2017

DIRECT SAMPLING METHODS FOR INVERSE SCATTERING PROBLEMS

Ala Mahmood Nahar Al Zaalig
Michigan Technological University, amalzaal@mtu.edu

DIRECT SAMPLING METHODS FOR INVERSE SCATTERING PROBLEMS

By

Ala M. Alzaalig

A DISSERTATION

Submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

In Mathematical Sciences

MICHIGAN TECHNOLOGICAL UNIVERSITY

2017

This dissertation has been approved in partial fulfillment of the requirements for the Degree of DOCTOR OF PHILOSOPHY in Mathematical Sciences.

Department of Mathematical Sciences

Dissertation Advisor:	*Dr. Jiguang Sun*
Committee Member:	*Dr. Mark S. Gockenbach*
Committee Member:	*Dr. Xiaodong Liu*
Committee Member:	*Dr. Yang Yang*
Department Chair:	*Dr. Mark S. Gockenbach*

Dedication

To my father

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Acknowledgments

I would like to express my gratitude to all those who have made this thesis possible. My deepest gratitude goes to my advisor **Prof. Jiguang Sun** for his untiring encouragement and guidance. He has walked with me through all the stages of this thesis, and without his consistent and enlightening instruction, this thesis could not have reached its present form.

I am also deeply indebted to my thesis examination committee, **Prof. Mark S. Gockenbach**, **Dr. Xiaodong Liu**, and **Dr. Yang Yang**, for their encouragement and priceless suggestions. Their academic support and input and personal cheering are greatly appreciated. Thank you. Also, my special thanks to the Department of Mathematical Sciences at Michigan Technological University.

Finally, thanks go to my parents, sisters, and my brothers, Dr. Baha Alzalg and Dr. Lewa Alzaleq, for almost unbelievable support. They are the most important people in my world and I dedicate this thesis to them.

Abstract

Recently, direct sampling methods became popular for solving inverse scattering problems to estimate the shape of the scattering object. They provide a simple tool to directly reconstruct the shape of the unknown scatterer. These methods are based on choosing an appropriate indicator function f on $\mathbb{R}^d$, $d = 2$ *or* 3, such that $f(z)$ decides whether z lies inside or outside the scatterer. Consequently, we can determine the location and the shape of the unknown scatterer.

In this thesis, we first present some sampling methods for shape reconstruction in inverse scattering problems. These methods, which are described in Chapter 1, include Multiple Signal Classification (MUSIC) by Devaney [20], the Linear Sampling Method (LSM) by Colton and Kirsch [14], the Factorization Method by Kirsch [36], and the Direct Sampling Method by Ito et al [32]. In Chapter 2, we introduce some direct sampling methods, including Orthogonality Sampling by Potthast [56] and a direct sampling method using far field measurements for shape reconstruction by Liu [45].

In Chapter 3, we generalize Liu's method for shape reconstruction in inverse electromagnetic scattering problems. The method applies in an inhomogeneous isotropic medium in $\mathbb{R}^3$ and uses the far field measurements. We study the behavior of the new indicator for the sampling points both outside and inside the scatterer.

In Chapter 4, we propose a new sampling method for multifrequency inverse source problem for time-harmonic acoustics using a finite set of far field data. We study the theoretical foundation of the proposed sampling method, and present some numerical experiments to demonstrate the feasibility and effectiveness of the method.

Final conclusions of this thesis are summarized in Chapter 5. Recommendations for possible future works are also given in this chapter.

Chapter 1

Introduction

Inverse scattering problems are of central importance in many areas of science and technology, such as geophysical exploration, radar and sonar, non-destructing testing and medical imaging (see, e.g., [1], [2], [6], [7], [8], [9], [12], [14], [16], [18], [21], [25], [26], [27], [28], [29], [53] [35], [36], [44], [47], [46], [49], [54], [58], [60], [61], [62], [64]).

Usually, a wave is directed into a region of space to be investigated [56], as a result a scattered wave is generated due the existence of obstacles or the structure of the unknown area, this scattered wave is detected and measured away from the region. By studying these scattered waves, the properties of the unknown obstacles/inhomogeneities, such as the shape, the size and the internal constitution, can be found. This is known as the *inverse scattering problem*.

One approach for solving inverse scattering problems is a *Sampling Method*. The basic idea of a sampling method is to design an indicator f on $\mathbb{R}^d$, $d = 2$ *or* 3, such that its value $f(z)$ can be used to decide whether a point z lies inside or outside the scatterer. According to the value of $f(z)$ we can determine the location and the shape of the unknown scatterer. These methods have the advantage of requiring less prior information than iterative methods, it is not necessary to know the boundary conditions satisfied by the total field or the topology of the unknown scatterer. In addition, they are very fast in general, since no scattering problem need to be solved.

In this chapter, we briefly describe multiple signal classification (MUSIC) [20] and how to use it to estimate the location of a number of pointlike scatterers. After that we discuss the Linear Sampling Method (LSM) by Colton and Kirsch [14] and the *Factorization Method* by Kirsch [36] for sound soft obstacle. Then we move on to introduce the Direct Sampling Method by Ito et al. [32], and Orthogonality Sampling by Potthast [56] in Chapter 2. Several advantages of these direct sampling methods are inherited from the classical ones, including their independence on any prior information on the geometry and physical properties of the unknown objects. The key feature of these direct sampling methods is that the computation of the indicator involves only inner products of the measurements, with some suitably chosen functions. This makes them robust to noises and computationally faster than the classical sampling methods. Nevertheless, their theoretical foundation is still far less well developed than the classical sampling methods.

In Chapter 2 we discuss a direct sampling method for inverse acoustic scattering problems that uses the far field measurements, which was proposed by Liu [45] in 2016, and study the theoretical foundation of the proposed method.

In Chapter 3, we generalize Liu's method and propose a direct sampling method for inverse electromagnetic scattering problems in an inhomogeneous isotropic medium in $\mathbb{R}^3$ using far field measurements. We study the behavior of the new indicator for the sampling points both outside and inside the scatterer.

In Chapter 4, we propose a new sampling method for multifrequency inverse source problem for time-harmonic acoustic that uses a finite set of far field data. We develop some theory for the proposed sampling method, and present numerical experiments to demonstrate the feasibility and effectiveness of the method.

Final conclusions of this thesis are summarized in Chapter 5. Recommendations for possible future works are also given in this chapter.

1.1 MUSIC Algorithm

1.1.1 Introduction

MUSIC (multiple signal classification) is essentially a method of characterizing the range of a self-adjoint operator [10]. Suppose A is a self-adjoint operator and $\lambda_1 \geq \lambda_2 \geq \ldots$ are the eigenvalues of A corresponding to the eigenvectors $v_1, v_2, \ldots$. Suppose that $\lambda_{M+1} = \lambda_{M+2} = \ldots = 0$ so that the vectors $v_{M+1}, v_{M+2}, \ldots$ span the null space of A. In practice, $\lambda_{M+1}, \lambda_{M+2}, \ldots$ could be very small, i.e., below the noise level of A. So the range of A is spanned by the vectors $v_1, v_2, .., v_M$ and the noise subspace of A is spanned by the vectors $v_{M+1}, v_{M+2}, \ldots$ The projection onto the noise subspace is given explicitly by

$$P_{noise} = \sum_{j>M} v_j \overline{v_j}^T,$$

where the superscript T denotes transpose, the bar denotes complex conjugate, and v_j^T is the linear functional that maps a vector f to inner product $\langle v_j, f \rangle$.

Since A is self-adjoint, then the noise subspace is orthogonal to the range. Therefore, a vector $f \in R(A)$ if and only if $\|P_{noise} f\| = 0$ if and only if

$$\frac{1}{\|P_{noise} f\|} = \infty.$$

This equation is the MUSIC characterization of the range of A.

If A is not self-adjoint, we use the singular value decomposition to construct MUSIC algorithm.

1.1.2 The Use of MUSIC in Inverse Scattering Theory

MUSIC is generally used in signal processing problems as a method for estimating the individual frequencies of a multiple-harmonic [11], [37]. As Devaney pointed out in [20] it could also be used for imaging, i.e., it provides a method to determine the point-like scatterers from the matrix A_{lp}. This is the complex $N \times N$ matrix where A_{lp} is the measured field at the receiver number l for the antenna number j. The following is the outlines of his approach.

Consider the mathematical model for wave propagation which is modeled by the Helmholtz equation [10]

$$\Delta u + k^2 u = 0$$

where k is the wave number. Suppose we have N antennas, located at the points $R_1, R_2, \ldots, R_N$, which transmit spherically spreading waves. If the jth antenna is excited by an input voltage e_j, the incident field produced at the point x by the jth antenna is

$$u_j^{in}(x) = G(x, R_j)e_j,$$

where $G(x, R_j)$ denotes the outgoing Green's function for Helmholtz equation.

Assume we have an array of M point scatterers at locations $X_1, X_2, ..., X_M \in \mathbb{R}^d$, $d = 2$ or 3. In this model we neglect all multiple scattering between the scatterers. If $u^{in}(X_m)$ is the incident field on the mth scatterer, it produces at x the scattered field $G(x, X_M)\tau_m u^{in}(X_m)$, where τ_m (real constant) gives the strength of the mth scatterer. The total scattered field due to the field emanating from the jth antenna is

$$u_j^s(x) = \sum_{m=1}^{m=M} G(x, X_m)\tau_m G(X_m, R_j)e_j.$$

If this scattered field is measured at antenna l, it is given by

$$u_j^s(R_l) = \sum_{m=1}^{m=M} G(R_l, X_m)\tau_m G(X_m, R_j)e_j.$$

This expression gives the matrix A, whose $(l, j)th$ element is

$$A_{l,j} = \sum_{m=1}^{m=M} G(R_l, X_m)\tau_m G(X_m, R_j).$$

This matrix can be written as

$$A = \sum_{m=1}^{m=M} \tau_m g_m g_m^T,$$

where

$$g_m = (G(R_1, X_m), G(R_2, X_m), \ldots, G(R_N, X_m))^T.$$

For simplicity we consider only the case $N > M$, and this means more antennas than scatterers.

The Green's function is symmetric, for this reason, the matrix A is symmetric. However, A is not self-adjoint. We will form a self-adjoint matrix $F = A^* A = \overline{A} A$ where the star denotes the adjoint and the bar denotes the complex conjugate. The matrix F can be written as

$$F = \sum_{m=1}^{m=M} \overline{\tau_m}\, \overline{g_m} \overline{g_m^T} \sum_{m=1}^{m=M} \tau_m g_m g_m^T.$$

From this, we observe that the eigenvectors of F are $\overline{g_m}$. This implies that the range of F is spanned by the M vectors $\overline{g_m}$.

The MUSIC algorithm can be used to determine the location of the scatterers as follows [10]. Let p be any point and form the vector $g^p = (G(R_1, p), G(R_2, p), \ldots, G(R_N, p))^T$. In this case, the point p coincides with the location of a scatterer if and only if $g^p \in R(F)$. As a result, $P_{noise} g^p = 0$ if and only if

$$\frac{1}{||P_{noise}g^p||} = \infty.$$

Therefore, a plot of the function

$$W(p) = \frac{1}{||P_{noise}g^p||}, \qquad p \in \mathbb{R}^d$$

should have sharp peaks at the location of the scatterers $X_1, X_2, ..., X_M$.

Fig. 1.1 shows the result for the example where $d = 2$, number of scatters $M = 4$, number of transducers $N = 20$, wave number $k = 3$ and the values of τ is 0.8 for all scatterers. To the data, a uniform white noise has been added. More details about MUSIC can be found in [42], [43], [35] and [30].

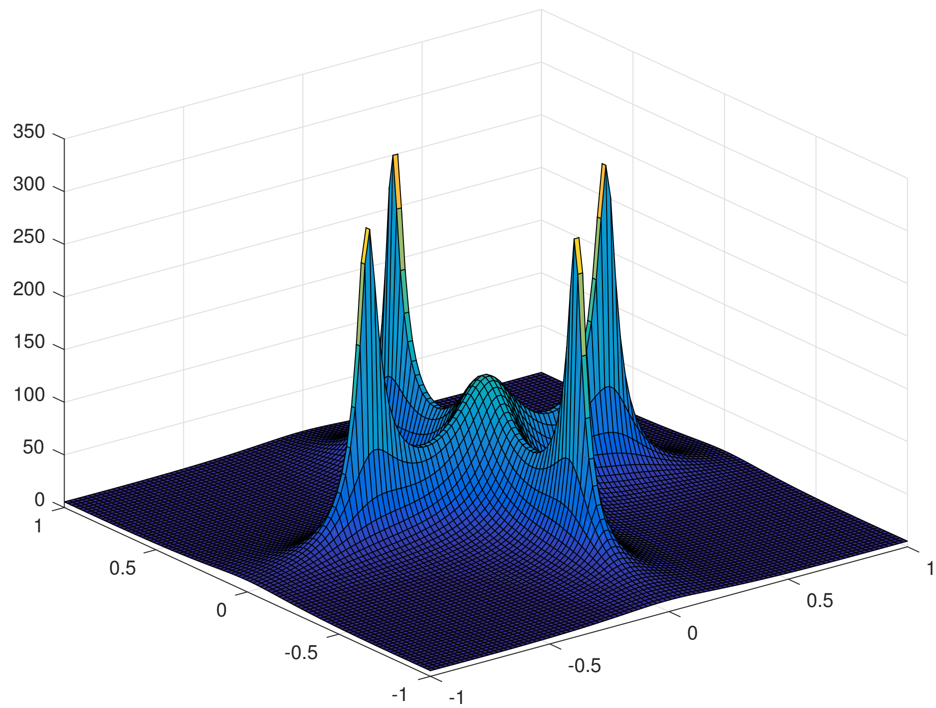

Figure 1.1: Plot of the function W

1.2 Linear Sampling Method

The aim of this section is to introduce the Linear Sampling Method for determining the scattering obstacle D from the knowledge of the far field pattern $u^\infty(\hat{x}, \hat{\theta})$ for all unit vectors $\hat{x}$ and $\hat{\theta}$ defined on the unit sphere. The linear sampling method was first proposed by Colton and Kirsch [14]. For more details, see [6], [17], [13] and [50].

Definition 1 [17] A Herglotz wave function is a function of the form

$$v(x) = \int_{\mathbb{S}^2} e^{ikx\cdot d} g_z(d) ds(d), \quad x \in \mathbb{R}^3,$$

where $\mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$ is the unit sphere in $\mathbb{R}^3$ and $g \in \mathbb{S}^2$. The function v is called the Herglotz function with kernel g.

The basic idea of this method is to find a Herglotz wave function v^i with kernel g, such that the corresponding scattered wave v^s approximates $\Phi(\cdot, z)$ in the interior of the scatterer D, where $\Phi(x, z)$ is the fundamental solution to the Helmholtz equation which is defined by

$$\Phi(x, z) = \begin{cases} \frac{i}{4} H_0^1(k|x-z|) & \text{for} \quad d = 2 \\ \\ \frac{1}{4\pi} \frac{e^{(ik|x-z|)}}{|x-z|} & \text{for} \quad d = 3 \end{cases} \tag{1.1}$$

Here, H_0^1 is the zeroth order Hankel function of the first kind. Define the space $L^2(\mathbb{S}^2)$ as the space of square integrable functions on the unit sphere $\mathbb{S}^2$. First we recall the following theorem

Theorem 1.1 (Theorem 3.26 of [17]) Let v^s be a radiating solution to the Helmholtz equation with far field pattern v_∞. Assume the bounded set D is the open complement of an unbounded domain of class C^2. Then the integral equation of the first kind

$$\int_{\mathbb{S}^2} u_\infty(\hat{x}, d) g(d) ds(d) = v_\infty(\hat{x}), \quad \hat{x} \in \mathbb{S}^2$$

possesses a solution $g \in L^2(\mathbb{S}^2)$ if and only if v^s is defined and continuous in $\mathbb{R}^3 \backslash \bar{D}$. Furthermore, the interior Dirichlet problem for the Helmholtz equation

$$\Delta v^i + k^2 v^i = 0 \quad in \quad D \tag{1.2}$$

with the boundary condition

$$v^i + v^s = 0 \quad on \quad \partial D \tag{1.3}$$

is solvable with any solution v_i being a Herglotz wave function.

In the linear sampling method [17] we have to find the kernel g_z as an approximate solution to the integral equation of the first kind

$$Fg_z = \Phi_\infty(\cdot, z), \tag{1.4}$$

where

$$\Phi_\infty(\hat{x}, z) = \frac{1}{4\pi} e^{-ik\hat{x}\cdot z}, \tag{1.5}$$

is the far field of the fundamental solution $\Phi(x, z)$, which is defined in (1.1).

F is the far field operator be defined as

$$F : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$$

$$Fg(\hat{x}) = \int_{\mathbb{S}^2} u^\infty(\hat{x}, \hat{\theta}) g(\hat{\theta}) ds(\hat{\theta}), \quad \hat{x} \in \mathbb{S}^2 \tag{1.6}$$

From Theorem 1.1, we conclude that g_z is a solution of (1.4) if and only if the Herglotz wave function

$$v(x) = \int_{\mathbb{S}^2} e^{ikx\cdot d} g_z(d) ds(d), \quad x \in \mathbb{R}^3,$$

solves the interior Dirichlet problem

$$\Delta v + k^2 v = 0 \quad in \quad D \tag{1.7}$$

with the boundary condition

$$v + \Phi(\cdot\,, z) = 0 \quad on \quad \partial D. \tag{1.8}$$

Hence, if a solution to the integral equation (1.4) exists for all $z \in D$, then from the boundary condition (1.8) the Herglotz wave function v and the fundamental solution $\Phi(\cdot, z)$ coincide [17]. So we conclude that $\|g_z\|_{L(\mathbb{S}^2)} \to \infty$ as the source point z approaches to the ∂D. Therefore , ∂D can be determined by solving (1.4) for z taken from a sufficiently fine grid in $\mathbb{R}^3$ and determining ∂D as the location where $\|g_z\|_{L^2(\mathbb{S}^2)}$ become large.

The solution to the interior Dirichlet problem (1.7) - (1.8) will have an extension as a Herglotz wave function across the boundary ∂D only in very special cases. Hence, the integral in (1.4) has no solution in general. The mathematical foundation of the linear sampling method is provided in the following theorems.

Theorem 1.2 (Corollary 5.31 of [17]) Assume that k^2 is not a Dirichlet eigenvalue for the negative Laplacian for D. Then the Herglotz operator $H : L^2(\mathbb{S}^2) \to H^{\frac{1}{2}}(\partial D)$

$$Hg(x) := \int_{\mathbb{S}^2} e^{ikx\cdot d} g(d) ds(d), \quad x \in \partial D,$$

is injective and has dense range.

Definition 1.3 (Definition 1.11 of [37]) Let the data-to-pattern operator $G : H^{\frac{1}{2}}(\partial D) \to L^2(\mathbb{S}^2)$ be defined by $Gf = u^\infty$ where $u^\infty \in L^2(\mathbb{S}^2)$ is the far field pattern of the solution u of the exterior Dirichlet problem with boundary value $f \in H^{\frac{1}{2}}(\partial D)$

$$\Delta u + k^2 u = 0 \ \ in \ \ \mathbb{R}^3 \backslash D \tag{1.9}$$

$$u = f \ \ on \ \partial D \tag{1.10}$$

$$\frac{\partial u}{\partial r} - iku = \mathcal{O}\Big(\frac{1}{r^2}\Big), \qquad r = |x| \to \infty \tag{1.11}$$

Theorem 1.4 (Corollary 5.32 of [17]) The operator $G : H^{\frac{1}{2}}(\partial D) \to L^2(\mathbb{S}^2)$ is bounded, injective and has dense range.

Theorem 1.5 (Theorem 5.17 of [17]) Assume that k^2 is not a Dirichlet eigenvalue for the negative Laplacian for D. Then the single-layer potential operator $S : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$ is defined by

$$S\varphi(x) := 2\int_{\partial D} \Phi(x, y)\varphi(y)ds(y), \quad x \in \partial D,$$

is a bijection with a bounded inverse.

Theorem 1.6 (Theorem 3.29 of [17]) The far field operator F defined in (1.6) has the factorization

$$F = -2\pi G S^* G^*, \tag{1.12}$$

where $G^* : L^2(\mathbb{S}^2) \to H^{-1/2}(\partial D)$ and $S^* : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$ are the adjoints of G and S, respectively.

Proof From the definition of the far field operator F and the Herglotz operator H. Note that Fg is far field pattern of the scattered wave corresponding to Herglotz operator Hg as incident field. So we have,

$$F = -GH. \tag{1.13}$$

The L^2 adjoint $H^* : H^{-1/2}(\partial D) \to L^2(\mathbb{S}^2)$ is defined by

$$H^*\varphi(\hat{x}) := \int_{\partial D} e^{-ik\hat{x}\cdot y}\varphi(y)ds(y), \quad \hat{x} \in \mathbb{S}^2.$$

The single-layer boundary operator $S : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$, defined by

$$S\varphi(x) := 2\int_{\partial D} \Phi(x,y)\varphi(y)ds(y), \quad x \in \partial D.$$

From the asymptotic behavior of the fundamental solution, note that $H^*\varphi$ is just the far field pattern of the single-layer potential S with density $4\pi\varphi$ and thus

$$H^* = 2\pi GS,$$

Consequently,

$$H = 2\pi S^* G^*.$$

Plug the factorization of H in (1.13) to get the result. $\square$

Theorem 1.7 (Theorem 1.12 of [37]) $\Phi_\infty(\cdot, z) \in R(G)$ if and only if $z \in D$.

Proof [37] Let $z \in D$. Define

$$u(x) := \Phi(x, z) = \frac{e^{ik|x-z|}}{4\pi|x-z|}, \quad x \notin D,$$

and $f := u|_{\partial D}$. Then $f \in H^{1/2}(\partial D)$ and the far field pattern of u is given by

$$u^\infty(\hat{x}) = \frac{1}{4\pi} e^{-ik\hat{x}\cdot z}, \quad \hat{x} \in \mathbb{S}^2,$$

which coincides with $\Phi_\infty(\cdot, z)$. So $Gf = u^\infty = \Phi_\infty(\cdot, z)$, i.e., $\Phi_\infty(\cdot, z) \in R(G)$. This ends the proof of the first part of the theorem.

Assume to the contrary, $z \notin D$. Since $\Phi_\infty(\cdot, z) \in R(G)$, then there exists $f \in$

$H^{1/2}(\partial D)$ with $Gf = \Phi_\infty(\cdot, z)$. Let u be the solution of the exterior Dirichlet problem with boundary data f and let $u^\infty = Gf$ be its far field pattern. Then by Rellich's lemma and analyticity the solution u to the exterior Dirichlet problem with boundary trace $u|_{\partial D} = f$ must coincide with $\Phi(\cdot, z)$ in $\mathbb{R}^3 \backslash (\overline{D} \cup \{z\})$. If $z \in \mathbb{R}^3 \backslash \overline{D}$, this contradicts the fact u is analytic in $R^3 \backslash \overline{D}$ and $\Phi(.\,, z)$ is singular at $x = z$.

If $z \in \partial D$, we have $\Phi(x\,, z) = f(x)$ for $x \in \partial D,\ x \neq z$. Since $f \in H^{1/2}(\partial D)$, then $\Phi(x\,, z)|_{\partial D} \in H^{1/2}(\partial D)$, which contradicts that $\Phi(x\,, z)|_{\partial D} \notin H^{1/2}(\partial D)$ since $\nabla \Phi(x, z) = \mathcal{O}(1/|x-z|^2)$ as $x \to z$. □

We are now ready to derive the traditional linear sampling approximation result. Note that the following theorem states the existence of particular solutions, which allow us to find the shape of D, but dose not provide a method to calculate those particular solutions.

Theorem 1.8 (Theorem 5.34 of [17]) Assume that k^2 is not a Dirichlet eigenvalue of the negative Laplacian in D and let F be the far field operator of the scattered field for Dirichlet boundary condition. Then the following hold:

1. For $z \in D$ and a given $\epsilon > 0$, there exists a function $g_z^\epsilon \in L^2(\mathbb{S}^2)$ such that

$$\|Fg_z^\epsilon - \Phi_\infty(\cdot, z)\|_{L^2(\mathbb{S}^2)} < \epsilon \tag{1.14}$$

and the Herglotz wave function $v_{g_z^\epsilon}$ with kernel g_z^ϵ converges to the solution $w \in H^1(D)$ of the Helmholtz equation with $w + \Phi(\cdot, z) = 0$ *on* ∂D as $\epsilon \to 0$.

2. For $z \notin D$, every $g_z^\epsilon \in L^2(\mathbb{S}^2)$ that satisfies (1.14) for a given $\epsilon > 0$ is such that

$$\lim_{\epsilon \to 0} \|v_{g_z^\epsilon}\|_{H^1(D)} = \infty.$$

which means $\lim_{\epsilon\to 0} \inf \left\{ ||v_{g_z^\epsilon}||_{H^1(D)} : g_z^\epsilon \text{ satisfies (1.14) } \right\} = \infty$.

Proof [17] Assume $z \in D$. Then by Theorem 1.7, $\Phi_\infty(\cdot, z) \in R(G)$. So $G\Phi(\cdot, z) = \Phi_\infty(\cdot, z)$. By Theorem 1.2, for a given arbitrary $\epsilon > 0$, there exists a Herglotz wave function with kernel $g_z^\epsilon \in L^2(\mathbb{S}^2)$ such that

$$\|Hg_z^\epsilon - (-\Phi(\cdot, z))\|_{H^{1/2}(\partial D)} < \frac{\epsilon}{||G||}.$$

Consequently

$$\|GHg_z^\epsilon + G\Phi(\cdot, z)\|_{L^2(\mathbb{S}^2)} < \epsilon.$$

Since $F = -GH$, we have

$$\|Fg_z^\epsilon - \Phi_\infty(\cdot, z)\|_{L^2(\mathbb{S}^2)} < \epsilon.$$

Since k^2 is not a Dirichlet eigenvalue of the negative Laplacian in D, from Theorem

1.5, we conclude that the interior Dirichlet problem in $H^1(D)$ is well-posed. Hence, if $z \in D$, then the convergence $Hg_z^\epsilon + \Phi(\cdot, z) \to 0 \, as \, \epsilon \to 0$ in $H^{1/2}(\partial D)$ holds, which implies convergence $v_{g^\epsilon} \to w$ as $\epsilon \to 0$ where $w \in H^1(D)$.

