$ decomposition for $v = 21$ and $v = 25$.

2.1 Dismantling Partial Parallel Classes

The following method was used in [1] to construct partial parallel classes decomposition of new sizes from already known decompositions. The idea behind this method is given in the following lemmas. The first one dismantles two partial parallel classes to form three partial parallel classes.

Lemma 11 [1] *Let n_1 and n_2 be positive integers such that $n_1 \leq n_2$, let V_1 and V_2 be disjoint sets such that $|V_1| = n_1$ and $|V_2| = n_2$, and let G be a bipartite graph with bipartition (V_1, V_2) such that G has maximum degree at most 3 and either G is connected or $n_2 \leq 2n_1$. Let x_1 and x_2 be integers such that either $x_1 \leq n_1$ and $2x_1 + x_2 < n_2$ or $x_1 = x_2 = \frac{n_1}{3} = \frac{n_2}{3}$. Then unless G is $K_{3,3}$ and $x_1 = x_2 = 1$, there are sets $X_1 \subseteq V_1$ and $X_2 \subseteq V_2$ such that $|X_1| = x_1, |X_2| = x_2$ and no edge of G has one end in X_1 and one end in X_2 .*

Lemma 12 [1] *Suppose a color type T' can be obtained from a color type T by iteratively applying the following operations:*

- (i) Take an entry y in the sequence and replace it with an entry a such that $a \leq y$.
- (ii) Take an entry y in the sequence and replace it with two entries a and b such that $a + b = y$.
- (iii) Take two entries y and z such that $y \leq z \leq 2y$ and replace them with three entries a , b , and c such that $a + b + c = y + z$, $a \leq y$, $b \leq z$, $(a, b, c) \neq (2, 2, 2)$ and either $2y < 2a + b$ or $a = b = c = \frac{2y}{3} = \frac{2z}{3}$.

Then, if (V, \mathcal{B}) is a PSTS that admits color type T , there is a PSTS (V, \mathcal{B}') with $\mathcal{B}' \subseteq \mathcal{B}$ that admits a color type T'' .

Lemma 12 may be used to obtain uniform decompositions when certain mixed decompositions are known. The following example illustrates the lemma by obtaining a decomposition of an $STS(25)$ into $5 - PPC$'s from the known decomposition with color type $8^6 7^6 5^2$.

Example 2.1.1 A decomposition of an $STS(25)$ into $5 - PPC$'s.

From the results of the paper we studied, [1], there is an $STS(25)$ that has a color type $T = 8^6 7^6 5^2$. We want to get a color type $T' = 5^{20}$, in other words we want 20 partial parallel classes of size 5. In order to get the second color type or the 20 $5 - PPC$'s of the $STS(25)$ we apply Lemma 12.

First we choose $y = 7$ and $z = 8$. Then we let $a = b = c = 5$. We check if the conditions stated in the lemma hold.

1. $7 \leq 8 \leq 2 \cdot 7 \Rightarrow y \leq z \leq 2y$
2. $5 + 5 + 5 = 7 + 8 \Rightarrow a + b + c = y + z$
3. $5 \leq 7 \Rightarrow a \leq y$
4. $5 \leq 8 \Rightarrow b \leq z$
5. $2 \cdot 7 \leq 2 \cdot 5 + 5 \Rightarrow 2y \leq 2a + b$

In this way, each time we have a partial parallel class of size 8 and a partial parallel class of size 7 we'll get 3 partial parallel classes of size 5. From the color type $T = 8^6 7^6 5^2$ that we have, we get 18 partial parallel classes of size 5 from the 6 partial parallel classes of size 8 and the 6 classes of size 7. There were already 2 partial parallel classes of size 5. Thus in total we have 20 partial parallel classes of size 5. This $STS(25)$ with color type $T' = 5^{20}$ has been constructed, i.e. there exists an $STS_5(25)$.

Example 2.1.2 A decomposition of an $STS(13)$ into 2 – PPC's.

In [1] we can find that there exists an $STS(13)$ that has a color type $T = 4^4 2^5$. We want to get a color type $T' = 2^{13}$, or 13 partial parallel classes of size 2. By partitioning the 4 partial parallel classes of size 4 of the color type $T = 4^4 2^5$ in two, we would get 8 partial parallel classes of size 2. We already have 5 other partial parallel classes of size 2. Thus we would have in total 13 partial parallel classes of size 2, or 13 2 – PPCs.

2.2 New Results

The main results of this chapter use 3-frames. Stinson solved the existence problem for 3-frames in 1987.

Lemma 13 [4] *There exists a 3-frame of type g^u if and only if g is even, $u \geq 4$, and $g(u-1) \equiv 0 \pmod{3}$.*

2.2.1 Partial Parallel classes of size $m = 2$

Lemma 14 *If there exists a 3-frame of type 2^u and $u \geq 7$, then there exists an $STS_2(2u+1)$, whenever $b = \frac{(2u+1)(2u)}{6} = \frac{u(2u+1)}{3}$ is even.*

Proof Let $v = 2u + 1$ and G_1, G_2, \dots, G_u , be the groups of a 3-frame of type 2^u . The blocks of the 3-frame along with the blocks formed as $G_i \cup \{\infty\}$, for $i = 1, 2, \dots, u$ forms a $STS(2u+1)$. To see this, we need to show that each pair is in exactly one block. In a 3-frame we have that each pair from distinct groups is in exactly one block, and no pairs from the same group are in the same block. From the blocks created as $G_i \cup \{\infty\}$, we have that the pairs in one group will be in exactly one block as well. All the pairs containing $\{\infty\}$ will be covered exactly once also. So every pair of the $STS(2u+1)$ is in exactly one block. The number of blocks in a $STS(v)$ is $\frac{v(v-1)}{6}$, so we will have $\frac{v(v-1)}{12}$ partial parallel classes.

Now we describe the partial parallel classes. Because a 3-frame of type 2^u exists, we must have $u \equiv 1 \pmod{3}$. But when $u \equiv 1 \pmod{6}$, $2 \nmid b$, hence $u \equiv 4 \pmod{6}$, and in particular u is even. We partition the groups into pairs:

$$\{G_1, G_2\}, \{G_3, G_4\}, \dots, \{G_{u-1}, G_u\}.$$

For each pair of groups G_i and G_j , consider the frame parallel classes F_i and F_j . In F_i there is at least one block B which is disjoint with G_j . In each frame parallel class

there are $\frac{v-3}{3}$ blocks. There are two blocks in F_i that intersect G_j . Thus we are left with $\frac{v-9}{3}$ blocks that do not intersect G_j . Let B be one of these blocks.

Remove B from F_i and add it to F_j . The number of blocks remaining in F_i now is:

$$\frac{2u-2}{3} - 1 = \frac{2u-5}{3} = \frac{2\frac{v-1}{2}-5}{3} = \frac{v-6}{3} = \frac{v}{3} - 2$$

Since $v \equiv 3 \pmod{6}$ this number is odd. Partition $G_i \cup \{\infty\}$ along with the remaining blocks of F_i into partial parallel classes of size 2. The number of blocks in F_j is:

$$\frac{2u-2}{3} = \frac{2\frac{v-1}{2}-2}{3} = \frac{v-3}{3}$$

which is even. Because $G_j \cap B = \emptyset$, we may take $G_j \cup \{\infty\}$ and B to be a partial parallel class of size 2. Partition the rest of F_j into $2 - PPC's$.

This accounts for: $\frac{\frac{v}{3}-2+1}{2} = \frac{v-3}{6}$ partial parallel classes in F_i and $(\frac{v-3}{3} + 2) \cdot \frac{1}{2} = \frac{v+3}{6}$ partial parallel classes in F_j . So in F_i and F_j together we have:

$$\frac{v-3}{6} + \frac{v+3}{6} = \frac{2v}{6} = \frac{v}{3}$$

partial parallel classes.

We have $\frac{u}{2}$ pairs of groups, and each of them gives us $\frac{v}{3}$ partial parallel classes. So in total we get:

$$\frac{v}{3} \cdot \frac{u}{2} = \frac{v}{3} \cdot \frac{v-1}{4} = \frac{v(v-1)}{12}$$

partial parallel classes, the number we were looking for.

Theorem 15 *If $2|b$, then there exists an $STS(v)$ whose blocks can be partitioned into $\frac{b}{2} 2 - PPC's$.*

Proof If $v \equiv 1 \pmod{6}$, then $b = \frac{v(v-1)}{3 \cdot 2} \equiv 0 \pmod{2}$ when $\frac{v-1}{6}$ is even. Let $v = 6t + 1$, then $b \equiv 0 \pmod{2}$ whenever t is even. By Theorem 3, there exists a $HATS(6t+1)$ for all such t , except for $t \in \{1, 2\}$. This is equivalent to an $STS(6t+1)$ with color type $(2t)^{3t^1}$. Because t is even, we can divide each partial parallel class into $2 - PPC's$. If $t = 1$, b is not even. If $t = 2$, then an $STS_2(13)$ exists by Lemma 12.

