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# On the Density of the Odd Values of the Partition Function

Samuel Judge
*Michigan Technological University*, sdjudge@mtu.edu

# ON THE DENSITY OF THE ODD VALUES OF THE PARTITION FUNCTION

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Samuel D. Judge

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Department of Mathematical Sciences

Dissertation Advisor: *Dr. Fabrizio Zanello*

Committee Member: *Dr. William J. Keith*

Committee Member: *Dr. Melissa S. Keranen*

Committee Member: *Dr. John A. Jaszczak*

Department Chair: *Dr. Mark S. Gockenbach*

# Dedication

To my mother, who was there at both the start and end of this journey.

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# Abstract

The purpose of this dissertation is to introduce a new approach to the study of one of the most basic and seemingly intractable problems in partition theory, namely the conjecture that the partition function $p(n)$ is equidistributed modulo 2. We provide a doubly-indexed, infinite family of conjectural identities in the ring of series $\mathbb{Z}_2[[q]]$, which relate $p(n)$ with suitable $t$-multipartition functions, and show how to, in principle, prove each such identity. We will exhibit explicit proofs for 32 of our identities. However, the conjecture remains open in full generality.

A striking consequence of these conjectural identities is that, under suitable existence conditions, for any $t$ coprime to 3, if the $t$-multipartition function is odd with positive density, then $p(n)$ is also odd with positive density. Additionally if *any* $t$-multipartition function is odd with positive density, then either $p(n)$ or the 3-multipartition function (or both) are odd with positive density. All of these facts appear virtually impossible to show unconditionally today.

Our arguments employ a combination of algebraic and analytic methods, including certain technical tools recently developed by Radu in his study of the parity of the Fourier coefficients of modular forms.

# Chapter 1

# Introduction

Let $\lambda$ be an unordered list of positive integers. We say that $\lambda$ is a *partition* of a positive integer $n$ if the elements of $\lambda$ (called *parts*) sum to $n$ (e.g., $\lambda = (3, 2, 1)$ is a partition of 6). Below are all the partitions of 5:

$$(5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1).$$

Partitions have been of interest in mathematical research dating back to 1674, when Gottfried Leibniz wrote to Johann Bernoulli regarding the "divulsions" of integers [37]. The first formal work on partitions was by Leonhard Euler in 1775, in response to a letter from Philip Naudé [21]. During his presentation to the St. Petersburg

Academy, Euler proved what is now called the *Pentagonal Number Theorem* (Equation (1.1) below). In order to describe this result, we define the *partition function*, denoted by $p(n)$, which indicates the total number of partitions of $n$ (for example, per above, $p(5) = 7$). By convention, we say the only partition of 0 is the empty partition; in particular, $p(0) = 1$.

Given a sequence $\{a_0, a_1, a_2, \dots\}$, we call the series

$$\sum_{n=0}^{\infty} a_n q^n$$

the *generating function* of the sequence. In particular,

$$\sum_{n=0}^{\infty} p(n) q^n$$

is the generating function of the partition numbers, commonly referred to as the *partition series*. This will allow us to use the powerful machinery of abstract algebra, specifically the theory of *rings*. We refer the reader to [23, 36] for a rigorous definition of a ring and other standard facts of abstract algebra. Throughout this work, all rings are assumed to be commutative with unity.

In general, if the terms of the sequence $\{a_0, a_1, a_2, \dots\}$ are elements of some ring $R$, the generating function is an element of its *ring of series*, $R[[q]]$. This is defined as the ring of all series with variable $q$ and coefficients in $R$, where addition is degree-wise and multiplication is the *convolution* (or *Cauchy product*). Specifically, given two series in $R[[q]]$, $f(q) = \sum_{n=0}^{\infty} a_n q^n$ and $g(q) = \sum_{n=0}^{\infty} b_n q^n$, the product of $f$ and $g$ is $(f \cdot g)(q) = \sum_{n=0}^{\infty} c_n q^n$, where $c_n = \sum_{i=0}^{n} a_i b_{n-i}$ for all $n$.

In ring theory, we say that an element is *invertible* if it has a multiplicative inverse in the ring. A well-known fact in abstract algebra that will prove useful is stated below without proof.

**Proposition 1.0.1.** *Let $R$ be a ring and let $f(x) = \sum_{i=0}^{\infty} a_i q^i$ be a series in $R[[q]]$. Then $f(x)$ is invertible in $R[[q]]$ if and only if $a_0$ is invertible in $R$.*

The assumption that $p(0) = 1$, in conjunction with the above proposition, gives us that the partition series is invertible in $R[[q]]$, for any ring $R$. For convenience, specifically when using modular forms, we will often pick $R = \mathbb{Z}$. However, a majority of this work focuses on the ring of series $\mathbb{Z}_p[[q]]$ (especially for $p = 2$), where $p$ is a prime number and $\mathbb{Z}_p = \{0, 1, 2, \ldots, p-1\}$ is a field with the operations of modular addition and multiplication.

By Proposition 1.0.1, $(1 - q^j)$ is invertible in $R[[q]]$ for any ring $R$, and its inverse is commonly denoted by

$$(1 - q^j)^{-1} = \frac{1}{1 - q^j}.$$

Additionally, by definition, a ring is closed under its operations. This, combined with the fact that in $\mathbb{Z}[[q]]$, $(1 - q^j)^{-1} = 1 + q^j + q^{2j} + q^{3j} + \cdots$, easily implies that the generating function of $p(n)$, as an element of $\mathbb{Z}[[q]]$, can be written as the following infinite product (see, for example, [4]):

$$\begin{aligned}\sum_{n=0}^{\infty} p(n)q^n &= (1 + q^1 + q^{1+1} + q^{1+1+1} + \cdots)(1 + q^1 + q^{2+2} + q^{2+2+2} + \cdots)\cdots \\ &= \prod_{j=1}^{\infty}(1 + q^j + q^{2j} + q^{3j} + \cdots) \\ &= \prod_{j=1}^{\infty} \frac{1}{1 - q^j}.\end{aligned}$$

Note that this coincides with a result that can be proven using analysis (under suitable assumptions of convergence), where the partition series becomes the *Taylor Series* centered at 0 of $\prod_{j=1}^{\infty}(1 - q^j)^{-1}$, for $|q| < 1$. It is important to remark that studying the convergence of our series is not necessary when we view them as elements of $R[[q]]$. However, as we evoke the theory of modular forms and complex analysis, we may need to ensure that our series converge. In those cases, we will consider values of $q = e^{2\pi i\tau}$, where $\tau \in \mathbb{C}$, such that convergence is guaranteed.

Euler's Pentagonal Number Theorem [21] shows that the inverse of the partition series in $\mathbb{Z}[[q]]$ has a special form, specifically:

$$\prod_{j=1}^{\infty}(1 - q^j) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}, \tag{1.1}$$

where the $n(3n - 1)/2$ are called the *generalized pentagonal numbers*. We note that the right side of (1.1) is in $\mathbb{Z}[[q]]$, since $n(3n+1)/2$ is a non-negative integer for all $n \in \mathbb{Z}$.

Since Euler's seminal result, many prominent mathematicians have contributed to the development of partition theory, including (but certainly not limited to) Andrews, Dyson, Hardy, Rademacher, Ramanujan, Selberg, Stanley, and Sylvester. This continued interest in partitions exists, in part, since the theory lies at the intersection of several major fields of mathematics. Euler proved a majority of his initial results by employing algebraic and combinatorial arguments, while a large portion of recent progress has taken place via the analytic machinery of modular forms, algebra, and most recently also analytic number theory.

A very natural generalization of a partition is to allow the parts of the partition to be assigned one of $t$ colors. We call these the *$t$-multipartitions* (or *$t$-colored* partitions) of $n$. For example, the 2-multipartitions of 3 are as follows:

$$(3), (\bar{3}), (2,1), (2,\bar{1}), (\bar{2},1), (\bar{2},\bar{1}), (1,1,1), (1,1,\bar{1}), (1,\bar{1},\bar{1}), (\bar{1},\bar{1},\bar{1}),$$

where $\bar{a}$ and $a$ have different colors. We denote the number of $t$-multipartitions of $n$ by $p_t(n)$ (for the above example, $p_2(3) = 10$); specifically, $p_1(n) = p(n)$. With reasoning similar for that of the partition series, it follows that the generating function for $p_t(n)$

is given by:

$$\sum_{n=0}^{\infty} p_t(n)q^n = \frac{1}{\prod_{i=1}^{\infty}(1-q^i)^t}. \tag{1.2}$$

The goal of this dissertation is to provide additional insight into the long-standing question of estimating the number of odd values of the partition function, $p(n)$. We do this by relating it to the number of odd values of the $t$-multipartition function, $p_t(n)$, for many values of $t$. Specifically, we conjecture a new framework through which to view the partition series and how it connects, in $\mathbb{Z}_2[[q]]$, to the generating function for the $t$-multipartitions for $t > 3$ (Equation (1.2)). This work is based on results in [33] and [34], written by the author, William Keith, and Fabrizio Zanello. The former paper has been accepted for publication in *Annals of Combinatorics* while the latter is published in the *Journal of Number Theory*, which serves to further emphasize the diversity of the subject.

The current available methods to analyze $\sum_{n=0}^{\infty} p(n)q^n$ in $\mathbb{Z}_2[[q]]$ are minimal. This is due, in large part, to the fact that modular forms prove most effective when viewing the partition series in $\mathbb{Z}_p[[q]]$ for primes $p > 3$, and the traditional techniques essentially cease to work for $p \leq 3$. In fact, significant progress has been achieved regarding partition identities in $\mathbb{Z}_p$ for $p > 3$ (see, among others, [3, 41, 47, 53, 56]).

The most famous of these are the so-called *Ramanujan's identities*, namely,

$$p(5k+4) = 0 \text{ in } \mathbb{Z}_5;$$

$$p(7k+5) = 0 \text{ in } \mathbb{Z}_7;$$

$$p(11k+6) = 0 \text{ in } \mathbb{Z}_{11},$$

for all (non-negative) integers $k$. Other researchers [5, 6, 9, 13, 20, 24, 55] have provided additional, *non-nested* arithmetic sequences where the partition function is equal to 0 in $\mathbb{Z}_p$ for $p = 5, 7, 11$, and powers of these primes. Analogous results were shown for small powers of certain other primes [7, 10]. In this context, *non-nested* means that $\{A'n + B', n \in \mathbb{Z}^+\}$ is not contained in $\{An + B, n \in \mathbb{Z}^+\}$; for example, $\{25n + 24, n \in \mathbb{Z}^+\}$ is a nested subset of $\{5n + 4, n \in \mathbb{Z}^+\}$, while $\{7n + 5, n \in \mathbb{Z}^+\}$ is not. These results for $p = 5, 7, 11$ followed, in part, from identities (in $\mathbb{Z}[[q]]$) for the partition function.