For $z \notin D$, assume on the contrary that there exists a sequence $\{\epsilon_n\} \to 0$ and the corresponding $g_n = g_z^{\epsilon_n}$ satisfies $\|Fg_n - \Phi_\infty(\cdot, z)\|_{L^2(\mathbb{S}^2)} < \epsilon_n$ such that $||v_n||_{H^1(D)}$ remains bounded, where $v_n := v_{g_n}$ is the Herglotz wave function with kernel g_n. Since $||v_n||_{H^1(D)}$ is bounded, without loss of generality we may assume $v_n \to v \in H^1(D)$ weakly as $n \to \infty$. Denote by $v^s \in H^1_{loc}(\mathbb{R}^3 \backslash \overline{D})$ the solution to the exterior Dirichlet problem for the Helmholtz equation with $v^s = v$ on ∂D and by v_∞ its far field pattern. Since Fg_n is the far field pattern of the scattered wave for the incident field $-v_n$, from (1.14) we conclude $v_\infty = -\Phi_\infty(\cdot, z)$ and hence $\Phi_\infty(\cdot, z) \in R(G)$, which contradicts Theorem (1.7). $\square$

In the linear sampling method we can numerically determine the function g_z in the above theorem and hence the scattering object D. Tikhonov regularization [17] can be used to solve (1.4). Generalization of the linear sampling method in inverse electromagnetic scattering can be found in [8]. For further details on the convergence of the linear sampling method we refer [6], [37] and [4].

1.3 Factorization Method

In general, there is no solution exists for (1.4) for noise-free data, hence, it is not clear what solution will be obtained by using Tikhonov regularization. To avoid this problem, Kirsch introduced in [36] and [34] the factorization method for solving inverse scattering problem for both the obstacle and non-absorbing inhomogeneous medium.

The idea of the factorization method [37] is to replace the far field operator in (1.4) by the operator $(F^*F)^{1/4}$. One can then show that

$$(F^*F)^{1/4}g_z = \Phi_\infty(\cdot,\, z) \tag{1.15}$$

has a solution if and only if $z \in D$. This method is called the factorization method since it relies on the factorization of the far field operator from Theorem 1.6.

Recall that the far field operator F has the following factorization

$$F = -2\pi G S^* G^*,$$

where G is data to pattern operator defined in Definition 1.3, S is the single-layer operator, $G^* : L^2(\mathbb{S}^2) \to H^{-1/2}(\partial D)$ and $S^* : H^{-1/2}(\partial D) \to H^{1/2}(\partial D)$ are the adjoints of G and S, respectively. Note that From the factorization of the far field operator F, the range of F is contained in the range of G [37]. From Theorem 1.7, we conclude that there is an explicit relationship between the range of the operator G and the shape of D, that is, $z \in D$ if and only if $\Phi_\infty(\cdot, z) \in R(G)$.

In the following theorem we summarize some well-known properties of the operator F.

Theorem 1.9 (Theorem 1.8 of [37])

1. The far field operator F satisfies

$$F - F^* = \frac{ik}{2\pi} F^* F,$$

where F^* is the adjoint operator of F.

2. The scattering operator $S = I + \frac{ik}{2\pi} F$ is unitary, i.e., $SS^* = S^*S = I$.

3. The far field operator F is compact and normal, i.e., $FF^* = F^*F$.

In the following lemma we summarize some well-known properties of S.

Lemma 1.10 (Lemma 1.14 of [37]) Assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D. Then the following holds.

1. $\Im\langle\varphi, S\varphi\rangle \neq 0$ for all $\varphi \in H^{-1/2}(\partial D)$ with $\varphi \neq 0$, where $\Im$ denotes for Im.

2. Let S_i be the single layer boundary operator of S corresponding to the wave number $k = i$. The operator S_i is self-adjoint and coercive as an operator from $H^{-1/2}(\partial D)$ onto $H^{1/2}(\partial D)$, i.e., there exits $c_0 > 0$ depends on i such that

$$\langle\varphi, S_i\varphi\rangle \geq c_0||\varphi||^2_{H^{-1/2}(\partial D)} \quad for\ all\ \varphi \in H^{-1/2}(\partial D).$$

3. The difference $S - S_i$ is compact from $H^{-1/2}(\partial D)$ onto $H^{1/2}(\partial D)$.

Since the far field operator F is normal [37], then there exits a complete set of orthogonal eigenfunctions $\psi_j \in L^2(\mathbb{S}^2)$ with corresponding eigenvalues $\lambda_j \in L^2(\mathbb{C}),\ j = 1, 2, 3, \ldots$ The spectral theorem for compact normal operators yields that F has the form

$$F\psi = \sum_{j=1}^{\infty} \lambda_j(\psi,\, \psi_j)_{L^2(\mathbb{S}^2)}\psi_j, \quad \psi \in L^2(\mathbb{S}^2).$$

As a conclusion, the far field operator F has a second factorization in the form

$$F = (F^*F)^{1/4} A_1\, (F^*F)^{1/4}, \tag{1.16}$$

where the operator $(F^*F)^{1/4} : L^2(\mathbb{S}^2) \longrightarrow L^2(\mathbb{S}^2)$ is given by

$$(F^*F)^{1/4}\psi = \sum_{j=1}^{\infty} \sqrt{|\lambda_j|}(\psi,\, \psi_j)_{L^2(\mathbb{S}^2)}\, \psi_j, \quad \psi \in L^2(\mathbb{S}^2),$$

and $A_1 : L^2(\mathbb{S}^2) \longrightarrow L^2(\mathbb{S}^2)$ of F is given by

$$A_1\psi = \sum_{j=1}^{\infty} \frac{\lambda_j}{|\lambda_j|}(\psi,\, \psi_j)_{L^2(\mathbb{S}^2)}\, \psi_j, \quad \psi \in L^2(\mathbb{S}^2).$$

The factorization method is based on the following result from functional analysis.

Theorem 1.11 (Theorem 1.23 of [37]) Let H be a Hilbert space, X a reflexive Banach space and let the compact operator $F : H \longrightarrow H$ have a factorization of the form

$$F = BAB^*$$

with operators $B : X \longrightarrow H$ and $A : X^* \longrightarrow X$, such that

1. $\Im\langle\varphi, A\varphi\rangle \neq 0$ for all $\varphi \in \overline{R(B^*)}$ with $\varphi \neq 0$.

2. The middle operator A has the form $A = A_0 + C$ for some compact operator C and some self-adjoint operator A_0 which is coercive on $R(B^*)$.

3. The far field operator F is one to one and $I + irF$ is unitary for some $r > 0$.

Then the ranges of B and $(F^*F)^{1/4}$ coincide.

Let $A : X \longrightarrow Y$ be a compact linear operator. The nonnegative square roots of the eigenvalues μ_n for all $n \in \mathbb{N}$ of the nonnegative self-adjoint compact operator $A^*A : X \longrightarrow X$ are called singular values. Consider the orthonormal sequences φ_n in X and g_n in Y such that $A\varphi_n = \mu_n g_n,\ A^* g_n = \mu_n \varphi_n$ for all $n \in \mathbb{N}$, then (μ_n, φ_n, g_n) is called the singular system of A.

Theorem 1.12 (Picard Theorem [17]) Let $A : X \longrightarrow Y$ be a compact linear operator in the Hilbert spaces X and Y with singular system (μ_n, φ_n, g_n). Then equation of the first kind

$$A\varphi = f$$

is solvable if and only if f belongs to the $N(A^*)^{\perp}$ and satisfies

$$\sum_{n=1}^{\infty} \frac{1}{\mu_n^2} |(f,\, g_n)|^2 < \infty.$$

In this case the solution is given by

$$\varphi = \sum_{n=1}^{\infty} \frac{1}{\mu_n} (f,\, g_n) \varphi_n.$$

Theorem 1.13 (Theorem 1.24 of [37]) Assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D. Then the ranges of G and $(F^*F)^{1/4}$ coincide.

Proof. Apply Theorem 1.11 to the factorization of F in (1.12) where $H = L^2(\mathbb{S}^2)$, $X = H^{1/2}(\partial D)$, $B = G$, and $A = -2\pi S^*$. Then Lemma 1.10 shows that $A = -2\pi S^*$ satisfies the conditions of Theorem 1.10. If k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D, then F is one to one. From Lemma 1.9, $I + \frac{ik}{2\pi}F$ is unitary. Thus the ranges of G and $(F^*F)^{1/4}$ coincide. $\square$

Theorem 1.14 (Theorem 1.25 of [37]) Assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D. For any $z \in \mathbb{R}^3$, the following are equivalent

1. $z \in D$.

2. $(F^*F)^{1/4} g_z = \Phi_\infty(\cdot,\, z)$ is solvable in $L^2(\mathbb{S}^2)$.

3. $W(z) := \left[\sum_j \frac{|(\Phi_\infty, \psi_j)_{L^2(\mathbb{S}^2)}|^2}{|\lambda_j|} \right]^{-1} > 0$. Here $\lambda_j \in \mathbb{C}$ are the eigenvalues of the normal operator F with corresponding normalized eigenfunctions $\psi_j \in L^2(\mathbb{S}^2)$.

Proof [37] By Theorem 1.7, $z \in D$ if and only if $\Phi_\infty \in R(G)$. By Theorem 1.13 the ranges of G and $(F^*F)^{1/4}$ coincide. So, $z \in D$ if and only if $\Phi_\infty \in R((F^*F)^{1/4})$, that is, if and only if the equation

$$(F^*F)^{1/4}g_z = \Phi_\infty(\cdot,\, z)$$

is solvable in $L^2(\mathbb{S}^2)$.

We write $\Phi_\infty(\cdot,\, z)$ in spectral form as

$$\Phi_\infty(\cdot,\, z) = \sum_j (\Phi_\infty(\cdot,\, z), \psi_j)_{L^2(\mathbb{S}^2)}\psi_j.$$

By the Picard Theorem, $(F^*F)^{1/4}g_z = \Phi_\infty(\cdot,\, z)$ is solvable in $L^2(\mathbb{S}^2)$, if and only if

$$\sum_j \frac{\Big|(\Phi_\infty(\cdot,\, z), \psi_j)\Big|^2_{L^2(\mathbb{S}^2)}}{|\lambda_j|} < \infty.$$

In this case,

$$g_z = \sum_j \frac{(\Phi_\infty(\cdot,\, z), \psi_j)_{L^2(\mathbb{S}^2)}}{\sqrt{|\lambda_j|}}\psi_j$$

is the solution of $(F^*F)^{1/4}g_z = \Phi_\infty(\cdot,\, z)$. Therefore, a point $z \in \mathbb{R}^3$ belongs to D if and only if the series

$$\sum_j \frac{\Big|(\Phi_\infty(\cdot,\, z), \psi_j)\Big|^2_{L^2(\mathbb{S}^2)}}{|\lambda_j|} < \infty.$$

$\square$

1.4 A Direct Sampling Method for Inverse Scattering Problems

1.4.1 Introduction

In this section we present a direct sampling method for time harmonic inverse medium scattering problems (IMSP) introduced by Ito, et al. [32]. The method directly estimates the shape of the unknown scatterers and based on a scattering analysis. It involves only computing the inner product of the fundamental solutions with the measured scattered field u^s located at the sampling points over the curve/surface Γ. Ito, et al. [33] extended the method in [32] for electromagnetic scattering problems.

Li et al. [39] developed three inverse scattering schemes for locating multiple multi-scale acoustic scatterers. Only one single far-field measurement is used for all of the three locating schemes. Each scatterer component is allowed to be an inhomogeneous medium with an unknown content or an impenetrable obstacle. The number of the multiple scatterer components may be unknown.Futhermore, the scatterers could be multi-scale.

Li et al. [38] developed two inverse scattering schemes for locating multiple electromagnetic scatterers by using the electric far field measurement. The first scheme is for locating scatterers of small size compared to the wavelength. The second scheme is for locating multiple perfectly conducting compared to the incident electromagnetic wavelength.

Song et al. [59] introduced a multi-dimensional sampling method to locate small scatterers. The indicator function is based on multi-static response matrix which is defined on a set of combinatorial sampling nodes inside the domain of interest. Bektas and Ozdemir [5] extended the use of conventional direct sampling method (DSM), which is only applicable to the multi-static measurement data, to the mono-static measurement data for radar imaging applications. They define a testing function which can be used in the indicator function of DSM with mono-static data. Li et al. [48] employed a direct sampling method to reconstruct the support of the potential for stationary Schrdinger equation.

1.4.2 A Direct Sampling Method to an Inverse Medium Scattering Problem

In this subsection, we introduce a direct sampling method to determine the shape of the scatterers/inhomogeneities [32]. Suppose that a bounded domain Ω in the homogeneous background space $\mathbb{R}^d$ (d = 2, 3) is occupied by some inhomogeneous media. Assume that the incident field is given by $u^i = e^{ikx\cdot d}$, where d is the direction of the plane wave and k is the wave number. Then the total field is defined as $u = u^i + u^s$, where u^s is the scattered field due to the inhomogeneous medium. The total field u induced by the inhomogeneous media satisfies the Helmholtz equation

$$\Delta u + k^2 q^2(x) u = 0, \tag{1.17}$$

where the function $q(x)$ refers to the refractive index, i.e. the ratio of the wave speed in the homogeneous background to that in the inhomogeneous medium.

Define $\eta(x) = k^2(q^2(x) - 1)$, which combines the relative refractive index $q^2 - 1$ with the wave number k, to characterize the inhomogeneity. The function $\eta(x)$ vanishes outside the inhomogeneous medium. Define the current induced by the inhomogeneous medium as $I = \eta u$. The scattered field can be written as [15]

$$u^s = \int_\Omega G(x,y)I(y)dy, \tag{1.18}$$

which makes the total field satisfy

$$u = u^i + \int_\Omega G(x,y)I(y)dy. \tag{1.19}$$

Here $G(x,y)$ is the fundamental solution for the open field given by

$$G(x,y) = \begin{cases} \frac{i}{4}H_0^1(k|x-y|) & \text{for} \quad d=2, \\ \\ \frac{1}{4\pi}\frac{e^{(ik|x-y|)}}{|x-y|} & \text{for} \quad d=3, \end{cases} \tag{1.20}$$

where H_0^1 is the zeroth order Hankel function of the first kind.

Multiplying both sides of (1.19) by η, we get the second-kind integral equation for the induced current I:

$$I(x) = \eta u^i + \eta \int_\Omega G(x,y)I(y)dy. \tag{1.21}$$

Consider a curve Γ which encloses the inhomogeneous medium. The fundamental solution $G(x, x_p)$ for the Helmholtz equation in the homogeneous background is given by

$$\Delta G(x, x_p) + k^2 G(x, x_p) = -\delta(x - x_p), \tag{1.22}$$

where $\delta(x - x_p)$ is the Dirac delta function located at the point $x_p \in \Omega_\Gamma$ (the domain enclosed by Γ) such that $\delta(x - x_p) = 0$ for all $x \neq x_p$.

Let x_q be another point in Ω_Γ. Multiplying both sides of (1.22) by the conjugate $\overline{G}(x, x_q)$ of the fundamental solution $G(x, x_q)$:

$$[\Delta G(x, x_p) + k^2 G(x, x_p)]\overline{G}(x, x_q) = -\delta(x - x_p)\overline{G}(x, x_q). \tag{1.23}$$

Integrating both sides over the domain Ω_Γ, we obtain

$$\begin{aligned} &\int_{\Omega_\Gamma} [\Delta G(x, x_p) + k^2 G(x, x_p)]\overline{G}(x, x_q) dx \\ &\quad - \int_{\Omega_\Gamma} \delta(x - x_p)\overline{G}(x, x_q) dx = -\overline{G}(x_p, x_q). \end{aligned} \tag{1.24}$$

Next we consider equation (1.22) at x_q and take its conjugate to get

$$\Delta \overline{G}(x, x_q) + k^2 \overline{G}(x, x_q) = -\delta(x - x_q). \tag{1.25}$$

Multiplying both sides of the resulting equation by $G(x, x_p)$ and integrating over the domain Ω_Γ, we get

$$\begin{aligned}&\int_{\Omega_\Gamma}[\Delta\overline{G}(x,x_q)+k^2\overline{G}(x,x_q)]G(x,x_p)dx\\&\quad=-\int_{\Omega_\Gamma}\delta(x-x_q)G(x,x_p)dx=-G(x_p,x_q).\end{aligned}\tag{1.26}$$

Subtracting (1.26) from (1.24)

$$G(x_p,x_q)-\overline{G}(x_p,x_q)$$

$$\begin{aligned}&=\int_{\Omega_\Gamma}\Big[\Delta G(x,x_p)\overline{G}(x,x_q)+k^2G(x,x_p)\overline{G}(x,x_q)\\&\qquad-\Delta\overline{G}(x,x_q)G(x,x_p)-k^2\overline{G}(x,x_q)G(x,x_p)\Big]dx\\&=\int_{\Omega_\Gamma}\Big[\overline{G}(x,x_q)\Delta G(x,x_p)-\Delta\overline{G}(x,x_q)G(x,x_p)\Big]dx.\end{aligned}\tag{1.27}$$

Applying Green's Second Theorem to (1.27), we get

$$G(x_p,x_q)-\overline{G}(x_p,x_q)=\int_\Gamma\Big[\overline{G}(x,x_q)\frac{\partial G(x,x_p)}{\partial n}-G(x,x_p)\frac{\partial\overline{G}(x,x_q)}{\partial n}\Big]ds.\tag{1.28}$$

The Sommerfeld radiation condition for Helmholtz equation states that

$$\frac{\partial G(x,x_p)}{\partial n}=ikG(x,x_p)+higher\,order\,terms.$$

Thus if we use the approximations

$$\frac{\partial G(x,x_p)}{\partial n} \approx ikG(x,x_p) \;\; and \;\; \frac{\partial \overline{G}(x,x_q)}{\partial n} \approx -ik\overline{G}(x,x_q), \tag{1.29}$$

which are valid if the points x_p and x_q are not close to the boundary Γ and substitute (1.29) in the right side of (1.28), we get

$$\int_\Gamma \Big[ikG(x,x_p)\overline{G}(x,x_q) + ik\overline{G}(x,x_q)G(x,x_p)\Big]ds \approx 2ik\int_\Gamma G(x_p,x_q)\overline{G}(x,x_q)ds. \tag{1.30}$$

But, for any complex number z, $z - \overline{z} = 2i\Im(z)$. So the left hand side of (1.28) equals to

$$G(x_p,x_q) - \overline{G}(x_p,x_q) = 2i\Im(G(x_p,x_q)). \tag{1.31}$$

From (1.29) , (130) and (1.31), we get

$$2ik\int_\Gamma G(x,x_p)\overline{G}(x,x_q)ds \approx 2i\Im(G(x_p,x_q)),$$

i.e.,

$$\int_\Gamma G(x,x_p)\overline{G}(x,x_q)ds \approx k^{-1}\Im(G(x_p,x_q)). \tag{1.32}$$

Consider the sampling domain $\overline{\Omega}$, where $\Omega \subset \overline{\Omega}$. Dividing the domain $\overline{\Omega}$ into a set of small elements τ_j and applying a rectangular quadrature rule, we get the approximation

$$u^s = \int_{\overline{\Omega}} G(x,y)I(y)dy \approx \sum_j w_j G(x,y_j), \tag{1.33}$$

where $y_j \in \tau_j$ and the weight w_j is given by $w_j = |\tau_j|I(y_j)$. Here $|\tau_j|$ is the volume of the element τ_j. Since the induced current I vanishes outside the support Ω, the summation in (1.33) is only over the elements intersecting Ω.

Multiplying both sides of (1.33) by $\overline{G}(x,x_p)$, where $x_p \in \overline{\Omega}$, and integrating over the boundary Γ, we get

$$\int_\Gamma u^s(x)\overline{G}(x,x_p)ds \approx \sum_j w_j \int_\Gamma G(x,y_j)\overline{G}(x,x_p)ds.$$

Therefore,

$$\int_\Gamma u^s(x)\overline{G}(x,x_p) \approx k^{-1}\sum_j w_j \Im(G(y_j,x_p)). \tag{1.34}$$

Eqn.(1.34) is valid under the assumption that the point scatterers y_j and the sampling points x_p are far away from Γ , and the elements τ_j are sufficiently small. Note that if the point x_p is close to some point $y_j \in \Omega$, then $G(y_j,x_p)$ is nearly singular and therefore (1.34) makes the summation very large. Conversely, if x_p is far away from all physical point scatterers, then due to the decay property of $G(x,y)$, the sum in (1.34) will be very small.

These observation leads to define the following index function for any point $x_p \in \overline{\Omega}$,

$$\Phi(x_p) = \frac{| < u^s(x), G(x, x_p) >_{L^2(\Gamma)} |}{||u^s(x)||_{L^2(\Gamma)} ||G(x, x_p)||_{L^2(\Gamma)}}. \tag{1.35}$$

In practice,

† If $|\Phi(x_p)| \approx 1, \quad x_p \in \Omega.$

† If $|\Phi(x_p)| \approx 0, \quad x_p \notin \Omega.$

Hence, $\Phi(x_p)$ serves as a characteristic function of Ω and thus we can identify Ω from the values of $\Phi(x_p)$ when they are close to 1. Numerical experiments for this method can be found in [32] and [65].

Chapter 2

Direct Sampling Methods for Shape Reconstruction in Inverse Acoustic Scattering Problems

This chapter introduces two direct sampling methods for shape reconstruction in inverse acoustic scattering problems (IASP). The first method is called the orthogonality sampling method and was proposed by Potthast in 2010 [56]. The basic idea of this method is to design an indicator function which is relatively small inside and outside the unknown scatterer D and large on the boundary ∂D. The second method was proposed by Liu in 2016 [45]. The basic idea of his method is to design an indicator which is big inside the scatterer and relatively small outside. The method is very simple to implement since only the inner products of the measurements with some suitably chosen functions are involved in computation of the indicator function. This method uses the factorization of the far field operator to give a lower bound on the proposed indicator function for sampling points inside the scatterer. For the sampling points outside the scatterer, Liu shows that the indicator decays as sampling point goes away from the boundary of the scatterer.

Several advantages of these direct sampling methods are inherited from the classical ones, including their independence on any prior information on the geometry and physical properties of the unknown objects. The key feature of these direct sampling methods is that the computation of the indicator involves only inner products of the measurements, with some suitably chosen functions. This makes them robust to noises and computationally faster than the classical sampling methods.

We will generalize this method in Chapter 3 to the case of inverse electromagnetic scattering problems in an inhomogeneous isotropic medium in $\mathbb{R}^3$. Moreover, in Chapter 4, we will generalize the method to the case of multifrequency inverse source problem for time-harmonic acoustic with a finite set of far field data.

2.1 Orthogonality Sampling Method

In this section we introduce the orthogonality sampling method by Potthast [56]. Consider the scattering of acoustic wave u^i by an impenetrable scatterer D with the Dirichlet boundary condition in two or three dimensions. The scattered field is denoted by u^s and the total field $u = u^i + u^s$ satisfies the Helmholtz equation

$$\begin{aligned} \Delta u + k^2 u = 0 \quad in \ \mathbb{R}^n \backslash \overline{D}, \\ u = 0 \quad on \ \partial D. \end{aligned} \tag{2.1}$$

The scattered field is assumed to satisfy the Sommerfeld radiation condition

$$\lim_{r:=|x|\to\infty} r^{\frac{n-1}{2}} (\frac{\partial u^s}{\partial r} - iku^{u^s}) = 0.$$

2.1.1 Orthogonality Sampling

Potthast proposed orthogonality sampling based on the following indicator [55]

$$\mu(z,k,\theta) := \left| \int_{\mathbb{S}^{n-1}} u^\infty(\hat{x},\theta) e^{ik\hat{x}\cdot z} ds(\hat{x}) \right|, \qquad z \in \mathbb{R}^n, \tag{2.2}$$

where $u^\infty(\hat{x},\theta)$ denotes the far field pattern for the scattering of an incident plane wave with the direction θ, evaluated at the direction $\hat{x} \in \mathbb{S}^{n-1},\ n = 2,3$.

Definition 2.1 [56] The orthogonality sampling indicator functional for the fixed wave number k, is defined as

$$\mu(\breve{y},k) = \left| \int_{\mathbb{S}^{n-1}} e^{ik\hat{x}\cdot\breve{y}}\, u^\infty(\hat{x})ds(\hat{x}) \right| \tag{2.3}$$

on a grid G of points $\breve{y} \in \mathbb{R}^n$ from the knowledge of the far field pattern $u^\infty(\hat{x})$ on $\mathbb{S}^{n-1}$.

Let $Y_\alpha^\beta(\cdot)$ for $\alpha \in \mathbb{N} \cup \{0\}$ and $\beta = -\alpha, ..., \alpha$ be the spherical harmonics which form a complete orthonormal system in $L^2(\mathbb{S}^{n-1})$ [17]. We recall the spherical harmonics of order $\alpha = 0,1$ for $\hat{x} = (\hat{x})_{l=1}^n \in \mathbb{S}^{n-1}$. In three dimensional case,

$$Y_0^0(\hat{x}) = \sqrt{\frac{1}{4\pi}}, \qquad Y_1^{-1}(\hat{x}) = \sqrt{\frac{3}{8\pi}}(\hat{x}^1 - i\hat{x}^2),$$

$$Y_1^0(\hat{x}) = \sqrt{\frac{3}{4\pi}}\hat{x}^3, \qquad Y_1^1(\hat{x}) = \sqrt{\frac{3}{8\pi}}(\hat{x}^1 + i\hat{x}^2).$$

In two dimensional case,

$$Y_0^0(\hat{x}) = \sqrt{\frac{1}{2\pi}}, \qquad Y_1^{-1}(\hat{x}) = \sqrt{\frac{1}{2\pi}}(\hat{x}^1 - i\hat{x}^2), \qquad Y_1^1(\hat{x}) = \sqrt{\frac{1}{2\pi}}(\hat{x}^1 + i\hat{x}^2).$$

The Funk-Hecke formula is defined as [17]

$$\int_{\mathbb{S}^{n-1}} e^{-ikz\cdot\hat{x}} Y_\alpha^\beta(\hat{x}) ds(\hat{x}) = \mu_\alpha f_\alpha(k|z|) Y_\alpha^\beta\Big(\frac{z}{|z|}\Big),$$

where

$$\mu_\alpha = \begin{cases} 2\pi & , \quad n=2; \\ \frac{4\pi}{i^\alpha} & , \quad n=3, \end{cases} \qquad \textit{and} \qquad f_\alpha(t) = \begin{cases} J_\alpha(t) & , \quad n=2; \\ j_\alpha(t) & , \quad n=3, \end{cases}$$

with J_α and j_α being the Bessel functions and spherical Bessel functions of order α, respectively.