If $v \equiv 3 \pmod{6}$ and $v \geq 15$, then $\frac{v-1}{2} \equiv 1, 4 \pmod{6}$, and $\frac{v}{3} \equiv 1, 3, 5 \pmod{6}$. Therefore, $b = \frac{v(v-1)}{3 \cdot 2} \equiv 0 \pmod{2}$ only when $\frac{v-1}{2} \equiv 4 \pmod{6}$. So let $u = \frac{v-1}{2} \equiv 4 \pmod{6}$. Then $u - 1 \equiv 0 \pmod{3}$, and by Lemma 13, there exists a 3-frame of type 2^u . Now we apply Lemma 14 to construct the $2 - PPC's$.

If $v = 7$ and $v = 9$, then by Theorem 10 neither the $STS(7)$ nor the $STS(9)$ can be partitioned into $2 - PPC's$.

2.2.2 Partial parallel classes of size $m = 4$

Lemma 16 *If there exists a 3-frame of type 2^u , and $u \geq 16$, then there exists an $STS_4(2u + 1)$, whenever $4|b$.*

Proof Let $v = 2u + 1$ and G_1, G_2, \dots, G_u , be the groups of a 3-frame of type 2^u . The blocks of the 3-frame along with the blocks formed as $G_i \cup \{\infty\}$, for $i = 1, 2, \dots, u$ forms a $STS(2u + 1)$. The proof for this is identical to the proof in Lemma 14. Again this $STS(2u + 1)$ has $\frac{v(v-1)}{6}$ blocks, so we will require $\frac{v(v-1)}{24}$ partial parallel classes. Because of the existence of a 3-frame of type 2^u , we must have that $u \equiv 1 \pmod{3}$. This means that $u \equiv 1, 4 \pmod{6}$. But when $u \equiv 1 \pmod{6}$, $b \not\equiv 0 \pmod{2}$. So $u \equiv 4 \pmod{6}$, and in particular u is even. However u being even is not enough, since we want that $4|b$. If $u \equiv 4 \pmod{6}$, then $u \equiv 0, 2 \pmod{4}$. $u \equiv 4 \pmod{6}$, $u \equiv 0 \pmod{4}$ by the Chinese Remainder Theorem, $u \equiv 4 \pmod{12}$. We partition the groups into sets of 4:

$$\{G_1, G_2, G_3, G_4\}, \{G_5, G_6, G_7, G_8\}, \dots, \{G_{u-3}, G_{u-2}, G_{u-1}, G_u\}.$$

For each set of 4 groups G_i, G_j, G_k, G_s for instance, consider the corresponding frame parallel classes F_i, F_j, F_k , and F_s . In each frame parallel class there are $\frac{v-3}{3}$ blocks. There are two blocks in F_i that intersect G_j . Thus we are left with $\frac{v-9}{3} \geq 8$ blocks that do not intersect with G_j . Let B_j be one of these blocks. Remove B_j from F_i and add it to F_j . There are 3 blocks in F_j that intersect B_j . Pick two blocks B_{j_1} and B_{j_2} that are not any of these.

So we would get this partial parallel class of size 4; $\{G_j \cup \{\infty\}, B_j, B_{j_1}, B_{j_2}\}$. The blocks B_{j_1} , and B_{j_2} are in frame F_j . We take two more blocks from F_i and add them to F_k and F_s respectively. Following the same reasoning as above, we can find these two more blocks that do not intersect with G_k and G_s . So we also remove B_k and B_s from F_i and add them to F_k and F_s respectively. In this way we would form two more partial parallel classes of size 4. After removing these 3 blocks from F_i the number of remaining blocks is:

$$\frac{2u-2}{3} - 3 = \frac{2^{\frac{v-1}{2}} - 2}{3} - 3 = \frac{v-3-9}{3} = \frac{v}{3} - 4$$

Because $v \equiv 3 \pmod{6}$, we have that $\frac{v}{3} - 4$ is odd.

Partition $G_i \cup \{\infty\}$ along with the remaining blocks of F_i into partial parallel classes of order 4. The number of partial parallel classes in F_i is:

$$\frac{\frac{v}{3} - 4 + 1}{4} = \frac{\frac{v}{3} - 3}{4} = \frac{v-9}{12}$$

The number of blocks in F_j, F_k , and F_s after adding up the blocks we took from frame F_i will be

$$\frac{2u-2}{3} + 1 = \frac{2^{\frac{v-1}{2}} - 2}{3} + 1 = \frac{v-3+3}{3} = \frac{v}{3}$$

We partition $G_j \cup \{\infty\}, G_k \cup \{\infty\}$, and $G_s \cup \{\infty\}$ along with the rest of the blocks in their respective frames, and we get

$$\left(\frac{v}{3} + 1\right) \cdot \frac{1}{4} = \frac{v+3}{12}$$

partial parallel classes. In these 4 frames together, F_i, F_j, F_k , and F_s we have in total

$$\frac{v-9}{12} + 3 \cdot \frac{v+3}{12} = \frac{v-9+3v+9}{12} = \frac{4v}{12} = \frac{v}{3}$$

partial parallel classes. We have $\frac{u}{4}$ sets of groups, each of them gives us $\frac{v}{3}$ partial parallel classes. Thus the total number of partial parallel classes is

$$\frac{v}{3} \cdot \frac{u}{4} = \frac{v}{3} \cdot \frac{v-1}{8} = \frac{v(v-1)}{24}$$

partial parallel classes, the number we were looking for.

Theorem 17 *If $4|b$, then there exists an $STS(v)$ whose blocks can be partitioned into $\frac{b}{4} 4 - PPCs$.*

Proof If $v \equiv 1 \pmod{6}$, then we should have that $\left(\frac{v-1}{6}\right) \equiv 0 \pmod{4}$, in order that $4|b$. Let $v = 6t + 1$, then we need $t \equiv 0 \pmod{4}$. From Theorem 3 there exists a $HATS(6t + 1)$ for all such t . This $HATS(6t + 1)$ is equivalent to an $STS(6t + 1)$

that admits a color type $(2t)^{3t}t^1$. Now since $t \equiv 0 \pmod{4}$ we can divide each of these partial parallel classes into $4 - PPC'$ s.

If $v \equiv 3 \pmod{6}$ then we have that $\frac{v-1}{2} \equiv 1, 4 \pmod{6}$ and $\frac{v}{3} \equiv 1, 3, 5 \pmod{6}$ and so $\frac{v}{3}$ is odd. Therefore because $4|b$ when $\frac{v-1}{2} \equiv 0 \pmod{4}$ and thus $u = \frac{v-1}{2} \equiv 4 \pmod{6}$. Consequently $u - 1 \equiv 0 \pmod{3}$, and so by Lemma 13, there exists a 3-frame of type 2^u . Now apply Lemma 16 to construct the $4 - PPC$ s.

2.3 Special Cases

Theorems 15 and 17 cover the decomposition of many cases. However there are cases when we cannot use these results. We discuss some of those cases now. The case of $v = 21$ and $m = 5$ has been covered in [1], but the case of $v = 33$ and $m = 8$ has not. We give our own construction for each of these.

2.3.1 When $v = 21$ and $m = 5$

Lemma 18 *There exists an $STS_5(21)$.*

In this case we have an $STS(21)$, which is actually a $KTS(21)$. We will have to decompose the 70 blocks into 14 partial parallel classes of size 5. We partition the points into 3 groups; $G_0 = \{0_0, 1_0, 2_0, 3_0, 4_0, 5_0, 6_0\}$, $G_1 = \{0_1, 1_1, 2_1, 3_1, 4_1, 5_1, 6_1\}$, $G_2 = \{0_2, 1_2, 2_2, 3_2, 4_2, 5_2, 6_2\}$. Since we are working on 21 points, $21 = 7 \cdot 3$, we can put an $STS(7)$ in each of the groups. We work on each group cyclically (mod 7).

Because we are working (mod 7) in each of the 3 groups, then we will have these differences, $\{1, 2, 3\}$ between the points in each of the groups. We will also consider the differences between the points of one group with the points of 2 other groups. We will call these type of differences, "cross differences".

The blocks of the partial parallel classes will be formed in the way that the differences between the points in each block cover all the differences among the 7 points in each group, i.e. $\{1, 2, 3\}$. The rest of the blocks will be formed such that the points of each block cover the cross differences between the groups G_0 , G_1 , and G_2 , i.e. $\{0, 1, 2, 3, 4, 5, 6\}$. Below we give all the 14 partial parallel classes of size 5.