Two such identities, the first of which Hardy described as "Ramanujan's most beautiful" [56], are:

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{i=1}^{\infty} \frac{(1-q^{5i})^5}{(1-q^i)^6}; \tag{1.3}$$

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\prod_{i=1}^{\infty} \frac{(1-q^{7i})^3}{(1-q^i)^4} + 49q\prod_{i=1}^{\infty} \frac{(1-q^{7i})^7}{(1-q^i)^8}. \tag{1.4}$$

Some of the most comprehensive results available today on partition identities modulo

$p$ are due to Ahlgren and Ono, who proved, using modular forms, that if $m$ is any integer coprime to 6, there are infinitely many, non-nested sequences $\{An+B, n \in \mathbb{Z}^+\}$ such that $p(An+B) = 0$ in $\mathbb{Z}_m$ [2, 3, 47]. Mahlburg [41] later explained combinatorially all such identities by means of a celebrated partition statistic called *crank*.

Many papers have been published on $p(n)$ over $\mathbb{Z}_2$ (see for example [1, 29, 46, 51]), but research in this direction has been comparably less successful. A natural and important question to ask is:

**Question 1.0.2.** *Given $x > 0$, for how many values of $n \leq x$ is $p(n)$ odd?*

In order to study this seemingly simple question, we consider the probability that $p(n)$ is odd for an $n \leq x$, and then let $x \to \infty$. We call this the *density*, $\delta_1$, of the odd values of $p(n)$. Thus, if the limit exists,

$$\delta_1 = \lim_{x \to \infty} \frac{\#\{n \leq x : p(n) \text{ is odd}\}}{x}.$$

Generalizing this to $t$-multipartitions, we define

$$\delta_t = \lim_{x \to \infty} \frac{\#\{n \leq x : p_t(n) \text{ is odd}\}}{x},$$

again under the assumption that the limit exists. We note that, for parity reasons, it is sufficient to restrict our attention to the case of $t$ odd. To justify this, let $t = 2^k s$

be a positive integer, with $s$ odd, and consider the following in $\mathbb{Z}_2[[q]]$:

$$\sum_{n=0}^{\infty} p_t(n)q^n = \sum_{n=0}^{\infty} p_{2^k s}(n)q^n = \frac{1}{\prod_{i=1}^{\infty}(1-q^i)^{2^k s}} = \frac{1}{\prod_{i=1}^{\infty}(1-q^{2^k i})^s} = \sum_{n=0}^{\infty} p_s(n)q^{2^k n}.$$

Therefore, $\delta_t = \delta_s/2^k$. It is a conjecture of Parkin and Shanks [51] (which is widely believed by experts in the field) that $\delta_1 = 1/2$. In fact, the parity of $p(n)$ appears to be equidistributed, with no known infinite sequences of odd or even values. Even more, there is significant computational evidence to suggest that for any positive, integer-valued polynomial $h(n)$, the odd values of $p(h(n))$ also have density $1/2$ [1, 15, 16, 50, 63].

However, the current state of the art is still far away from showing that $\delta_t > 0$ (much less, $\delta_t = 1/2$), for any given value of $t$. Indeed, we have still not shown that $\delta_t$ exists for any $t$. As mathematician Paul Monsky once pointedly stated, "the best minds of our generation haven't gotten anywhere with understanding the parity of $p(n)$" [42].

Due to a large amount of computational evidence, we generalize the Parkin-Shanks conjecture to *every* $t \geq 1$. Namely, we have:

**Conjecture 1.0.3** ([33], Conjecture 1)**.** *The density $\delta_t$ exists and is equal to $\frac{1}{2}$ for any positive odd integer $t$. Equivalently, if $t = 2^k t_0$ with $t_0 \geq 1$ odd, then $\delta_t$ exists and equals $1/2^{k+1}$.*

Currently, the best result on the odd values of the partition function, which has improved on the work of multiple authors [1, 12, 22, 45, 49], is due to Bellaïche, Green, and Soundararajan [11] and states that the number of odd values of $p(n)$ for $n \leq x$ is $\gg \frac{\sqrt{x}}{\log \log x}$, where we say $f \gg g$ if $f/g \geq c$ for $x \to \infty$ (with $c$ a positive constant). In fact, their bound holds for any $t$-multipartition function, $p_t(n)$, which improves on the work of Zanello [68]. In particular, it is quite significant that it is not even known that the number of odd values is $\gg \sqrt{x}$, for any $t$.

Conversely, we note that the current record lower bound on the number of *even* values of $p(n)$ is $\gg \sqrt{x} \log \log x$ (see [12]). Unlike for the odd values, it is trivial to show that the number of even values of $p(n)$ is $\gg \sqrt{x}$. We will give a proof of this below. However, the lower bound of $\sqrt{x} \log \log x$ is highly nontrivial and was obtained by means of modular forms.

**Proposition 1.0.4.** $\#\{0 \leq n \leq x : \ p(n) \ \textit{is even}\} \gg \sqrt{x}$.

**Proof:** Consider the following identity in $\mathbb{Z}_2[[q]]$:

$$\left[\frac{1}{1-q} + \frac{1}{\prod_{i=1}^{\infty}(1-q^i)}\right] \prod_{i=1}^{\infty}(1-q^i) = 1 + \frac{\sum_{n=-\infty}^{\infty} q^{\frac{n(3n-1)}{2}}}{1-q}. \tag{1.5}$$

It is straightforward to see that the nonzero coefficients on the right side of (1.5) have density 2/3; in particular, there are $\gg x$ of them in degrees $n \leq x$. This, combined with the observation that $\prod_{i=1}^{\infty}(1-q^i)$ has $\ll \sqrt{x}$ nonzero coefficients in degrees

$n \leq x$ (in fact, Equation (1.1) implies that, asymptotically, there are roughly $\sqrt{\frac{8}{3}\,x}$ such coefficients), leads to the conclusion that

$$\frac{1}{1-q} + \frac{1}{\prod_{i=1}^{\infty}(1-q^i)} \tag{1.6}$$

must have $\gg x/\sqrt{x} = \sqrt{x}$ nonzero coefficients in degrees $n \leq x$. From here, it is sufficient to note that the nonzero coefficients of (1.6) are precisely the even coefficients of the partition series. ■

This dissertation provides a novel way to study the parity of $p(n)$, by relating its generating function in $\mathbb{Z}_2[[q]]$ to that of the $t$-multipartitions in a nontrivial way. As one anonymous reviewer of [34] said, this work is "very nice and [introduces] non-standard ideas which might very well shed some light on [the Parkin and Shanks] conjecture about the density of the odd values of the partition function." Our work shows that $\delta_1 > 0$ (assuming it exists) if there is any $\delta_t > 0$ with $a$ coprime to 3; and that $\delta_1 + \delta_3 > 0$ (always assuming existence) if there exists any $\delta_t > 0$.

We will prove the above *delta implications* as a consequence of a doubly-indexed, infinite family of conjectural identities for $p_t(An + B)$ (for certain $A$, $B$, and $t$) in $\mathbb{Z}_2[[q]]$, which appear to be entirely new. We will then show how the modular form machinery, in particular that introduced by Radu [54], can, in principle, be used to prove *any* given case of our conjecture. After stating thirty-two specific cases of our conjecture (see Theorems 2.0.3, 2.0.6, 2.0.7, 2.0.8, 2.0.9, 2.0.10, and 4.2.1), we will provide an explicit proof of twenty-two of them via modular forms. Finally, we will exhibit completely algebraic proofs for the final ten identities. It remains an interesting open problem to find algebraic or combinatorial arguments for most of our other conjectural identities.

# Chapter 2

# Formulation of Results

In this chapter, we conjecturally relate the positivity of $\delta_t$ to that of $\delta_1 + \delta_3$ for all $t$, and more specifically, the positivity of $\delta_t$ to that of $\delta_1$ for any $t$ coprime to 3. We explicitly prove this relationship for several $t$, as stated below. As we demonstrated in Chapter 1, it is sufficient to consider the case of $t$ odd.

**Theorem 2.0.1.** *Let all of the following $\delta_i$ exist. Then $\delta_t > 0$ implies $\delta_1 > 0$ for*

$$t = 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37, 41, 43, 47, 49.$$

**Theorem 2.0.2.** *Let all of the following $\delta_i$ exist. Then $\delta_t > 0$ implies $\delta_3 > 0$ for*

$$t = 9, 15, 21, 27, 33, 39, 45.$$

To prove these delta implications, we first provide several identities in $\mathbb{Z}_2[[q]]$, which will be proved in full in Chapter 3. Recall that $\sum_n a(n)q^n = \sum_n b(n)q^n$ in $\mathbb{Z}_p[[q]]$ if $a(n) = b(n)$ in $\mathbb{Z}_p$ for all $n$. Unless specified otherwise, all identities from here on will be in $\mathbb{Z}_2[[q]]$. For sake of ease throughout this document, we will use the common notation

$$f_k = f_k(q) = \prod_{i=1}^{\infty}(1 - q^{ki}).$$

**Theorem 2.0.3.** *The following sixteen identities hold in* $\mathbb{Z}_2[[p]]$*:*

$$q\sum_{n=0}^{\infty} p(5n+4)q^n = \frac{1}{(f_1)^5} + \frac{1}{f_5}, \tag{2.1}$$

$$q\sum_{n=0}^{\infty} p(7n+5)q^n = \frac{1}{(f_1)^7} + \frac{1}{f_7}, \tag{2.2}$$

$$q\sum_{n=0}^{\infty} p(11n+6)q^n = \frac{1}{(f_1)^{11}} + \frac{1}{f_{11}}, \tag{2.3}$$

$$q\sum_{n=0}^{\infty} p(13n+6)q^n = \frac{1}{(f_1)^{13}} + \frac{1}{f_{13}}, \tag{2.4}$$

$$q\sum_{n=0}^{\infty} p(17n+5)q^n = \frac{1}{(f_1)^{17}} + \frac{1}{f_{17}}, \tag{2.5}$$

$$q\sum_{n=0}^{\infty} p(19n+4)q^n = \frac{1}{(f_1)^{19}} + \frac{1}{f_{19}}, \tag{2.6}$$

$$q\sum_{n=0}^{\infty} p(23n+1)q^n = \frac{1}{(f_1)^{23}} + \frac{1}{f_{23}}, \tag{2.7}$$

$$q^2\sum_{n=0}^{\infty} p(25n+24)q^n = \frac{1}{(f_1)^{25}} + \frac{q}{f_1} + \frac{1}{(f_5)^5}, \tag{2.8}$$

$$q^2\sum_{n=0}^{\infty} p(29n+23)q^n = \frac{1}{(f_1)^{29}} + \frac{q}{(f_1)^5} + \frac{1}{f_{29}}, \tag{2.9}$$

$$q^2\sum_{n=0}^{\infty} p(31n+22)q^n = \frac{1}{(f_1)^{31}} + \frac{q}{(f_1)^7} + \frac{1}{f_{31}}, \tag{2.10}$$

$$q^2\sum_{n=0}^{\infty} p(35n+19)q^n = \frac{1}{(f_1)^{35}} + \frac{q}{(f_1)^{11}} + \frac{1}{f_{35}} + \frac{1}{(f_7)^5} + \frac{1}{(f_5)^7}, \tag{2.11}$$

$$q^2\sum_{n=0}^{\infty} p(37n+17)q^n = \frac{1}{(f_1)^{37}} + \frac{q}{(f_1)^{13}} + \frac{1}{f_{37}}, \tag{2.12}$$

$$q^2\sum_{n=0}^{\infty} p(41n+12)q^n = \frac{1}{(f_1)^{41}} + \frac{q}{(f_1)^{17}} + \frac{1}{f_{41}}, \tag{2.13}$$

$$q^2\sum_{n=0}^{\infty} p(43n+9)q^n = \frac{1}{(f_1)^{43}} + \frac{q}{(f_1)^{19}} + \frac{1}{f_{43}}, \tag{2.14}$$

$$q^2\sum_{n=0}^{\infty} p(47n+2)q^n = \frac{1}{(f_1)^{47}} + \frac{q}{(f_1)^{23}} + \frac{1}{f_{47}}, \tag{2.15}$$

$$q^3\sum_{n=0}^{\infty} p(49n+47)q^n = \frac{1}{(f_1)^{49}} + \frac{q}{(f_1)^{25}} + \frac{q^2}{f_1} + \frac{1}{(f_7)^7}. \tag{2.16}$$