We start with a representation of the scattered field u^s for a sound-soft scatterer D [50]

$$u^s(x) = \int_{\partial D} \Phi(x,y) \frac{\partial u}{\partial \nu}(y)\, ds(y), \quad x \in \mathbb{R}^n \backslash \overline{D},$$

where $\Phi(x,y),\ x \neq y$, is the fundamental solution of the Helmholtz equation given by

$$\Phi(x,y) = \begin{cases} \frac{i}{4} H_0^1(k|x-y|) & \text{for} \quad n=2, \\ \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|} & \text{for} \quad n=3. \end{cases} \tag{2.4}$$

The far field pattern of the scattered field u^s is given by

$$u^\infty(\hat{x}) = \gamma \int_{\partial D} e^{-ik\hat{x}\cdot y} \frac{\partial u}{\partial \nu}(y)\, ds(y), \quad \hat{x} \in \mathbb{S}^{n-1},$$

where γ is a constant given by

$$\gamma := \begin{cases} \frac{e^{i\pi/4}}{\sqrt{8\pi k}} & n = 2, \\ \frac{1}{4\pi} & n = 3. \end{cases}$$

Multiplying $u^\infty(\hat{x})$ by $\overline{f_z}$ with $f_z(\hat{x}) := e^{-ik\hat{x}\cdot z}$, $z \in \mathbb{R}^n$, and integrating over $\mathbb{S}^{n-1}$ yeilds

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} u^\infty(\hat{x}) e^{ik\hat{x}\cdot z}\, ds(\hat{x}) &= \gamma \int_{\mathbb{S}^{n-1}} \int_{\partial D} e^{-ik\hat{x}\cdot(y-z)} \frac{\partial u}{\partial \nu}(y)\, ds(y)\, ds(\hat{x}), \\ &= \gamma \int_{\partial D} \left(\int_{\mathbb{S}^{n-1}} e^{-ik\hat{x}\cdot(y-z)}\, ds(\hat{x}) \right) \frac{\partial u}{\partial \nu}(y)\, ds(y). \end{aligned}$$

For the three-dimensional case, we have

$$\int_{\mathbb{S}^{n-1}} u^\infty(\hat{x}) e^{-ik\hat{x}\cdot z}\, ds(\hat{x}) = 4\pi\gamma \int_{\partial D} j_0(k|y-z|) \frac{\partial u}{\partial \nu}(y)\, ds(y).$$

The integral

$$u^s_{red}(z) := 4\pi\gamma \int_{\partial D} j_0(k|y-z|) \frac{\partial u}{\partial \nu}(y)\, ds(y)$$

is called the reduced scattered field.

Lemma 2.2 [56] The indicator function $\mu(\breve{y}, k)$ given in (2.3) is equal to the modulus of the field

$$u^s_{red}(\breve{y}) := \gamma\mu_0 \int_{\partial D} f_0(k|\breve{y} - y|)\frac{\partial u}{\partial \nu}(y)\, ds(y), \quad \breve{y} \in \mathbb{R}^n.$$

For the Neumann boundary condition, the indicator function in (2.3) is given by

$$u^s_{red,N}(\breve{y}) := \gamma\mu_0 \int_{\partial D} \frac{\partial}{\partial \nu(y)} f_0(k|\breve{y} - y|)u(y)\, ds(y), \quad \breve{y} \in \mathbb{R}^n.$$

The relation between the reduced scattered field and the shape of the scatterer is an open problem and needs further investigation.

Recall that

$$\mu(\breve{y}, \hat{\theta}, k) := \left| \int_{\mathbb{S}^{n-1}} u^\infty(\hat{x}, \hat{\theta}, k) e^{ik\hat{x}\cdot\breve{y}} ds(\hat{x}) \right|$$

is the indicator function for fixed frequency k and fixed direction $\hat{\theta}$. Numerical examples show the feasibility and effectiveness of the indicator $\mu(\breve{y}, \hat{\theta}, k)$ for location reconstruction, in particular for small objects. But, $\mu(\breve{y}, \hat{\theta}, k)$ does not work for shape reconstruction of extended scatterers. To solve this difficulty, Potthast [56] suggested the following indicator function with respect to the all the incident directions

$$
\begin{aligned}
\mu[MD](\breve{y},k) &:= \int_{\mathbb{S}^{n-1}} [\mu(\breve{y},\hat{\theta},k)]^{\rho} ds(\hat{\theta}) \\
&= \int_{\mathbb{S}^{n-1}} \left| \int_{\mathbb{S}^{n-1}} u^{\infty}(\hat{x},\hat{\theta},k)\, e^{ik\hat{x}\cdot z} ds(\hat{x}) \right|^{\rho} ds(\hat{\theta}), \qquad z \in \mathbb{R}^n,
\end{aligned} \tag{2.5}
$$

where $\rho = 1$ *or* $\rho = 2$, for $\breve{y} \in \mathbb{R}^n$ and fixed $k \in \mathbb{R}^+$ for the fixed frequency case.

Numerical examples show that the indicator function $\mu[MD]$ is a good indicator for shape reconstruction for extended scatterers.

Potthast also extended the indicator function in (2.3) to one-wave multi-frequency situation. When several frequencies are taken into account the results are significantly improved . Assume that u^{∞} depends on the wave number k and is given for $k_1, , k_2, \ldots, k_M$ with some $M \in \mathbb{N}$. Define the multi-frequency functional by

$$
\mu[MF](\breve{y}) := \int_{k_1}^{k_M} \left| \int_{\mathbb{S}^{n-1}} e^{ik\hat{x}\cdot\breve{y}}\, u^{\infty}(\hat{x},\hat{\theta},k) ds(\hat{x}) \right|^{\rho} dk, \tag{2.6}
$$

where $\rho = 1$ *or* $\rho = 2$, for $\breve{y} \in \mathbb{R}^n$ and fixed direction $\hat{\theta}$.

For the full multi-direction multi-frequency (MDMF) the indicator function is

$$
\mu[MDMF](\breve{y}) := \int_{k_1}^{k_M} \int_{\mathbb{S}^{n-1}} \left| \int_{\mathbb{S}^{n-1}} e^{ik\hat{x}\cdot\breve{y}}\, u^{\infty}(\hat{x},\hat{\theta},k) ds(\hat{x}) \right|^{\rho} ds(\hat{\theta})\, dk, \tag{2.7}
$$

where $\rho = 1$ *or* $\rho = 2$.

There is no theoretical analysis established for the behavior of all of these indicator defined in (2.5), (2.6) and (2.7).

2.1.2 Numerical Study of the Indicator Functions

In this subsection, we implement the indicator functions of orthogonality sampling for a kite [50]. In Figures 2.1 - 2.6, we provide a numerical study of the above functionals, including fixed frequency one wave $\mu(\breve{y}, k)$, multi-direction fixed-frequency $\mu[MD]$, multi-frequency fixed direction $\mu[MF]$ and multi-direction multi-frequency $\mu[MDMF]$ setup.

Fig. 2.1 and Fig. 2.2 show the behavior of the indicator function $\mu(\breve{y}, k)$ for one wave fixed frequency when $k = 1$ and $k = 3$, respectively. From these figures, the location of the unknown scatterer can be clearly seen, but no information about the shape of the scatterer. In Fig. 2.3 and Fig. 2.4, the indicator function $\mu[MD]$ already provides a lot of information about the shape of the scatterer, although it is still rather wavy. The same applies for the case $\mu[MF]$ when we use multi-frequency fixed direction, as shown in Fig. 2.5. For the case of multi-frequency multi-direction, see Fig. 2.6, the indicator $\mu[MDMF]$ provides stable and reliable shape reconstructions.

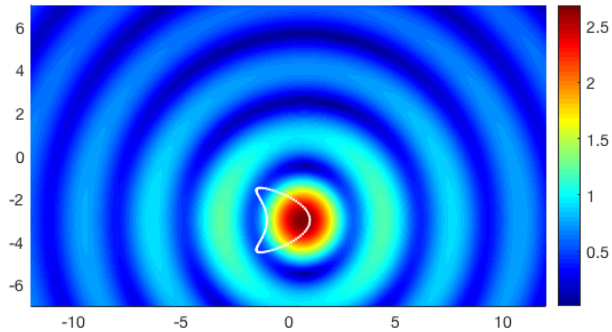

Figure 2.1: Constructing orthogonality sampling $\mu(\breve{y}, k)$ for fixed frequency $k = 1$, and incident wave angle $\hat{\theta} = \pi$.

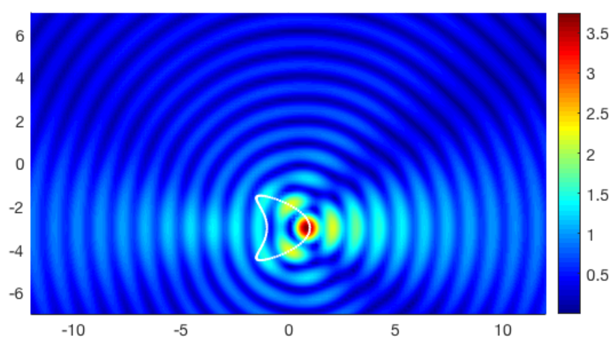

Figure 2.2: Constructing orthogonality sampling $\mu(\breve{y}, k)$ for fixed frequency $k = 3$, and incident wave angle $\hat{\theta} = \pi$.

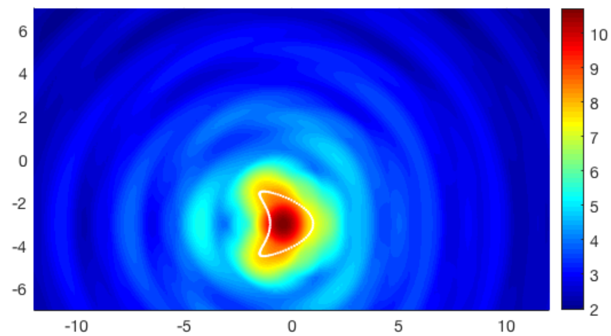

Figure 2.3: Behavior of the multi-directions functional $\mu[MD]$ for $k = 1$ and for six different incident waves with the angle of incidence being multiples of $\pi/3$.

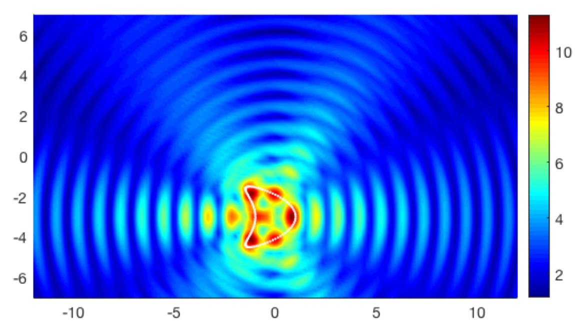

Figure 2.4: Behavior of the multi-directions functional $\mu[MD]$ for $k = 3$ and for six different incident waves with the angle of incidence being multiples of $\pi/3$.

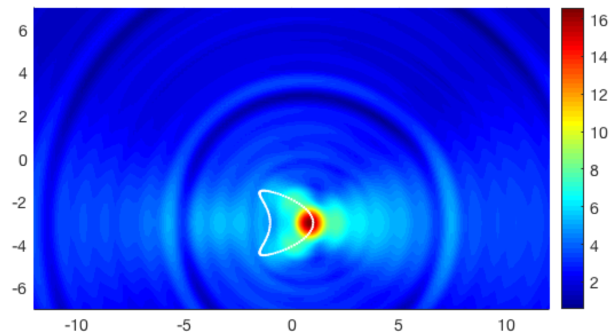

Figure 2.5: Behavior of the multi-frequency functional $\mu[MF]$ for $k = \{1, 1.5, 2, 2.5, 3\}$ with single incident wave angle $\hat{\theta} = \pi/3$.

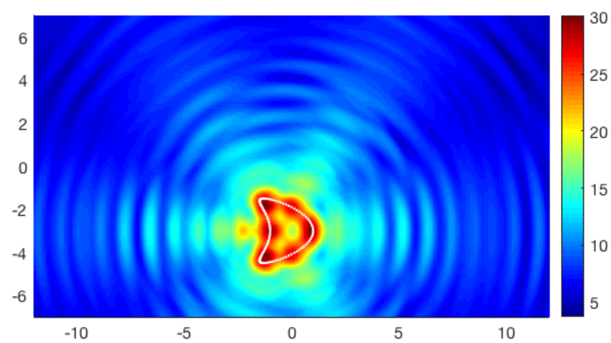

Figure 2.6: Behavior of the multi-frequency functional $\mu[MDMF]$ for $k = \{1, 1.5, 2, 2.5, 3\}$ for six different incident waves with the angle of incidence being multiples of $\pi/3$.

2.2 A Direct Sampling Method for Shape Reconstruction in Inverse Acoustic Scattering Problems

2.2.1 Introduction

In this section we introduce a direct sampling method for shape reconstruction in inverse acoustic scattering problems (IASP), proposed by Liu in 2016 [45], using the far field measurements. The basic idea of this method is to design an indicator which is big inside the scatterer and relatively small outside.

With the help of the factorization of the far field, Liu established a lower bound estimate for the sampling points inside the scatterer. For the sampling points outside the scatterer, the indicator function decays as Bessel functions when the sampling point goes away from the boundary of the scatterer. At the end of the section, we will discuss the stability of the method. For more details, see [45].

Let $\Omega \subset \mathbb{R}^n$, where $n = 2, 3$, be an open and bounded domain with Lipschitz boundary

$\partial\Omega$ such that $\mathbb{R}^n \backslash \overline{\Omega}$ is connected. The incident field is of the form

$$u^i(x, \hat{\theta}) = e^{ikx\cdot\hat{\theta}}, \ x \in \mathbb{R}^n, \tag{2.8}$$

where $\hat{\theta} \in \mathbb{S}^{n-1}$ denotes the direction of the incident wave and k is the wave number. Then the scattering problem for the inhomogeneous medium is to find the total field $u = u^i + u^s$ such that

$$\Delta u + k^2(1+q)u = 0 \ \ in \ \ \mathbb{R}^n, \tag{2.9}$$

$$\lim_{r:=|x|\to\infty} r^{\frac{n-1}{2}} \Big(\frac{\partial u^s}{\partial r} - iku^{u^s} \Big) = 0, \tag{2.10}$$

where $q \in L^\infty(\mathbb{R}^n)$ such that $\Im(q) \geqslant 0$, $q = 0$ in $\mathbb{R}^n \backslash \overline{\Omega}$ and (2.10) is the Sommerfeld radiating condition that holds uniformly with respect to all directions $\hat{x} := x/|x| \in \mathbb{S}^{n-1}$.

If the scatterer Ω is impenetrable, the direct scattering is to find the total field $u = u^i + u^s$ such that

$$\Delta u + k^2 u = 0 \ \ in \ \ \mathbb{R}^n \backslash \overline{\Omega}, \tag{2.11}$$

$$B(u) = 0 \ on \ \partial\Omega, \tag{2.12}$$

$$\lim_{r:=|x|\to\infty} r^{\frac{n-1}{2}} \Big(\frac{\partial u^s}{\partial r} - iku^{u^s} \Big) = 0, \tag{2.13}$$

where B denotes one of the following three boundary conditions:

$$(1)B(u) := u \text{ on } \partial\Omega; \qquad (2)B(u) := \frac{\partial u}{\partial \nu} \text{ on } \partial\Omega; \quad (3)B(u) := \frac{\partial u}{\partial \nu} + \lambda u \ \text{ on}\, \partial\Omega.$$

These correspond, to the case when the scatterer Ω is sound-soft, sound-hard, and of impedance type, respectively. Here, ν is the unit outward normal to $\partial\Omega$ and $\lambda \in L^\infty(\partial\Omega)$ is the impedance function such that $\Im(\lambda) \geqslant 0$ almost everywhere on $\partial\Omega$.

Every radiating solution of the Helmholtz equation has the following asymptotic behavior at infinity:

$$u^s(x,\hat{\theta}) = \frac{e^{i\frac{\pi}{4}}}{\sqrt{8k\pi}} \left[e^{-i\frac{\pi}{4}} \sqrt{\frac{k}{2\pi}} \right]^{n-2} \frac{e^{ikr}}{r^{\frac{n-1}{2}}} \left\{ u^\infty(\hat{x},\hat{\theta}) + \mathcal{O}\left(\frac{1}{r}\right) \right\} \quad as \quad r := |x| \to \infty, \tag{2.14}$$

with $\hat{x} = \frac{x}{|x|} \in \mathbb{S}^{n-1}$. The inverse problem we consider in this section is to determine Ω from a knowledge of the far field pattern $u^\infty(\hat{x},\hat{\theta})$ for $\hat{x}$, $\hat{\theta} \in \mathbb{S}^{n-1}$.

Liu in his paper [45] proposed a new direct sampling method by using the following indicator

$$I_{new}(z) := \left| \int_{\mathbb{S}^{n-1}} e^{-ik\hat{\theta}\cdot z} \int_{\mathbb{S}^{n-1}} u^\infty(\hat{x},\hat{\theta}) e^{ik\hat{x}\cdot z} ds(\hat{x}) ds(\hat{\theta}) \right|, \quad z \in \mathbb{R}^n \tag{2.15}$$

2.2.2 Theoretical Foundation of the Proposed Sampling Method

For the sampling points inside the scatterer Ω, it is shown that there is a lower bound for the indicator $I_{new}(z)$, and for the sampling points outside the scatterer the indicator function, $I_{new}(z)$ starts to decay as z go away from the boundary of Ω. Moreover, the method is stable with respect to the noise in the data.

The indicator function $I_{new}(z)$ given by (2.15) can be written in the form

$$I_{new}(z) := |(F\phi_z, \phi_z)|, \qquad z \in \mathbb{R}^n. \tag{2.16}$$

Here, we denote by $(\cdot\,,\,\cdot)$ the inner product of $L^2(\mathbb{S}^{n-1})$, and F is the far field operator defined as

$$F : L^2(\mathbb{S}^{n-1}) \to L^2(\mathbb{S}^{n-1}),$$

$$Fg(\hat{x}) = \int_{\mathbb{S}^{n-1}} u^\infty(\hat{x}, \hat{\theta}) g(\hat{\theta}) ds(\hat{\theta}), \qquad \hat{x} \in \mathbb{S}^{n-1}. \tag{2.17}$$

Also ϕ_z is the test function defined as

$$\phi_z(\vartheta) = e^{-ikz\cdot\vartheta}, \qquad \vartheta \in \mathbb{S}^{n-1}. \tag{2.18}$$

2.2.2.1 Resolution analysis for the sampling points inside the scatterer

The method is based on the following result from functional analysis.

Theorem 2.3 (Theorem 1.16 of [37]) Let X, Y be (complex) reflexive Banach spaces with dual X^*, Y^*, respectively, and the dual forms $\langle \cdot, \cdot \rangle$ in $\langle X^*, X \rangle$ and $\langle Y^*, Y \rangle$. Let $F : Y^* \to Y$ and $B : X \to Y$ be linear operators with

$$F = BAB^*$$

for some linear and bounded operator $A : X^* \to X$, which satisfies a coercivity assumption, i.e., there exists $c > 0$ such that

$$|\langle \varphi, A\varphi \rangle| \geqslant c||\varphi||_{X^*}^2 \text{ for all } \varphi \in R(B^*) \subset X^*.$$

Then , for any $\phi \in Y$, $\phi \neq 0$,

$$\phi \in R(B) \quad \text{if and only if} \quad \inf\{|(\psi, F\psi)| : \psi \in Y^*, (\psi, \phi) = 1\} > 0$$

Moreover, if $\phi = B\varphi_0 \in R(B)$ for some $\varphi_0 \in X$ then

$$\inf\{|(\psi, F\psi)| : \psi \in Y^*, (\psi, \phi) = 1\} \geq \frac{c}{||\varphi_0||_X^2}.$$

Proof [37] First, note that,

$$|\langle \psi, F\psi\rangle| = |\langle \psi, BAB^*\psi\rangle| = |\langle B^*\psi, AB^*\psi\rangle| \geq c||B^*\psi||^2_{X^*}, \quad \text{for all } \psi \in Y^*.$$

Let $\phi = B\varphi_0$ for some $\varphi_0 \in X$. For $\psi \in Y^*$ with $\langle \psi, \phi\rangle = 1$, we have

$$|\langle \psi, F\psi\rangle| \geq c||B^*\psi||^2_{X^*} = \frac{c}{||\varphi_0||^2_X}||B^*\psi||^2_{X^*}||\varphi_0||^2_X \geq \frac{c}{||\varphi_0||^2_X}|\langle B^*\psi, \varphi_0\rangle|^2$$

$$= \frac{c}{||\varphi_0||^2_X}|\langle \psi, B\varphi_0\rangle|^2 = \frac{c}{||\varphi_0||^2_X}|\langle \psi, \phi\rangle|^2 = \frac{c}{||\varphi_0||^2_X}.$$

This proves

$$\inf\{|(\psi, F\psi)| : \psi \in Y^*, (\psi, \phi) = 1\} \geq \frac{c}{||\varphi_0||^2_X}.$$

Assume that $\phi \notin R(B)$. Define $V := \{\psi \in Y^* \,:\, \langle \psi, \phi\rangle = 0\}$. We want to show $B^*(V)$ is dense in $R(B^*) \subset X^*$. This is equivalent to show that $[B^*(V)]^\perp$ and $[R(B^*)]^\perp = N(B)$ coincide. Let $\varphi \in [B^*(V)]^\perp$. Then $\langle B^*\psi, \varphi\rangle = 0$ for all $\psi \in V$, which implies that $\langle \psi, B\varphi\rangle = 0$ for all $\psi \in V$, and hence that $B\varphi \in V^\perp = span\{\phi\}$. Since $\phi \notin R(B)$, this implies $B\varphi = 0$, i.e., $\varphi \in N(B)$.

By Hahn-Banach Theorem [51] one can find $\hat{\phi} \in Y^*$ with $\langle \hat{\phi}, \phi\rangle = 1$. Since $B^*(v)$ is dense in $R(B^*)$, we can choose a sequence $\{\hat{\psi_n}\}$ in V such that

$$B^*\hat{\psi}_n \to -B^*\hat{\phi} \quad as \quad n \to \infty.$$

Set $\psi_n = \hat{\psi}_n + \hat{\phi}$. Then $\langle \psi_n, \phi \rangle = \langle \hat{\psi}_n, \phi \rangle + \langle \hat{\phi}, \phi \rangle = 0 + 1 = 1$, and

$$B^* \psi_n = B^* \hat{\psi}_n + B^* \hat{\phi} \to -B^* \hat{\phi} + B^* \hat{\phi} = 0 \qquad n \to \infty.$$

Since $|\langle \psi, F\psi \rangle| = |\langle B^* \psi, AB^* \psi \rangle|$. By using the Cauchy-Schwartz inequality implies that

$$|\langle \psi_n, F\psi_n \rangle| \leq ||A|| \, ||B^* \psi_n||^2_{X^*}$$

and thus $|\langle \psi_n, F\psi_n \rangle| \to 0, \; n \to \infty$. Consequently,

$$\inf\{|(\psi, F\psi)| : \psi \in Y^*, (\psi, \phi) = 1\} = 0,$$

which is a contrapositive. $\square$

For all $z \in \mathbb{R}^n$, define $A_z \subset L^2(\mathbb{S}^{n-1})$ by

$$A_z := \{\psi \in L^2(\mathbb{S}^{n-1}) : (\psi, \phi_z) = 1\}.$$

First we consider the case of scattering by a impenetrable scatterer, as modeled by (2.11) - (2.13). This is discussed in Kirsch's book [37].

Lemma 2.4 (Lemma 1.17 of [37]) Let X be a Banach space and $A, \; A_0 : X^* \to X$ be a linear and bounded operators such that

1. $\Im\langle\varphi, A\varphi\rangle \neq 0$ for all $\varphi \in \overline{R(B^*)}$ with $\varphi \neq 0$.

2. $\langle\varphi, A_0\varphi\rangle$ is real-valued for all $\varphi \in R(B^*)$, and there exists $c_0 > 0$ such that

$$\langle\varphi, A_0\varphi\rangle \geq c_0||\varphi||^2_{X^*} \quad \text{for all } \varphi \in R(B^*),$$

3. $A - A_0$ is compact.

Then there exists $c > 0$ such that

$$|\langle\varphi, A\varphi\rangle| \geqslant c||\varphi||^2_{X^*} \quad \text{for all } \varphi \in R(B^*) \subset X^*.$$

Lemma 2.5 (Lemma 1.20 of [37]) Assume that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in Ω. For any $z \in \mathbb{R}^3$, $z \in \Omega$ if and only if

$$\inf\{|(F\psi, \psi)| : \psi \in A_z\} > 0.$$

where F is defined by (2.17) . Furthermore, for $z \in \Omega$ we have the estimate :

$$\inf\{|(F\psi, \psi)| : \psi \in A_z\} \geq \frac{c}{||\Phi(\cdot, z)||^2_{H^{\frac{1}{2}}(\partial\Omega)}}$$

for some constant $c > 0$ which is independent of z. $\Phi(., z)$ is the fundamental solution of Helmholtz equation.

Proof From Theorem 1.6, F has the following factorization

$$F = -2\pi G S^* G^*.$$

From Theorem 1.10, the middle operator S^* satisfies the conditions of Lemma 2.4. Therefore, there exists $c > 0$ such that

$$|\langle S^*\varphi, \varphi\rangle| \geqslant c||\varphi||^2_{H^{1/2}(\partial\Omega)} \quad \text{for all } \varphi \in R(G^*) \subset H^{1/2}(\partial\Omega).$$

From Theorem 1.7, $z \in \Omega$ if and only if $\Phi_\infty(\cdot,\, z) \in R(G)$.