<i>PPC1</i>	<i>PPC2</i>	<i>PPC3</i>	<i>PPC4</i>
$0_0 \ 1_0 \ 3_0$	$1_0 \ 2_0 \ 4_0$	$2_0 \ 3_0 \ 5_0$	$3_0 \ 4_0 \ 6_0$
$0_1 \ 1_1 \ 3_1$	$1_1 \ 2_1 \ 4_1$	$2_1 \ 3_1 \ 5_1$	$3_1 \ 4_1 \ 6_1$
$0_2 \ 1_2 \ 3_2$	$1_2 \ 2_2 \ 4_2$	$2_2 \ 3_2 \ 5_2$	$3_2 \ 4_2 \ 6_2$
$2_0 \ 2_1 \ 2_2$	$3_0 \ 3_1 \ 3_2$	$4_0 \ 4_1 \ 4_2$	$5_0 \ 5_1 \ 5_2$
$4_0 \ 5_1 \ 6_2$	$5_0 \ 6_1 \ 0_2$	$6_0 \ 0_1 \ 1_2$	$0_0 \ 1_1 \ 2_2$

<i>PPC5</i>	<i>PPC6</i>	<i>PPC7</i>	<i>PPC8</i>
0 ₀ 4 ₀ 5 ₀	1 ₀ 5 ₀ 6 ₀	0 ₀ 2 ₀ 6 ₀	0 ₀ 2 ₁ 4 ₂
0 ₁ 4 ₁ 5 ₁	1 ₁ 5 ₁ 6 ₁	0 ₁ 2 ₁ 6 ₁	1 ₀ 3 ₁ 5 ₂
0 ₂ 4 ₂ 5 ₂	1 ₂ 5 ₂ 6 ₂	0 ₂ 2 ₂ 6 ₂	2 ₀ 4 ₁ 6 ₂
6 ₀ 6 ₁ 6 ₂	0 ₀ 0 ₁ 0 ₂	1 ₀ 1 ₁ 1 ₂	3 ₀ 5 ₁ 0 ₂
1 ₀ 2 ₁ 3 ₂	2 ₀ 3 ₁ 4 ₂	3 ₀ 4 ₁ 5 ₂	4 ₀ 6 ₁ 1 ₂
<i>PPC9</i>	<i>PPC10</i>	<i>PPC11</i>	<i>PPC12</i>
0 ₀ 3 ₁ 6 ₂	0 ₀ 4 ₁ 1 ₂	1 ₀ 5 ₁ 2 ₂	0 ₀ 5 ₁ 3 ₂
1 ₀ 4 ₁ 0 ₂	3 ₀ 6 ₁ 2 ₂	2 ₀ 6 ₁ 3 ₂	1 ₀ 6 ₁ 4 ₂
2 ₀ 5 ₁ 1 ₂	4 ₀ 0 ₁ 3 ₂	3 ₀ 0 ₁ 4 ₂	2 ₀ 0 ₁ 5 ₂
5 ₀ 0 ₁ 2 ₂	5 ₀ 1 ₁ 4 ₂	4 ₀ 1 ₁ 5 ₂	3 ₀ 1 ₁ 6 ₂
6 ₀ 1 ₁ 3 ₂	6 ₀ 2 ₁ 5 ₂	5 ₀ 2 ₁ 6 ₂	6 ₀ 3 ₁ 0 ₂
	<i>PPC13</i>		<i>PPC14</i>
	0 ₀ 6 ₁ 5 ₂		2 ₀ 1 ₁ 0 ₂
	1 ₀ 0 ₁ 6 ₂		3 ₀ 2 ₁ 1 ₂
	4 ₀ 2 ₁ 0 ₂		4 ₀ 3 ₁ 2 ₂
	5 ₀ 3 ₁ 1 ₂		5 ₀ 4 ₁ 3 ₂
	6 ₀ 4 ₁ 2 ₂		6 ₀ 5 ₁ 4 ₂

Figures 2.1 - 2.5 illustrate the 14 partial parallel classes of size 5 of the $STS(21)$. The blocks $\{0_0, 1_0, 3_0\}$, $\{0_1, 1_1, 3_1\}$, and $\{0_2, 1_2, 3_2\}$ colored in red in Figure 2.1 cover

the differences 1, 2, and 3 in group G_0 , G_1 , and G_2 . After developing these blocks (mod 7) all edges with differences 1, 2, and 3 have been covered.

The block $\{2_0, 2_1, 2_2\}$ colored in green in Figure 2.1 covers the cross differences (0, 0, 0) between the group G_0 , G_1 , and G_2 . When we develop (mod 7) this base block we will have covered all edges with cross difference 0.

This block $\{4_0, 5_1, 6_2\}$, colored in blue in Figure 2.1 covers edges with cross differences (1, 1, 5) between the groups G_0 and G_1 , G_1 and G_2 , G_2 and G_0 respectively. Developing this base block (mod 7) we cover all the (1, 1, 5) cross differences between the three groups.

On the second partial parallel class in Figure 2.3 we cover the cross differences (2, 2, 3) between the groups G_0 and G_1 , G_1 and G_2 , G_2 and G_0 respectively. The block

$\{0_0, 2_1, 4_2\}$ developed $(\text{mod } 7)$ will give us all such differences. Two more blocks that cover these cross differences are given in red, on the third partial parallel class in Figure 2.3.

Continuing on Figure 2.3, the rest of the blocks on the third partial parallel class, given in blue cover the cross differences $(3, 3, 1)$ between the groups G_0 and G_1 , G_1 and G_2 , G_2 and G_0 . We have four more such blocks on the first partial parallel class of Figure 2.4, colored in blue. By developing $(\text{mod } 7)$ the block $\{0_0, 3_1, 6_2\}$ we will get all such differences between the three groups.

The next cross differences being covered by the green blocks in Figure 2.4 are $(4, 4, 6)$. Considering the block $\{0_0, 4_1, 1_2\} (\text{mod } 7)$, we will get all these differences between the groups G_0 and G_1 , G_1 and G_2 , G_2 and G_0 .

The third partial parallel class in Figure 2.4 has four blocks, that are colored in purple, which cover the cross differences $(5, 5, 4)$ between the groups G_0 and G_1 , G_1 and G_2 , G_2 and G_0 . We have three more such blocks on the first partial parallel class of Figure 2.5. The base block of these differences is $\{0_0, 5_1, 3_2\}$.

The last cross differences that are being covered in Figure 2.5 between the groups G_0 and G_1 , G_1 and G_2 , G_2 and G_0 , are the differences $(6, 6, 2)$. All these blocks are give in blue. The base block of these differences is $\{0_0, 6_1, 5_2\}$.

As we can see all the differences within the groups and between the groups, the cross differences, are covered the same number of times. Having all the differences means that we have all the blocks of the $STS(21)$, and all of them are being partitioned into partial parallel classes of size 5.