We now frame the above identities in a much broader context. From this generalization, we conjecturally provide an *infinite class* of delta implications (under the assumption all relevant $\delta_i$ exist) analogous to those given in Theorem 2.0.1: specifically, Corollary to Conjecture 2.0.4 below. As usual, we denote by $\lfloor a \rfloor$ (resp. $\lceil a \rceil$) the nearest integer to $a$ that is less (resp. greater) than or equal to $a$.

**Conjecture 2.0.4** ([34], Conjecture 2.3)**.** *Fix any positive integer $a$ coprime to* 6*. Let $b = 24^{-1}$ in $\mathbb{Z}_a$ (or in the trivial case $a = 1$, let $b = 0$) and $k = \lceil \frac{a^2-1}{24a} \rceil$. Then, in $\mathbb{Z}_2[[q]]$,*

$$q^k\sum_{n=0}^{\infty} p(an+b)q^n = \sum_{d|a}\sum_{j=0}^{\lfloor k/d \rfloor} \frac{\epsilon^1_{a,d,j}\ q^{dj}}{\prod_{i\ge 1}(1-q^{di})^{\frac{a}{d}-24j}}, \tag{2.17}$$

*for a suitable choice of the* $\epsilon^1_{a,d,j} \in \{0,1\}$, *where*

$$\epsilon^1_{a,d,j} = \begin{cases} 1, & \textit{if } \; d = 1, j = 0; \\ 0, & \textit{if } \; \frac{a}{d} - 24j < 0. \end{cases}$$

**Corollary to Conjecture 2.0.4.** *Fix any positive integer* $a$ *not divisible by* 3, *and suppose all necessary* $\delta_i$ *exist. If* $\delta_a > 0$, *then* $\delta_1 > 0$.

Something of note is what we mean by "necessary $\delta_i$" in the above corollary. Given any $a$ coprime to 6, it is certainly sufficient to assume that for all $1 \leq i \leq a$ where $i$ is coprime to 6, $\delta_i$ exists, simply because those values of $i$ correspond to the only multipartition functions that may appear on the right side of the equation. However, in general, a small quantity of such $\delta_i$ is actually needed in order to obtain a given delta implication (since $\epsilon^1_{a,d,j} = 0$ for many values of $a, d, j$), though determining explicitly which seems extremely hard. This will be discussed further in Chapter 4.

We now give a proof of Corollary to Conjecture 2.0.4. This result, combined with Theorem 2.0.3 (to be proven in Chapter 3), shows Theorem 2.0.1. Theorem 2.0.2 can be proven in a similar fashion, as an easy application of Corollary to Conjecture 2.0.5 and Theorem 2.0.6.

**Proof:** We proceed by induction on $a$. The base case $a = 1$ is clear. We additionally provide a proof of the case $a = 5$, as this will give insight into the logic we will use for the remainder of the argument. We use the corresponding Identity (2.1) from Theorem 2.0.3 (to be proved later), namely,

$$q\sum_{n=0}^{\infty} p(5n+4)q^n = \frac{1}{\prod_{i\geq 1}(1-q^i)^5} + \frac{1}{\prod_{i\geq 1}(1-q^{5i})}.$$

Suppose that $\delta_5 > 0$ and $\delta_1 = 0$. Therefore, $\#\{n \leq x : p_5(n) \text{ is odd}\} = \delta_5 x + o(x)$, while the number of odd coefficients up to $x$ of $1/\prod_{i\geq 1}(1-q^{5i}) = \sum_{n=0}^{\infty} p(n)q^{5n}$ is $o(x)$. Hence, the number of odd coefficients up to $x$ of $\sum_{n=0}^{\infty} p(5n+4)q^n$ is also $\delta_5 x + o(x)$, which yields $\delta_1 \geq \delta_5/5 > 0$, a contradiction. This shows the case $a = 5$.

Now suppose the result holds for all cases up to and including an arbitrary $a-4 \geq 5$, and assume $\delta_1 = 0$. We note that the next case is either $a-2$ or $a$. We assume it is $a$ and, consequently, that $\delta_a > 0$; the proof for the case $a-2$ is entirely similar. By Conjecture 2.0.4, there exists some identity of the form

$$q^k \sum_{n=0}^{\infty} p(an+b)q^n = \sum_{d|a} \sum_{j=0}^{\lfloor k/d \rfloor} \frac{\epsilon^1_{a,d,j}\ q^{dj}}{\prod_{i\geq 1}(1-q^{di})^{\frac{a}{d}-24j}},$$

where $\epsilon^1_{a,1,0} = 1$. By assumption, the number of odd coefficients of $q^k \sum_{n=0}^{\infty} p(an+b)q^n$ up to $x$ must be $o(x)$, and the number of odd coefficients of $1/\prod_{i\geq 1}(1-q^i)^a$ is $\delta_a x + o(x)$. Therefore, at least one additional term in the finite double sum on the right side must give positive density. It is sufficient from here to note that an additional term giving positive density must be of the form

$$\frac{q^{d_0 j_0}}{\prod_{i\geq 1}(1-q^{d_0 i})^{\frac{a}{d_0}-24j_0}},$$

for suitable $d_0$ and $j_0$ with $(d_0, j_0) \neq (1, 0)$, and that $a/d_0 - 24j_0$ is always coprime to 6. By the inductive assumption, since $\delta_{a/d_0-24j_0} > 0$, then $\delta_1 > 0$. This contradiction gives the result. ■

Conjecture 2.0.4 is a special case ($t = 1$) of Conjecture 2.0.5, which posits that for *any odd integers* $a$ and $t$ where $3|t$ if $3|a$, an identity similar to (though usually more complicated than) Identity (2.1) exists. In this context, "more complicated" means that computational evidence suggests that as $t$ and $a$ increase, so does the number of terms in the finite double sum.

**Conjecture 2.0.5** ([34], Conjecture 2.4)**.** *Fix any positive odd integers* $a$ *and* $t$*, where* $3|t$ *if* $3|a$*. Let* $k = \lceil \frac{t(a^2-1)}{24a} \rceil$*. Then, in* $\mathbb{Z}_2[[q]]$*,*

$$q^k \sum_{n=0}^{\infty} p_t(an+b)q^n = \sum_{d|a} \sum_{j=0}^{\lfloor k/d \rfloor} \frac{\epsilon^t_{a,d,j}\ q^{dj}}{\prod_{i\geq 1}(1-q^{di})^{\frac{at}{d}-24j}}, \tag{2.18}$$

*where*

$$b = \begin{cases} 0, & \text{if } \ a = 1; \\ \frac{t}{3} \cdot 8^{-1} \ \text{in } \mathbb{Z}_a, & \text{if } \ 3|t; \\ t \cdot 24^{-1} \ \text{in } \mathbb{Z}_a, & \text{otherwise,} \end{cases}$$

*for a suitable choice of the* $\epsilon^t_{a,d,j} \in \{0,1\}$, *where*

$$\epsilon^t_{a,d,j} = \begin{cases} 1, & \text{if } d = 1, j = 0; \\ 0, & \text{if } \frac{a}{d} - 24j < 0. \end{cases}$$

**Corollary to Conjecture 2.0.5.** *Fix any positive integer* $A$ *and suppose all necessary* $\delta_i$ *exist. If* $\delta_A > 0$, *then* $\delta_1 + \delta_3 > 0$.

The proof for Corollary to Conjecture 2.0.5 is similar to that of Corollary to Conjecture 2.0.4, and therefore is omitted in the fullest detail. However, in short, if $A$ is even, divide out all the factors of 2, and then proceed from the remaining odd part. Given that all relevant $\delta_i$ exist, we will assume that $\delta_A$ is positive for $A = at$ (with $a$ and $t$ as in Conjecture 2.0.5) and $\delta_t = 0$. This implies that one additional term in the finite double sum must also give positive density, which inductively yields the positivity of either $\delta_1$ or $\delta_3$.

We will now list several specific cases of Conjecture 2.0.5, namely for $t = 3, 5, 7, 9, 15$. Theorem 2.0.6 gives the delta implications in Theorem 2.0.2. The subsequent four theorems do not give any new delta implications, but provide additional insight into the intrinsic beauty and patterns that appear to be at the heart of Conjecture 2.0.5.