Now, in Theorem 2.3, by choosing $Y = L^2(\mathbb{S}^2)$, $X = H^{1/2}(\partial\Omega)$, $B = G$ and $A = -2\pi S^*$, we have

$$z \in D \quad \Longleftrightarrow \quad \inf\{|(F\psi, \psi)| : \psi \in A_z\} > 0.$$

Since $\Phi_\infty = G\Phi(.,\, z)|_{\partial\Omega}$ for $z \in \Omega$, we have for $z \in D$

$$\inf\{|(F\psi, \psi)| : \psi \in A_z\} \geq \frac{c}{||\Phi(\cdot,\, z)||^2_{H^{\frac{1}{2}}(\partial\Omega)}}$$

for some constant $c > 0$ independent of z. $\square$

For the case $B(u)$ is not sound soft, the same lower bound can be obtained as in Lemma 2.5 (See Theorem 2.8 of [37]).

For the case of scattering by an inhomogeneous medium, as modeled by (2.9) - (2.10), Liu [45] established a lower bound for the scatterer points inside Ω. First we list some of the results on the factorization of the far field operator for inhomogeneous medium [37].

Assumption 2.6 [37] Let $q \in L^\infty(\mathbb{R}^n)$ satisfy

1. $\Im(q) \geq 0$ and $q = 0$ in $\mathbb{R}^n \backslash \overline{\Omega}$.
2. There exists $c_1 > 0$ such that $1 + \Re(q) \geq c_1$ for almost all $x \in \Omega$.
3. $|q|$ is locally bounded below, i.e., for every compact subset $D \subset \Omega$ there exists $c_2 > 0$ (depending on D) such that $|q| \geq c_2$ for almost all $x \in D$.
4. There exists $t \in [0, \pi]$ and $c_3 > 0$ such that $\Re[e^{-it}q(x)] \geq c_3|q|$ for almost all $x \in \Omega$.

The inner product of $L^2(\Omega)$ is denoted by $(\cdot\,,\,\cdot)_\Omega$. Define the weighted space $L^2(\Omega, |q|)$ as the completion of $L^2(\Omega)$ with respect to the norm corresponding to the inner product

$$(\phi, \psi)_{L^2(\Omega,|q|)} = \iint_\Omega \phi\overline{\psi}|q|\, dx.$$

The homogenous interior transmission problem is defined as finding the solutions $v, w \in H^2(\Omega)$ such that

$$\Delta v + k^2(1+q)v = 0 \text{ in } \Omega, \quad \Delta w + k^2 w = 0 \text{ in } \Omega$$

and

$$v = w \ \text{ on } \ \partial\Omega \quad \textit{and} \quad \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} \ \text{ on } \ \partial\Omega,$$

Let $u = v - w$. Then u vanishes on $\partial\Omega$ and $u = v - w$ satisfies the differential equation

$$\Delta u + k^2(1+q)u = -k^2\, qw \ \text{ in } \ \Omega.$$

Definition 2.7 [37] We say that k^2 is an interior transmission eigenvalue if there exists $(u, w) \in H_0^1(\Omega) \times L^2(\Omega, |q|)$ with $(u, w) \neq (0, 0)$ and a sequence $\{w_j\}$ in $H^2(\Omega)$ such that $w_j \to w$ in $L^2(\Omega, |q|)$, $\Delta w_j + k^2 w_j = 0$ in Ω, and

$$\int_\Omega [\nabla u \cdot \nabla \psi - k^2(1+q)u\psi] dx = k^2 \int_\Omega qw\psi dx \quad \text{for all } \ \psi \in H^1(\Omega).$$

Lemma 2.8 (Theorems 4.5, 4.6 and 4.8 of [37]) Assume that the conditions of Assumption 2.6 hold and F be the far field operator define by (2.17). Then

1. The far field operator F has the factorization

$$F = H^* T H,$$

where $H : L^2(\mathbb{S}^{n-1}) \to L^2(\Omega)$ is defined by

$$(Hg)(x) = \sqrt{|q(x)|} \int_{\mathbb{S}^{n-1}} g(\theta) e^{ikx\cdot\theta} ds(\theta), \qquad x \in \Omega,$$

the adjoint operator $H^* : L^2(\Omega) \to L^2(\mathbb{S}^{n-1})$ is defined by

$$(H^*\varphi)(\hat{x}) = \iint_D \varphi(y)\, e^{ik\hat{x}\cdot\theta} \sqrt{|q(y)|} ds(y), \qquad \hat{x} \in \mathbb{S}^{n-1},$$

and $T : L^2(\Omega) \to L^2(\Omega)$ is defined by

$$Tf = k^2 \Big(\frac{q}{|q|} \Big) (f + \sqrt{|q|} v_\Omega), \qquad f \in L^2(\Omega).$$

where $v \in H^1_{loc}(\mathbb{R}^n)$ is the radiating solution of

$$\Delta v + k^2(1+q)v = -k^2 \Big(\frac{q}{|q|} \Big) f \quad \text{ in } \mathbb{R}^n.$$

2. Define $T_0 : L^2(\Omega) \to L(\Omega)$ by $T_0 f = k^2 \Big(\frac{q}{|q|} \Big) f$ for $f \in L^2(\Omega)$. Then $T - T_0$ is

compact and $\Re[e^{-it}T_0]$ is coercive, i.e., there exits $c > 0$ such that

$$\Re[e^{-it}(T_0 f, f)_\Omega] \geq c||f||^2_{L^2(\Omega)}, \qquad f \in L^2(\Omega).$$

3. Assume that k^2 is not an interior transmission eigenvalue. Then

$$\Im(Tf, f)_\Omega > 0 \quad \text{for all} \quad f \in \overline{R(H)}, \quad f \neq 0.$$

Theorem 2.9 (Lemma 2.4 of [45]) Assume that the conditions of Assumption 2.6 holds and k^2 is not an interior transmission eigenvalue. Then the middle operator $T : L^2(\Omega) \to L^2(\Omega)$, which is defined in Lemma 2.8, part 1, satisfies the coercivity condition, i.e., there exists a constant $c > 0$ such that

$$|(Tf, f)_\Omega| > c||f||^2_{L^2(\Omega)}, \quad \text{for all} \quad f \in R(H).$$

Proof [45] Suppose to the contrary there is no $c > 0$ such that

$$|(Tf, f)_\Omega| > c||f||^2_{L^2(\Omega)} \quad \text{for all} \quad f \in R(H),$$

Then there exits a sequence $\{f_j\} \in R(H)$ such that

$$||f_j||_{L^2(\Omega)} = 1 \quad and \quad (Tf_j, f_j)_\Omega \to 0 \quad as \quad j \to \infty.$$

Since the unit ball in $L^2(\Omega)$ is weakly compact there exists a subsequence $\{f_j\}$ which converges weakly to some $f \in \overline{R(H)}$. From Lemma 2.8 part (2) the operator $T - T_0$ is compact, which implies

$$(T - T_0)f_j \to (T - T_0)f \quad in \quad L^2(\Omega).$$

Hence

$$((T - T_0)(f - f_j), f_j)_\Omega \to 0 \quad as \quad j \to \infty.$$

Since T is linear, we can rewrite $(Tf,\, f_j)_\Omega$ as

$$(Tf,\, f_j)_\Omega = (Tf_j,\, f_j)_\Omega + ((T - T_0)(f - f_j),\, f_j)_\Omega$$

$$+(T_0(f - f_j),\, f)_\Omega - (T_0(f - f_j),\, f - f_j)_\Omega$$

Note that the left hand side $(Tf,\, f_j)_\Omega$ converges to $(Tf,\, f)_\Omega$, while the first three terms on the right hand side converge to zero. By definition of T_0 and the assumption that $\Im(q) \geq 0$, we deduce that $\Im(T_0(f - f_j),\, f - f_j)_\Omega \geq 0$. From this fact and part 3 of Lemma 2.8, we have $f = 0$. By using part 2 of Lemma 2.8, $\Re[e^{-it}T_0]$ is coercive. Thus,

$$c||f_j||^2_\Omega \leq \Re[e^{-it}(T_0 f_j,\, f_j)_\Omega] \leq |e^{-it}(T_0 f_j,\, f_j)_\Omega|$$

$$= |(T_0 f_j,\, f_j)_\Omega| \leq |((T_0 - T)f_j,\, f_j)_\Omega| + |(Tf_j,\, f_j)_\Omega|,$$

which tends to zero as $j \to \infty$. Therefore, $f_j \to 0$, which contradicts to the assumption that $||f_j||_\Omega = 1$. $\square$

Lemma 2.10 (Theorem 4.6 of [37]) Assume that the conditions of Assumption 2.6 hold. For $z \in \mathbb{R}^3$, $z \in \Omega$ if and only if $\phi_z \in R(H^*)$, where ϕ_z is defined in (2.18).

Proof Let $z \in \Omega$. We need to show $\phi_z \in R(H^*)$. For any $\epsilon > 0$, choose a function $\chi \in C^\infty(\mathbb{R})$ with $\chi(t) = 1$ for $|t| \geq \epsilon$ and $\chi(t) = 0$ for $|t| \leq \epsilon/2$. Let $B_\epsilon(z) \subset D$ be any closed ball with center z and radius $\epsilon > 0$ that is completely contained in Ω. Define $v \in C^\infty(\mathbb{R}^3)$ by $v(x) = \chi(|x - z|)\Phi(x, z)$ *in* $\mathbb{R}^3$. Then $v = \Phi(\cdot, z)$ on $\partial\Omega$ and $\frac{\partial v}{\partial \nu} = \partial\Phi(\cdot, z)/\partial\nu$ on $\partial\Omega$ and $\Delta v + k^2 v = 0$ for $|x - z| \geq \epsilon$. From representation theorem (Theorem 2.1 of [17]) we have for $x \in \Omega$

$$v(x) = \int_{\partial\Omega} \Big\{ \Phi(x, y)\frac{\partial v(y)}{\partial \nu} - v(y)\frac{\partial \Phi(x, y)}{\partial \nu(y)} \Big\} ds(y) - \iint_\Omega \{\Delta v(y) + k^2 v(y)\}\Phi(x, y) dy$$

$$= \int_{\partial\Omega} \Big\{ \Phi(x, y)\frac{\partial \Phi(y, z)}{\partial \nu(y)} - \Phi(y, z)\frac{\partial \Phi(x, y)}{\partial \nu(y)} \Big\} ds(y)$$
$$- \iint_{|y-z|<\epsilon} \{\Delta v(y) + k^2 v(y)\}\Phi(x, y) dy$$

$$= \int_\Omega \Big\{ \Phi(x, y)\Delta\Phi(y, z) - \Phi(y, z)\Delta\Phi(x, z) \Big\} ds(y)$$
$$- \iint_{|y-z|<\epsilon} \{\Delta v(y) + k^2 v(y)\}\Phi(x, y) dy$$

$$= \int_{\Omega} \Big\{ - k^2\Phi(x,y)\Phi(y,z) + k^2\Phi(y,z)\Phi(x,z) \Big\} ds(y)$$

$$- \iint_{|y-z|<\epsilon} \{\Delta v(y) + k^2 v(y)\}\Phi(x,y)dy$$

$$= - \iint_{|y-z|<\epsilon} \{\Delta v(y) + k^2 v(y)\}\Phi(x,y)dy$$

Since $\Phi(\cdot,z)$ and v coincide outside $\overline{\Omega}$ we conclude that

$$\phi_z(\hat{x}) = v^\infty(\hat{x}) = - \iint_{|y-z|<\epsilon} \{\Delta v(y) + k^2 v(y)\} e^{-ik\hat{x}\cdot y} dy \quad \text{for } \hat{x} \in \mathbb{S}^2.$$

Set

$$w = \begin{cases} -(\Delta v + k^2 v)/\sqrt{|q|} & \text{in} \quad B_\epsilon(z); \\ 0 & \text{in} \quad \Omega\backslash\overline{B_\epsilon(z)}. \end{cases} \tag{2.19}$$

Then $w \in L^2(\Omega)$ since $|q|$ is bounded below on $B_\epsilon(z)$ and $\phi_z = H^* w$; thus $\phi_z \in R(H^*)$, which ends first part of the proof.

Let now $z \notin \Omega$ and assume that there exists $w \in L^2(\Omega)$ with $\phi_z = H^* w$ on $\mathbb{S}^2$. Then, by Rellich's Lemma (Theorem 2.13 of [17]) and the unique continuation,

$$\iint_{\Omega} w(y)\Phi(x,y)\sqrt{|q(y)|}dy = \Phi(x,z) \quad \text{for all } x \text{ in the exterior of } \Omega \cup \{z\}.$$

Note that the right hand side has singularity at $z \notin \Omega$ while the left hand side is a C^1 function in $\mathbb{R}^3$ (Lemma 4.1 of [24]), because it is a solution to the Helmholtz equation in the exterior of D. This is a contradiction, and the proof is complete. $\square$

Using Theorem 2.3 and the previous three Lemmas 2.8, 2.9 and 2.10, we formulate the following result.

Lemma 2.11 (Lemma 2.5 of [45]) Consider the scattering by inhomogeneous medium, as modeled by (2.9)- (2.10). Assume that the conditions of Assumption 2.6 hold and k^2 is not an interior transmission eigenvalue. Then $z \in \Omega$ if and only if

$$\inf\{|(F\psi, \psi)| : \psi \in A_z\} > 0.$$

Furthermore, for $z \in \Omega$ we have the estimate

$$\inf\{|(F\psi, \psi)| : \psi \in A_z\} \geq \frac{c}{||w(\cdot, z)||^2_{L^2(\Omega)}} \tag{2.20}$$

for some constant $c > 0$ which is independent of z and w is defined by (2.19).

Proof From Lemma 2.8, part 1, F has the following factorization

$$F = H^*TH.$$

Since k^2 is not an interior transmission eigenvalue, then from Theorem 2.9, there exists $c > 0$ such that

$$|(Tf,\, f)| > c||f||^2_{L^2(\Omega)} \quad \text{for all } \ \varphi \in R(H).$$

From Theorem 2.10, $z \in \Omega$ if and only if $\phi_z(\cdot,\, z) \in R(H^*)$.

Now apply Theorem 2.3 by choosing $Y = L^2(\mathbb{S}^2)$, $X = L^2(\Omega)$, $B = H^*$, and $A = T$. We have

$$z \in D \Longleftrightarrow \quad \inf\{|(\psi, F\psi)| : \psi \in A_z\} > 0.$$

Since $\phi_z = H^*w$, for $z \in D$, we have

$$\inf\{|(F\psi, \psi)| : \psi \in A_z\} \geq \frac{c}{||w(\cdot, z)||^2_{L^2(\Omega)}}$$

for some constant $c > 0$ that is independent of z. □

Note that

$$\gamma := (\phi_z, \phi_z) = \int_{\mathbb{S}^{n-1}} |\phi_z|^2 ds = \int_{\mathbb{S}^{n-1}} 1 ds = \begin{cases} 2\pi & \text{in} \quad n = 2; \\ & \\ 4\pi & \text{in} \quad n = 3. \end{cases} \tag{2.21}$$

This implies that $\psi_z := \phi_z/\gamma \in A_z$. By the linearity of the far field operator F and using the estimate (2.20) or Lemma 2.5, we have

$$
\begin{aligned}
I_{new}(z) &= |(F\phi_z, \phi_z)| \\
&= \gamma |(F\psi_z, \phi_z)| \\
&\geq \gamma \ \inf \{ |(F\psi, \psi)| : \psi \in A_z \} \\
&\geq \frac{c\gamma}{M_z}, \qquad z \in \Omega.
\end{aligned}
$$

for some constant $c > 0$ which is independent of z. Here M_z is defined by

$$
M_z = \begin{cases} ||\Phi(\cdot, z)||^2_{H^{\frac{1}{2}}(\partial\Omega)} & \text{for the scattering by impenetrable scatterers;} \\ ||w(\cdot, z)||^2_{L^2(\Omega)} & \text{for the scattering by inhomogeneous medium.} \end{cases} \tag{2.22}
$$

The main result is summarized by the following Theorem

Theorem 2.12 (Theorem 2.6 of [45]) Under the assumptions of Lemmas 2.5 and 2.11, we have

$$
I_{new}(z) \geq \frac{c\gamma}{M_z}, \qquad z \in \Omega,
$$

for some constant $c > 0$ which is independent of z. Here, M_z is defined by (2.22) and γ is defined by (2.21).

2.2.2.2 Resolution analysis for the sampling points outside the scatterer

In this subsection we study the behavior of I_{new} outside the scatterer Ω [45]. First we need to introduce the following Lemma

Lemma 2.13 (Riemann-Lebesgue Lemma [3]) If f is L^1-integrable on $\mathbb{R}^d$, i.e., if the Lebesgue integral of $|f|$ is finite, then the Fourier transform of f satisfies

$$\hat{f}(z) := \int_{\mathbb{R}^d} f(x) e^{-iz\cdot x}\, dx \to 0, \;\; as \;\; |z| \to \infty.$$

As we know the far field pattern has the following form

$$u^{\infty}(\hat{x}, \hat{\theta}) = \int_{\partial\Omega} \left\{ u^s(y, \hat{\theta}) \frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \nu(y)} - \frac{\partial u^s(y, \hat{\theta})}{\partial \nu} e^{-ik\hat{x}\cdot y} \right\} ds(y), \;\; \hat{x} \in \mathbb{S}^{n-1}$$

Substituting $u^{\infty}(\hat{x}, \hat{\theta})$ into I_{new}, yields [45]

$$\begin{aligned} I_{new}(z) &:= \left| \int_{\mathbb{S}^{n-1}} e^{-ik\hat{\theta}\cdot z} \int_{\mathbb{S}^{n-1}} u^{\infty}(\hat{x}, \hat{\theta}) e^{ik\hat{x}\cdot z} ds(\hat{x}) ds(\hat{\theta}) \right| \\ &= \left| \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \int_{\partial\Omega} \left\{ u^s(y, \hat{\theta}) \frac{\partial e^{-ik\hat{x}\cdot(y-z)}}{\partial \nu(y)} \right.\right. \\ &\qquad \left.\left. - \frac{\partial u^s(y, \hat{\theta})}{\partial \nu} e^{-ik\hat{x}\cdot(y-z)} \right\} ds(y) ds(\hat{x}) e^{-ik\hat{\theta}\cdot z} ds(\hat{\theta}) \right| \end{aligned}$$

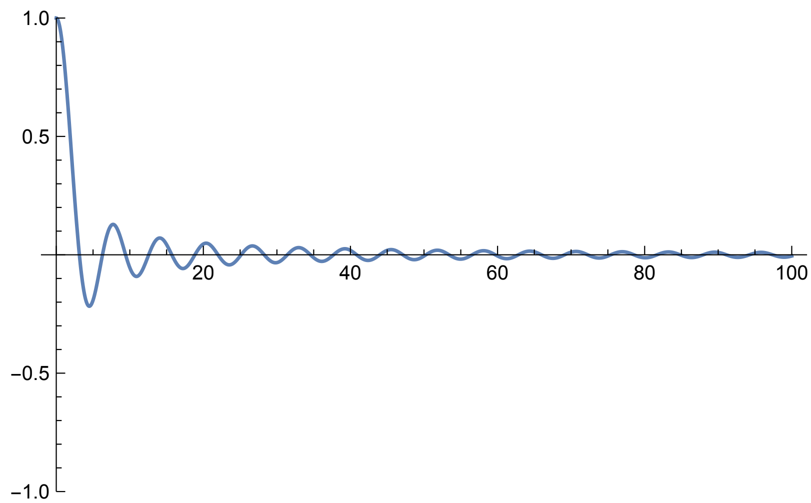

Figure 2.7: Decay behavior of Spherical Bessel function $j_0(x)$ in two dimensions.

$$
\begin{aligned}
&= \Big| \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \int_{\Omega} \Big\{ -iku^s(y,\hat{\theta})\nu(y) \cdot \hat{x} e^{-ik\hat{x}\cdot(y-z)} \\
&\qquad - \frac{\partial u^s(y,\hat{\theta})}{\partial \nu} e^{-ik\hat{x}\cdot(y-z)} \Big\} ds(\hat{x}) ds(y) e^{-ik\hat{\theta\cdot z}} ds(\hat{\theta}) \Big| \\
&:= \Big| \int_{\mathbb{S}^{n-1}} G(z,\hat{\theta}) e^{-ik\hat{\theta}\cdot z} ds(\hat{\theta}) \Big|,
\end{aligned}
$$

with

$$
\begin{aligned}
G(z,\hat{\theta}) := \int_{\partial\Omega} \Big\{ &- iku^s(y,\hat{\theta})\nu(y). \int_{\mathbb{S}^{n-1}} \hat{x} e^{-ik\hat{x}\cdot(y-z)} ds(\hat{x}) \\
&- \frac{\partial u^s(y,\hat{\theta})}{\partial \nu} \int_{\mathbb{S}^{n-1}} e^{-ik\hat{x}\cdot(y-z)} ds(\hat{x}) \Big\} ds(y).
\end{aligned} \tag{2.23}
$$

Substituting the Funk-Hecke formula onto (2.23), we get

$$
G(z,\hat{\theta}) = \int_{\partial\Omega} \Big\{ -ik\mu_1 u^s(y,\hat{\theta})\nu(y). \frac{y-z}{|y-z|} f_1(k|y-z|) - \mu_0 \frac{\partial u^s(y,\hat{\theta})}{\partial \nu} f_0(k|y-z|) \Big\} ds(y).
$$

We conclude that $G(z,\hat{\theta})$ is a superposition of the Bessel functions f_0 and f_1. The

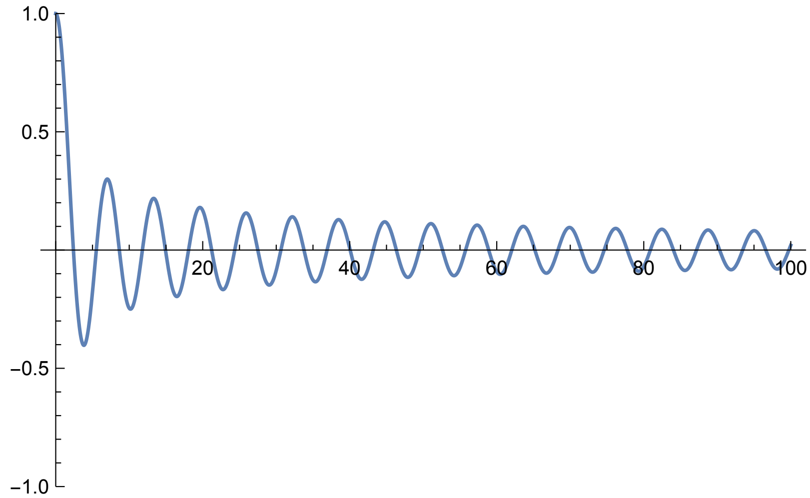

Figure 2.8: Decay behavior of Bessel function $J_0(x)$ in two dimensions.

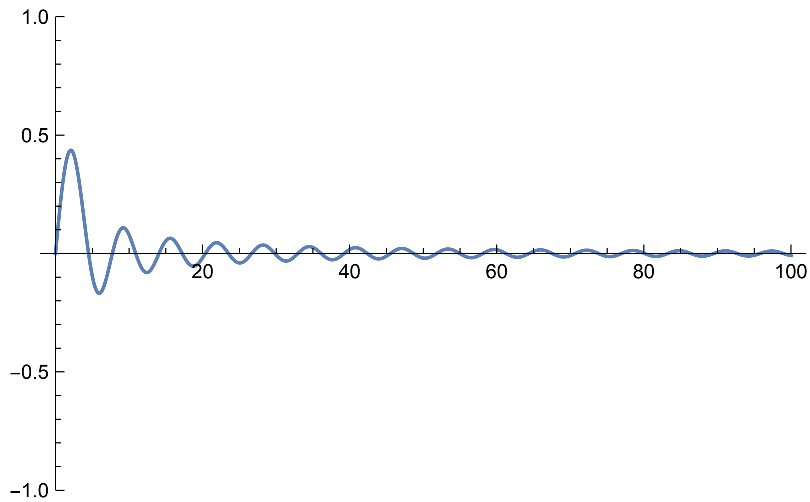

Figure 2.9: Decay behavior of Spherical Bessel function $j_1(x)$ in two dimensions.

following asymptotic formulas for the Bessel and spherical Bessel functions hold

$$j_0(t) = \frac{\sin t}{t}\Big\{1 + \mathcal{O}\Big(\frac{1}{r}\Big)\Big\},\ as\ t \to \infty,$$

$$j_1(t) = \frac{\cos t}{t}\Big\{-1 + \mathcal{O}\Big(\frac{1}{r}\Big)\Big\},\ as\ t \to \infty,$$

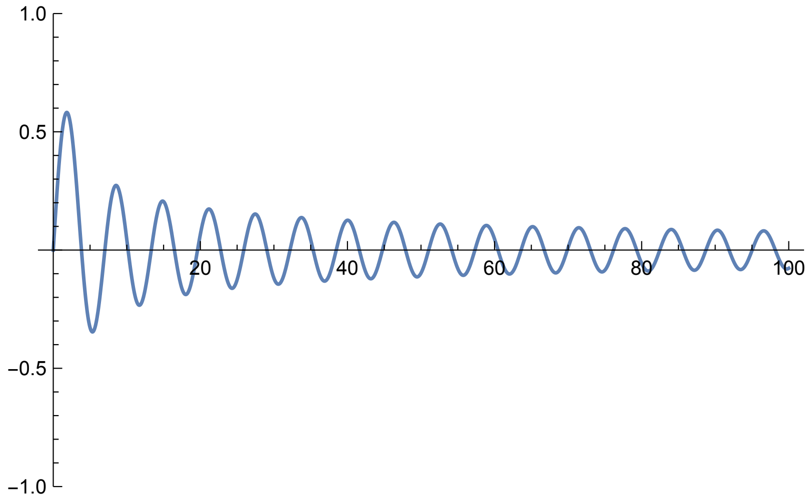

Figure 2.10: Decay behavior of Bessel function $J_1(x)$ in two dimensions.

$$J_0(t) = \frac{\sin t + \cos t}{\sqrt{\pi t}}\Big\{1 + \mathcal{O}\Big(\frac{1}{r}\Big)\Big\},\ \ as\ t \to \infty,$$

$$J_1(t) = \frac{\cos t - \sin t}{\sqrt{\pi t}}\Big\{-1 + \mathcal{O}\Big(\frac{1}{r}\Big)\Big\},\ \ as\ t \to \infty.$$

See Fig. 2.7, 2.8, 2.9 and 2.10 for the behavior of these four functions. This further implies that $G(z, \hat{\theta})$ decays as the sampling points z go away from the boundary $\partial\Omega$. By Riemann-Lebesgue Lemma, we obtain that

$$I_{new}(z) \to 0,\ \ as\ \ |z| \to \infty.$$

2.2.2.3 Stability statement

We end this section by a stability statement, which shows that the lower bound of the indicator function I_z is bounded above for all sampling points inside the scatterer.