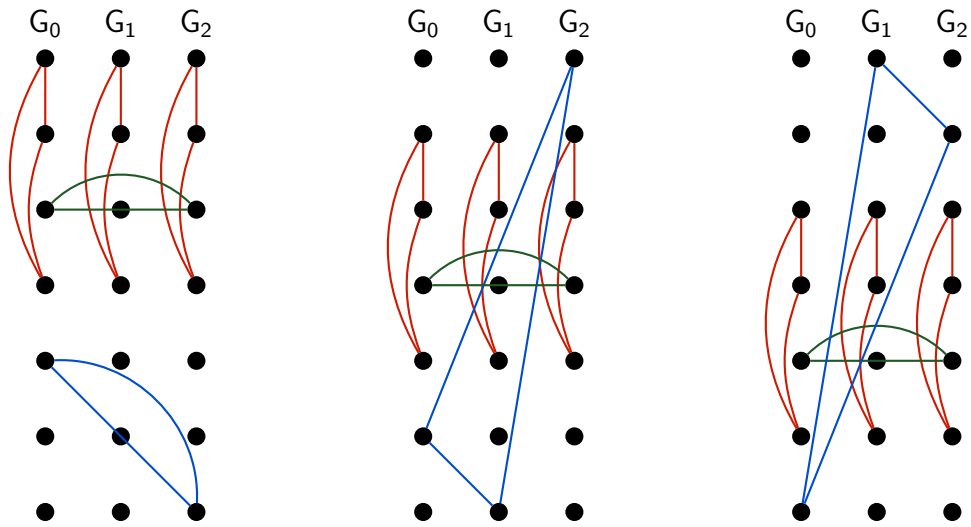


Figure 2.1: Partial parallel classes of size 5

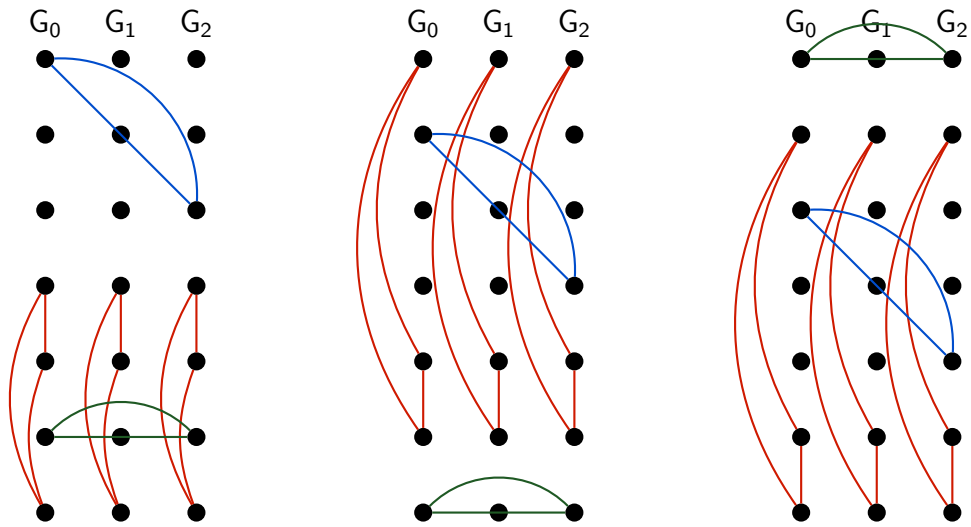


Figure 2.2: Partial parallel classes of size 5

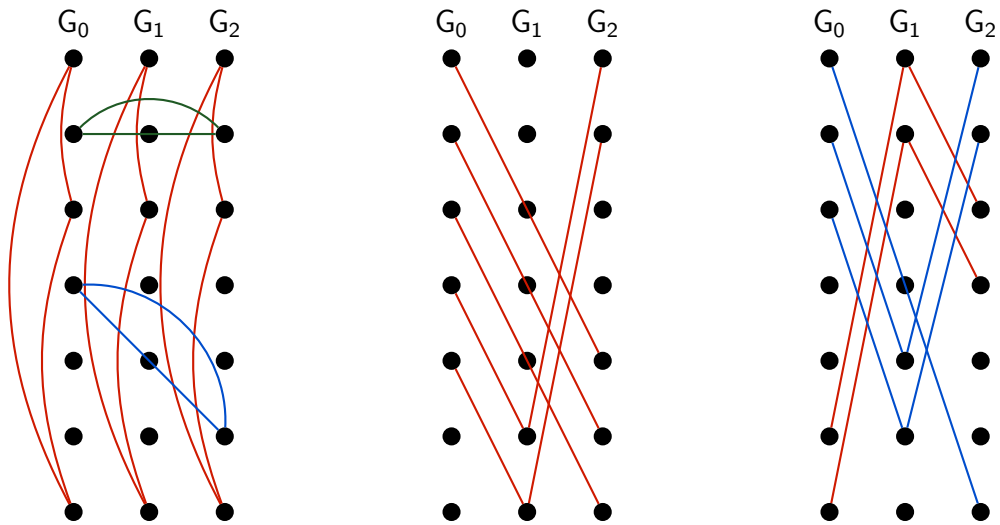


Figure 2.3: Partial parallel classes of size 5

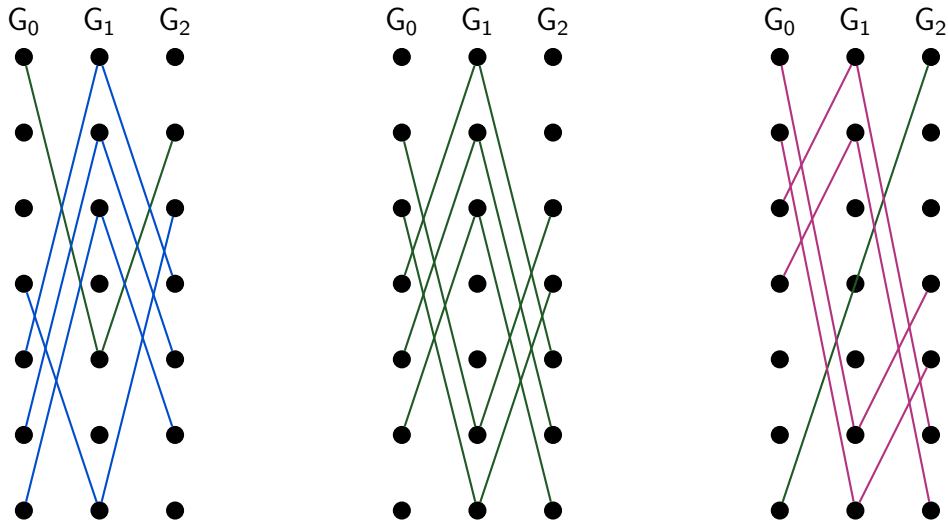


Figure 2.4: Partial parallel classes of size 5

2.3.2 When $v = 33$ and $m = 8$

In this construction, we use a resolvable GDD. Rees solved the existence problem for RGDD's with block size 3.

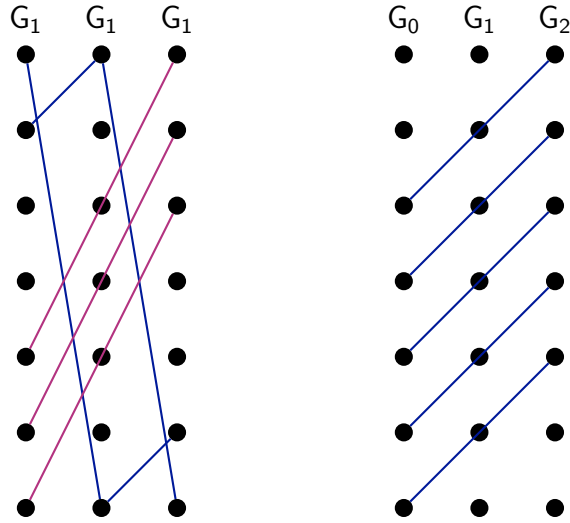


Figure 2.5: Partial parallel classes of size 5

Lemma 19 [10] *A 3-RGDD(g^u) exists if and only if $u \geq 3$, $g(u - 1)$ is even, $gu \equiv 0 \pmod{3}$.*

Lemma 20 *There exists an $STS_8(33)$.*

Proof

The four groups of size 8 are:

$$G_0 = \{1_0, 2_0, 3_0, 4_0, 5_0, 6_0, 7_0, 8_0\}, G_1 = \{1_1, 2_1, 3_1, 4_1, 5_1, 6_1, 7_1, 8_1\},$$

$G_2 = \{1_2, 2_2, 3_2, 4_2, 5_2, 6_2, 7_2, 8_2\}$, and $G_3 = \{1_3, 2_3, 3_3, 4_3, 5_3, 6_3, 7_3, 8_3\}$. The elements of these groups will be the points of the $STS(33)$. The 22 partial parallel classes of size 8 are as follows.

PPC1			PPC2			PPC3			PPC4		
7_0	8_0	∞	3_0	6_0	∞	1_0	6_2	6_3	1_0	5_2	5_3
2_1	2_2	8_3	6_1	6_2	4_3	2_0	5_2	7_3	2_0	6_2	8_3
3_1	3_2	6_3	7_1	7_2	2_3	3_0	8_2	5_3	3_0	7_2	6_3
4_1	4_2	7_3	8_1	8_2	3_3	7_1	8_1	∞	4_0	8_2	7_3
5_1	6_2	2_3	1_1	2_2	6_3	5_0	1_2	1_3	5_0	2_2	2_3
6_1	5_2	3_3	2_1	1_2	7_3	6_0	2_2	4_3	6_0	1_2	3_3
7_1	8_2	1_3	3_1	4_2	5_3	7_0	3_2	2_3	7_0	4_2	1_3
8_1	7_2	4_3	4_1	3_2	8_3	8_0	4_2	3_3	3_1	6_1	∞

<i>PPC5</i>			<i>PPC6</i>			<i>PPC7</i>			<i>PPC8</i>		
1_0	4_1	4_3	1_0	3_1	3_3	1_0	7_1	3_2	1_0	8_1	4_2
2_0	3_1	1_3	2_0	4_1	2_3	7_3	8_3	∞	2_0	7_1	1_2
7_2	8_2	∞	3_0	1_1	4_3	3_0	5_1	4_2	3_0	6_1	3_2
4_0	1_1	2_3	4_0	2_1	1_3	4_0	6_1	1_2	4_0	5_1	2_2
5_0	7_1	7_3	5_0	8_1	8_3	5_0	4_1	8_2	5_0	3_1	7_2
6_0	8_1	6_3	6_0	7_1	5_3	6_0	3_1	5_2	3_3	6_3	∞
7_0	5_1	8_3	3_2	6_2	∞	7_0	2_1	7_2	7_0	1_1	8_2
8_0	6_1	5_3	8_0	5_1	6_3	8_0	1_1	6_2	8_0	2_1	5_2

<i>PPC9</i>			<i>PPC10</i>			<i>PPC11</i>			<i>PPC12</i>		
1_1	1_2	5_3	1_0	2_0	3_0	1_0	4_0	7_0	1_1	4_2	8_3
5_1	5_2	1_3	4_0	5_0	6_0	2_0	5_0	8_0	2_1	3_2	5_3
4_0	7_2	8_3	1_1	2_1	3_1	1_1	4_1	7_1	1_0	5_0	∞
8_0	3_2	4_3	4_1	5_1	6_1	2_1	5_1	8_1	4_1	1_2	6_3
3_0	2_1	3_3	1_2	2_2	3_2	1_2	4_2	7_2	5_1	7_2	3_3
7_0	6_1	7_3	4_2	5_2	6_2	2_2	5_2	8_2	6_1	8_2	2_3
2_0	8_1	2_2	1_3	2_3	3_3	1_3	4_3	7_3	7_1	5_2	4_3
6_0	4_1	6_2	4_3	5_3	6_3	2_3	5_3	8_3	8_1	6_2	1_3

<i>PPC13</i>			<i>PPC14</i>			<i>PPC15</i>			<i>PPC16</i>		
1_1	3_2	7_3	1_0	7_2	7_3	1_0	8_2	8_3	1_2	5_2	∞
2_1	4_2	6_3	1_1	5_1	∞	2_0	7_2	5_3	2_0	2_1	4_3
3_1	1_2	8_3	3_0	5_2	8_3	3_0	6_2	7_3	3_0	3_1	2_3
4_1	2_2	5_3	4_0	6_2	5_3	4_0	5_2	6_3	4_0	4_1	3_3
5_1	8_2	4_3	5_0	4_2	4_3	5_0	3_2	3_3	5_0	6_1	6_3
6_1	7_2	1_3	6_0	3_2	1_3	2_1	4_1	∞	6_0	5_1	7_3
2_0	4_0	∞	7_0	2_2	3_3	7_0	1_2	4_3	7_0	8_1	5_3
8_1	5_2	2_3	8_0	1_2	2_3	8_0	2_2	1_3	8_0	7_1	8_3