**Theorem 2.0.6.** *The following seven identities hold in* $\mathbb{Z}_2[[q]]$*:*

$$q\sum_{n=0}^{\infty} p_3(3n+2)q^n = \frac{1}{(f_1)^9} + \frac{1}{(f_3)^3}, \tag{2.19}$$

$$q\sum_{n=0}^{\infty} p_3(5n+2)q^n = \frac{1}{(f_1)^{15}} + \frac{1}{(f_5)^3}, \tag{2.20}$$

$$q\sum_{n=0}^{\infty} p_3(7n+1)q^n = \frac{1}{(f_1)^{21}} + \frac{1}{(f_7)^3}, \tag{2.21}$$

$$q^2\sum_{n=0}^{\infty} p_3(9n+8)q^n = \frac{1}{(f_1)^{27}} + \frac{q}{(f_1)^3} + \frac{1}{(f_3)^9}, \tag{2.22}$$

$$q^2\sum_{n=0}^{\infty} p_3(11n+7)q^n = \frac{1}{(f_1)^{33}} + \frac{q}{(f_1)^9} + \frac{1}{(f_{11})^3}, \tag{2.23}$$

$$q^2\sum_{n=0}^{\infty} p_3(13n+5)q^n = \frac{1}{(f_1)^{39}} + \frac{q}{(f_1)^{15}} + \frac{1}{(f_{13})^3}, \tag{2.24}$$

$$q^2\sum_{n=0}^{\infty} p_3(15n+2)q^n = \frac{1}{(f_1)^{45}} + \frac{q}{(f_1)^{21}} + \frac{1}{(f_{15})^3} + \frac{1}{(f_5)^9} + \frac{1}{(f_3)^{15}}. \tag{2.25}$$

**Theorem 2.0.7.** *The following two identities hold in* $\mathbb{Z}_2[[q]]$*:*

$$q\sum_{n=0}^{\infty} p_5(5n)q^n = \frac{1}{(f_1)^{25}} + \frac{1}{(f_5)^5}, \tag{2.26}$$

$$q^2\sum_{n=0}^{\infty} p_5(7n+4)q^n = \frac{1}{(f_1)^{35}} + \frac{q}{(f_1)^{11}} + \frac{1}{(f_7)^5}. \tag{2.27}$$

**Theorem 2.0.8.** *The following two identities hold in* $\mathbb{Z}_2[[q]]$*:*

$$q^2 \sum_{n=0}^{\infty} p_7(5n+3)q^n = \frac{1}{(f_1)^{35}} + \frac{q}{(f_1)^{11}} + \frac{1}{(f_5)^7}, \tag{2.28}$$

$$q^2 \sum_{n=0}^{\infty} p_7(7n)q^n = \frac{1}{(f_1)^{49}} + \frac{q}{(f_1)^{25}} + \frac{1}{(f_7)^7}. \tag{2.29}$$

**Theorem 2.0.9.** *The following two identities hold in* $\mathbb{Z}_2[[q]]$*:*

$$q \sum_{n=0}^{\infty} p_9(3n)q^n = \frac{1}{(f_1)^{27}} + \frac{1}{(f_3)^9}, \tag{2.30}$$

$$q^2 \sum_{n=0}^{\infty} p_9(5n+1)q^n = \frac{1}{(f_1)^{45}} + \frac{q}{(f_1)^{21}} + \frac{1}{(f_5)^9}. \tag{2.31}$$

**Theorem 2.0.10.** *The following identity holds in* $\mathbb{Z}_2[[q]]$*:*

$$q \sum_{n=0}^{\infty} p_{15}(3n+1)q^n = \frac{1}{(f_1)^{45}} + \frac{q}{(f_1)^{25}} + \frac{1}{(f_3)^{15}}. \tag{2.32}$$

# Chapter 3

# Proofs of Results

## 3.1 Modular Form Proofs

### 3.1.1 Preliminaries

We now present the necessary facts and notation coming from modular forms, along with the main theorem of Radu [54]; for basic facts, proofs, and further details, we refer the reader to [35, 48]. The notation used below mainly follows that of Radu [54].

Fix $N \geq 1$ and let $\mathbb{Z}^{d(N)}$ denote the set of integer tuples with entries $r_\delta$ indexed by the positive divisors $\delta$ of $N$. For a given $r = (r_\delta) \in \mathbb{Z}^{d(N)}$, define

$$\omega(r) = \sum_{\delta|N} r_\delta\,; \quad \sigma_\infty(r) = \sum_{\delta|N} \delta r_\delta\,; \quad \sigma_0(r) = \sum_{\delta|N} \frac{N}{\delta} r_\delta\,; \quad \Pi(r) = \prod_{\delta|N} \delta^{|r_\delta|}.$$

Additionally, let $\sum_{n\geq 0} \alpha_r(n) q^n = \prod_{\delta|N} \prod_{i\geq 1} (1-q^{\delta i})^{r_\delta}$, and define

$$g_{a,b}(q) = q^{\frac{24b+\sigma_\infty(r)}{24a}} \sum_{n=0}^{\infty} \alpha_r(an+b) q^n.$$

Now let $\Delta^*$ be the set of tuples $(a, N, M, b, (r_\delta)) \in (\mathbb{N}^+)^3 \times \{0, 1, \ldots, a-1\} \times \mathbb{Z}^{d(N)}$ such that the following hold: for all primes $p$, if $p|a$ then $p|M$; if $r_\delta \neq 0$, then $\delta | aM$; and if $\kappa = \gcd(a^2-1, 24)$, then

$$24 \mid \kappa \frac{aM^2}{N} \sigma_0(r); \quad 8 \mid \kappa M \omega(r); \quad \frac{24a}{\gcd(\kappa(24b+\sigma_\infty(r)), 24a)} \mid M.$$

We define $\Gamma_0(N)$ in the following way:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} v & w \\ x & y \end{pmatrix} \in SL_2(\mathbb{Z}) : x = 0 \text{ in } \mathbb{Z}_N \right\}.$$

For $a, N \in \mathbb{N}^+$, $b \in \{0, 1, \ldots, a-1\}$, and $r \in \mathbb{Z}^{d(N)}$, let $P_{a,r}(b)$ be the set of residues modulo $a$ that can be written as $b\omega^2 + (\omega^2 - 1)\sigma_\infty(r)/24$ for some integer $\omega$ with

$\gcd(\omega, 6) = 1$. Further, let

$$\chi_{a,r}(b) = \prod_{\ell \in P_{a,r}(b)} e^{2\pi i \frac{(1-a^2)(24\ell+\sigma_\infty(r))}{24a}}.$$

Set $f$ to be a modular form of weight $k$ and some character $\chi$ for $\Gamma_0(N)$. We call the *level* of $f$ the least value of $N$ for which $f$ is a modular form of weight $k$ for $\Gamma_0(N)$. An *$\eta$-quotient* is a quotient of powers of the Dedekind $\eta$-function, $\eta(\tau)$, and magnifications thereof, $\eta(C\tau)$, where

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{i=1}^{\infty} (1 - q^i), \quad q = e^{2\pi i \tau}.$$

The following theorem of Gordon, Hughes, and Newman gives sufficient conditions for any given $\eta$-quotient to be a modular form.

**Theorem 3.1.1** ([26, 43])**.** *Let $N \geq 1$ be an integer and $f(\tau) = \prod_{\delta|N} \eta^{r_\delta}(\delta\tau)$, with $r = (r_\delta) \in \mathbb{Z}^{d(N)}$. If*

$$\sigma_\infty(r) = \sigma_0(r) = 0 \text{ in } \mathbb{Z}_{24},$$

*then $f$ is a modular form of weight $k = \frac{1}{2}\sum r_\delta$, level dividing $N$, and character $\chi(d) = \left(\frac{(-1)^k \cdot \beta}{d}\right)$, where $\beta = \prod_{\delta|N} \delta^{r_\delta}$ and $\left(\frac{\cdot}{d}\right)$ denotes the Jacobi symbol.*

Radu's main theorem can be phrased as follows:

**Theorem 3.1.2** ([54])**.** *Let $(a, N, M, b, r = (r_\delta)) \in \Delta^*$, $s = (s_\delta) \in \mathbb{Z}^{d(M)}$, and let $\nu$*

*be an integer such that* $\chi_{a,r}(b) = e^{\frac{2\pi i\nu}{24}}$ *(such an integer is guaranteed to exist by [54], Lemma 43). Define*

$$F(s,r,a,b)(\tau) = \prod_{\delta|M} \eta^{s_\delta}(\delta\tau) \prod_{u \in P_{a,r}(b)} g_{a,u}(q).$$

*Then* $F$ *is a weakly holomorphic modular form of weight zero and trivial character for* $\Gamma_0(M)$ *if and only if the following conditions hold:*

$$|P_{a,r}(b)| \cdot \omega(r) + \omega(s) = 0; \tag{3.1}$$

$$\nu + |P_{a,r}(b)| \cdot a\sigma_\infty(r) + \sigma_\infty(s) = 0 \text{ in } \mathbb{Z}_{24}; \tag{3.2}$$

$$|P_{a,r}(b)| \cdot \frac{aM}{N}\sigma_0(r) + \sigma_0(s) = 0 \text{ in } \mathbb{Z}_{24}; \tag{3.3}$$

$$\Pi(s) \cdot \prod_{\delta|N} (a\delta)^{|r_\delta| \cdot |P_{a,r}(b)|} \text{ is a perfect square in } \mathbb{Z}. \tag{3.4}$$

(We note that in the theorem above, $|r_\delta|$ refers to absolute value while $|P_{a,r}(b)|$ refers to the cardinality of the set.) We now present theorems of Ligozat and Radu that bound the order of the cusps for an $\eta$-quotient and the modular function $F(s,r,a,b)$, respectively. Our statements differ slightly from those in the original papers.

**Theorem 3.1.3** ([35, 39, 48])**.** *Let* $N$ *be a positive integer. If* $f(\tau) = \prod_{\delta|N} \eta^{r_\delta}(\delta\tau)$ *is an* $\eta$*-quotient satisfying the conditions of Theorem 3.1.1 for* $\Gamma_0(N)$*, then* $f$ *can only have cusps at rational numbers of the form* $\frac{\gamma}{\mu}$*, where* $\mu|N$ *and* $\gcd(\gamma,\mu) = 1$*. When*

*$f$ has a cusp $\frac{\gamma}{\mu}$, its order is the absolute value of*

$$\frac{N}{24}\sum_{\delta|N}\frac{\left(\gcd(\mu,\delta)\right)^2\cdot r_\delta}{\gcd\left(\mu,\frac{N}{\mu}\right)\cdot\mu\delta}. \tag{3.5}$$

**Theorem 3.1.4** ([54], Theorem 47 and Equations (56-57))**.** *For $F(s,r,a,b)$ and the corresponding $r\in\mathbb{Z}^{d(N)}$ and $s\in\mathbb{Z}^{d(M)}$, as constructed above and satisfying the conditions of Theorem 3.1.2, the order of $F$ at any cusp is uniformly bounded from above by the absolute value of*

$$\min_{n\in\mathbb{N}}\frac{M}{\gcd(n^2,M)}\left(|P_{a,r}(b)|\min_{\substack{m|a,\\ \gcd(m,n)=1}}\frac{1}{24}\sum_{\delta|N}r_\delta\frac{(\gcd(\delta m,an))^2}{\delta a}+\frac{1}{24}\sum_{\delta|M}s_\delta\frac{(\gcd(\delta,n))^2}{\delta}\right).$$

Finally, a classical result of Sturm [62] gives sufficient conditions to equate two holomorphic modular forms in $\mathbb{Z}_p[[q]]$ for some given prime $p$. We phrase it in the following way.