Theorem 2.14 (Theorem 2.7 of [45]) Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 2, 3$, and denote by $BC(\Omega)$ the space of bounded continuous functions on Ω with sup norm. Then

$$||I_{new}(\cdot)||^2_{BC(\Omega)} \leq \gamma^2 \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} |u^\infty(\hat{x}, \hat{\theta})|^2 ds(\hat{x}) ds(\hat{\theta}), \tag{2.24}$$

where γ is given by (2.21).

Proof [45] By using the Cauchy-Schwartz inequality, we have, for all $z \in \Omega$

$$|I_{new}(z)|^2 := \left| \int_{\mathbb{S}^{n-1}} e^{-ik\hat{\theta}\cdot z} \int_{\mathbb{S}^{n-1}} u^\infty(\hat{x}, \hat{\theta}) e^{ik\hat{x}\cdot z} ds(\hat{x}) ds(\hat{\theta}) \right|^2$$

$$\leq \int_{\mathbb{S}^{n-1}} |e^{-ik\hat{\theta}.z}|^2 \, ds(\hat{\theta}) \quad \int_{\mathbb{S}^{n-1}} \left| \int_{\mathbb{S}^{n-1}} u^\infty(\hat{x}, \hat{\theta}) e^{ik\hat{x}\cdot z} ds(\hat{x}) \right|^2 ds(\hat{\theta})$$

$$= \int_{\mathbb{S}^{n-1}} 1 \, ds(\hat{\theta}) \quad \int_{\mathbb{S}^{n-1}} \left| \int_{\mathbb{S}^{n-1}} u^\infty(\hat{x}, \hat{\theta}) e^{ik\hat{x}\cdot z} ds(\hat{x}) \right|^2 ds(\hat{\theta})$$

$$\leq \gamma \int_{\mathbb{S}^{n-1}} \quad \int_{\mathbb{S}^{n-1}} |u^\infty(\hat{x}, \hat{\theta})|^2 ds(\hat{x}) \int_{\mathbb{S}^{n-1}} |e^{ik\hat{x}\cdot z}|^2 \, ds(\hat{x}) \quad ds(\hat{\theta})$$

$$= \gamma \int_{\mathbb{S}^{n-1}} \quad \int_{\mathbb{S}^{n-1}} |u^\infty(\hat{x}, \hat{\theta})|^2 ds(\hat{x}) \int_{\mathbb{S}^{n-1}} 1 \, ds(\hat{x}) \quad ds(\hat{\theta})$$

$$= \gamma^2 \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} |u^\infty(\hat{x}, \hat{\theta})|^2 ds(\hat{x}) ds(\hat{\theta}),$$

We have used the fact that $|e^{-ik\hat{\theta}\cdot z}| = |e^{ik\hat{x}\cdot z}| = 1$ for all $\hat{x}$, $\hat{\theta} \in \mathbb{S}^{n-1}$ and $z \in \Omega$. $\square$

2.2.3 The Relation Between I_{new} and $\mu[MD]$

It is shown that there is a relation between I_{new} and $\mu[Md]$ when $\rho = 2$, which is discussed in Section 2.1 of this chapter. To find this relation, we need the following lemma.

Lemma 2.14 (Lemma 2.8 of [45])

1. The far field pattern satisfies the reciprocity relation

$$u^\infty(\hat{x}, \hat{\theta}) = u^\infty(-\hat{\theta}, -\hat{x}) \quad for\ all\ \ \hat{x}, \hat{\theta} \in \mathbb{S}^{n-1}.$$

2. The far field operator satisfies

$$F - F^* - \frac{i}{4\pi}\left(\frac{k}{2\pi}\right)^{n-2} F^* F = 2iR, \tag{2.25}$$

where F^* is the L^2-adjoint of F and $R : L^2(\mathbb{S}^{n-1}) \longrightarrow L^2(\mathbb{S}^{n-1})$ is a self-adjoint nonnegative operator. The operator R vanishes for the cases of Dirichlet or Neumann boundary conditions. For the impedance boundary conditions, the operator R is given by

$$Rh(\hat{x}) := \int_{\mathbb{S}^{n-1}} \Big(\int_{\partial\Omega} \Im(\lambda) u(y,\hat{\theta}) \overline{u(y,\hat{x})} ds(y) \Big) h(\hat{\theta}) ds(\hat{\theta}), \quad \hat{x} \in \mathbb{S}^{n-1}.$$

For the case of inhomogeneous medium, the operator R is given by

$$Rh(\hat{x}) := \int_{\mathbb{S}^{n-1}} \Big(\int_{\partial\Omega} k^2 \Im(\lambda) u(y,\hat{\theta}) \overline{u(y,\hat{x})} ds(y) \Big) h(\hat{\theta}) ds(\hat{\theta}), \quad \hat{x} \in \mathbb{S}^{n-1},$$

where $u(\cdot,\hat{\theta})$ is the total field in Ω corresponding to the incident plane wave $u^i(\cdot,\hat{\theta})$ with incident direction $\hat{\theta}$.

Proof See Theorems 1.8, 2.5 and 4.4 of [37].

By interchanging the roles of $\hat{x}$ and $\hat{\theta}$ and using reciprocity relation in the previous lemma, we have, for $\rho = 2$, [45]

$$\begin{aligned}
\mu[MD](z) &= \int_{\mathbb{S}^{n-1}} \left| \int_{\mathbb{S}^{n-1}} u^\infty(\hat{x},\hat{\theta})\, e^{ik\hat{x}\cdot z} ds(\hat{x}) \right|^2 ds(\hat{\theta}) \\
&= \int_{\mathbb{S}^{n-1}} \left| \int_{\mathbb{S}^{n-1}} u^\infty(-\hat{x},-\hat{\theta})\, e^{-ik\hat{x}\cdot z} ds(\hat{x}) \right|^2 ds(\hat{\theta}) \\
&= \int_{\mathbb{S}^{n-1}} \left| \int_{\mathbb{S}^{n-1}} u^\infty(\hat{\theta},\hat{x})\, e^{-ik\hat{x}\cdot z} ds(\hat{x}) \right|^2 ds(\hat{\theta}) \\
&= \int_{\mathbb{S}^{n-1}} \left| \int_{\mathbb{S}^{n-1}} u^\infty(\hat{x},\hat{\theta})\, e^{-ik\hat{\theta}\cdot z} ds(\hat{\theta}) \right|^2 ds(\hat{x})
\end{aligned}$$

$$= ||F\phi_z||^2_{L^2(\mathbb{S}^{n-1})}.$$

Since $F - F^* = 2i\Im(F)$, we can rewrite (2.25) as

$$2i\Im(F) = 2iR + \frac{i}{4\pi}\left(\frac{k}{2\pi}\right)^{n-2} F^*F.$$

From this, we have

$$\Im(Fg,\, g) = (Rg,\, g) + \frac{1}{8\pi}\left(\frac{k}{2\pi}\right)^{n-2}(Fg,\, Fg).$$

Taking $g = \phi_z$, we get

$$|(F\phi_z, \phi_z)| \geq \Im(F\phi_z,\, \phi_z) = (R\phi_z,\, \phi_z) + \frac{1}{8\pi}\left(\frac{k}{2\pi}\right)^{n-2}(F\phi_z,\, F\phi_z).$$

Since R is a nonnegative operator, we have

$$I_{new} = |(F\phi_z, \phi_z)| \geq \frac{1}{8\pi}\left(\frac{k}{2\pi}\right)^{n-2} ||F\phi_z||^2_{L^2(\mathbb{S}^{n-1})}. \tag{2.26}$$

By using the Cauchy-Schwartz inequality, we have

$$|(F\phi_z, \phi_z)|^2 \leq ||F\phi_z||^2_{L^2(\mathbb{S}^{n-1})}||\phi_z||^2_{L^2(\mathbb{S}^{n-1})} = 2^{n-1}\pi||F\phi_z||^2_{L^2(\mathbb{S}^{n-1})}. \tag{2.27}$$

Combination of the previous two inequalities (2.26) and (2.27) yields

$$\frac{1}{8\pi}\left(\frac{k}{2\pi}\right)^{n-2}||F\phi_z||^2_{L^2(\mathbb{S}^{n-1})} \leq |(F\phi_z, \phi_z)| \leq \sqrt{\pi}2^{\frac{n-1}{2}}||F\phi_z||_{L^2(\mathbb{S}^{n-1})},$$

$$\frac{1}{8\pi}\left(\frac{k}{2\pi}\right)^{n-2}||F\phi_z||^2_{L^2(\mathbb{S}^{n-1})} \leq I_{new}(z) \leq \sqrt{\pi}2^{\frac{n-1}{2}}||F\phi_z||_{L^2(\mathbb{S}^{n-1})}. \tag{2.28}$$

The inequalities (2.28) and the results given in the previous two subsections show why Potthast's reconstruction scheme by using the indicator $\mu[MD]$ for $\rho = 2$ works for shape reconstruction in inverse acoustic scattering problems.

Chapter 3

New Sampling Method for Shape Reconstruction in Inverse Electromagnetic Scattering Problems

3.1 Introduction

In this chapter, we propose a new direct sampling method for inverse electromagnetic scattering problems. We generalize Liu's method, which was discussed in Chapter

2 for inverse acoustic scattering problems to the case of electromagnetic scattering problems in isotropic and source free media. In this method we will propose an indicator function which is big when the sampling point lies inside the scatterer and when the sampling point moves away from the boundary of the scatterer the value of the indicator function decays and goes to zero. The method is very simple to implement since only the inner products of the measurements with some suitably chosen functions are involved in computation of the indicator function.

We consider electromagnetic wave [37] propagation in an inhomogeneous isotropic medium in $\mathbb{R}^3$ with electric permittivity $\epsilon = \epsilon(x) > 0$, constant magnetic permeability $\mu = \mu_0$, and electric conductivity $\sigma = \sigma(x)$. We assume that $\epsilon(x) = \epsilon_0$, where ϵ_0 is constant, and $\sigma(x) = 0$ for all x outside some sufficiently large ball. Let $k = \omega\sqrt{\epsilon_0\mu_0} > 0$ be the wave number with frequency ω. An incident electromagnetic field consists of a pair H^i and E^i which satisfy the time harmonic Maxwell system in vacuum, i.e.,

$$curl\, E^i - i\omega\mu_0 H^i = 0 \;\; in \;\; \mathbb{R}^3, \tag{3.1}$$

$$curl\, H^i + i\omega\epsilon_0 E^i = 0 \;\; in \;\; \mathbb{R}^3. \tag{3.2}$$

The total fields are superpositions of the incident and scattered fields, i.e., $E = E^i + E^s$

and $H = H^i + H^s$, and satisfy

$$curl\, E - i\omega\mu_0 H = 0 \;\; in \;\; \mathbb{R}^3, \tag{3.3}$$

$$curl\, H + i\omega\epsilon E = \sigma E \;\; in \;\; \mathbb{R}^3. \tag{3.4}$$

The tangential components of E and H are continuous on the interfaces where σ and ϵ are discontinuous. The scattered field E^s, H^s satisfies the Silver-Muller radiation condition

$$\sqrt{\frac{\mu_0}{\epsilon_0}}\, H^s(x) \times x - |x| E^s(x) = \mathcal{O}\left(\frac{1}{|x|}\right) \quad as\ |x| \to \infty \tag{3.5}$$

uniformly for all directions $\hat{x} = \frac{x}{|x|}$.

As seen from (3.3) the magnetic field is divergence free, i.e., $div(H) = 0$. So we will always work with the magnetic field H only. In general this is not the case for the electric field E. Uniqueness and existence of the scattering problems (3.1) - (3.5) is shown in chapter 9 of Colton and Kress 's book [17].

From (3.4), we have $E = \frac{1}{\sigma - i\omega\epsilon} curl H$. Substituting in (3.3) yields

$$curl\left[\frac{1}{\sigma - i\omega\epsilon} curl\ H\right] - i\omega\mu_0 H = 0. \tag{3.6}$$

Let ϵ_r denote the (complex-valued) relative permittivity

$$\epsilon_r(x) = \frac{\epsilon(x)}{\epsilon_0} + i\frac{\sigma(x)}{\omega\epsilon_0}.$$

Then (3.6) can be written as

$$curl\left[\frac{1}{\epsilon_r} curl\ H\right] - k^2 H = 0 \ \ in \ \ \mathbb{R}^3, \tag{3.7}$$

where again $k = \omega\sqrt{\epsilon_0\mu_0}$. Then the incident field H^i satisfies

$$curl^2 H^i - k^2 H^i = 0 \ \text{ in } \ \mathbb{R}^3. \tag{3.8}$$

The Silver-Muller radiation condition becomes

$$curl\, H^s(x) \times \hat{x} - ikH^s(x) = \mathcal{O}\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty. \tag{3.9}$$

Let $D \subset \mathbb{R}^3$ be open and bounded such that ∂D is C^2 and the complement $\mathbb{R}^3 \backslash \overline{D}$ is connected. Let $\epsilon_r \in L^\infty(D)$ satisfying $\Im(\epsilon_r) \geq 0 \ in \ D$, and $\epsilon_r = 1$ in $\mathbb{R}^3 \backslash \overline{D}$. We consider the special case where the incident waves H^i and E^i [17], [37] are

$$H^i(x,\theta;p) = p\, e^{ik\theta\cdot x} \ \ and \ \ E^i(x,\theta;p) = -\frac{1}{i\omega\epsilon_0} curl\, H^i(x,\theta;p) = -\sqrt{\frac{\mu_0}{\epsilon_0}}(\theta \times p)\, e^{ik\theta\cdot x}, \tag{3.10}$$

where $\theta \in \mathbb{S}^2$ is a unit vector giving the direction of incidence and $p \in \mathbb{C}^3$ is a constant vector giving the polarization. We assume that $p \cdot \theta = \sum_{j=1}^{3} p_j \theta_j = 0$ in order to ensure that H^i and E^i are divergence free.

Every radiating solution of the Maxwell equation has the following asymptotic form [17]

$$H^s(x,\theta;p) = \frac{e^{ik|x|}}{4\pi|x|}\left\{H^\infty(\hat{x},\theta;p) + \mathcal{O}\left(\frac{1}{|x|^2}\right)\right\} \quad as\ |x| \to \infty, \tag{3.11}$$

$$E^s(x,\theta;p) = \frac{e^{ik|x|}}{4\pi|x|}\left\{H^\infty(\hat{x},\theta;p) \times \hat{x} + \mathcal{O}\left(\frac{1}{|x|^2}\right)\right\} \quad as\ |x| \to \infty,$$

uniformly in all directions $\hat{x} = \frac{x}{|x|}$ where the vectors field H^∞ and E^∞ are defined on the unit sphere S^2 and are known as the magnetic and the electric far field patterns, respectively. H^∞ and E^∞ are tangential vector fields, i.e., $H^\infty(\hat{x},\theta;p) \cdot \hat{x} = 0$ and $E^\infty(\hat{x},\theta;p) \cdot \hat{x} = 0$ for all $\hat{x} \in S^2$ and all $\theta \in S^2$ and $p \in \mathbb{C}^3$ with $p \cdot \theta = 0$. Since $E^\infty(\hat{x},\theta;p) = H^\infty(\hat{x},\theta;p) \times \hat{x}$, it is sufficient to work only with one far field pattern, H^∞. The far field pattern depends on p linearly, i.e, we can write $H^\infty(\hat{x},\theta;p) = H^\infty(\hat{x},\theta)p$ for all $p \in \mathbb{C}^3$ with $p \cdot \theta = 0$, where $H^\infty(\hat{x},\theta) \in \mathbb{C}^{3\times 3}$ is a matrix.

Let $q(x) = 1 - 1/\epsilon_r(x)$. Then the function $q(x)$ vanishes outside the inhomogeneous medium, i.e., $q(x) = 0$ in $\mathbb{R}^3 \backslash \overline{D}$. The inverse problem we consider in this chapter is to determine the support $\overline{D}$ of q from a knowledge of the far field pattern $H^\infty(\hat{x},\theta;p)$ for all $\hat{x}$, $\theta \in S^2$ and all $p \in \mathbb{C}^3$ with $p \cdot \theta = 0$. We consider two cases of the inhomogeneous medium D. First, we will consider the case where D is absorbing everywhere with $\Im(\epsilon_r) > 0$ on D. Second case, we consider the general case where only parts of D may be absorbing, i.e., we allow general values for ϵ.

In this chapter, we propose a new direct sampling method using the indicator

$$I_z := k^2 \left| \int_{\mathbb{S}^2} (\hat{\theta} \times p)\, e^{-ik\theta \cdot z} \cdot \int_{\mathbb{S}^2} H^\infty(\hat{x}, \hat{\theta})(\hat{x} \times p)\, e^{ik\hat{x}\cdot z} ds(\hat{x}) ds(\hat{\theta}) \right|, \;\; z \in \mathbb{R}^3, \quad (3.12)$$

where, $\hat{x} \in S^2$, $\theta \in S^2$ and $p \in \mathbb{C}^3$. The theoretical foundation of the proposed reconstruction scheme will be established in the next section. By using the factorization of the far field operator which discussed in [37], we show a lower bound of the indicator I_z for the sampling points inside the scatterer.

3.2 Theoretical Foundation of the Proposed Sampling Method

The aim of this section is to establish the mathematical basis of our sampling method. We introduce the subspace $L^2_t(\mathbb{S}^3)$ of $L^2(\mathbb{R}^3, \mathbb{C}^3)$ consisting of all tangential fields on the unit sphere [37], i.e.,

$$L_t^2(\mathbb{S}^2) := \Big\{g : \mathbb{S}^2 \to \mathbb{C}^3 : g \in L^2(\mathbb{S}^2),\ g(\hat{x}) \cdot \hat{x} = 0,\ \hat{x} \in \mathbb{S}^2\Big\}.$$

$F : L_t^2(\mathbb{S}^2) \to L_t^2(\mathbb{S}^2)$ is the far field operator defined as

$$Fp(\hat{x}) := \int_{\mathbb{S}^2} H^\infty(\hat{x}, \hat{\theta}; p(\hat{\theta}))\, ds(\hat{\theta}) = \int_{\mathbb{S}^2} H^\infty(\hat{x}, \hat{\theta})\, p(\hat{\theta})\, ds(\hat{\theta}), \quad \hat{x} \in \mathbb{S}^2. \tag{3.13}$$

For all sampling point $z \in \mathbb{R}^3$, define a test function $\phi_z \in L_t^2(\mathbb{S}^2)$ as

$$\phi_z(\vartheta) := ik(\vartheta \times p)\, e^{-ikz\cdot\vartheta}, \quad \vartheta \in \mathbb{S}^2. \tag{3.14}$$

We can rewrite the indicator function I_z, which is given by (3.12), in the form

$$I_z := |(F\phi_z, \phi_z)|, \quad z \in \mathbb{R}^3. \tag{3.15}$$

Here, we denote by $(\,\cdot\,,\cdot\,)$ the inner product of $L_t^2(\mathbb{S}^2)$. Define the spaces [37]

$$H_0(curl, D) = \{v \in H(curl, D) : \nu \times v = 0 \quad \text{on} \ \ \partial D\},$$

$$H_{loc}(curl, \mathbb{R}^3) = \Big\{v : \mathbb{R}^3 \longrightarrow \mathbb{C}^3 : v|_D \in H(curl, D), \ \text{ where } \ D \subset \mathbb{R}^3\Big\},$$

where

$$H(curl, D) = \Big\{v \in L^2(D, \mathbb{C}^3) : curl\, v \ \in L^2(D, \mathbb{C}^3)\Big\}.$$

The electromagnetic interior transmission problem is defined as finding the solutions $v,\ w \in H(curl, D)$ such that

$$curl\left[\tfrac{1}{\epsilon_r} curl\ v\right] - k^2 v = 0 \quad \text{in} \quad D, \qquad\qquad curl^2 w - k^2 w = 0 \quad \text{in} \quad D,$$

$$\nu \times v = \nu \times w \quad on \quad \partial D \qquad \text{and} \qquad \tfrac{1}{\epsilon_r}\, \nu \times curl\, v = \nu \times curl\ w \quad \text{on} \quad \partial D.$$

Let $u = v - w$. Then u and w satisfy

$$curl\left[\tfrac{1}{\epsilon_r} curl\ u\right] - k^2 u = curl[q\ curl\ w] \quad \text{in} \quad D, \qquad\qquad curl^2 w - k^2 w = 0 \quad in \quad D,$$

$$\nu \times u = 0 \quad on \quad \partial D \qquad and \qquad \tfrac{1}{\epsilon_r}\, \nu \times curl\, u = q\nu \times curl\ w \quad \text{on} \ \partial D.$$

We say that the wave number k^2 is an interior transmission eigenvalue [37] if there exists a non-vanishing pair $(u, w) \in H_0(curl, D) \times L^2(D, \mathbb{C}^3, |q|)$ and a sequence $\{w_i\}$ in $H(curl, D)$ with $w_j \to w$ in $L^2(D, \mathbb{C}^3, |q|)$ such that

$$curl\Big[\frac{1}{\epsilon_r}\, curl\, u\Big] - k^2 u = curl\,[\,qw\,] \ \ \text{in} \ \ D, \quad curl^2 w_j - k^2 w_j = 0 \ \ in \ \ D,$$

and

$$\frac{1}{\epsilon_r}\, \nu \times curl\ u = q\nu \times w \quad \text{on} \quad \partial D,$$

where $L^2(D, \mathbb{C}^3, |q|)$ denotes the weighted L^2-space of vector fields on D.

3.2.1 Resolution Analysis for the Sampling Points Inside the Scatterer

For all $z \in \mathbb{R}^3$, define $A_z \subset L_t^2(\mathbb{S}^2)$ by

$$A_z := \{\psi \in L_t^2(\mathbb{S}^2) : (\psi, \phi_z) = 1\},$$

where ϕ_z is the test function given by (3.14). To establish a lower bound for the indicator function inside the scatterers, we first make the following general assumptions on the ϵ_r.

Assumption 1 [37] Let $\epsilon_r \in L^\infty(D)$ satisfy

1. $\Im(\epsilon_r) \geq 0$ in D and $\epsilon_r = 1$ in $\mathbb{R}^3 \backslash \overline{D}$.

2. There exists $c_1 > 0$ with $\Re(\epsilon_r) \geq c_1$ on D.

3. For all $f \in L^2(\mathbb{R}^3, \mathbb{C}^3)$ with compact support there exists a unique radiating solution v of $\;curl\left[\frac{1}{\epsilon_r} curl\ v\right] - k^2 v = curl\, f \;\; in \;\; \mathbb{R}^3.$

4. $|\epsilon_r - 1|$ is locally bounded below, i.e., for every compact subset $M \subset D$, there exists $c > 0$ (depending on M) with $|\epsilon_r - 1| \geq c$ for almost all $x \in M$.

Lemma 3.1 (Theorems 5.10, 5.11, 5.12 , 5.15 of [37])

Assume that the conditions of Assumption 1 hold. Let $T : L^2(D, \mathbb{C}^3) \to L^2(D, \mathbb{C}^3)$ be given by

$$Tf = k^2(sign\, q)\Big[f + \sqrt{|q|}\; curl\, v\Big],$$

where the contrast is $q = 1 - 1/\epsilon_r$, $sign\, q := q/|q|$ and $v \in H_{loc}(curl, \mathbb{R}^n)$ is the radiating solution of $curl\left[\frac{1}{\epsilon_r} curl\ v\right] - k^2 v = curl\left[\frac{q}{\sqrt{q}} f\right]$ in $\mathbb{R}^3$. Then we have

1. Let F be the far field operator defined by (3.13) and $H : L^2_t(\mathbb{S}^2) \to L^2(D, \mathbb{C}^3)$ defined by

   $$(Hp)(x) = \sqrt{|q(x)|}\, curl \int_{\mathbb{S}^2} p(\theta) e^{ikx\cdot\theta} ds(\theta), \quad x \in D.$$

   Then F has the factorization

   $$F = H^* T H,$$

   where $H^* : L^2(D, \mathbb{C}^3) \to L^2_t(\mathbb{S}^2)$ denotes the adjoint of H, which is given by

   $$(H^*\varphi)(x) = ik\hat{x} \times \iint_D \varphi(y) e^{-ik\hat{x}\cdot y} \sqrt{|q(y)|}\; dy, \quad \hat{x} \in S^2.$$

2. For any $\epsilon > 0$. Choose a function $\chi \in C^\infty(\mathbb{R})$ with $\chi(t) = 1$ for $|t| \geq \epsilon$ and $\chi(t) = 0$ for $|t| \leq \epsilon/2$. Let $B_\epsilon(z) \subset D$ be any closed ball with center z

and radius $\epsilon > 0$ that is completely contained in D. Define $w_0 \in C^\infty(\mathbb{R}^3)$ by $w_0 = \chi(|x-z|)\Phi(x,z)$ in $\mathbb{R}^3$, where $\Phi(x,z) = \frac{1}{4\pi}\frac{e^{ik|x-z|}}{|x-z|}$, $x \neq z$. Set

$$w = \begin{cases} -p(\Delta w_0 + k^2 w_0)/\sqrt{|q|} & \text{in} \quad B_\epsilon(z); \\ 0 & \text{in} \quad D \backslash \overline{B_\epsilon(z)}. \end{cases} \tag{3.16}$$

Then $w \in L^2(D, \mathbb{C}^3)$ and $\phi_z = H^* w$ where ϕ_z is defined in (3.14).

3. Assume that k^2 is not an eigenvalue of the interior transmission eigenvalue problem. Then $\Im(Tf, f)_{L^2(D)} > 0$ for all $f \in \overline{R(H)} \subset L^2(D, \mathbb{C}^3)$ with $f \neq 0$.

   Here $\overline{R(H)}$ denotes the closure of $R(H)$ in $L^2(D, \mathbb{C}^3)$.

4. Assume that there exists a constant $\gamma_0 > 0$ such that $\Im(q) \geq \gamma_0 |q|$ almost everywhere in D. Then there exists $\gamma_1 > 0$ such that $\Im(Tf,\ f)_{L^2(D)} \geq \gamma_1 ||f||^2_{L^2(D)}$ for all $f \in L^2(D, \mathbb{C}^3)$.