<i>PPC17</i>	<i>PPC18</i>	<i>PPC19</i>	<i>PPC20</i>
1 ₀ 2 ₁ 7 ₃	1 ₀ 6 ₁ 6 ₂	1 ₀ 5 ₁ 1 ₂	3 ₁ 2 ₂ 7 ₃
2 ₀ 1 ₁ 3 ₃	2 ₀ 5 ₁ 3 ₂	2 ₀ 6 ₁ 4 ₂	7 ₁ 6 ₂ 3 ₃
3 ₀ 4 ₁ 1 ₃	1 ₃ 5 ₃ ∞	3 ₀ 7 ₁ 2 ₂	2 ₀ 8 ₂ 6 ₃
4 ₀ 3 ₁ 4 ₃	4 ₀ 7 ₁ 4 ₂	4 ₀ 8 ₁ 3 ₂	6 ₀ 4 ₂ 2 ₃
2 ₂ 4 ₂ ∞	5 ₀ 1 ₁ 5 ₂	5 ₀ 2 ₁ 6 ₂	1 ₀ 1 ₁ 1 ₃
6 ₀ 6 ₁ 8 ₃	6 ₀ 2 ₁ 8 ₂	6 ₀ 1 ₁ 7 ₂	5 ₀ 5 ₁ 5 ₃
7 ₀ 7 ₁ 6 ₃	7 ₀ 3 ₁ 6 ₂	2 ₃ 4 ₃ ∞	3 ₀ 8 ₁ 1 ₂
8 ₀ 8 ₁ 7 ₃	8 ₀ 4 ₁ 7 ₂	8 ₀ 3 ₁ 8 ₂	7 ₀ 4 ₁ 5 ₂

<i>PPC21</i>	<i>PPC22</i>
2 ₀ 6 ₀ 7 ₀	1 ₀ 6 ₀ 8 ₀
3 ₀ 4 ₀ 8 ₀	3 ₀ 5 ₀ 7 ₀
2 ₁ 6 ₁ 7 ₁	1 ₁ 6 ₁ 8 ₁
3 ₁ 4 ₁ 8 ₁	3 ₁ 5 ₁ 7 ₁
2 ₂ 6 ₂ 7 ₂	1 ₂ 6 ₂ 8 ₂
3 ₂ 4 ₂ 8 ₂	3 ₂ 5 ₂ 7 ₂
2 ₃ 6 ₃ 7 ₃	1 ₃ 6 ₃ 8 ₃
3 ₃ 4 ₃ 8 ₃	3 ₃ 5 ₃ 7 ₃

The following discussion describes the procedure used to obtain the decomposition given in Lemma 20.

An $STS(33)$ has 176 blocks. If we want to partition the blocks into partial parallel classes of size 8, then we will have 22 such classes.

In order to do that we start with a 3-frame of type 2^4 , which we know exists from Lemma 13. A 3-frame of type 2^4 was given previously in Example 1.1.3. We illustrate it again in Figure 2.6. We give each point of this frame weight 4, so we will have 4 groups of size 8 and we add to it the $\{\infty\}$ point, so we get in total 33 points. Then we use a resolvable group divisible design, $RGDD(4^3)$ on each of the blocks of the 3-frame. The frame parallel classes will tell us which groups to use to get the blocks of the partial parallel classes. The first frame parallel class indicates us that we are not going to use G_0 and so on. When we miss G_0 , we join to it the $\{\infty\}$ point and put on it a $KTS(9)$. And we do the same when we miss the other three groups G_1, G_2, G_3 .

Since we gave to each point weight 4, then each block of the frame would give us 12 points, so we put an $RGDD(4^3)$ on these points. Without loss of generality we assume that parallel class 1, (PC1) of the $RGDD(4^3)$ looks like in Figure 2.6. So we consider this parallel class on the $RGDD(4^3)$ that we put on block 1, $\{3, 5, 8\}$ of frame parallel class 1, (FPC1). On the $RGDD(4^3)$ that we put on block 2, $\{4, 6, 7\}$ of frame parallel class 1, (FPC1) we take in consideration parallel class 2, (PC2). We take out one block from PC1. The block we take out is given in red in Figure 2.7, and we replace it with the block that contains the $\{\infty\}$ point on PC1 of the $KTS(9)$ that we put on $G_0 \cup \{\infty\}$. This other block is given in green in Figure 2.7. This gives us 8 blocks, since we used two parallel classes of two $RGDDs(4^3)$. So we have constructed the first 8 – PPC of an $STS(33)$.

We repeat the same procedure again in these two blocks of FPC1, but now on the $RGDD(4^3)$ that we put on block 1, we work with PC2, and on the $RGDD(4^3)$ that we put on block 2, we work with PC1. Again we take out one block from PC1. This block is given in red in Figure 2.8, and we replace it with the block that contains the $\{\infty\}$ point on PC2 of the $KTS(9)$. This block is the one given in green color on 2.7. In this way we get another 8 – PPC.

On FPC2 we miss G_1 . On $G_1 \cup \{\infty\}$ we put a $KTS(9)$. Again we put an $RGDD(4^3)$ on block 1, $\{1, 6, 8\}$ and one other on block 2, $\{2, 5, 7\}$ of this frame parallel class. We consider on block 1, PC2 of the $RGDD(4^3)$, and on block 2, PC1 of the $RGDD(4^3)$. We take out the red block in Figure 2.8 from PC2, and we replace it with the green block that contains the $\{\infty\}$ point on PC1 of the $KTS(9)$. Thus we have get PPC3 of $STS(33)$.

We continue to work on these two blocks of FPC2. Now we change the parallel classes that we consider on the $RGDDs(4^3)$ that we put on them. On block 1, we consider PC1 and on block 2, PC2. Again we take out one block from PC2, the red block in Figure 2.8, and replace it with the green block that contains the $\{\infty\}$ point on PC2 of the $KTS(9)$. This counts for the fourth 8 – PPC of the $STS(33)$.

The same procedure is followed when working on FPC3 and FPC4. In these two last frames, we work with PC3 and PC4 of the $RGDDs(4^3)$ that we put on their respective blocks.

As we can see from each frame parallel class we get 2 partial parallel classes of size 8. From all the 4 frame parallel classes we get 8 partial parallel classes of size 8. This is illustrated Figures 2.8, 2.9, and 2.10. Another partial parallel class is created with all the red blocks that we took out from the $RGDDs(4^3)$ before. This is illustrated in

Figure 2.11. Now the number of $8 - PPCs$ is 9.

The blocks we took out from the $RGDDs(4^3)$ were replaced with blocks from the $KTS(9)$ (the green blocks). In each parallel class of a $KTS(9)$ there are 3 blocks. Taking out one block we are left with 2. So we have 2 blocks left in PC1 and 2 blocks left in PC2 of the $KTS(9)$ that we used on G_0 , which was the group that was missed by FPC1. Each frame parallel class gives us these 4 blocks of the $KTS(9)$, and furthermore none of these blocks contains $\{\infty\}$. Thus we may take the 2 leftover blocks on G_0, G_1, G_2 , and G_3 from PC1 to make a partial parallel class of size 8. We may do the same with the left over blocks from PC2 to make another partial parallel class. So far we have in total 11 $8 - PPCs$.

As we mentioned above we need to have 22 $8 - PPCs$ all together. To construct the 11 partial parallel classes we used only PC1 and PC2 of the $RGDDs(4^3)$ we put on the blocks of FPC1 and FPC2. On FPC3 and FPC4 we used only PC3 and PC4 of $RGDDs(4^3)$ that we put on their blocks. On all the FPCs we used PC1 and PC2 of the $KTS(9)$. Now on the $RGDDs(4^3)$ that we put on the blocks of FPC1 and FPC2 we consider PC3 and PC4. On the $RGDDs(4^3)$ that we put on the blocks of FPC3 and FPC4 we consider PC1 and PC2. This time on all the frame parallel classes we consider PC3 and PC4 of the $KTS(9)$. This we get 8 more $8 - PPCs$ from the frame parallel classes. We get one more from the blocks we take out from parallel classes of the $RGDDs(4^3)$. Two other $8 - PPCs$ are obtained from the left over block on PC3 and PC4 of the $KTS(9)$. Hence 11 more partial parallel classes of size 8. This completes the 22 $8 - PPCs$ of the $STS(33)$.

We illustrate the whole process of constructing the partial parallel classes in Figures 2.7 - 2.16.