**Theorem 3.1.5** ([62])**.** *Let $p$ be a prime number, and $f(\tau)=\sum_{n=n_0}^{\infty}a(n)q^n$ and $g(\tau)=\sum_{n=n_1}^{\infty}b(n)q^n$ be holomorphic modular forms of weight $k$ for $\Gamma_0(N)$ of characters $\chi_1$ and $\chi_2$, respectively, where $n_0,n_1\in\mathbb{N}$. If either $\chi_1=\chi_2$ and*

$$a(n)=b(n)\ \textit{in}\ \mathbb{Z}_p\quad\textit{for all}\quad n\leq\frac{kN}{12}\cdot\prod_{d\ \text{prime};\ d|N}\left(1+\frac{1}{d^2}\right),$$

*or* $\chi_1 \neq \chi_2$ *and*

$$a(n) = b(n) \text{ in } \mathbb{Z}_p \quad \text{for all} \quad n \leq \frac{kN^2}{12} \cdot \prod_{d \text{ prime; } d|N} \left(1 - \frac{1}{d^2}\right),$$

*then* $f(\tau) = g(\tau)$ *in* $\mathbb{Z}_p[[q]]$ *(i.e.,* $a(n) = b(n)$ *in* $\mathbb{Z}_p$ *for all* $n$*).*

### 3.1.2 Proofs

We now show, as a sample, one special case of Conjecture 2.0.4, namely when $a = 31$ and $t = 1$ (see also Theorem 2.0.3, Equation (2.10)). In principle, *any given identity* from Conjectures 2.0.4 and 2.0.5 can be shown using the exact same strategy, as we will explicitly outline in the proof.

We will employ a technique introduced by Radu [54]. Our approach can be summarized as follows: we start with the identity for $q^k \sum_{n=0}^{\infty} p_t(an+b)q^n$ given by Conjecture 2.0.5. Then we consider the subset of $\Delta^*$ where $N = 1$ and $r = (r_1) = (-t)$, and determine an integer $M \geq 1$ and an $s$-vector $s$ such that the corresponding function $F$ satisfies the conditions of Theorem 3.1.2. If in the process, we encounter modular forms that are weakly holomorphic, we clear the poles by multiplying by another modular form of sufficiently high order (specifically, some $\eta(4\tau)^{24i}$ always works), according to Theorems 3.1.3 and 3.1.4. As a final step, we use Sturm's bound (Theorem

3.1.5) to complete the proof.

Note that as $a$ increases, a much higher power of $\eta(4\tau)^{24}$ is needed to clear out the poles, thus making the computational cost necessary to verify Sturm's bound significantly greater. Further, there appears to be no infinite family of identities coming from either Conjecture 2.0.4 or 2.0.5 for which one can apply a *uniform* Sturm bound, nor any family with a bounded number of terms. In fact, based on our computations, the theorems proved in this dissertation appear to include every identity consisting of three $t$-multipartition terms on the right side. In light of the above reasons, we have not yet been able to prove our conjectures in any form beyond a case-by-case basis.

**Theorem 3.1.6.** *The following identity holds in* $\mathbb{Z}_2[[q]]$*:*

$$q^2 \sum_{n=0}^{\infty} p(31n+22)q^n = \frac{1}{f_{31}} + \frac{1}{(f_1)^{31}} + \frac{q}{(f_1)^7}. \tag{2.10}$$

**Proof:** We will provide as broad a proof as possible for arbitrary $a$ and $t$ as in Conjecture 2.0.5, and only look at the specific case of the statement ($a = 31$ and $t = 1$) in the second part of the argument. We begin by using the function $\alpha_r$, defined at the beginning of the previous subchapter, and set

$$\sum_{n\geq 0} \alpha_r(n)q^n = \sum_{n\geq 0} p_t(n)q^n = \frac{1}{\prod_{i\geq 1}(1-q^i)^t}. \tag{3.6}$$

Therefore, $N = 1$ and $r = (r_1) = (-t)$ . Now we consider $P_{a,r}(b)$. We notice that $\sigma_\infty(r) = -t$, and so

$$b\omega^2 + \frac{\omega^2 - 1}{24}\sigma_\infty(r) = b\omega^2 + \frac{t(1-\omega^2)}{24}.$$

Note that $24b = t$ in $\mathbb{Z}_a$, and as such, the above becomes (in $\mathbb{Z}_a$)

$$b\omega^2 + \frac{24b(1-\omega^2)}{24} = b\omega^2 + b - b\omega^2 = b.$$

Therefore $P_{a,r}(b) = \{b\}$, for any $a$, $b$, and $t$ satisfying the conditions of Conjecture 2.0.5. We now note that

$$g_{a,b}(q) = q^{\frac{24b-t}{24a}} \sum_{n\geq 0} p_t(an+b)q^n;$$

$$\chi_{a,r}(b) = e^{\frac{2\pi i(24b-t)(1-a^2)}{24a}};$$

$$\nu = \frac{(1-a^2)(24b-t)}{a}.$$

Finally, notice that since $|P_{a,r}(b)| = 1$, by Theorem 3.1.2 we have

$$F(s,r,a,b)(\tau) = \prod_{\delta|M} \eta^{s_\delta}(\delta\tau) \quad q^{\frac{24b-t}{24a}} \sum_{n\geq 0} p_t(an+b)q^n,$$

for some $M$ and $s$-vector $s = (s_\delta)$. Consider now the specific case of the statement, where $a = 31$, $t = 1$, and $b = 22$. We have $N = 1$, and $r = (r_1) = (-1)$. Additionally, we choose $M = 2 \cdot 31 = 62$. As proved above, $P_{31,r}(22) = \{22\}$. Moving to the conditions of Theorem 3.1.2, standard computations imply that we are looking for an $s$-vector $s$ that satisfies

$$\omega(s) = 1;$$

$$\sigma_\infty(s) = 7 \text{ in } \mathbb{Z}_{24};$$

$$\sigma_0(s) = 2 \text{ in } \mathbb{Z}_{24};$$

$$31 \cdot \Pi(s) \text{ is a perfect square in } \mathbb{Z}.$$

It is not hard to check that $s = (s_1, s_2, s_{31}, s_{62}) = (3, 1, 4, -7)$ satisfies these conditions. This $s$-vector results in

$$F(s, r, 31, 22)(\tau) = q^{\frac{17}{24}} \frac{\eta(\tau)^3 \eta(2\tau) \eta(31\tau)^4}{\eta(62\tau)^7} \sum_{n=0}^{\infty} p(31n + 22) q^n. \tag{3.7}$$

This is a modular form of weight zero, by Theorem 3.1.2. Thus, in order to obtain $F(s, r, 31, 22)(q)$ on the left side of Equation (2.10), we have to multiply this latter by

$$q^{\frac{-31}{24}} \frac{\eta(\tau)^3 \eta(2\tau) \eta(31\tau)^4}{\eta(62\tau)^7}. \tag{3.8}$$

Now consider the right side of (2.10). Expressing it in terms of $\eta$-quotients, it becomes:

$$\frac{q^{\frac{31}{24}}}{\eta(\tau)^{31}} + \frac{q^{\frac{31}{24}}}{\eta(\tau)^{7}} + \frac{q^{\frac{31}{24}}}{\eta(31\tau)}. \tag{3.9}$$

By multiplying Formulas (3.8) and (3.9), we obtain

$$\frac{\eta(2\tau)\eta(31\tau)^4}{\eta(\tau)^{28}\eta(62\tau)^7} + \frac{\eta(2\tau)\eta(31\tau)^4}{\eta(\tau)^4\eta(62\tau)^7} + \frac{\eta(\tau)^3\eta(2\tau)\eta(31\tau)^3}{\eta(62\tau)^7}. \tag{3.10}$$

At this point, in order to match the weight of the modular form $F$, we want to turn the last displayed formula into a weight-zero modular form as well. Using the fact that $\eta(2\tau) = \eta(\tau)^2$ in $\mathbb{Z}_2[[q]]$, we rewrite (3.10) as

$$\frac{\eta(\tau)^{32}\eta(31\tau)^4}{\eta(2\tau)^{29}\eta(62\tau)^7} + \frac{\eta(\tau)^8\eta(31\tau)^4}{\eta(2\tau)^5\eta(62\tau)^7} + \frac{\eta(\tau)^3\eta(2\tau)\eta(31\tau)^3}{\eta(62\tau)^7}. \tag{3.11}$$

This is a modular form of weight zero, by Theorem 3.1.1 and the fact that the sum of weight zero modular forms is a weight zero modular form. It will be sufficient to show that the right side of (3.7) and (3.11) are equal in $\mathbb{Z}_2[[q]]$. Before using Sturm's Theorem, we must first clear all possible cusps. To do so, we notice that the bound given in Theorem 3.1.4, uniform on all cusps of $F(s, r, 31, 22)$, is

$$\left| \min_{n\in\mathbb{N}} \frac{62}{\gcd(n^2, 62)} \left( \min_{\substack{m|31 \\ \gcd(m,n)=1}} \frac{1}{24} \sum_{\delta|1} r_\delta \frac{(\gcd(\delta m, 31n))^2}{31\delta} + \frac{1}{24} \sum_{\delta|62} s_\delta \frac{(\gcd(\delta, n))^2}{\delta} \right) \right|$$

$$= \frac{567}{8} = 70.875.$$

Theorem 3.1.3 allows us to make a similar calculations to clear the possible cusps of (3.11), resulting in a bound on the order of 34. Thus a power of $\eta(4\tau)$ sufficient to simultaneously clear cusps of (3.7) and (3.11) is $24 \cdot 14 = 336$. Hence the identity to be checked in $\mathbb{Z}_2[[q]]$ is

$$q^{\frac{17}{24}}\eta(4\tau)^{336}\frac{\eta(\tau)^3\eta(2\tau)\eta(31\tau)^4}{\eta(62\tau)^7}\sum_{n=0}^{\infty}p(31n+22)q^n \stackrel{?}{=}$$

$$\eta(4\tau)^{336}\left[\frac{\eta(\tau)^{32}\eta(31\tau)^4}{\eta(2\tau)^{29}\eta(62\tau)^7}+\frac{\eta(\tau)^8\eta(31\tau)^4}{\eta(2\tau)^5\eta(62\tau)^7}+\frac{\eta(\tau)^3\eta(2\tau)\eta(31\tau)^3}{\eta(62\tau)^7}\right],$$

where both sides are holomorphic modular forms of weight $k = 0 + \frac{336}{2} = 168$. A straightforward application of Theorem 3.1.5 gives a Sturm bound of at most $416,640$. A Mathematica calculation verifies that this identity holds in $\mathbb{Z}_2[[q]]$ up to that point, completing the proof. ■

We now provide a table of sufficient $M$ and $s$-vectors needed to prove the Conjecture for nineteen cases, stated in Theorems 2.0.3,2.0.6, 2.0.7, 2.0.8, 2.0.9, and 2.0.10.