5. Define the operator T_0 from $L^2(D, \mathbb{C}^3)$ to itself by $T_0 f = (sign\, q) f$ for $f \in L^2(D, \mathbb{C}^3)$. Then $T - T_0$ is compact in $L^2(D, \mathbb{C}^3)$.

6. Assume that there exists $r > 0$ such that

$$\left| \epsilon_r(x) - \frac{1}{2}(1 - ri) \right| \geq \frac{\sqrt{1+r^2}}{2} \quad \text{for almost all } x \in D.$$

   Choose $t \in (0, 2\pi)$ such that $\cos t \leq 1/\sqrt{1+r^2}$. Then $\Re[e^{-it}\, T_0]$ is coercive, i.e., there exists $c > 0$ such that

$$\Re[e^{-it}(T_0 f,\, f)_{L^2(D)}] \geq c\,||f||^2_{L^2(D)}, \quad f \in L^2(D).$$

Lemma 3.2 Assume that the conditions of Assumption 1 hold and there exists a constant $\gamma_0 > 0$ such that $\Im(q) \geq \gamma_0|q|$ almost everywhere in D. Then the middle operator $T : L^2(D, \mathbb{C}^3) \to L^2(D, \mathbb{C}^3)$ which is defined in Lemma 3.1 satisfies the coercivity condition, i.e., there exists a constant $c > 0$ such that

$$|(Tf, f)_{L^2(D)}| \geq c||f||^2_{L^2(D)}, \text{ for all } f \in R(H) \subset L^2(D, \mathbb{C}^3).$$

Proof. From previous lemma part 4, we can find a constant $\gamma_1 > 0$ such that

$$\Im(Tf,\, f)_{L^2(D)} \geq \gamma_1||f||^2_{L^2(D)} \text{ for all } f \in L^2(D, \mathbb{C}^3).$$

Since $R(H) \subset L^2(D, \mathbb{C}^3)$, we can find a constant $c > 0$ such that

$$\Im(Tf,\, f)_{L^2(D)} \geq c||f||^2_{L^2(D)} \text{ for all } f \in R(H) \subset L^2(D, \mathbb{C}^3).$$

Since $|(Tf, f)_{L^2(D)}| \geq \Im(Tf,\, f)_{L^2(D)}$, we have

$$|(Tf,\, f)_{L^2(D)}| \geq c||f||^2_{L^2(D)} \text{ for all } f \in R(H) \subset L^2(D, \mathbb{C}^3). \qquad \square$$

Lemma 3.3 Assume that the conditions of Assumption 1 hold and there exists $r > 0$ such that

$$\left|\epsilon_r(x) - \frac{1}{2}(1 - ri)\right| \geq \frac{\sqrt{1+r^2}}{2} \text{ for almost all } \quad x \in D.$$

Furthermore, assume that k^2 is not an interior transmission eigenvalue. Then the middle operator $T : L^2(D, \mathbb{C}^3) \to L^2(D, \mathbb{C}^3)$ which is defined in Lemma 3.1 satisfies the coercivity condition, i.e., there exists a constant $c > 0$ such that

$$|(Tf, f)_{L^2(D)}| > c||f||^2_{L^2(D)}, \text{ for all } f \in R(H).$$

Proof. Suppose to the contrary there is no c such that $|(Tf, f)_{L^2(D)}| > c||f||^2_{L^2(D)}$ for all $f \in R(H)$. Then there exists a sequence $\{f_j\} \in R(H)$ such that

$$||f_j||_{L^2(D)} = 1 \quad \text{and} \quad (Tf_j, f_j)_{L^2(D)} \to 0 \text{ as } j \to \infty.$$

Since the unit ball in $L^2(D)$ is weakly compact, there exists a subsequence $\{f_j\}$ which converges weakly to some $f \in \overline{R(H)}$. From part 5 of Lemma 3.1 the operator $T - T_0$ is compact, which implies that

$$(T - T_0)f_j \to (T - T_0)f \quad \text{in} \quad L^2(D, \mathbb{C}^3).$$

Thus

$$((T - T_0)(f - f_j), f_j)_{L^2(D)} \to 0 \quad \text{as} \quad j \to \infty.$$

Since T is linear, we can rewrite $(Tf,\, f_j)_{L^2(D)}$ as

$$(Tf,\, f_j)_{L^2(D)} = (Tf_j,\, f_j)_{L^2(D)} + ((T - T_0)(f - f_j),\, f_j)_{L^2(D)}$$

$$+T_0(f - f_j,\, f)_{L^2(D)} - (T_0(f - f_j),\, f - f_j)_{L^2(D)}$$

The left hand side converges to $(Tf,\, f)_{L^2(D)}$. The first three terms on the right hand side converge to zero. Since by assumption $\Im(\epsilon_r) \geq 0$, $q = 1 - 1/\epsilon_r = 1 - \overline{\epsilon_r}/|\epsilon_r|^2$, so $\Im(q) = \Im\,[\epsilon_r/|\epsilon_r|^2]$. Hence $\Im(q) \geq 0$ and we deduce that $\Im(T_0(f - f_j),\, f - f_j)_{L^2(D)} \geq 0$. From this fact and part 3 of Lemma 3.1, we have $f = 0$. Since from part 6 of Lemma 3.1, $\Re[e^{-it}T_0]$ is coercive, we have

$$c||f_j||^2_{L^2(D)} \leq \Re[e^{-it}(T_0 f_j,\, f_j)_{L^2(D)}] \leq |e^{-it}(T_0 f_j,\, f_j)_{L^2(D)}|$$

$$= |(T_0 f_j,\, f_j)_{L^2(D)}| \leq |((T_0 - T)f_j,\, f_j)_{L^2(D)}| + |(Tf_j,\, f_j)_{L^2(D)}|,$$

which tends to zero as $j \to \infty$. Therefore, $f \to 0$ which contradicts to the assumption that $|f_j||_{L^2(D)} = 1$. □

Theorem 3.4 (Theorem 5.11 of [37]) Assume that the conditions of Assumption 1 hold. Then $z \in D$ if and only if $\phi_z \in R(H^*)$, where the adjoint $H^* : L^2(D, \mathbb{C}^3) \to$

$L_t^2(\mathbb{S}^2)$ of H is given by Lemma 3.1 (part 1) and ϕ_z is given by (3.14).

After these preparations we are able to give a characterization of the support $\overline{D}$ of q where all of D is absorbing, i.e, $\Im(\epsilon_r) > 0$ on D, and for more general case where only parts of D may be absorbing, i.e., we allow quite general values of ϵ. For both cases we will give a lower bound of the proposed indicator function for sampling points inside the scatters.

We formulate and prove the first result of this chapter in which we treat the absorbing medium.

Lemma 3.5 Consider the inverse scattering by an inhomogeneous medium. Assume that the conditions of Assumption 1 hold and there exists $\gamma_0 > 0$ such that $\Im(\epsilon_0(x)) \geq \gamma_0$ for almost all $x \in D$. Then $z \in D$ if and only if

$$\inf\{|(F\psi, \psi)| : \psi \in A_z\} > 0.$$

Furthermore, for $z \in D$ we have the estimate

$$\inf\{|(F\psi, \psi)| : \psi \in A_z\} \geq \frac{c}{||w(\cdot, z)||^2_{L^2(D)}} \tag{3.17}$$

for some constant $c > 0$ which is independent of z. Here w is defined by (3.16).

Proof Since $q = 1 - 1/\epsilon_r = 1 - \overline{\epsilon_r}/|\epsilon_r|^2$, $\Im(q) = \Im\left[\epsilon_r/|\epsilon_r|^2\right]$. By our assumption on ϵ_r, we conclude $\Im(q) = \Im(\epsilon_r)/|\epsilon_r|^2 \geq \gamma_0/||\epsilon_r||_\infty^2$. Hence the assumption of part 4 in Lemma 3.1 is satisfied. By using Lemma 3.2, the middle operator $T : L^2(D, \mathbb{C}^3) \to L^2(D, \mathbb{C}^3)$ satisfies the coercivity condition, i.e., there exists a constant $c > 0$ such that

$$|(Tf, f)_{L^2(D)}| \geq c||f||^2_{L^2(D)}, \text{ for all } \quad f \in R(H) \subset L^2(D, \mathbb{C}^3).$$

Let F be the far field operator, which is defined by (3.13), and $H : L^2_t(\mathbb{S}^2) \to L^2(D, \mathbb{C}^3)$, which is defined in Lemma 3.1. Since Assumption 1 is satisfied, then by using Lemma 3.1 (part 1), F has the factorization $F = H^*TH$.

Let w be defined by (3.16). Then by using Lemma 3.1 (part 2), $w \in L^2(D, \mathbb{C}^3)$ and $\phi_z = H^*w$, where ϕ_z is defined in (3.14). In Theorem 2.3, choose $B^* = H$, $A = T$, $Y = L^2_t(\mathbb{S}^2)$, and $X = L^2(D, \mathbb{C}^3)$. Then $\phi_z \in R(H^*)$ if and only if

$$\inf\{|(F\psi, \psi)| : \psi \in A_z\} > 0.$$

Furthermore, we have the estimate

$$\inf\{|(F\psi, \psi)| : \psi \in A_z\} \geq \frac{c}{||w(\cdot, z)||^2_{L^2(D)}}$$

for some constant $c > 0$ which is independent of z. Now by using Theorem 3.4, $z \in D$ if and only if $\phi_z \in R(H^*)$. So $z \in D$ if and only if

$$\inf\{|(F\psi, \psi)| : \psi \in A_z\} > 0$$

also for all $z \in D$, we have

$$\inf\{|(F\psi, \psi)| : \psi \in A_z\} \geq \frac{c}{||w(\cdot, z)||^2_{L^2(D)}}.$$

$\square$

In the second situation we consider more general electric permittivities ϵ_r, where only parts of D may be absorbing.

Lemma 3.6 Consider the inverse scattering by an inhomogeneous medium. Assume that the conditions of Assumption 1 hold and there exists $r > 0$ such that

$$\left|\epsilon_r - \frac{1}{2}(1 - ri)\right| \geq \frac{\sqrt{1 + r^2}}{2} \text{ for almost all } x \in D.$$

Furthermore, assume that k^2 is not an interior transmission eigenvalue. Then $z \in D$ if and only if

$$\inf\{|(F\psi, \psi)| : \psi \in A_z\} > 0$$

Furthermore, for $z \in D$ we have the estimate

$$\inf\{|(F\psi, \psi)| : \psi \in A_z\} \geq \frac{c}{||w(\cdot, z)||^2_{L^2(D)}} \tag{3.18}$$

for some constant $c > 0$ which is independent of z. Here w is defined by (3.16).

Proof By using Lemma 3.3, the middle operator $T : L^2(D,\mathbb{C}^3) \to L^2(D,\mathbb{C}^3)$ satisfies the coercivity condition, i.e., there exists a constant $c > 0$ such

$$|(Tf, f)_{L^2(D)}| \geq c||f||^2_{L^2(D)}, \text{ for all } \quad f \in R(H) \subset L^2(D,\mathbb{C}^3).$$

Let F be the far field operator defined by (3.13) and $H : L^2_t(\mathbb{S}^2) \to L^2(D,\mathbb{C}^3)$, which is defined in Lemma 3.1. Since Assumption 1 is satisfied, then by Lemma 3.1 (part 1), F has the factorization $F = H^*TH$.

Let w be defined by (3.16). Then by using Lemma 3.1 (part 2), $w \in L^2(D,\mathbb{C}^3)$ and $\phi_z = H^*w$, where ϕ_z is defined in (3.14). In Theorem 2.3, choose $B^* = H$, $A = T$, $Y = L^2_t(\mathbb{S}^2)$, and $X = L^2(D,\mathbb{C}^3)$. Then $\phi_z \in R(H^*)$ if and only if

$$inf\{|(F\psi,\psi)| : \psi \in A_z\} > 0$$

Furthermore, we have the estimate

$$inf\{|(F\psi,\psi)| : \psi \in A_z\} \geq \frac{c}{||w(\cdot,z)||^2_{L^2(D)}}$$

for some constant $c > 0$ which is independent of z. Now by using Theorem 3.4, $z \in D$ if and only if $\phi_z \in R(H^*)$. So $z \in D$ if and only if

$$inf\{|(F\psi,\psi)| : \psi \in A_z\} > 0.$$

also for all $z \in D$, we have

$$inf\{|(F\psi, \psi)| : \psi \in A_z\} \geq \frac{c}{||w(\cdot, z)||^2_{L^2(D)}}. \qquad \square$$

The estimation (3.17) and (3.18) in Lemma 3.5 and Lemma 3.6 provides some insight to our indicator I_z for sampling points $z \in D$.

Note that

$$\begin{aligned} \gamma := (\phi_z, \phi_z) &= \int_{\mathbb{S}^2} |\phi_z(\hat{x})|^2 ds = \int_{\mathbb{S}^2} k^2 |\hat{x} \times p|^2 |e^{-ik\hat{x}\cdot z}|^2 ds \\ &= \int_{\mathbb{S}^2} k^2 |\hat{x} \times p|^2 ds = \frac{8}{3}\pi\, k^2 ||p||^2. \end{aligned} \tag{3.19}$$

This implies that $\psi_z := \phi_z/\gamma \in A_z$. By the linearity of the far field operator F and using the estimate in (3.17) or the estimate in (3.18), we have

$$I_z = |(F\phi_z, \phi_z)| = \gamma |(F\psi_z, \phi_z)|$$

$$\geq \gamma\ inf\ \{|(F\psi, \psi)| : \psi \in A_z\}$$

$$\geq \frac{c\gamma}{||w(\cdot, z)||^2_{L^2(D)}}, \quad z \in D,$$

for some constant $c > 0$ which is independent of z.

The main result is summarized by the following Theorem

Theorem 3.7 Under the assumptions of Lemma 3.5 or Lemma 3.6, we have

$$I_z \geq \frac{c\gamma}{||w(\cdot, z)||^2_{L^2(D)}}, \qquad z \in D,$$

for some constant $c > 0$ independent of z. Here, γ is defined by (3.19).

This result characterizes the support $\overline{D}$ of q. Again D can be absorbing everywhere, or only parts of D may be absorbing.

3.2.2 Resolution Analysis for the Sampling Points Outside the Scatterer

In this subsection we study the behavior of I_z outside the scatterer. For the subsequent analysis, we need the well known Funk-Hecke formula

$$\int_{\mathbb{S}^2} e^{-ikz\cdot\hat{x}} Y_\alpha^\beta(\hat{x}) ds(\hat{x}) = \kappa_\alpha\, j_\alpha(k|z|) Y_\alpha^\beta(\hat{z}), \tag{3.20}$$

where $\kappa_\alpha = 4\pi/i^\alpha$ and j_α is the the Spherical Bessel functions of order α.

It is well known that the far field pattern for scattering problem (3.7) - (3.9) has the following form [17]

$$H^{\infty}(\hat{x},\hat{\theta};p) = \hat{x} \ \times \int_{\partial D} \Big\{ ik \, [\nu(y) \times H^s(y,\hat{\theta};p)]$$

$$+\sqrt{\frac{\mu_0}{\epsilon_0}}[\nu(y) \times curl \ H^s(y,\hat{\theta};p)] \times \hat{x} \Big\} e^{-ik\hat{x}\cdot y} ds(y).$$

Inserting it into our indictor I_{new}, we get,

$$I_z := k^2 \Bigg| \int_{\mathbb{S}^2}\int_{\mathbb{S}^2}\int_{\partial D} (\hat{\theta} \times p) \ \cdot \Big\{ ik \, \hat{x} \times \ [\nu(y) \times H^s(y,\hat{\theta})(\hat{x}\times p)] \ e^{-ik\hat{x}\cdot(y-z)}$$

$$+\sqrt{\frac{\mu_0}{\epsilon_0}}\hat{x} \times ([\nu(y) \times curl \ H^s(y,\hat{\theta})(\hat{x}\times p)] \times \hat{x}) \ e^{-ik\hat{x}\cdot(y-z)} \ \Big\} ds(y) ds(\hat{x}) \ e^{-ik\hat{\theta}\cdot z} ds(\hat{\theta}) \Bigg|,$$

$$= k^2 \left| \int_{\mathbb{S}^2} G(z,\hat{\theta}) e^{-ik\hat{\theta}\cdot z} ds(\hat{\theta}) \right|$$

where

$$G(z,\hat{\theta}) := (\hat{\theta} \times p) \cdot \int_{\mathbb{S}^2}\int_{\partial D} \Big\{ ik \ \ \hat{x} \times \ [\nu(y) \times H^s(y,\hat{\theta})(\hat{x}\times p)] \ e^{-ik\hat{x}\cdot(y-z)}$$

$$+\sqrt{\frac{\mu_0}{\epsilon_0}}\hat{x} \times ([\nu(y) \times curl \ H^s(y,\hat{\theta})(\hat{x}\times p)] \times \hat{x}) \ e^{-ik\hat{x}\cdot(y-z)} \Big\} \ \ ds(y) ds(\hat{x}). \tag{3.21}$$

Since $a \times b \times c = b(a \cdot c) - c(a \cdot b)$ for any vectors a, b and c, $G(z,\hat{\theta})$ can be written as

$$G(z,\hat{\theta}) = (\hat{\theta} \times p) \cdot \Big\{ \ ik \int_{\mathbb{S}^2}\int_{\partial D} \ \Big[\nu(y)(\hat{x} \cdot \ H^s(y,\hat{\theta})(\hat{x}\times p))$$

$$- H^s(y,\hat{\theta})(\hat{x}\times p)(\hat{x}.\ \nu(y)\Big]\ e^{-ik\hat{x}\cdot(y-z)}\ ds(y)ds(\hat{x})$$

$$+\sqrt{\frac{\mu_0}{\epsilon_0}}\int_{\mathbb{S}^2}\int_{\partial D}\hat{x}\times\Big[curl\ H^s(y,\hat{\theta})(\hat{x}\times p)(\nu(y)\cdot\ \hat{x})$$

$$-\hat{x}\ (\nu(y)\ \cdot curl\ H^s(y,\hat{\theta})(\hat{x}\times p))\Big]e^{-ik\hat{x}\cdot(y-z)}\ ds(y)ds(\hat{x})\Big\}.$$

$G(z,\hat{\theta})$ can be written as

$$G(z,\hat{\theta}) = (\hat{\theta}\times p)\cdot\int_{\partial D}\{ik\ \int_{\mathbb{S}^2}\ \nu(y)(\hat{x}.\ H^s(y,\hat{\theta})(\hat{x}\times p))e^{-ik\hat{x}\cdot(y-z)}\ ds(\hat{x})$$

$$-ik\ \int_{\mathbb{S}^2}\ H^s(y,\hat{\theta})(\hat{x}\times p)(\hat{x}.\ \nu(y))\ e^{-ik\hat{x}\cdot(y-z)}\ ds(\hat{x})$$

$$+\sqrt{\frac{\mu_0}{\epsilon_0}}\ \int_{\mathbb{S}^2}[\hat{x}\times curl\ H^s(y,\hat{\theta})(\hat{x}\times p)(\nu(y).\ \hat{x})]\ e^{-ik\hat{x}\cdot(y-z)}\ ds(\hat{x})$$

$$-\sqrt{\frac{\mu_0}{\epsilon_0}}\ \int_{\mathbb{S}^2}[\hat{x}\times\ \hat{x}(\nu(y)\ \cdot\ curl\ H^s(y,\hat{\theta})(\hat{x}\times p))]e^{-ik\hat{x}\cdot(y-z)}\ ds(\hat{x})\}ds(y).$$

By using Funk-Hecke formula in (3.20), we get

$$G(z,\hat{\theta}) = (\hat{\theta}\times p)\cdot\int_{\partial D}\{ik\kappa_2\ \ \nu(y)(\frac{y-z}{|y-z|}.\ H^s(y,\hat{\theta})(\frac{y-z}{|y-z|}\times p))\ j_2(k|y-z|)$$

$$-\ ik\kappa_2\ \ H^s(y,\hat{\theta})(\frac{y-z}{|y-z|}\times p)(\frac{y-z}{|y-z|}.\ \nu(y))\ j_2(k|y-z|)$$

$$+\sqrt{\frac{\mu_0}{\epsilon_0}}\kappa_3\ \frac{y-z}{|y-z|}\times curl\ H^s(y,\hat{\theta})(\frac{y-z}{|y-z|}\times p)(\nu(y)\cdot\ \frac{y-z}{|y-z|})\ \ j_3(k|y-z|)$$

$$-\sqrt{\frac{\mu_0}{\epsilon_0}}\kappa_3\ \frac{y-z}{|y-z|}\times\ \frac{y-z}{|y-z|}(\nu(y)\ \cdot curl\ H^s(y,\hat{\theta})(\frac{y-z}{|y-z|}\times p))\ j_3(k|y-z|)\}ds(y).$$

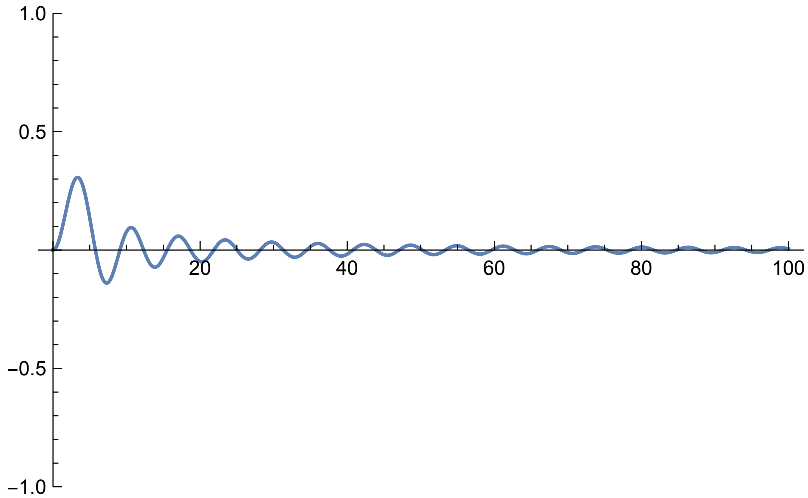

Figure 3.1: Decay behavior of Spherical Bessel function $j_2(x)$ in two dimensions.

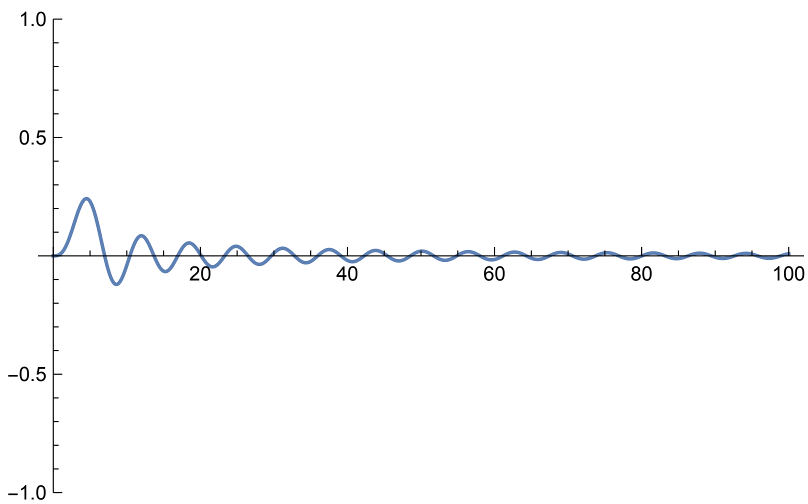

Figure 3.2: Decay behavior of Spherical Bessel function $j_3(x)$ in two dimensions.

We note that $G(z, \hat{\theta})$ is a superposition of the Spherical Bessel functions j_2 and j_3. As we see from Fig. 3.1 and Fig. 3.2, for large argument, these two functions decay as the sampling points z goes away from the boundary ∂D. By Riemann-Lebesgue Lemma, we obtain that

$$I_{new}(z) \to 0, \;\; as \;\; |z| \to \infty.$$

3.2.3 Stability Statement

We end this section by a stability statement, which shows that the lower bound of the indicator function I_z is bounded above for all sampling points inside the scatterer.

Theorem 3.10 (Stability statement) Let D be a bounded domain in $\mathbb{R}^3$, and denote by $BC(D)$ the space of bounded continuous functions on D with sup norm. Then

$$||I_z(\cdot)||^2_{BC(D)} \leq 4\pi\gamma k^2 \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} |H^\infty(\hat{x},\hat{\theta})(\hat{x}\times p)|^2 ds(\hat{x})ds(\hat{\theta}), \tag{3.22}$$

where γ is given by (3.19).

Proof Using the Cauchy-Schwartz inequality, we find that, for all $z \in D$,

$$|I_z|^2 = \left|\int_{\mathbb{S}^2} k(\hat{\theta}\times p)\, e^{-ik\hat{\theta}\cdot z} \int_{\mathbb{S}^2} k\ H^\infty(\hat{x},\hat{\theta})(\hat{x}\times p)\, e^{ik\hat{x}\cdot z} ds(\hat{x})ds(\hat{\theta})\right|^2$$

$$\leq \int_{\mathbb{S}^2} k^2|\hat{\theta}\times p|^2 |e^{-ik\hat{\theta}\cdot z}|^2 ds(\hat{\theta}) \int_{\mathbb{S}^2}\left|\int_{\mathbb{S}^2} k\ H^\infty(\hat{x},\hat{\theta})(\hat{x}\times p)\, e^{ik\hat{x}\cdot z} ds(\hat{x})\right|^2 ds(\hat{\theta})$$

$$= \gamma \int_{\mathbb{S}^2}\left|\int_{\mathbb{S}^2} k\ H^\infty(\hat{x},\hat{\theta})(\hat{x}\times p)\, e^{ik\hat{x}\cdot z} ds(\hat{x})\right|^2 ds(\hat{\theta})$$

$$\leq \gamma \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} k^2 |H^\infty(\hat{x},\hat{\theta})(\hat{x}\times p)|^2 ds(\hat{x}) \int_{\mathbb{S}^2} |e^{ik\hat{x}\cdot z}|^2 ds(\hat{x})\ \ ds(\hat{\theta})$$

$$= \gamma \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} k^2 |H^\infty(\hat{x},\hat{\theta})(\hat{x}\times p)|^2 ds(\hat{x}) \int_{\mathbb{S}^2} 1\ ds(\hat{x})\ \ ds(\hat{\theta})$$

$$= 4\pi\, k^2 \gamma \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} |H^\infty(\hat{x}, \hat{\theta})(\hat{x} \times p)|^2 ds(\hat{x}) \;\; ds(\hat{\theta}),$$

where we have used the fact that $\int_{\mathbb{S}^2} 1 \; ds(\hat{x}) = 4\pi$, for all $\hat{x}, \hat{\theta} \in \mathbb{S}^2$ and $|e^{ik\hat{x}\cdot z}| = |e^{-ik\hat{\theta}\cdot z}| = 1$ for $z \in D$. $\square$

Chapter 4

New Sampling Method for Multifrequency Inverse Source Problems with Sparse Far Field Measurements

4.1 Introduction

Inverse source problems (ISPs) have attracted the attention of many researcher because of their applications, such as identification of pollution sources in the environment [22] [23], sound source localization [57], and determination of the source current distribution in the brain from boundary measurements [19].