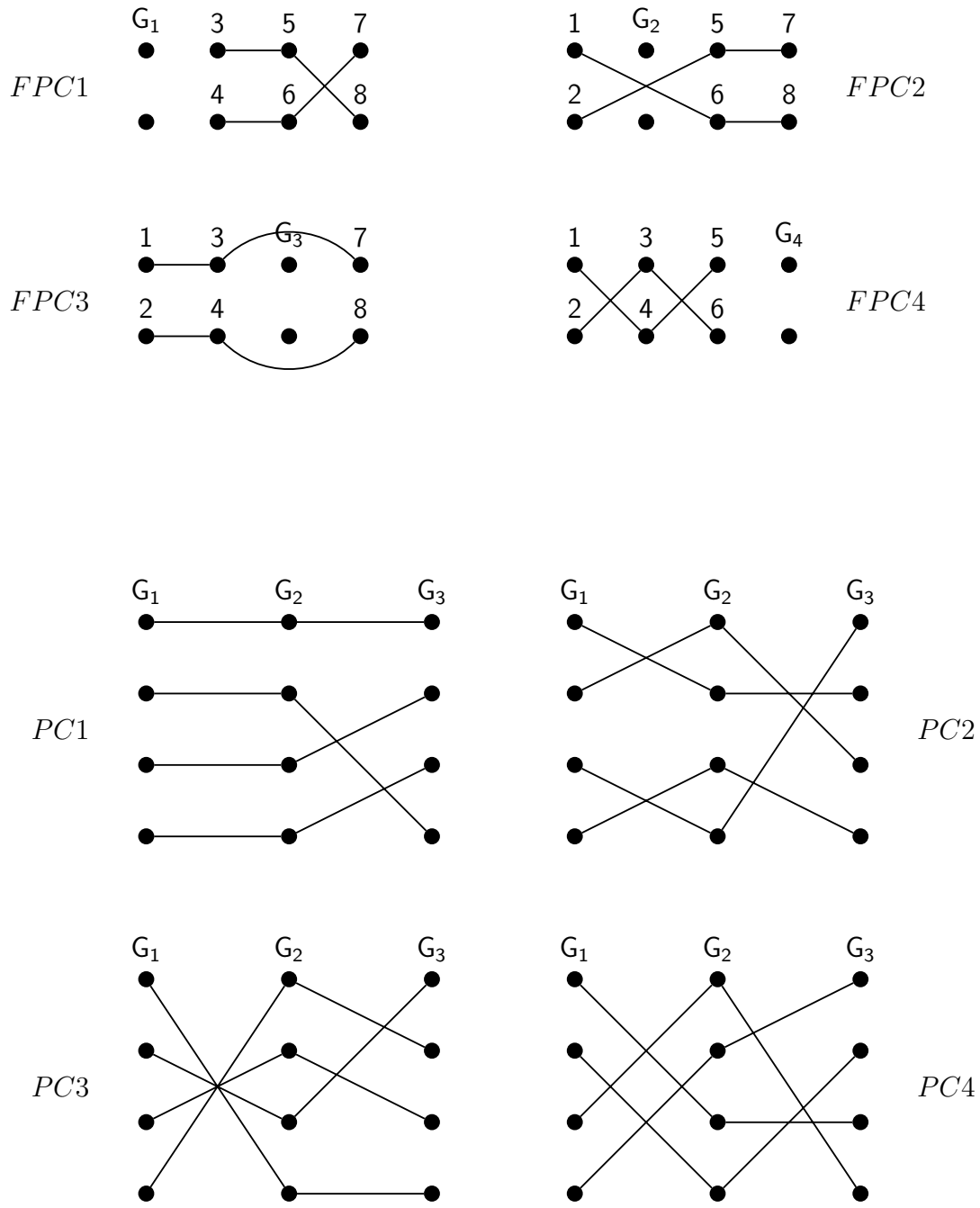


Figure 2.6: 3-frame of type 2^4 and an $RGDD(4^3)$

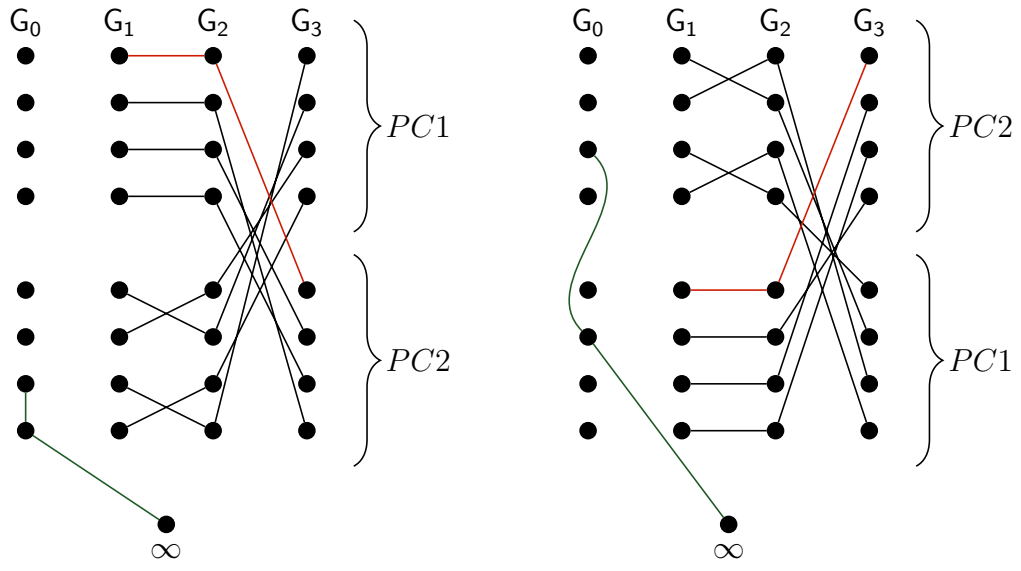


Figure 2.7: Two 8-PPC's using FPC1

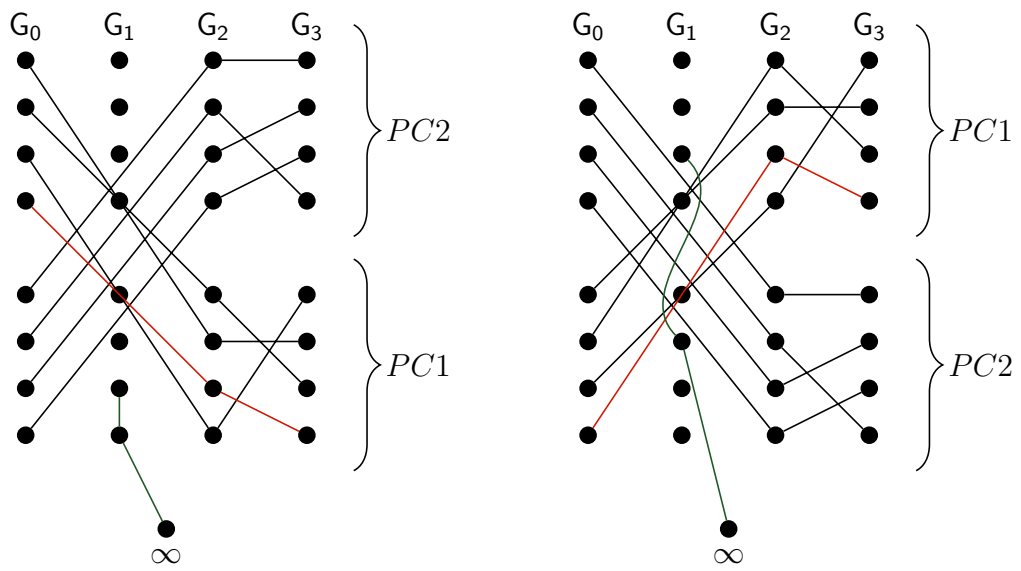


Figure 2.8: Two 8-PPC's using FPC2

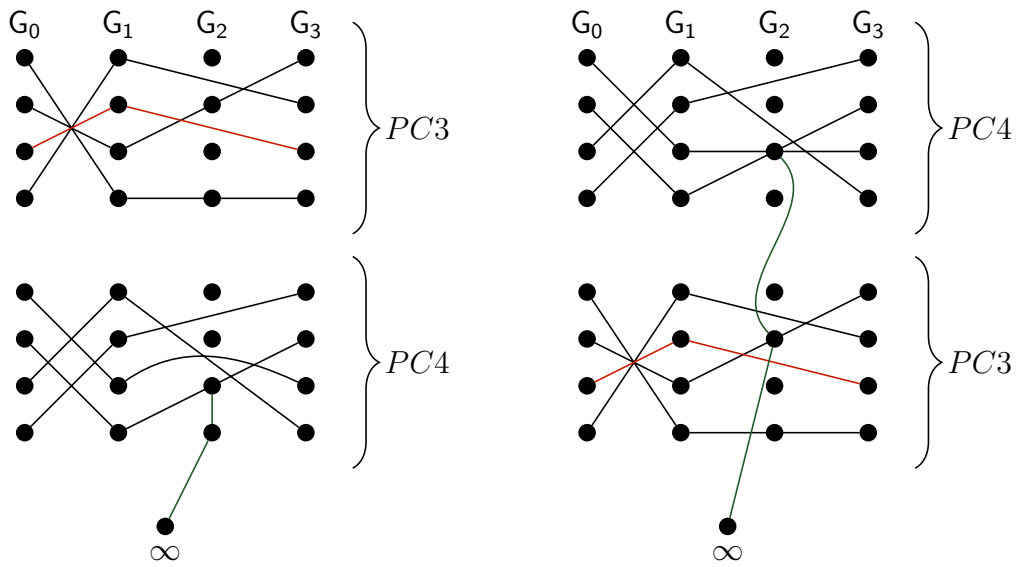


Figure 2.9: Two 8-PPC's using FPC3

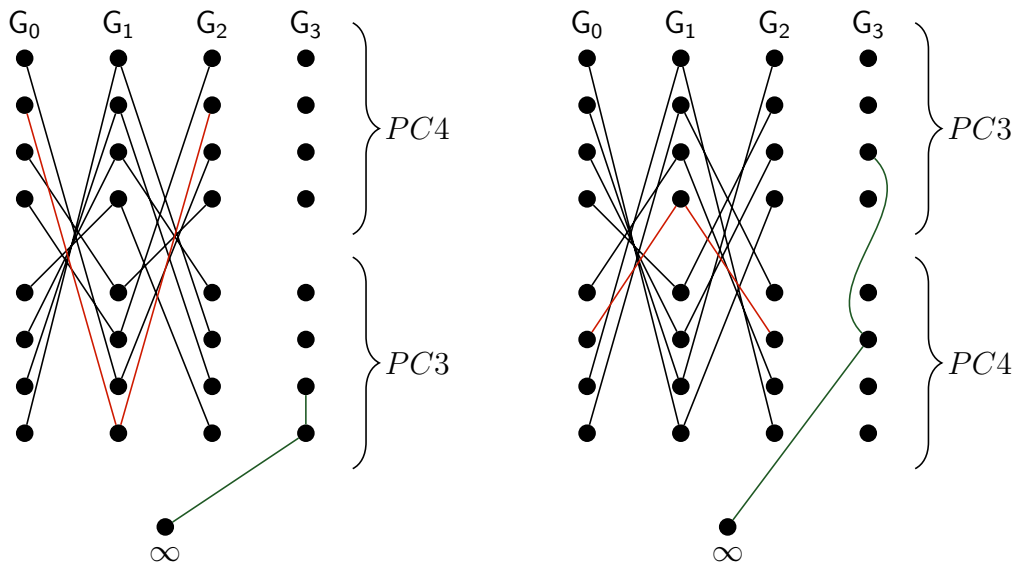


Figure 2.10: Two 8-PPC's using FPC4

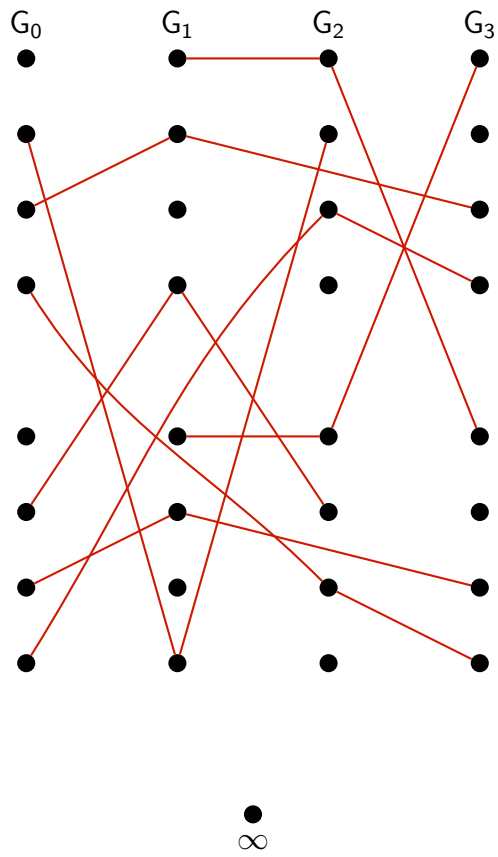


Figure 2.11: The δ -PPC created by all the blocks that we took out from the RGDD's.