**Table 3.1**
This table provides a $M$ and $s$-vector sufficient to prove the listed equation number of its corresponding theorem.

| **Theorem and Equation** | $M$ **and** $s$**-vector** |
|---|---|
| Theorem 2.0.3, Equation 2.3 | $M = 22,\ s = (10, 2, 11, -22)$ |
| Theorem 2.0.3, Equation 2.5 | $M = 34,\ s = (16, 2, 17, -34)$ |
| Theorem 2.0.3, Equation 2.6 | $M = 38,\ s = (18, 2, 19, -38)$ |
| Theorem 2.0.3, Equation 2.7 | $M = 46,\ s = (22, 2, 23, -46)$ |
| Theorem 2.0.3, Equation 2.9 | $M = 48,\ s = (6, 4, 3, -12)$ |
| Theorem 2.0.3, Equation 2.11 | $M = 70,\ s = (2, 2, 2, 8, 8, 1, 5, -27)$ |
| Theorem 2.0.3, Equation 2.12 | $M = 74,\ s = (7, 1, 6, -13)$ |
| Theorem 2.0.3, Equation 2.13 | $M = 82,\ s = (4, 2, 5, -10)$ |
| Theorem 2.0.3, Equation 2.14 | $M = 86,\ s = (8, 4, 17, -28)$ |
| Theorem 2.0.3, Equation 2.15 | $M = 94,\ s = (8, 4, 9, -20)$ |
| Theorem 2.0.6, Equation 2.21 | $M = 14,\ s = (8, 4, 19, -28)$ |
| Theorem 2.0.6, Equation 2.23 | $M = 22,\ s = (8, 4, 11, -20)$ |
| Theorem 2.0.6, Equation 2.24 | $M = 26,\ s = (4, 4, 23, -28)$ |
| Theorem 2.0.6, Equation 2.25 | $M = 30,\ s = (2, 2, 2, 4, 8, 4, 7, -26)$ |
| Theorem 2.0.7, Equation 2.27 | $M = 14,\ s = (4, 4, 1, -4)$ |
| Theorem 2.0.8, Equation 2.28 | $M = 10,\ s = (8, 4, 15, -20)$ |
| Theorem 2.0.8, Equation 2.29 | $M = 74,\ s = (4, 4, 3, -4)$ |
| Theorem 2.0.9, Equation 2.30 | $M = 6,\ s = (8, 8, 1, -8)$ |
| Theorem 2.0.9, Equation 2.31 | $M = 10,\ s = (4, 4, 5, -4)$ |
| Theorem 2.0.10, Equation 2.32 | $M = 6,\ s = (8, 8, 7, -8)$ |

## 3.2 Algebraic Proofs

We will now prove ten of the identities from Theorems 2.0.3, 2.0.6, 2.0.7, and 2.0.9 algebraically. As a reminder, all equalities are in the ring of series $\mathbb{Z}_2[[q]]$.

**Theorem 2.0.3.**

$$q\sum_{n=0}^{\infty} p(5n+4)q^n = \frac{1}{(f_1)^5} + \frac{1}{f_5}, \tag{2.1}$$

$$q\sum_{n=0}^{\infty} p(7n+5)q^n = \frac{1}{(f_1)^7} + \frac{1}{f_7}, \tag{2.2}$$

$$q\sum_{n=0}^{\infty} p(13n+6)q^n = \frac{1}{(f_1)^{13}} + \frac{1}{f_{13}}, \tag{2.4}$$

$$q^2\sum_{n=0}^{\infty} p(25n+24)q^n = \frac{1}{(f_1)^{25}} + \frac{q}{f_1} + \frac{1}{(f_5)^5}, \tag{2.8}$$

$$q^3\sum_{n=0}^{\infty} p(49n+47)q^n = \frac{1}{(f_1)^{49}} + \frac{q}{(f_1)^{25}} + \frac{q^2}{f_1} + \frac{1}{(f_7)^7}, \tag{2.16}$$

$$q\sum_{n=0}^{\infty} p_3(3n+2)q^n = \frac{1}{(f_1)^9} + \frac{1}{(f_3)^3}, \tag{2.19}$$

$$q\sum_{n=0}^{\infty} p_3(5n+2)q^n = \frac{1}{(f_1)^{15}} + \frac{1}{(f_5)^3}, \tag{2.20}$$

$$q^2\sum_{n=0}^{\infty} p_3(9n+8)q^n = \frac{1}{(f_1)^{27}} + \frac{q}{(f_1)^3} + \frac{1}{(f_3)^9}, \tag{2.22}$$

$$q\sum_{n=0}^{\infty} p_5(5n)q^n = \frac{1}{(f_1)^{25}} + \frac{1}{(f_5)^5}, \tag{2.26}$$

$$q\sum_{n=0}^{\infty} p_9(3n)q^n = \frac{1}{(f_1)^{27}} + \frac{1}{(f_3)^9}. \tag{2.30}$$

**Proof: Identity (2.1)**:

$$q\sum_{n=0}^{\infty} p(5n+4)q^n = \frac{1}{(f_1)^5} + \frac{1}{f_5}.$$

We will use Ramanujan's identity (Equation (1.3)), which in $\mathbb{Z}_2[[q]]$, reduces to:

$$\sum_{n=0}^{\infty} p(5n+4)q^n = \frac{(f_5)^5}{(f_1)^6}.$$

Using an identity of Blecksmith-Brillhart-Gerst (see [14], p. 301; cf. also Hirschhorn's Equation (13) in [29]), we see that

$$\frac{f_1}{\prod_{i=1}^{\infty}(1-q^{10i-5})} = f_1 f_5 = \sum_{n=1}^{\infty} q^{n^2-n} + \sum_{n=1}^{\infty} q^{5n^2-5n+1}. \tag{3.12}$$

A famous result (see [32], page 90) on the inverse of the 3-multipartition function states:

$$(f_1)^3 = \sum_{n=0}^{\infty} q^{n(n+1)/2},$$

Therefore, since we are working in $\mathbb{Z}_2[[q]]$,

$$\sum_{n=1}^{\infty} q^{n^2-n} + \sum_{n=1}^{\infty} q^{5n^2-5n+1} = \sum_{n=0}^{\infty} \left(q^{n(n+1)/2}\right)^2 + q\sum_{n=0}^{\infty} \left(q^{5n(n+1)/2}\right)^2$$

$$= \left(\sum_{n=0}^{\infty} q^{n(n+1)/2}\right)^2 + \left(q\sum_{n=0}^{\infty} q^{5n(n+1)/2}\right)^2$$

$$= (f_1)^6 + q(f_5)^6$$

Thus, we have transformed Equation (3.12) into

$$f_1 f_5 = (f_1)^6 + q(f_5)^6. \tag{3.13}$$

Dividing across (3.13) by $(f_1)^6 f_5$, we obtain

$$\frac{1}{(f_1)^5} = \frac{1}{f_5} + q\frac{(f_5)^5}{(f_1)^6}. \tag{3.14}$$

Finally, substituting Equation (3.14) into (1.3) gives

$$q\sum_{n=0}^{\infty} p(5n+4)q^n = q\frac{(f_5)^5}{(f_1)^6} = \frac{1}{(f_1)^5} + \frac{1}{f_5},$$

which is (2.1).

**Identity (2.2)**:

$$q\sum_{n=0}^{\infty} p(7n+5)q^n = \frac{1}{f_1^7} + \frac{1}{f_7}.$$

We will start with Identity (1.4) in $\mathbb{Z}_2[[q]]$, namely:

$$\sum_{n=0}^{\infty} p(7n+5)q^n = \frac{(f_7)^3}{(f_1)^4} + q\frac{(f_7)^7}{(f_1)^8}, \tag{1.4}$$

along with the following, which is equivalent to an identity of Lin ([40], Equation 2.4):

$$f_1 f_7 = (f_1)^8 + q(f_1)^4(f_7)^4 + q^2(f_7)^8. \tag{3.15}$$

Dividing across (3.15) by $(f_1)^8 f_7$ yields

$$\frac{1}{(f_1)^7} = \frac{1}{f_7} + q\frac{(f_7)^3}{(f_1)^4} + q^2\frac{(f_7)^7}{(f_8)^8}.$$

Substituting the above into (1.4) gives

$$q\sum_{n=0}^{\infty} p(7n+5)q^n = q\frac{(f_7)^3}{(f_1)^4} + q^2\frac{(f_7)^7}{(f_1)^8} = \frac{1}{(f_1)^7} + \frac{1}{f_7}.$$

**Identity (2.4)**:

$$q\sum_{n=0}^{\infty} p(13n+6)q^n = \frac{1}{f_1^{13}} + \frac{1}{f_{13}}.$$

We employ a classical result of Zuckerman [69] for $p(13n+6)$, which can be stated in

$\mathbb{Z}_2[[q]]$ as:

$$\sum_{n=0}^{\infty} p(13n+6)q^n = \frac{f_{13}}{(f_1)^2} + q^5\frac{(f_{13})^{11}}{(f_1)^{12}} + q^6\frac{(f_{13})^{13}}{(f_1)^{14}}. \tag{3.16}$$

Additionally, we see from Calkin *et al.* [17] that

$$\frac{f_{13}}{f_1} = (f_1)^{12} + q(f_1)^{10}(f_{13})^2 + q^6(f_{13})^{12} + q^7\frac{(f_{13})^{14}}{(f_1)^2}. \tag{3.17}$$

Thus after dividing the terms of (3.17) by $(f_1)^{12}f_{13}$, we obtain:

$$\frac{1}{(f_1)^{13}} = \frac{1}{f_{13}} + q\frac{f_{13}}{(f_1)^2} + q^6\frac{(f_{13})^{11}}{(f_1)^{12}} + q^7\frac{(f_13)^{13}}{(f_1)^{14}}.$$

Substituting into Equation (3.16) yields:

$$\sum_{n=0}^{\infty} p(13n+6)q^n = \frac{f_{13}}{(f_1)^2} + q^5\frac{(f_{13})^{11}}{(f_1)^{12}} + q^6\frac{(f_{13})^{13}}{(f_1)^{14}} = \frac{1}{(f_1)^{13}} + \frac{1}{f_{13}}.$$

**Identity (2.8)**:

$$q^2\sum_{n=0}^{\infty} p(25n+24)q^n = \frac{1}{(f_1)^{25}} + \frac{q}{f_1} + \frac{1}{(f_5)^5}.$$

To prove this, we will again use an identity of Zuckerman [69] (or alternatively, [13], Equation (21.1)) in $\mathbb{Z}_2[[q]]$, which gives us that

$$\sum_{n=0}^{\infty} p(25n+24)q^n = \frac{(f_5)^6}{(f_1)^7} + q^2\frac{(f_5)^{18}}{(f_1)^{19}} + q^4\frac{(f_5)^{30}}{(f_1)^{31}}, \tag{3.18}$$

along with a repeated application of Equation (3.14). We start by multiplying (3.18) by $q^2$:

$$q^2\sum_{n=0}^{\infty} p(25n+24)q^n = q^2\frac{(f_5)^6}{(f_1)^7} + q^4\frac{(f_5)^{18}}{(f_1)^{19}} + q^6\frac{(f_5)^{30}}{(f_1)^{31}}. \tag{3.19}$$

Factoring the right side of Equation (3.19) appropriately, we obtain:

$$q\frac{f_5}{f_1}\left[q\frac{(f_5)^5}{(f_1)^6}\right] + \frac{(f_1)^5}{(f_5)^2}\left[q\frac{(f_5)^5}{(f_1)^6}\right]^4 + (f_1)^5\left[q\frac{(f_5)^5}{(f_1)^6}\right]^6. \tag{3.20}$$