In this chapter we propose a new sampling method for multifrequency inverse source problems for time-harmonic acoustic waves with a finite set of far field data. The method is based on the factorization method for multifrequency inverse source problems with sparse far field measurements waves discussed by Griesmaier and Scmiedecke in 2017 [31]. We approximate the position and the convex geometry of the support of the source f of the time harmonic acoustic waves from the far field data. In addition, we assume that far field measurements of the wave radiated by a collection of compactly supported sources are available across a frequency band $(0, k_{max}) \subseteq \mathbb{R}$ but only at a few (finitely many) of linearly independent observation directions

$$\{\theta_1, \ \ldots, \theta_J\} = \Theta \subseteq S^{d-1},$$

where $d = 2, 3$ denotes the dimension.

The main feature of this method is that the indicator function is based on the inner product and therefore the method is very simple to implement. With the help of the factorization of the corresponding far field operator [31], a lower bound estimate will be established for the sampling points inside the support of the source. Moreover, we will show that the indicator function decays as the sampling point moves away from the support of the source, and thus gives a characterization of the support of the source.

In order to reduce the number of sensor locations required to obtain a useful reconstruction of the support of the sources [31], we develop a reconstruction method that efficiently utilizes multifrequency information. Let $k = w/c > 0$ be the wave number of a time harmonic wave where $w > 0$ and $c > 0$ denote the frequency and sound speed, respectively. Let

$$D := \bigcup_{m=1}^{M} D_m \subseteq \mathbb{R}^d$$

be an ensemble of finitely many well-separated bounded domains in $\mathbb{R}^d$, $d = 2, 3$, i.e., $\overline{D_j} \cap \overline{D_l} = \emptyset$ for $j \neq l$. Suppose $f \in L^\infty(D)$ represent the acoustic source with compact support in D. Then the time-harmonic wave $u \in H^1_{loc}(\mathbb{R}^d)$ radiated by f solves the Helmholtz equation

$$-\Delta u - k^2 u = f \qquad in \ \ \mathbb{R}^d \tag{4.1}$$

and satisfies the Sommerfeld radiation condition

$$\lim_{r \longrightarrow \infty} r^{\frac{d-1}{2}} \left(\frac{\partial u}{\partial r} - iku \right) = 0, \qquad r = |x|. \tag{4.2}$$

Furthermore, u also satisfies the asymptotic behavior

$$u(x; k) = C_k \frac{e^{ik|x|}}{|x|^{\frac{d-1}{2}}} u^\infty(\theta_x; k) + \mathcal{O}(|x|^{-\frac{d+1}{2}}), \quad \theta_x = \frac{x}{|x|} \in S^{d-1},$$

as $r = |x| \longrightarrow \infty$, where $C_k = e^{i\pi/4}/\sqrt{8\pi k}$ if $d = 2$ and $C_k = k^2/4\pi$ if $d = 3$, and the far field pattern $u^\infty(.\,; k)$ is the far field radiated by f at wave number k given by

$$u^\infty(\theta_x; k) = \int_D e^{-ik\theta_x \cdot y} f(y)\, dy = \widehat{f}(k\theta_x) \qquad \theta_x \in S^{d-1}. \tag{4.3}$$

We assume that the far field is observed at only a few observation directions

$$\{\theta_1,\ \ldots, \theta_J\} := \Theta \subseteq S^{d-1}, \tag{4.4}$$

but across a whole band of wave numbers $k \in (0, k_{max})$ for some $k_{max} > 0$. Hence, the measured data set is equivalent to

$$\big\{u^\infty(\theta_j, k) \mid \theta_j \in \Theta,\ k \in (0, k_{max})\big\}. \tag{4.5}$$

The inverse problem that we consider here is to deduce information on the location of the support of the source f from these data. We remark that Sylvester [63] shows the uniqueness of this inverse problem.

The convex hull of a subset $\Omega \subseteq R^d$ [31] is the intersection of all closed half spaces

$$H_{s,\theta} := \{x \in \mathbb{R}^d \mid x \cdot \theta \le s\}, \qquad \theta \in S^{d-1},\ s \in \mathbb{R},$$

which contain Ω, i.e,

$$ch(\Omega) := \bigcap_{\theta \in S^{d-1}} \{x \in \mathbb{R}^d \mid x \cdot \theta \le s_\Omega(\theta)\},$$

where

$$s_\Omega(\theta) := \sup_{x \in \Omega} x \cdot \theta, \qquad \theta \in S^{d-1,}$$

is the supporting function of Ω.

The θ-convex hull for a single direction $\theta \in S^{d-1}$ is defined as

$$K_{s_\Omega}(\theta) := \{x \in \mathbb{R}^d \mid s_\Omega(-\theta) \le x \cdot \theta \le s_\Omega(\theta)\},$$

which is the smallest strip (intersection of two parallel half spaces) with normals in the directions $\pm\theta$ that contains $\overline{\Omega}$.

The Θ-convex hull of Ω is defined as

$$K_{s_\Omega}(\Theta) := \bigcap_{\theta \in \Theta} \{x \in \mathbb{R}^d \mid s_\Omega(-\theta) \le x \cdot \theta \le s_\Omega(\theta)\},$$

Note that, for $D = \bigcup_{m=1}^{m=M} D_m$, we have

$$\bigcup_{m=1}^{M} ch(D_m) \subsetneq K_D(\Theta) := \bigcap_{\theta \in \Theta} \bigcup_{m=1}^{M} K_{s_{D_m}}(\theta) \subsetneq ch(D).$$

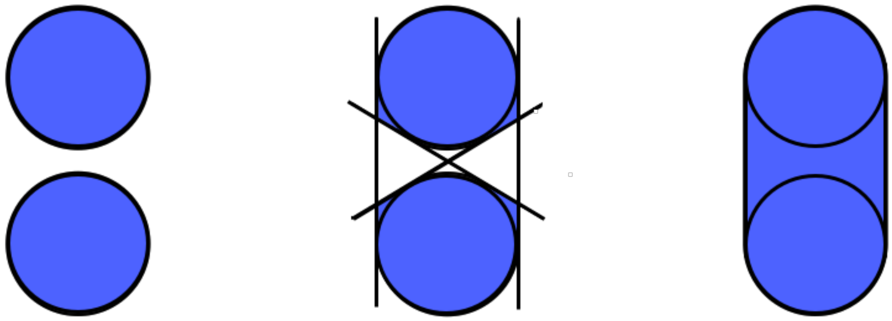

Figure 4.1: Example with two disjoint disks D_1, D_2. Left: $ch(D_1) \cup ch(D_2)(blue)$. Center: $\cap_{\theta \in S^1}(K_{S_{D_1}}(\theta) \cup K_{S_{D_2}}(\theta))(blue)$. Right: $ch(D_1 \cup D_2)$(blue).

Fig. 4.1 [31] shows the above inequality for the case when $M = 2$ and D_1 and D_2 are two disjoint balls in $\mathbb{R}^d$.

Recently, Griesmaier [31] shows that under certain conditions on the source f, the middle set $K_{s_D}(\Theta)$ can be reconstructed by the data given in (4.5). We study the behaviors of the indicator function for each single observation direction. Then we combine them for all observation directions in Θ to give a fast and effect reconstruction of $K_{s_D}(\Theta)$.

Hence, we assume that the real part of a complex multiple of the sources is bounded away from zero on their support, i.e., we assume that $f \in L^\infty(D)$ is such that there exist $\tau \in R$ and $c_0 > 0$ such that $\Re(e^{i\tau} f(x)) \geq c_0$ for almost every $x \in D$. We call this a coercivity assumption on the sources.

First we propose an indicator function for a single observation direction in terms of the far field data $u^\infty(\theta_j; k)$, measured in the observation directions θ_j. We will show, according to values of this indicator function, whether z lies in the strip $K_{s_D}(\theta_j)$ or not. This can be used to determine the smallest union of strips (intersections of parallel half spaces) with normals in the observation directions $\pm\theta_j$ that contains the support of the sources. The proposed indicator function for a single observation direction θ_j is characterized by the following

$$I_{z,\epsilon}^{\theta_j} := \left| \frac{1}{|B_\epsilon(z)|^2} \int_0^{k_{max}} \int_0^{k_{max}} u^\infty(\theta_j, t-s) \int_{B_\epsilon(z)} \int_{B_\epsilon(z)} e^{i\theta_j \cdot (ty-sx)} dydx \; ds \; dt \right|, \quad z \in \mathbb{R}^d, \tag{4.6}$$

where $B_\epsilon(z)$ denotes the ball of radius ϵ centered at z, and $|B_\epsilon(z)|$ its volume. We will discuss the theoretical foundation of the proposed reconstruction scheme shortly. With the help of the inf-criterion characterization obtained by using the factorization method, we will show, if $z \in \bigcup_{m=1}^M int(K_{S_{D_m}}(\theta_j))$, then there exists $\epsilon > 0$ such that $I_{z,\epsilon}^{\theta_j} > 0$. We will give a lower bound for $I_{z,\epsilon}^{\theta_j}$ for this case. If $z \notin \bigcup_{m=1}^M int(K_{S_{D_m}}(\theta_j))$, then there exists $\epsilon_0 > 0$ such that the behavior of $I_{z,\epsilon}^{\theta_j}$ decays and goes to 0 for any $0 < \epsilon \leq \epsilon_0$. Here we use $int(\Omega)$ to denote the interior of Ω.

By combining this test for all observation directions $\{\theta_1, \; . \; . \; . \; , \theta_J\} = \Theta \subseteq S^{d-1}$, we introduce the following indicator

$$I_{z,\epsilon}^{\Theta} := \sum_{\theta_j \in \Theta} \left| \frac{1}{|B_\epsilon(z)|^2} \int_0^{k_{max}} \int_0^{k_{max}} u^\infty(\theta_j, t-s) \int_{B_\epsilon(z)} \int_{B_\epsilon(z)} e^{i\theta_j \cdot (ty-sx)} dydx \; ds \; dt \right|, \tag{4.7}$$

where $z \in \mathbb{R}^d$.

This indicator recovers a union of the intersections of all strips obtained for each pair of observation directions $\pm\theta_j$. It gives a characterization of a subset of the Θ-convex hull of the support D of the source in terms of the measured data (4.5).

4.2 Theoretical Foundation of the Proposed Sampling Method

The aim of this section is to establish the mathematical basis of the sampling method. We consider the case of the far field data $u^\infty(\theta_j; k)$, $k \in (0, k_{max})$ for a single observation direction θ_j, $1 \leq j \leq J$. Define the convolution operator $F^{\theta_j} : L^2(0, k_{max}) \to L^2(0, k_{max})$ by

$$(F^{\theta_j}\phi)(t) := \int_0^{k_{max}} u^\infty(\theta_j; t - s)\phi(s)ds, \qquad t \in (0, k_{max}). \tag{4.8}$$

For all sampling points $z \in \mathbb{R}^d$ and $\epsilon > 0$, define a test function $\phi_{z,\epsilon}^{\theta_j} \in L^2(0, k_{max})$ by

$$\phi_{z,\epsilon}^{\theta_j}(t) := \frac{1}{|B_\epsilon(z)|} \int_{B_\epsilon(z)} e^{-it\theta j \cdot y} dy, \quad t \in (0, k_{max}), \tag{4.9}$$

where again $B_\epsilon(z)$ denotes the ball of radius ϵ centered at z, and $|B_\epsilon(z)|$ its volume. Then we may write our indicator function $I_z(t)$ given in (4.6) in the simple form

$$I_{z,\epsilon}^{\theta_j} := \left|(\phi_{z,\epsilon}^{\theta_j},\ F^{\theta_j}\phi_{z,\epsilon}^{\theta_j})\right|, \quad z \in \mathbb{R}^d. \tag{4.10}$$

Here, we denote by $(\cdot\,,\cdot)$ the inner product of $L^2(0, k_{max})$.

For all $z \in \mathbb{R}^d$ and $\epsilon > 0$, define $A_z \subset L^2(0, k_{max})$ by

$$A_{z,\epsilon} := \{\psi \in L^2(0, k_{max}) : (\psi, \phi_{z,\epsilon}^{\theta_j}) = 1\},$$

where $\phi_{z,\epsilon}^{\theta_j}$ is the test function defined by (4.9). To establish a lower bound for our indicator function for $z \in \bigcup_{m=1}^{M} int(K_{S_{D_m}}(\theta_j))$, we need to use the following lemmas in the factorization method for multifrequency inverse source problems with sparse far field measurements discussed in [31].

Lemma 4.1 (Theorem 3.1 of [31]) Let $F^{\theta_j} : L^2(0, k_{max}) \to L^2(0, k_{max})$ be defined by (4.8). Then F^{θ_j} has the following factorization

$$F^{\theta j} = L_D^{\theta j} T_D (L_D^{\theta j})^*,$$

where the operator $L_D^{\theta j} : L^2(D) \to L^2(0, k_{max})$ is defined by

$$(L_D^{\theta j}\psi)(t) := \int_D e^{-it\theta_j \cdot y}\, \psi(y)dy, \quad t \in (0, k_{max}).$$

Its adjoint $(L_D^{\theta_j})^* : L^2(0, k_{max}) \to L^2(D)$ is defined by

$$((L_D^{\theta_j})^*\phi)(y) := \int_0^{k_{max}} e^{is\theta_j \cdot y}\, \phi(s)ds, \quad y \in D,$$

The operator $T_D : L^2(D) \to L^2(D)$ is a multiplication operator given by $T_D g = fg$, where $f \in L^\infty(D)$ denotes the source radiating the far field u^∞ as in (4.3).

The dependence of the range of the operator $L_D^{\theta j}$ on the projection $(\theta_j \cdot D)\theta_j$ of the domain D on the one-dimensional subspace of $\mathbb{R}^d$ spanned by the observation directions $\pm\theta_j$ is discussed in Lemma 3.3 of [31]. The next lemma characterizes the projection $(\theta_j \cdot D)\theta_j$ of the domain D on the one-dimensional subspace of $\mathbb{R}^d$ spanned by the observation direction θ_j in terms of the range of the operator $L_D^{\theta j}$

Lemma 4.2 (Lemma 3.4 of [31]) Consider the test function defined in (4.9). For any $z \in \mathbb{R}^d$.

1. If $\theta_j \cdot z \in \theta_j \cdot D$, then there exists $\epsilon > 0$ such that $\phi_{z,\epsilon}^{\theta_j} \in R(L_D^{\theta_j})$.

2. For any $\epsilon > 0$,

$$\theta_j \cdot B_\epsilon(z) \bigcap \theta_j \cdot D = \emptyset$$

implies that $\phi_{z,\epsilon}^{\theta_j} \notin R(L_D^{\theta_j})$.

Proof [31]

1. If $\theta_j \cdot z \in \theta_j \cdot D$, then there exists $y \in D$ such that $\theta_j \cdot z = \theta_j \cdot y$. Since D is open, there exists $\epsilon > 0$ such that $B_\epsilon(y) \subseteq D$. Hence, $B_\epsilon(z) \cdot \theta_j = B_\epsilon(y) \cdot \theta_j$. Define

$$w_\epsilon = \chi_{B_\epsilon(y)}, \tag{4.11}$$

where $\chi_{B_\epsilon(y)}$ denotes the characteristic function on $B_\epsilon(y)$. Then we have

$$\phi_{z,\epsilon}^{\theta_j} = L_D^{\theta_j} w_\epsilon.$$

Therefore, $\phi_{z,\epsilon}^{\theta_j} \in R(L_D^{\theta_j})$.

2. If $\theta_j \cdot B_\epsilon(z) \bigcap \theta_j \cdot D = \emptyset$, then Lemma 3.3 of [31] shows that

$$R(L_{B_\epsilon(z)}^{\theta_j}) \cap R(L_D^{\theta_j}) = \{0\}.$$

Since $0 \neq \phi_{z,\epsilon}^{\theta_j} = L_{B_\epsilon(z)}^{\theta_j} \chi_{B_\epsilon(z)} \in R(L_{B_\epsilon(z)}^{\theta_j})$, this implies $\phi_{z,\epsilon}^{\theta_j} \notin R(L_D^{\theta_j})$. $\square$

Lemma 4.3 Assume that there exists $\tau \in \mathbb{R}$ and a constant $c_0 > 0$ such that $\Re(e^{i\tau} f(x)) \geq c_0$ for almost every $x \in D$. Then the middle operator $T_D : L^2(D) \to L^2(D)$, which is defined in Lemma 4.1, satisfies the coercivity condition, i.e., there exists a constant $c > 0$ such that

$$|\langle g, T_D\, g\rangle_D| \geq c||g||^2_{L^2(D)} \quad \text{for all} \quad g \in L^2(D).$$

Proof. Since $\Re(e^{i\tau} f(x)) \geq c_0$, then

$$\Re(e^{i\tau}\langle\, g, T_D\, g\,\rangle_D) = \int_D \Re(e^{i\tau} f(x))|g(x)|^2\, dx \geq c_0||g||^2_{L^2(D)},$$

$$|\langle\, g, T_D\, g\rangle_D| = |e^{i\tau}|\, |\langle g, T_D g\rangle_D| = |e^{i\tau}\langle g, T_D g\rangle_D|$$

$$\geq Re(e^{i\tau}\langle g, T_D\, g\rangle_D) \geq c_0||g||^2_{L^2(D)}. \qquad \square$$

By combining Theorem 3.6 and Theorem 3.7 in [31], we have the following lemma

Lemma 4.4 (Theorems 3.6 and 3.7 of [31]) Consider the test function $\phi^{\theta_j}_{z,\epsilon}$, which is defined in (4.9). For any $z \in \mathbb{R}^d$, $z \in \bigcup_{m=1}^{M} int(K_{S_{D_m}}(\theta_j))$ if and only if there exists $\epsilon > 0$ such that $\phi^{\theta_j}_{z,\epsilon} \in R(L^{\theta_j}_D)$.

Lemma 4.5 Consider the inverse source problem defined in (4.1)- (4.2). Assume that there exists $\tau \in \mathbb{R}$ and a constant $c_0 > 0$ such that $\Re(e^{i\tau} f(x)) \geq c_0$ for almost all $x \in D$. For any $z \in \mathbb{R}^d,\ z \in \bigcup_{m=1}^{M} int(K_{S_{D_m}}(\theta_j))$ if and only if there exists $\epsilon > 0$ such that

$$\inf\{|(\psi,\, F^{\theta_j}\psi)| : \psi \in A_{z,\epsilon}\} > 0.$$

Furthermore, for $z \in \bigcup_{m=1}^{M} int(K_{S_{D_m}}\{\pm\theta_j\})$ we have the estimate

$$\inf\{|((\psi,\, F^{\theta_j}\psi)| : \psi \in A_{z,\epsilon}\} \geq \frac{c}{||w_\epsilon||^2_{L^2(D)}} \tag{4.12}$$

for some constant $c > 0$ independent of z. Here w_ϵ is defined by (4.11).

Proof. By using Lemma 4.1, F^{θ_j} has the factorization

$$F^{\theta j} = L_D^{\theta j} T_D (L_D^{\theta j})^*,$$

where $L_D^{\theta_j}, T_D$ and $(L_D^{\theta_j})^*$ are defined in Lemma 4.1. Since there there exits $\tau \in \mathbb{R}$ and a constant $c_0 > 0$ such that $\Re(e^{i\tau} f(x)) \geq c_0$ for almost all $x \in D$, then by using Lemma 4.3, there exists a constant $c > 0$ with

$$|(g, T_D\, g)_D| \geq c||g||^2_{L^2(D)} \quad \text{for all} \quad g \in L^2(D).$$

In Theorem 2.3, choose $B = L_D^{\theta_j}$, $A = T_D$, $Y = L^2(0, k_{max})$, and $X = L^2(D)$. Then for some $\epsilon > 0$, $\phi_{z,\epsilon}^{\theta_j} \in R(L_D^{\theta_j})$ if and only if

$$\inf\{|(\psi,\, F^{\theta_j}\psi)| : \psi \in A_{z,\epsilon}\} > 0.$$

Furthermore, for $\phi_{z,\epsilon}^{\theta_j} \in R(L_D^{\theta_j})$, we have the estimate

$$\inf\{|(\psi,\, F^{\theta_j}\psi)| : \psi \in A_{z,\epsilon}\} \geq \frac{c}{||w_\epsilon||^2_{L^2(D)}}$$

for some constant $c > 0$ independent of z. By using Lemma 4.4, $z \in \bigcup_{m=1}^{M} int(K_{S_{D_m}}(\theta_j))$ if and only if there exists $\epsilon > 0$ such that $\phi_{z,\epsilon}^{\theta_j} \in R(L_D^{\theta_j})$. Therefore, $z \in \bigcup_{m=1}^{M} int(K_{S_{D_m}}(\theta_j))$ if and only if

$$\inf\{|(\psi,\, F^{\theta_j}\psi)| : \psi \in A_{z,\epsilon}\} > 0.$$

Furthermore, for $z \in \bigcup_{m=1}^{M} int(K_{S_{D_m}}(\theta_j))$, we have the estimate

$$\inf\{|(\psi,\, F^{\theta_j}\psi)| : \psi \in A_{z,\epsilon}\} \geq \frac{c}{||w_\epsilon||^2_{L^2(D)}}. \qquad \square$$

A straightforward calculation shows that

$$(\phi_{z,\epsilon}^{\theta_j}, \phi_{z,\epsilon}^{\theta_j}) = \int_0^{k_{max}} \left|\phi_{z,\epsilon}^{\theta_j}\right|^2 ds = \frac{1}{|B_\epsilon(z)|^2} \int_0^{k_{max}} \int_{B_\epsilon(z)} \int_{B_\epsilon(z)} 1 \, dy \, dw \, ds = k_{max}. \quad (4.13)$$

This implies that $\psi_{z,\epsilon} := \phi^{\theta_j}_{z,\epsilon}/k_{max} \in A_{z,\epsilon}$. By the linearity of the far field operator F^{θ_j} and using the estimate (4.12), we have

$$I^{\theta_j}_{z,\epsilon} = |(\phi^{\theta_j}_{z,\epsilon},\ F^{\theta_j}\phi^{\theta_j}_{z,\epsilon})|$$

$$= k_{max}\ |(\phi^{\theta_j}_{z,\epsilon},\ F^{\theta_j}\psi_{z,\epsilon},)|$$

$$\geq k_{max}\ \inf\{\ |(\psi,\ F^{\theta_j}\psi)| : \psi \in A_{z,\epsilon}\}$$

$$\geq \frac{c\,k_{max}}{||w_\epsilon||^2_{L^2(D)}}$$

for some constant $c > 0$ which is independent of z.

The main result is summarized in the following Theorem

Theorem 4.6 Under the assumptions of Lemma 4.5,

if $z \in \bigcup_{m=1}^{M} int(K_{S_{D_m}}(\theta_j))$, then there exists $\epsilon > 0$ such that

$$I^{\theta_j}_{z,\epsilon} \geq \frac{c\,k_{max}}{||w_\epsilon||^2_{L^2(D)}}$$

for some constant $c > 0$ which is independent of z. Here w_ϵ is defined by (4.11).

Now, we move on to study the behavior of the indicator function given in (4.6) for a single observation direction $\{\theta_j\}$ as the sampling points z move away from the strip $K_{S_{D_m}}(\theta_j)$. Let ϵ in (4.6) go to zero. Then we can rewrite the indicator function in

(4.6) as

$$I_z^{\theta_j} = \left| \int_0^{k_{max}} \int_0^{k_{max}} u^\infty(\theta_j, t-s) e^{i(t-s)\theta_j \cdot z} ds dt \right|.$$

By making the substitution $k = t - s$, $\eta = t + s$, the indicator can be written as

$$I_z^{\theta_j} = k_{max} \left| \int_0^{k_{max}} u^\infty(\theta_j, k) e^{ik\theta_j \cdot z} dk \right|.$$

Recall from (4.3) that the far field pattern has the following representation

$$u^\infty(\theta_j, k) = \int_D e^{-ik\theta_j \cdot y} f(y) dy, \quad \theta_j \in \Theta, \quad k \in (0, k_{max}).$$

Substituting this into the indicator $I_z^{\theta_j}$ we get

$$I_z^{\theta_j} = k_{max} \left| \int_0^{k_{max}} \int_D e^{-ik\theta_j \cdot y} f(y) dy e^{ik\theta_j \cdot z} dk \right|$$

$$= k_{max} \left| \int_D \int_0^{k_{max}} e^{ik\theta_j \cdot (z-y)} dk \, f(y) dy \right|$$

$$= k_{max} \left| \int_D \frac{e^{ik_{max}\theta_j \cdot (z-y)} - 1}{\theta_j \cdot (z-y)} f(y) dy \right|.$$

We observe that the indicator $I_z^{\theta_j}$ decays as $1/\theta_j \cdot (z-y)$ as the sampling point z moves away from the strip $K_{S_{D_m}}(\theta_j)$.