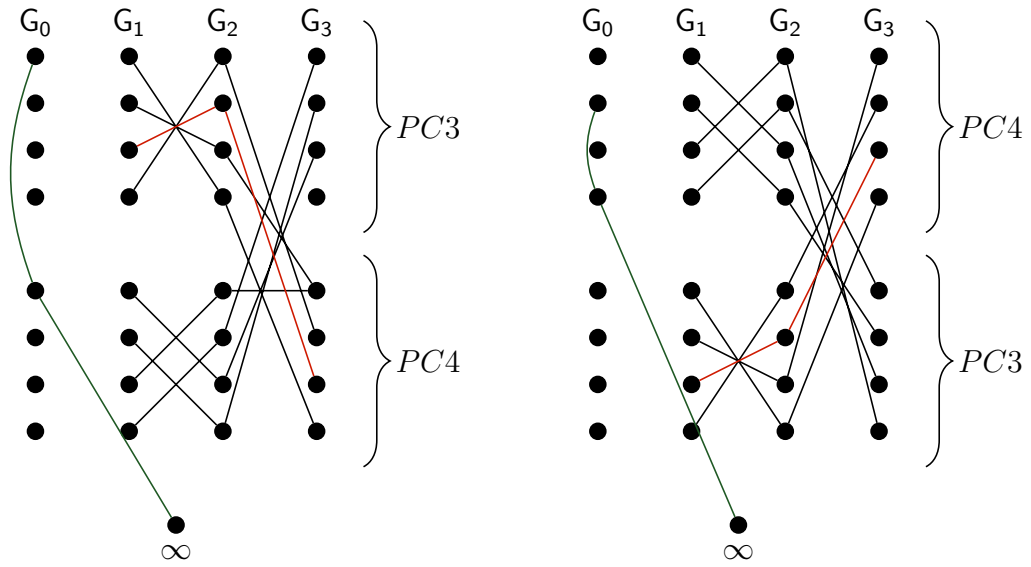


Figure 2.12: Two 8-PPC's using FPC1

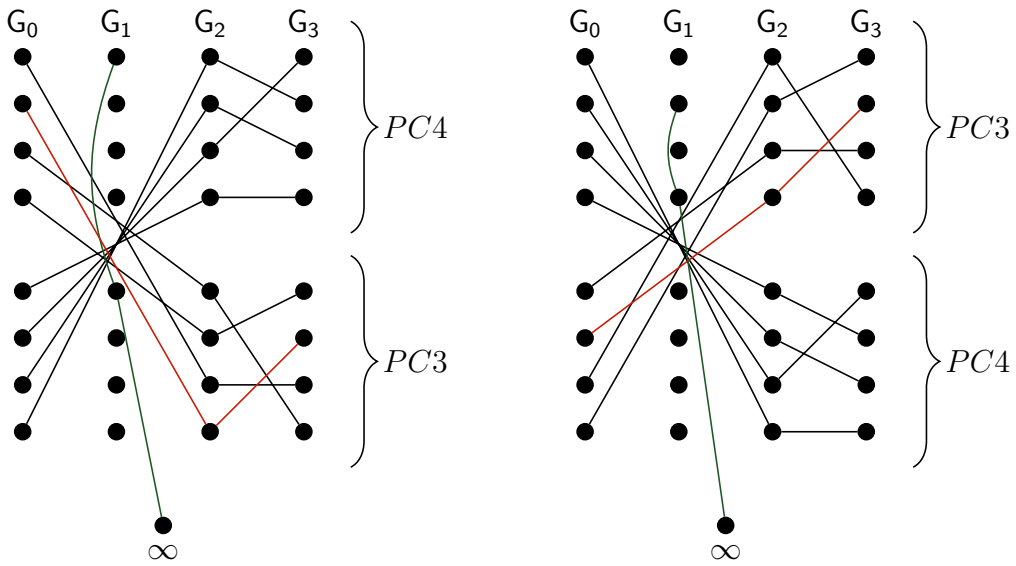


Figure 2.13: Two 8-PPC's using FPC2

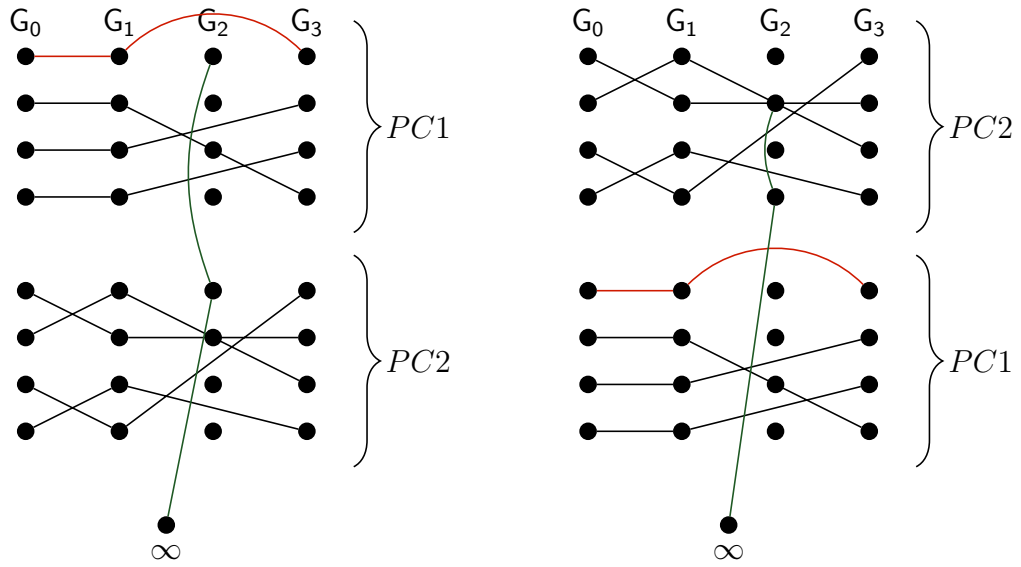


Figure 2.14: Two 8-PPC's using FPC3

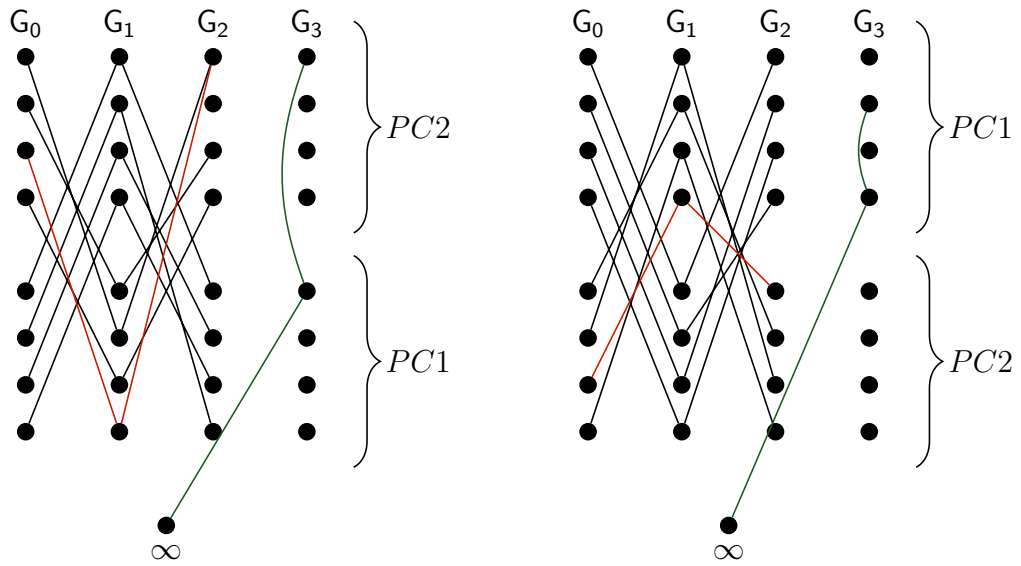


Figure 2.15: Two 8-PPC's using FPC4

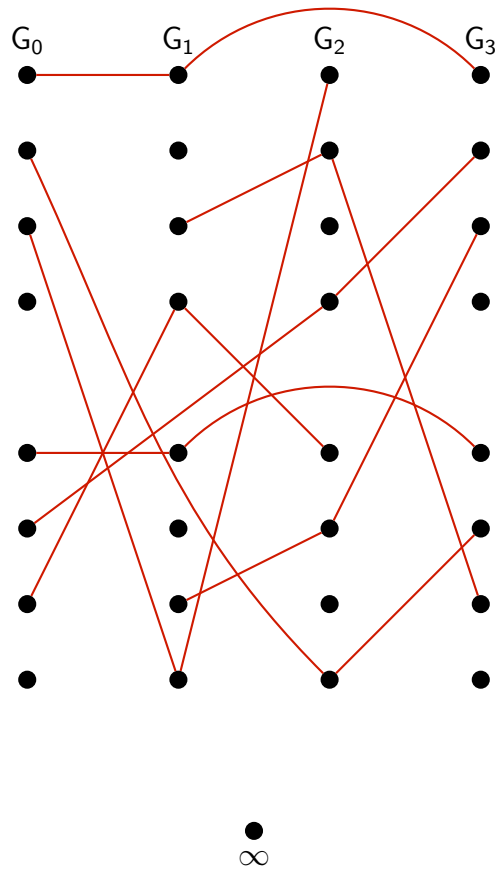


Figure 2.16: The δ -PPC created by all the blocks that we took out from the RGDD's.

Chapter 3

Summary, Conclusions and Further Research

3.1 Summary and Conclusions

In this report we studied the problem of decomposing Steiner triple systems into partial parallel classes of size m . We developed some constructions on how to solve this problem, namely the constructions in Lemma 14 and in Lemma 16. As in the paper [1] that motivated our study, we wanted to cover the cases $9 \leq v \leq 32$. We extended the order of the Steiner triple systems to $v = 43$. For orders $9 \leq v \leq 43$ we found ways to partition the triples into partial parallel classes of size m . So every time $v \equiv 3 \pmod{6}$ and the number of blocks is even we can apply Lemma 14 to obtain partial parallel classes of size 2. And equivalently for partial parallel classes of size 4, when $v \equiv 3 \pmod{6}$ and the number of blocks is divisible by 4 we use Lemma 16.

We obtain some other partitions into partial parallel classes by using already known results. For example we use Kirkman triple systems and Hanani triple systems to obtain the partitions in many cases.

There were three cases that we treated separately and we used different methods. For the Steiner triple system of order 21, to partition it into partial parallel classes of size 5 we used a Steiner triple system of order 7 in each of its three 7-points. We worked $\pmod{7}$ and so we covered all the differences the same number of times. To construct the 20 partial parallel classes of size 5 of a $STS(25)$ we used the construction described by Lemma 12 given in [1]. The last case was $STS(33)$ and its size 8 partial parallel classes. As we described in Chapter 2 in this case we used initially a 3-frame of type 2^4 . Giving weight 4 to each point of this frame and adding another point $\{\infty\}$, we had the necessary points to con-

struct the 22 partial parallel classes of size 8 of the $STS(33)$. Then we needed to use an $RGDD(4^3)$, and we also needed the use of a $KTS(9)$.

Hence because we were able to construct the partial parallel classes mentioned above, we conclude that there exist an $STS_5(21)$, an $STS_5(25)$, and an $STS_8(33)$.

As a brief partial summary of the results in this report we have the following theorems.