We now substitute Equation (3.14) in. Thus (3.20) becomes:

$$q\frac{f_5}{f_1}\left[\frac{1}{(f_1)^5} + \frac{1}{f_5}\right] + \frac{(f_1)^5}{(f_5)^2}\left[\frac{1}{(f_1)^5} + \frac{1}{f_5}\right]^4 + (f_1)^5\left[\frac{1}{(f_1)^5} + \frac{1}{f_5}\right]^6.$$

Multiplying out and canceling like terms yields:

$$q\frac{f_5}{(f_1)^6} + \frac{q}{f_1} + \frac{1}{(f_1)^{25}} + \frac{1}{(f_1)^5(f_5)^4}. \tag{3.21}$$

Again factoring, substituting Equation (3.14) in, distributing, and then combining like terms finally yields:

$$q^2\sum_{n=0}^{\infty}p(25n+24)q^n = \frac{1}{(f_5)^4}\left[q\frac{(f_5)^5}{(f_1)^6}\right]+\frac{q}{f_1}+\frac{1}{(f_1)^{25}}+\frac{1}{(f_1)^5(f_5)^4}$$

$$= \frac{1}{(f_5)^4}\left[\frac{1}{(f_1)^5}+\frac{1}{(f_5)}\right]+\frac{q}{f_1}+\frac{1}{(f_1)^{25}}+\frac{1}{(f_1)^5(f_5)^4}$$

$$= \frac{1}{(f_1)^{25}}+\frac{q}{f_1}+\frac{1}{(f_5)^5}.$$

**Identity (2.16)**:

$$q^3\sum_{n=0}^{\infty}p(49n+47)q^n = \frac{1}{(f_1)^{49}}+\frac{q}{(f_1)^{25}}+\frac{q^2}{f_1}+\frac{1}{(f_7)^7}.$$

We deduce, from an identity of Zuckerman [69], the following in $\mathbb{Z}_2[[q]]$:

$$\sum_{n=0}^{\infty}p(49n+47)q^n = x+qf_1x^2, \tag{3.22}$$

where we have set $x = q^2\frac{(f_7)^{12}}{(f_1)^{13}}+q^6\frac{(f_7)^{28}}{(f_1)^{29}}$.

We will also evoke Equation (2.2):

$$q\sum_{n=0}^{\infty}p(7n+5)q^n = \frac{1}{(f_1)^7}+\frac{1}{f_7}, \tag{3.23}$$

and Equation (1.4), which in $\mathbb{Z}_2[[q]]$ becomes:

$$\sum_{n=0}^{\infty}p(7n+5)q^n = \frac{(f_7)^3}{(f_1)^4}+q\frac{(f_7)^7}{(f_1)^8}. \tag{3.24}$$

By setting

$$y = \frac{q^{-1}}{(f_1)^7}+\frac{q^{-1}}{f_7} \tag{3.25}$$

and combining Equations (3.23) and (3.24), we obtain:

$$y = \frac{(f_7)^3}{(f_1)^4}+q\frac{(f_7)^7}{(f_1)^8}.$$

Now substituting into Equation (3.22), we have that

$$\sum_{n=0}^{\infty}p(49n+47)q^n = y^4(f_1)^3q^2+y^8(f_1)^7q^5.$$

By (3.25), we derive the following identity:

$$\sum_{n=0}^{\infty}p(49n+47)q^n = q^2(f_1)^3\left[\frac{q^{-1}}{(f_1)^7}+\frac{q^{-1}}{f_7}\right]^4+q^5(f_1)^7\left[\frac{q^{-1}}{(f_1)^7}+\frac{q^{-1}}{f_7}\right]^8. \tag{3.26}$$

Since we are working in $\mathbb{Z}_2[[q]]$, (3.26) can be rewritten as

$$\sum_{n=0}^{\infty} p(49n+47)q^n = (f_1)^3 q^2 \left[\frac{q^{-4}}{(f_1)^{28}} + \frac{q^{-4}}{(f_7)^4}\right] + (f_1)^7 q^5 \left[\frac{q^{-8}}{(f_1)^{56}} + \frac{q^{-8}}{(f_7)^8}\right].$$

Distributing and multiplying through by $q^3$ yields:

$$q^3 \sum_{n=0}^{\infty} p(49n+47)q^n = q\frac{1}{(f_1)^{25}} + q\frac{(f_1)^3}{(f_7)^4} + \frac{1}{(f_1)^{49}} + \frac{(f_1)^7}{(f_7)^8}. \tag{3.27}$$

The final piece needed is an algebraic result of Lin ([40], Equation 2.4), specifically:

$$f_1 f_7 + q^2 (f_7)^8 = (f_1)^8 + q(f_1)^4 (f_7)^4. \tag{3.28}$$

Dividing Equation (3.28) by $f_1(f_7)^8$ gives

$$\frac{1}{(f_7)^7} + \frac{q^2}{f_1} = \frac{(f_1)^7}{(f_7)^8} + q\frac{(f_1)^3}{(f_7)^4}.$$

Substituting into Equation (3.27), we obtain:

$$q^3 \sum_{n=0}^{\infty} p(49n+47)q^n = \frac{1}{(f_1)^{49}} + \frac{q}{(f_1)^{25}} + \frac{q^2}{f_1} + \frac{1}{(f_7)^7}.$$

**Identity (2.19)**:

$$q\sum_{n=0}^{\infty} p_3(5n+2)q^n = \frac{1}{(f_1)^{15}} + \frac{1}{(f_5)^3}. \qquad (2.20)$$

We start from an identity of Chan ([18], Theorem 1; see also Xiong [65], Theorem 1.1), which has the immediate corollary:

$$\sum_{n=0}^{\infty} p_3(3n+2)q^n = \frac{(f_3)^9}{(f_1)^{12}}. \qquad (3.29)$$

We can combine this with a result of Hirschorn and Sellers ([30], Theorem 2.1; note that a power of 2 is missing from the factor $(q^{12})_\infty$ in the published version of this paper), which yields

$$\frac{1}{(f_1)^9(f_3)^9} = \frac{q}{(f_1)^{12}} + \frac{1}{(f_3)^{12}}. \qquad (3.30)$$

Multiplying across this latter identity by $(f_3)^9$ and combining with (3.29), we easily obtain

$$q\sum_{n=0}^{\infty} p_3(3n+2)q^n = q\frac{(f_3)^9}{(f_1)^{12}} = \frac{1}{(f_1)^9} + \frac{1}{(f_3)^3}.$$

**Identity (2.20)**:

$$q\sum_{n=0}^{\infty} p_3(5n+2)q^n = \frac{1}{(f_1)^{15}} + \frac{1}{(f_5)^3}.$$

We begin with a result of Chan-Lewis ([19], Identity (1.11); see also Xiong [66]), which states:

$$q\sum_{n=0}^{\infty} p_3(5n+2)q^n = q\frac{(f_5)^3}{(f_1)^6} + q^2\frac{(f_5)^9}{(f_1)^{12}} + q^3\frac{(f_5)^{15}}{(f_1)^{18}}. \tag{3.31}$$

Using a similar proof technique to that for (2.8), we factor appropriately, and the right side of (3.31) becomes:

$$\frac{1}{(f_5)^2}\left[q\frac{(f_5)^5}{(f_1)^6}\right] + \frac{1}{f_5}\left[q\frac{(f_5)^5}{(f_1)^6}\right]^2 + \left[q\frac{(f_5)^5}{(f_1)^6}\right]^3.$$

Substituting in Equation (2.1), we obtain:

$$\frac{1}{(f_5)^2}\left[\frac{1}{(f_1)^5} + \frac{1}{f_5}\right] + \frac{1}{f_5}\left[\frac{1}{(f_1)^5} + \frac{1}{f_5}\right]^2 + \left[\frac{1}{(f_1)^5} + \frac{1}{f_5}\right]^3.$$

Multiplying out and combining like terms yields the intended result, specifically:

$$q\sum_{n=0}^{\infty} p_3(5n+2)q^n = \frac{1}{(f_1)^{15}} + \frac{1}{(f_5)^3}.$$

**Identity (2.22)**:

$$q^2\sum_{n=0}^{\infty} p_3(9n+8)q^n = \frac{1}{(f_1)^{27}} + \frac{q}{(f_1)^3} + \frac{q}{(f_3)^9}.$$

After 3-dissecting (3.29) (which is equivalent to a result of Xiong; [67] Theorem 1.2), and multiplying both sides by $q^2$, we obtain:

$$q^2 \sum_{n=0}^{\infty} p_3(9n+8)q^n = q^4 \frac{(f_3)^{36}}{(f_1)^{39}}.$$

We start by rewriting the right side of the identity above as:

$$(f_3)^{36}(f_1)^9 \left( \frac{q^4}{(f_4)^{12}} \right).$$

Then we apply Identity (3.30) (under the transformation of $q \to q^4$) and the last displayed formula becomes:

$$(f_3)^{36}(f_1)^9 \left( \frac{1}{(f_{12})^{12}} + \frac{1}{(f_4)^9 (f_{12})^9} \right). \tag{3.32}$$

Using the identity $f_2 = (f_1)^2$ in $\mathbb{Z}_2[[q]]$, we can transform (3.32) into

$$(f_3)^{36}(f_1)^9 \left( \frac{1}{(f_3)^{48}} + \frac{1}{(f_1)^{36} (f_3)^{36}} \right).$$

Distributing through the above, we obtain:

$$\frac{1}{(f_1)^{27}} + \frac{(f_1)^9}{(f_3)^{12}}.$$

Using Identity (3.30) one final time yields:

$$q^2 \sum_{n=0}^{\infty} p_3(9n+8)q^n = \frac{1}{(f_1)^{27}} + \frac{1}{(f_3)^9} + \frac{q}{(f_1)^3}.$$

**Identity (2.26)**:

$$q \sum_{n=0}^{\infty} p_5(5n)q^n = \frac{1}{(f_1)^{25}} + \frac{q}{(f_5)^5}. \tag{2.26}$$

Recall Identity (2.1), which states:

$$q \sum_{n=0}^{\infty} p(5n+4)q^n = \frac{1}{(f_1)^5} + \frac{1}{f_5}.$$

Extracting every power $q^{5n}$ from this identity and equating the resulting series in $\mathbb{Z}_2[[q]]$, we obtain

$$q \sum_{n=0}^{\infty} p(25n+24)q^{5n+4} = \sum_{n=0}^{\infty} p(n)q^{5n} + \sum_{n=0}^{\infty} p_5(5n)q^{5n}.$$

Making the substitution $q^5 \to q$, the above identity becomes:

$$q \sum_{n=0}^{\infty} p(25n+24)q^n = \sum_{n=0}^{\infty} p(n)q^n + \sum_{n=0}^{\infty} p_5(5n)q^n.$$

Multiplying the above by $q$ and then using Equation (2.8) yields:

$$q\sum_{n=0}^{\infty} p_5(5n)q^n = q^2\sum_{n=0}^{\infty} p(25n+24)q^n + q\sum_{n=0}^{\infty} p(n)q^n. \tag{3.33}$$