If we take $\Theta = S^{d-1}$ and let ϵ goes to zero then the indicator given in (4.7) can be written as

$$I_z^{\Theta} = \left| \int_0^{k_{max}} \int_0^{k_{max}} \int_{S^{d-1}} u^{\infty}(\theta, t-s)\, e^{i(t-s)\theta.\, z} d\theta\; dt\, ds \right|, \qquad z \in \mathbb{R}^d,\; \theta \in S^{n-1}.$$

By making the substitution $k = t - s$, $\eta = t + s$, such an indicator can be written in the form

$$I_z^{\Theta} = k_{max} \left| \int_0^{k_{max}} \int_{S^{d-1}} u^{\infty}(\theta, k)\, e^{ik\theta\cdot z} d\theta\; dk \right|, \qquad z \in \mathbb{R}^d.$$

Substituting the far field $u^{\infty}(\Theta;\, k)$, which is defined in (4.3), into I_z^{Θ}, yields

$$\begin{aligned}
I_z^{\Theta} &= k_{max} \left| \int_0^{k_{max}} \int_{S^{d-1}} \int_D e^{-ik\theta\cdot y}\, f(y) dy\, e^{ik\theta\cdot z} d\theta\; dk \right| \\
&= k_{max} \left| \int_0^{k_{max}} \int_D \int_{S^{d-1}} e^{-ik\theta\cdot y}\; e^{ik\theta\cdot z} d\theta\; f(y) dy dk \right| \\
&= k_{max} \left| \int_0^{k_{max}} \int_D \int_{S^{d-1}} e^{-ik\theta\cdot(y-z)}\, d\theta\; f(y) dy dk \right| \\
&= k_{max} \left| \int_0^{k_{max}} \mu \int_D f_0(k|y-z|) f(y)\; dy\; ds \right|,
\end{aligned}$$

where

$$\mu = \begin{cases} 2\pi & , \quad d = 2; \\ 4\pi & , \quad d = 3. \end{cases} \qquad \textit{and} \qquad f_0(t) = \begin{cases} J_0(t) & , \quad d = 2; \\ j_0(t) & , \quad d = 3, \end{cases}$$

with J_0 and j_0 being the Bessel functions and Spherical Bessel functions of order zero, respectively.

This means that I_z^Θ is a superposition of the Bessel functions f_0. For large arguments, we have the following asymptotic formulas for the Bessel and spherical Bessel functions

$$j_0(t) = \frac{\sin t}{t}\Big\{1 + \mathcal{O}\Big(\frac{1}{t}\Big)\Big\}, \qquad as \quad t \to \infty,$$

$$J_0(t) = \frac{\sin t + \cos t}{\sqrt{\pi t}}\Big\{1 + \mathcal{O}\Big(\frac{1}{t}\Big)\Big\}, \qquad as \quad t \to \infty.$$

See Fig. 2.7 and 2.8 for the behavior of these four functions. This further implies that I_z^Θ decays as the sampling points z goes away from the boundary ∂D. Therefore, I_z^Θ goes to zero as z goes far away from the boundary ∂D.

4.3 Numerical Implementation

Now we present some numerical examples of the new sampling method in two dimensions. The synthetic data is computed by solving the forward problem using the Lippmann-Schwinger equation, i.e., (4.3). Let D be the compact support of f. We generate a triangular mesh $\mathcal{T}$ with the mesh size $h \approx 0.01$. For the direction θ and fixed k, the far-field pattern is approximated by

$$u^{\infty}(\theta;k) \approx \sum_{T\in\mathcal{T}} e^{-ik\theta\cdot y_T} f(y_T)|T|, \tag{4.14}$$

where $T \in \mathcal{T}$ is a triangle, y_T is the center of T, and $|T|$ denotes the area of T.

For all examples, for $\theta_j \in \Theta$, we assume to have multiple frequency far field data

$$u^{\infty}(\theta_j;k_n), \quad n = 1, \ldots N,$$

where $N = 20$, $k_{min} = 0.5$, $k_{max} = 20$ such that

$$k_n := (n-0.5)\Delta k, \quad \Delta k := \frac{k_{max}}{N}.$$

Since the test function $\phi_z^{\theta_j}$ in (4.9) is continuous, we choose ϵ to be 0, we discretize $\phi_z^{\theta_j}$ by the test vector

$$\phi_z^{\theta_j} := [e^{-it_0\theta_j\cdot z}, \ldots e^{-it_{N-1}\theta_j\cdot z}]^T \in \mathbb{C}^N, \qquad z \in \mathbb{R}^d. \tag{4.15}$$

We assume that $\Delta k \leq \frac{\pi}{R}$, where R is the radius of the smallest ball centered at the origin that contains the support of the source f. So no two points in the region of interest $B_R(0)$ share the same test vector $\phi_z^{\theta_j}$.

Assume that the sampling domain is $S := [-4, 4]\times[-4, 4]$. Each direction is uniformly divided into 80 intervals and we end up with 81^2 sampling points uniformly distributed in S. We denote by Z the set of all sampling points.

4.3.1 One Observation Direction

First we consider the case for a single observation direction θj. From Theorem 4.6, the the far field data $u^\infty(\theta_j, k),\ k \in (0, k_{max})$, for a single observation directions $\theta_j \in \Theta$ uniquely determine the smallest union of strips perpendicular to $\pm\theta_j$ that contains all sources.

Corollary 4.7 Under the assumption of Theorem 4.6, we have, for any $z \in \mathbb{R}^d$,

1. If $z \in \bigcup_{m=1}^{M} int(K_{S_{D_m}}(\theta_j))$, then there exists $\epsilon > 0$ and $c > 0$ such that $I_{z,\epsilon}^{\theta_j} \geq \frac{c\,k_{max}}{||w_\epsilon||^2_{L^2(D)}} > 0$.

2. If $z \notin \bigcup_{m=1}^{M} int(K_{S_{D_m}}(\theta_j))$, then there exists $\epsilon_0 > 0$ such that $I_{z,\epsilon}^{\theta_j}$ goes to zero for any $0 < \epsilon \leq \epsilon_0$.

From Corollary 4.7 we expect that the value of the indicator function $I_{z,\epsilon}^{\theta_j}$ defined in (4.6) is much larger for $z \in \bigcup_{m=1}^{M} int(K_{S_{D_m}}(\theta_j))$ than for $z \notin \bigcup_{m=1}^{M} int(K_{S_{D_m}}(\theta_j))$.

We normalize the indicator function, i.e., the plot is for $I_z^\theta / M(I_z^\theta)$ where $M(I_z^\theta)$ the

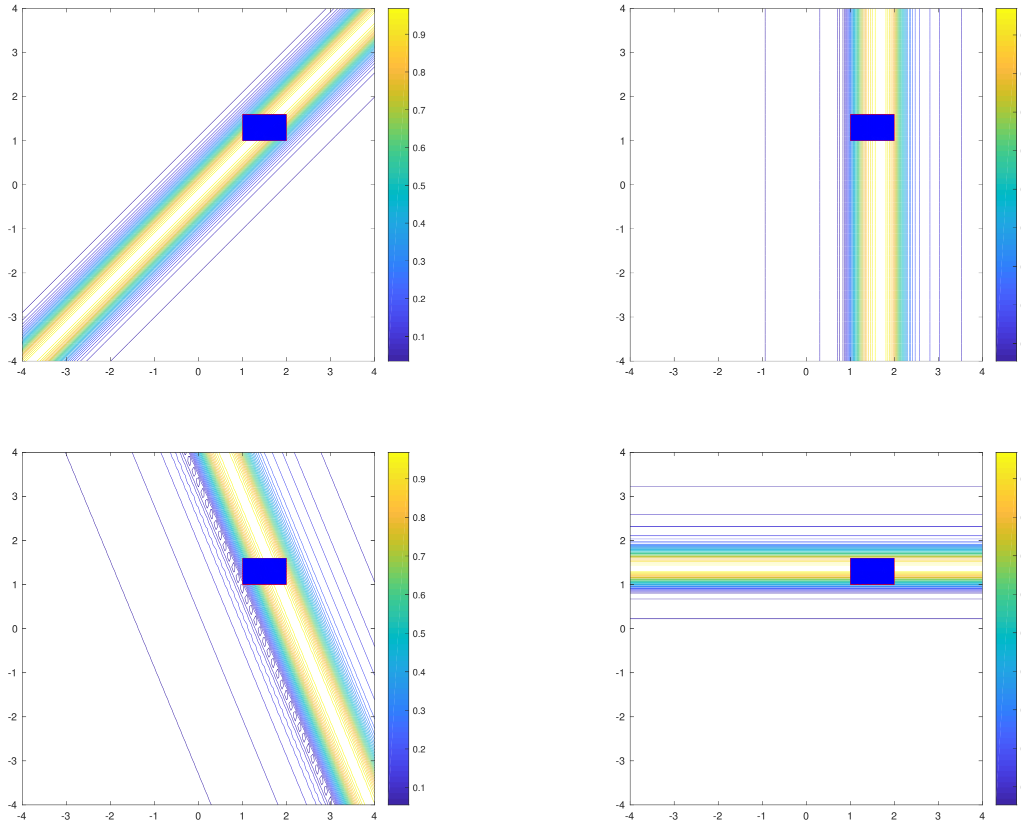

Figure 4.2: Indicators of different observation directions for one object. Top Left: $\theta = -\pi/4$. Top Right: $\theta = 0$. Bottom Left: $\theta = \pi/8$. Bottom Right: $\theta = \pi/2$.

largest element of $I_z^\theta, z \in Z$. Let $f = 5$ and assume that the support of f is a rectangle given by $(1,2)\times(1,1.6)$. In Fig. 4.2, we plot the indicators for four different observation directions $-\pi/4, 0, \pi/8$ and $\pi/2$. The picture clearly shows that the source lies in a strip, which is perpendicular to the observation direction.

In Fig. 4.3, we show the results when the support of source has two components. One is a rectangle given by $(1,1.6)\times(1,1.4)$. The other one is a disc with radius 0.2

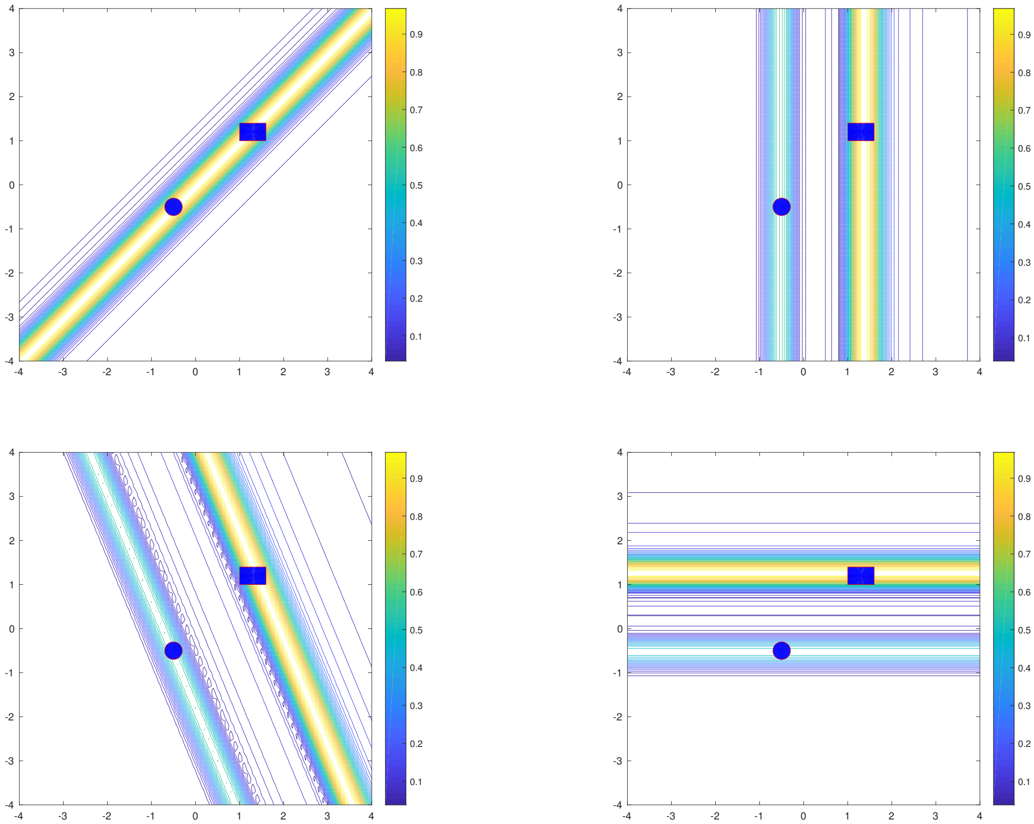

Figure 4.3: Indicators of different observation directions for two objects. Top Left: $\theta = -\pi/4$. Top Right: $\theta = 0$. Bottom Left: $\theta = \pi/8$. Bottom Right: $\theta = \pi/2$.

centered at $(-0.5, -0.5)$. For different observation directions, strips containing the objects are constructed effectively.

4.3.2 Two Observation Directions

Now we consider two observation directions: $\theta_1 = 0$ and $\theta_2 = \pi/2$. We compute the indicators and superimpose them in one picture. Since the observation directions are

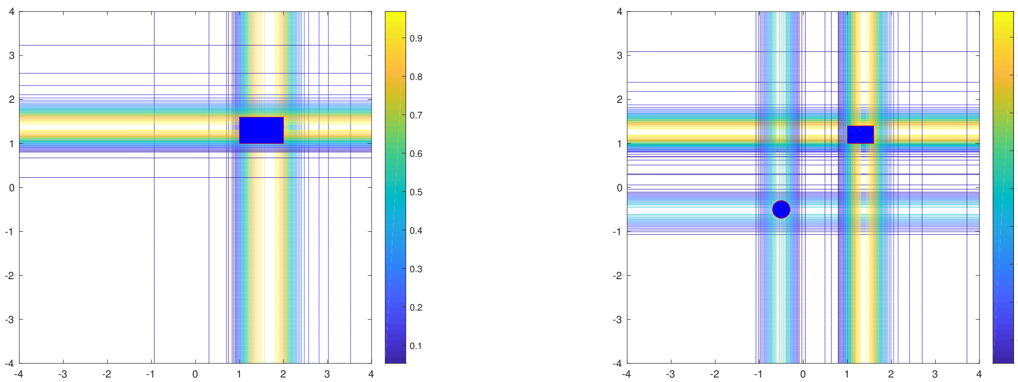

Figure 4.4: $\theta = \pi/2$ and $\theta = 0$. Left: Single object; Right: Two objects.

perpendicular to each other, the strips are perpendicular to each other in Fig. 4.4. For both one object and two objects, we see that intersection of the strips contains the support of f.

4.3.3 Multiple Observation Directions

Combining the characterization of the support of the source from Theorem 4.6 and Corollary 4.7 for all available receiver directions $\theta_1, \ . \ . \ ., \theta_j \in \Theta$, we obtain the following result

Corollary 4.8 Under the assumption of Theorem 4.6, we have for any $z \in \mathbb{R}^d$,

1. If $z \in \bigcap_{j=1}^{J} \bigcup_{m=1}^{M} int(K_{S_{D_m}}(\theta_j))$, then there exists $\epsilon > 0$ such that $I_{z,\epsilon} > 0$, for any $1 \leq j \leq J$.

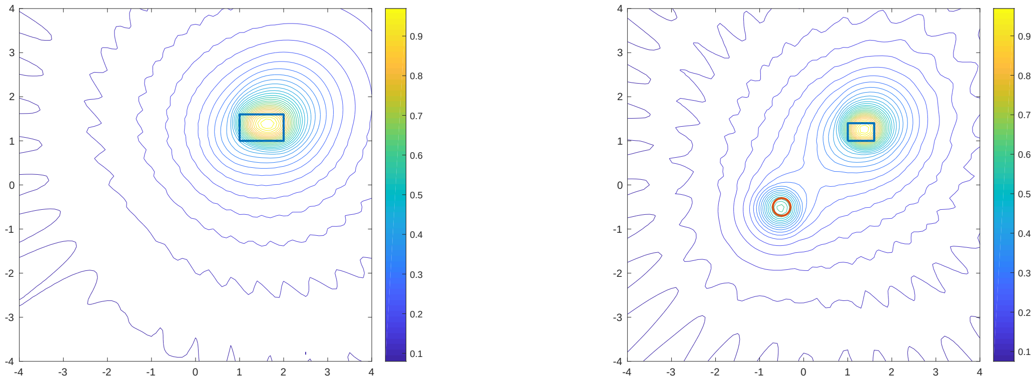

Figure 4.5: Reconstruction using multiple observation directions when $f = 5$. Left: single object. Right: Two objects.

2. If $z \notin \bigcap_{j=1}^{J} \bigcup_{m=1}^{M} int(K_{S_{D_m}}(\theta_j))$, then there exists $1 \leq j \leq J$, and $\epsilon_0 > 0$ such that $I_{z,\epsilon}$ goes to zero for any $0 < \epsilon \leq \epsilon_0$.

Corollary 4.8 gives a rigorous characterization of a subset of the $\Theta-$convex hull of the support D of the source. For the numerical implantation of Corollary 4.8, we compute the corresponding indicator function $I_z^{\theta_j}$ for each observation direction θ_j, $j = 1, \ldots J$. We expect that the value of I_z defined in (4.7) is much larger for $z \in \bigcap_{j=1}^{J} \bigcup_{m=1}^{M} int(K_{S_{D_m}}(\theta_j))$ than for $z \notin \bigcap_{j=1}^{J} \bigcup_{m=1}^{M} int(K_{S_{D_m}}(\theta_j))$. Hence, the plot I_z for any $z \in B_R(0)$ should yield a visualization of $\bigcap_{j=1}^{J} \bigcup_{m=1}^{M} int(K_{S_{D_m}}(\theta_j))$.

We use $N = 20$ observation directions $\theta_j, j = 1, \ldots, 20$ such that $\theta_j = -\pi/2 + j\pi/N$. We superimpose the indicators and show the results in Fig. 4.5. The locations and sizes of support of f are reconstructed correctly.

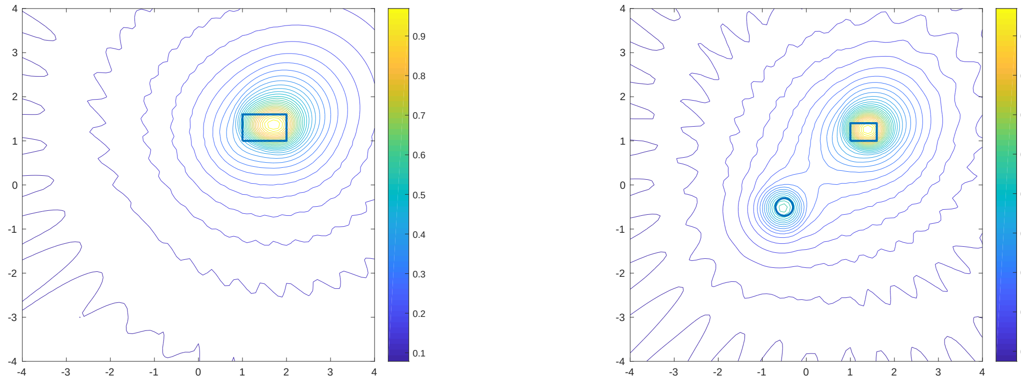

Figure 4.6: Reconstructions using multiple observation directions when $f(x, y) = x^2 - y^2 + 5$. Left: single object. Right: Two objects.

Next, we choose $f(x, y) = x^2 - y^2 + 5$, a function depending on the locations but independent of the wave number k. The reconstruction is shown in Fig. 4.6. Finally, we assume that f depends on k as well. Let

$$f_1(x, y; k) = k^2(x^2 - y^2 + 5),$$

and

$$f_2(x, y; k) = e^{ik(x \cos 3\pi/2 + y \sin 3\pi/2)}(x^2 - y^2 + 5).$$

The reconstructions are shown in Fig. 4.7. Note that this case is not covered by the theory.

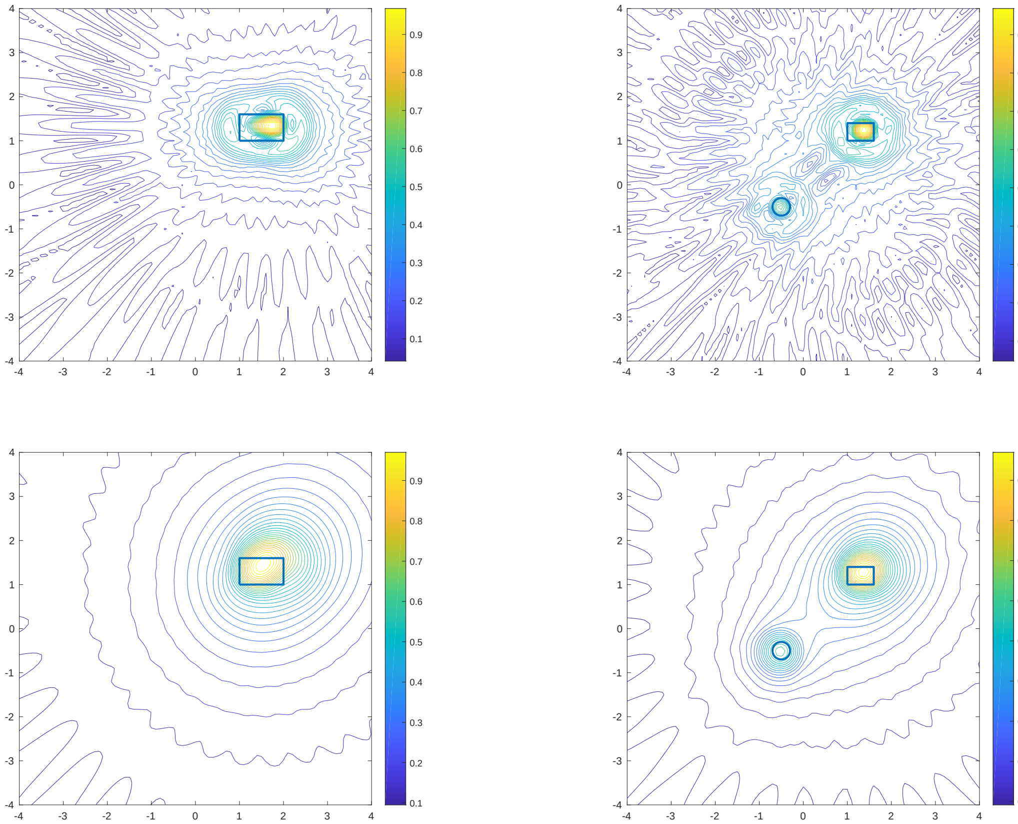

Figure 4.7: Reconstructions of sources depending on wavenumber k. Top: $f_1(x, y; k) = k^2(x^2 - y^2 + 5)$. Top Left: one object. Top Right: two objects. Bottom: $f_2(x, y; k) = e^{ik(x \cos 3\pi/2 + y \sin 3\pi/2)}(x^2 - y^2 + 5)$. Bottom Left: one object. Bottom Right: two objects.

4.3.4 Extended objects

The sizes of supports of f in the above examples are small compared with the wavelengths used. The smallest wavelength is $\lambda_{min} = 2\pi/10 \approx 0.628$. In Fig. 4.8, we show

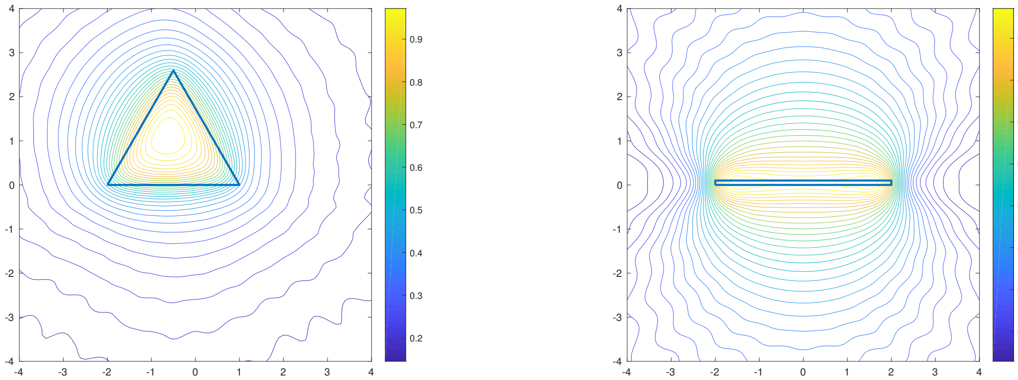

Figure 4.8: Reconstructions of larger objects when $f(x,y) = 5$. Left: triangle. Right: thin bar.

the reconstructions of larger objects. One is an equilateral triangle with vertices

$$(-2,0), \quad (1,0), \quad (-1/2, 3/2\sqrt{3}-1).$$

The second one is a thin slab given by $(-2,2)\times(0,0.1)$. The results indicate that shorter wavelength could lead to better reconstruction.

Chapter 5

Conclusions

5.1 Summary and Conclusions

In this thesis, we developed two new sampling methods for two inverse scattering problems.

We first generalized the indicator method in [45], which is discussed in Chapter 2, and proposed a new direct sampling method for inverse electromagnetic scattering problems in an inhomogeneous isotropic medium in $\mathbb{R}^3$ using the far field measurements. We considered two cases of the contrasts. First case, when all of D is absorbing. Second case, we considered the more general case whene only parts of D may be absorbing. In this method we proposed an indicator function which is big when the sampling point lies inside the scatterer and when the sampling point moves away from the boundary of the scatterer the value of the indicator function decays and goes to zero. The main feature of this method is that the indicator function is based on the inner product, and therefore the method is very simple to implement. With the help of the factorization of the corresponding far field operator, a lower bound established for sampling points inside the scatterers. Furthermore, we showed that the indicator function decays like Bessel function as the sampling point moves away from the boundary of the scatterers. Moreover, we showed that the proposed method is stable with respect to noises in the data.

As the second contribution, we proposed a new sampling method for multifrequency inverse source problem for time-harmonic acoustic waves using a finite set of far field data. The method is based on the factorization method for multifrequency inverse source problems with sparse far field measurements. The main feature of this method is that the indicator function is based on the inner product, and therefore the method is very simple to implement. We have developed a non-iterative reconstruction scheme of factorization-type to locate the support of the sources and studied the behavior of the indicator function, which gives a characterization of the support of the source. The method produces a union of convex polygons with normals in the observation directions that approximates the positions and the convex hull of well-separated source components.

5.2 Future Work

Based on the work accomplished in this thesis, we provide below a number of possible future works:

1. Study the orthogonality sampling [56] for the detection of the location and shape of objects from the far field pattern of scattered electromagnetic waves.

2. Study the theoretical foundation of the orthogonality sampling [56]. The theory of the orthogonality sampling is only partially resolved and the relation between indicator functions proposed in [56] and the shape of the scatterer is open and needs further investigation.

3. Study a factorization method for multifrequency inverse source problem for time-harmonic electromagnetic waves with a limited set of far field data.

4. Generalize the method proposed in Chapter 4 to the case of multifrequency inverse source problem for time-harmonic electromagnetic waves with a limited set of far field data.

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