Theorem 21 *If $2|b$, then there exists an $STS(v)$ whose blocks can be partitioned into $\frac{b}{2} 2 - PPCs$.*

Theorem 22 *If $4|b$, then there exists an $STS(v)$ whose blocks can be partitioned into $\frac{b}{4} 4 - PPCs$.*

The tables below, Table 3.1, 3.2, and 3.3 summarize our results, and give the current state of the problem up to $v = 99$. Up to $v = 43$ we could partition them for each value of m . After this order we used the constructions mentioned above to cover the partitions for some of the m values. New results in the summary tables are marked with an (*).

3.2 Future work

The methods used in Lemmas 14, and 16 could possibly be generalized. These two lemmas take care of cases when $m = 2$ and $m = 4$. We conjecture that this can always be done for large values of m , when some necessary conditions are met. The method used for the case of $v = 33$ and $m = 8$ also shows promise for generalization.

Table 3.1: Summary table

v	b	$\lfloor \frac{v}{3} \rfloor$	m	Construction
9	12	3	2	Theorem 10
			3	Lemma 6
13	26	4	2	Lemma 12
15	35	5	5	Lemma 6
19	54	6	3	Lemma 8
21	70	7	2	* Lemma 14
			5	* Lemma 18
			7	Lemma 6
25	100	8	2	Lemma 8
			4	Lemma 8
			5	Lemma 12
27	117	9	3	Lemma 7
			9	Lemma 6
31	155	10	5	Lemma 8
33	176	11	2	* Lemma 14
			4	* Lemma 16
			8	* Lemma 20
			11	Lemma 6
37	222	12	2	Lemma 8
			3	Lemma 8
39	247	13	13	Lemma 6
43	301	14	7	Lemma 8
45	330	15	2	* Lemma 14
			3	Lemma 7
			5	Lemma 7
			6	?
			10	?
			11	?
			15	Lemma 6
49	392	16	2	Lemma 8
			4	Lemma 8
			7	?
			8	Lemma 8
			14	?

Table 3.2: Summary table

v	b	$\lfloor \frac{v}{3} \rfloor$	m	Construction
51	425	17	5	?
			17	Lemma 6
55	495	18	3	Lemma 8
			5	Theorem 9
			9	Lemma 8
			11	?
			15	?
57	532	19	2	* Lemma 14
			4	* Lemma 16
			7	?
			14	?
			19	Lemma 6
61	610	20	2	Lemma 8
			5	Lemma 8
			10	Lemma 8
63	651	21	3	Lemma 7
			7	Lemma 7
			21	Lemma 6
67	737	22	11	Lemma 8
69	782	23	2	* Lemma 14
			17	?
			23	Lemma 6
73	876	24	2	Lemma 8
			3	Lemma 8
			4	Lemma 8
			6	Lemma 8
			12	Lemma 8
75	925	25	5	Lemma 7
			25	Lemma 6
79	1027	26	13	Lemma 8
81			12	?
			15	?
			18	?
			20	?
			24	?
			27	Lemma 6

Table 3.3: Summary table

v	b	$\lfloor \frac{v}{3} \rfloor$	m	Construction
81	1080	27	2	* Lemma 14
			3	Lemma 7
			4	* Lemma 16
			5	?
			6	?
			8	?
			9	Lemma 7
			10	?
85	1190	28	2	Lemma 8
			5	Theorem 9
			7	Lemma 8
			10	?
			14	Lemma 8
			17	?
87	1247	29	29	Lemma 6
91	1365	30	3	Lemma 8
			5	Lemma 8
			7	Theorem 9
	1365	30	13	?
			15	Lemma 8
			21	?
93	1426	31	2	* Lemma 14
			23	?
			31	Lemma 6
97	1552	32	2	Lemma 8
			4	Lemma 8
			8	Lemma 8
			16	Lemma 8
99	1617	33	3	Lemma 7
			7	Theorem 9
			11	Lemma 7
			21	?
			33	Lemma 6

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