The right side of Equation (3.33) can be expressed (as a result of (2.8)) as:

$$\left(\frac{1}{(f_1)^{25}} + \frac{q}{f_1} + \frac{1}{(f_5)^5}\right) + \frac{q}{f_1}.$$

Combining like terms gives the following:

$$q\sum_{n=0}^{\infty} p_5(5n)q^n = \frac{1}{(f_1)^{25}} + \frac{1}{(f_5)^5}.$$

**Identity (2.30)**:

$$q\sum_{n=0}^{\infty} p_9(3n)q^n = \frac{1}{(f_1)^{27}} + \frac{1}{(f_3)^9}. \tag{2.30}$$

We will prove this entirely similarly to the previous identity. Consider Equation (2.19):

$$q\sum_{n=0}^{\infty} p_3(3n+2)q^n = \frac{1}{(f_1)^9} + \frac{1}{(f_3)^3} = \sum_{n=0}^{\infty} p_9(n)q^n + \sum_{n=0}^{\infty} p_3(n)q^{3n}.$$

Extracting out the $q^{3n}$ terms from the above equation, we obtain

$$q\sum_{n=0}^{\infty} p_3(3(3n+2)+2)q^{3n+2} = \sum_{n=0}^{\infty} p_9(3n)q^{3n} + \sum_{n=0}^{\infty} p_3(n)q^{3n}.$$

Transforming $q^3 \to q$ and then multiplying through by $q$ yields:

$$q^2\sum_{n=0}^{\infty} p_3(9n+8)q^n = q\sum_{n=0}^{\infty} p_9(3n)q^n + q\sum_{n=0}^{\infty} p_3(n)q^n.$$

It is sufficient from here to note that, by Equation (2.22), this is equivalent to

$$\frac{1}{(f_1)^{27}} + \frac{q}{(f_1)^3} + \frac{1}{(f_3)^9} = q\sum_{n=0}^{\infty} p_9(3n)q^n + \frac{q}{(f_1)^3}.$$

Finally, combining like terms proves the result and completes the theorem. ■

# Chapter 4

# Conclusion

## 4.1 Discussion of Results

This dissertation provided a conjectural family of identities in $\mathbb{Z}_2[[q]]$, which relate the partition function and certain $t$-multipartition functions, as explicitly laid out in Conjecture 2.0.4. We additionally conjectured that this is a special case of an infinite, two-dimensional set of multipartition identities (stated in Conjecture 2.0.5). The purpose of these identities is to provide a new framework in which to view and approach Question 1.0.2. In short, given appropriate existence conditions, if $\delta_t > 0$ for any $t$, then $\delta_1 + \delta_3 > 0$ (Corollary to Conjecture 2.0.5). More specifically, if 3 does not divide $t$, then $\delta_t > 0 \implies \delta_1 > 0$ (Corollary to Conjecture 2.0.4). We proved this relationship explicitly for $t \leq 49$.

## 4.2 Future Work

A natural direction for future work is to establish a relationship between $\delta_1$ and $\delta_3$. The nicest possible such relationship would be $\delta_1 > 0 \iff \delta_3 > 0$, though the results in this dissertation do not appear to give either implication. Another intriguing direction would be to show a reverse implication between $\delta_t$ and $\delta_1$ for $t$ coprime to 3: specifically, $\delta_1 > 0$ implies that $\delta_t > 0$. In fact, it would be interesting to prove any delta implication of the form $\delta_a > 0 \implies \delta_b > 0$, for odd $b > a > 0$. Again, our identities as described do not appear to help in this direction.

Finally, an exciting future research project would, of course, consist of proving in full Conjecture 2.0.5. This appears to require an entirely new approach, for reasons discussed in this work. An obvious difficulty that might need to be overcome is finding for which tuples $\{a, d, j, t\}$ we have $\epsilon^t_{a,d,j} = 1$. One starting point could be to relate, in $\mathbb{Z}_2[[q]]$,

$$\sum_{n=0}^{\infty} p_t(an+b)q^n \text{ and } \sum_{n=0}^{\infty} p_T(An+B)q^n,$$

where $TA = ta$. For example, consider the following theorem and corollary, of which we omit a complete proof.

**Theorem 4.2.1.** *The following two identities hold in* $\mathbb{Z}_2[[q]]$*:*

$$q^6 \sum_{n=0}^{\infty} p(125n+99)q^n = \frac{1}{(f_1)^{125}} + \frac{q}{(f_1)^{101}} + \frac{q^2}{(f_1)^{77}} + \frac{q^4}{(f_1)^{29}} + \frac{q^5}{(f_1)^5} + \frac{1}{(f_5)^{25}}; \quad (4.1)$$

$$q^6 \sum_{n=0}^{\infty} p_5(25n+20)q^n = \frac{1}{(f_1)^{125}} + \frac{q}{(f_1)^{101}} + \frac{q^2}{(f_1)^{77}} + \frac{q^4}{(f_1)^{29}} + \frac{1}{(f_5)^{25}} + \frac{q^5}{f_5}. \quad (4.2)$$

**Sketch of Proof:** For Equation (4.1), set $M = 250$ and $s = (2, 2, 2, 4, 8, 4, 5, -26)$ in Theorem 3.1.2, and for Equation (4.2), set $M = 50$ and $s = (2, 2, 9, 2, 2, -12)$. From here, the proof is similar to that of Theorem 3.1.6, given in Chapter 3. ■

**Corollary 4.2.2.** *The following identity holds in* $\mathbb{Z}_2[[q]]$*:*

$$\sum_{n=0}^{\infty} p(125n+99)q^n + \sum_{n=0}^{\infty} p_5(25n+20)q^n + \sum_{n=0}^{\infty} p(5n+4)q^n = 0. \quad (4.3)$$

The fact that most of the terms for $\sum_{n=0}^{\infty} p(125n+99)q^n$ and $\sum_{n=0}^{\infty} p_5(25n+20)q^n$ are the same is hardly a coincidence. Understanding how Identity (4.3) generalizes could provide additional insight into further identities of Conjecture 2.0.5 and how they might relate on a deeper level.

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# Appendix A

# Sample Code

Below is the code that was used to find most of the congruences, as well as enough evidence to suggest the conjectures. The comments throughout the code will make the purpose of each function clear.

## A.1 PartitionCongruences.py

```
## Pentagonal Numbers ##

def pent(n):
    pentNums = []
    i = 0
    p1 = 0
```

```
    p2 = 0
    while p1 <= n and p2 <= n:
        i+=1
        p1 = int(i*(3*i-1)/2)
        j = -i
        p2 = int(j*(3*j-1)/2)
        if p1 <= n:
            pentNums.append(p1)
        if p2 <= n:
            pentNums.append(p2)
    return pentNums

## this is the generating function of p(n) ##

def GeneratingFunctionP(limit):
    PentNums = pent(limit)[:]
    GF = [0,1,3,4,5]
    if limit <= 5:
        return GF
    for i in range(GF[-1]+1,limit+1):
        indicator = 0
        for j in PentNums:
            if i - j in GF:
                indicator += 1
        if indicator % 2 == 1:
            GF.append(i)
    return GF

## this is the generating function of fa ##

def GeneratingFunctionPa(a,limit):
    x = GeneratingFunctionP(int(limit/a))[:]
    GF = []
    for aterm in x:
```

```
        GF.append(aterm*a)
    return GF

## Triangular Numbers ##

def tri(n):
    triNums = []
    i = 0
    t1 = 0
    while t1 <= n:
        i+=1
        t1 = int(i*(i+1)/2)
        if t1 <= n:
            triNums.append(t1)
    return triNums

## this is the generating function of p3(n) ##

def GeneratingFunctionP3(limit):
    TriNums = tri(limit)[:]
    GF = [0,1,2,4]
    if limit < 5:
        return GF
    for i in range(GF[-1]+1,limit+1):
        indicator = 0
        for j in TriNums:
            if i - j in GF:
                indicator += 1
        if indicator % 2 == 1:
            GF.append(i)
    return GF

## this is the generating function of fa^3 ##
```

```
def GeneratingFunctionPa3(a,limit):
    x = GeneratingFunctionP3(int(limit/a))[:]
    GF = []
    for aterm in x:
        GF.append(aterm*a)
    return GF

## Left side of Conjecture .... ##
## Choose an a, t, and number of terms of the conjecture↩
    ##
## function will return b as a print and the nonzero ↩
   powers of q ##

def generalTerm(a,t,terms):
    k = int(ceil(t*(a**2-1)/(24*a)))
    b = int(t*(1-a**2)/24) % a
    t1 = t % 3
    t2 = int(t/3)
    threes = []
    ones = []
    for i in range(t1):
        ones.append(GeneratingFunctionPa(a,terms))
    for i in range(t2):
        threes.append(GeneratingFunctionPa3(a,terms))
    x = multOfSeries(threes,terms)[:]
    y = multOfSeries(ones,terms)[:]
    GT = multOfSeries([x,y],terms)
    return GT




## random functions needed ##
```

```
def floor(n):
    return (int(n))

def ceil(n):
    if int(n) == n:
        return n
    else:
        return int(n)+1

## sum a list of power series together ##

def sumOfSeries(series,terms):
    SUM = series[0][:]
    for i in range(1,len(series)):
        newSeries = series[i][:]
        for aterm in newSeries:
            if aterm in SUM:
                SUM.remove(aterm)
            if aterm not in SUM:
                if aterm < terms:
                    SUM.append(aterm)
    SUM.sort()
    return SUM

## multiply a list of power series together ##

def multOfSeries(series,terms):
    if len(series) == 1:
        return series[0]
    if len(series) == 0:
        return [0]
    MULT = series[0][:]
    for i in range(1,len(series)):
        M = MULT[:]
```

```
        MULT = []
        newSeries = series[i][:]
        for aterm in M:
            for anotherTerm in newSeries:
                if aterm+anotherTerm <= terms:
                    if aterm+anotherTerm in MULT:
                        MULT.remove(aterm+anotherTerm)
                    else:
                        MULT.append(aterm+anotherTerm)
    MULT.sort()
    return MULT

## shifts by some power of q in front ##

def shift(k,series,terms):
    newSeries = []
    for aterm in series:
        if aterm + k <= terms:
            newSeries.append(aterm + k)
    return newSeries
```

# Appendix B

# Letters of Permission

*On the Density of the Odd Values of the Partition Function* [33] has been accepted for publication in the *Annals of Combinatorics* (Springer). As it has not yet been published (at the time of this dissertation's publication), we (Samuel Judge, William Keith, and Fabrizio Zanello) still retain the copyright and as such, there is no need to obtain letters of permission.

*On the Density of the Odd Values of the Partition Function, II: An Infinite Conjectural Framework* [34] was published in the *Journal of Number Theory* (Elsevier). Per the Author's Rights, we (Samuel Judge and Fabrizio Zanello) retain the right to publish this work again in the form of a dissertation (see below for details).