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# Block-avoiding sequencings of points in Steiner triple systems

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## Abstract

Given an  $\text{STS}(v)$ , we ask if there is a permutation of the points of the design such that no  $\ell$  consecutive points in this permutation contain a block of the design. Such a permutation is called an  $\ell$ -good sequencing. We prove that 3-good sequencings exist for any  $\text{STS}(v)$  with  $v > 3$  and 4-good sequencings exist for any  $\text{STS}(v)$  with  $v > 71$ . Similar results also hold for partial  $\text{STS}(v)$ . Finally, we determine the existence or nonexistence of 4-good sequencings for all the nonisomorphic  $\text{STS}(v)$  with  $v = 7, 9, 13$  and  $15$ .

## 1 Introduction

A Steiner triple system of order  $v$  is a pair  $(X, \mathcal{B})$ , where  $X$  is a set of  $v$  points and  $\mathcal{B}$  is a set of 3-subsets of  $X$  (called blocks), such that every pair of points occur in exactly one block. We will abbreviate the phrase “Steiner triple system of order  $v$ ” to  $\text{STS}(v)$ .

It is well-known that an  $\text{STS}(v)$  contains exactly  $v(v-1)/6$  blocks, and an  $\text{STS}(v)$  exists if and only if  $v \equiv 1, 3 \pmod{6}$ . The definitive reference for Steiner triple systems is the book [4] by Colbourn and Rosa.

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Suppose  $(X, \mathcal{B})$  is an  $\text{STS}(v)$ . We ask if there is a permutation (or *sequencing*) of the points in  $X$  so that no three consecutive points in the sequencing comprise a block in  $\mathcal{B}$ . That is, can we find a sequencing  $\pi = [x_1 x_2 \cdots x_v]$  of  $X$  such that  $\{x_i, x_{i+1}, x_{i+2}\} \notin \mathcal{B}$  for all  $i$ ,  $1 \leq i \leq v - 2$ ? Such a sequencing will be termed a *3-good* sequencing for the given  $\text{STS}(v)$ .

More generally, for an integer  $\ell \geq 3$ , we could ask if there is a sequencing of the points such that no  $\ell$  consecutive points in the sequencing contain a block in  $\mathcal{B}$ . Such a sequencing will be termed  $\ell$ -good for the given  $\text{STS}(v)$ . Obviously an  $\ell$ -good sequencing is also  $m$ -good if  $3 \leq m \leq \ell$ .

As an example, consider the  $\text{STS}(7)$   $(X, \mathcal{B})$ , where  $X = \mathbb{Z}_7$  and  $\mathcal{B} = \{013, 124, 235, 346, 450, 451, 562\}$ . The sequencing  $[0 1 2 3 4 5 6]$  is easily seen to be 3-good. However, it is not 4-good, as the block 013 is contained in the first four points of the sequencing. (Note that, here and elsewhere, we might write blocks  $\{x, y, z\}$  as  $xyz$  if the context is clear.)

Actually, it is not difficult to see that the unique (up to isomorphism)  $\text{STS}(7)$  does not have a 4-good sequencing. By relabelling points if necessary, suppose there is a 4-good sequence for an  $\text{STS}(7)$  that begins  $[0 1 2 3]$ . There cannot be a block contained in  $\{0, 1, 2, 3\}$ . Hence, without loss of generality,  $\{0, 1, 4\}$ ,  $\{0, 2, 5\}$  and  $\{0, 3, 6\}$  are blocks. This forces the remaining blocks to be  $\{1, 2, 6\}$ ,  $\{1, 3, 5\}$ ,  $\{2, 3, 4\}$  and  $\{4, 5, 6\}$ . In particular,  $\{4, 5, 6\}$  is a block, so there is no way to complete the sequence beginning  $[0 1 2 3]$  to a 4-good sequence.

A *partial Steiner triple system of order  $v$*  is a pair  $(X, \mathcal{B})$ , where  $X$  is a set of  $v$  points and  $\mathcal{B}$  is a set of 3-subsets of  $X$  (called *blocks*), such that every pair of points occur in at most one block. We will abbreviate the phrase “partial Steiner triple system of order  $v$ ” to partial  $\text{STS}(v)$  or  $\text{PSTS}(v)$ . There are no congruential restrictions on the values  $v$  for which  $\text{PSTS}(v)$  exist. We will also consider  $\ell$ -good sequencings of  $\text{PSTS}(v)$ .

The main results we prove in this paper are that every  $\text{STS}(v)$  with  $v > 3$  has a 3-good sequencing, and every  $\text{STS}(v)$  with  $v > 71$  has a 4-good sequencing. Similar results are obtained for  $\text{PSTS}(v)$  as well. We also study 4-good sequencings of  $\text{STS}(v)$  for  $v \leq 15$ . We show that there is no 4-good sequencing of the  $\text{STS}(7)$  or  $\text{STS}(9)$ , but all  $\text{STS}(13)$  and  $\text{STS}(15)$  have 4-good sequencings.

We will use the following notation. Suppose  $(X, \mathcal{B})$  is an  $\text{STS}(v)$ . Then, for any pair of points  $x, y$ , let  $\text{third}(x, y) = z$  if and only if  $\{x, y, z\} \in \mathcal{B}$ . The function  $\text{third}$  is well-defined because every pair of points occurs in a unique block in  $\mathcal{B}$ .

### 1.1 Background and motivation

Brian Alspach gave a talk entitled “Strongly Sequenceable Groups” at the 2018 Kliakhandler Conference, which was held at Michigan Technological University. In this talk, among other things, the notion of sequencing diffuse posets was introduced and the following research problem was posed:

“Given a triple system of order  $n$  with  $\lambda = 1$ , define a poset  $P$  by letting its elements be the triples and any union of disjoint triples. This poset is not diffuse in general, but it is certainly possible that  $P$  is sequenceable.”

A *sequenceable* STS( $v$ ) (or PSTS( $v$ )) is an STS( $v$ ) in which the points can be ordered (i.e., sequenced) so that no  $t$  consecutive points can be partitioned into  $t/3$  blocks, for any  $t \equiv 0 \pmod 3$ ,  $t < v$ . The problem is studied in Alspach, Kreher and Pastine [1] and in Kreher and Stinson [3]. In [3], it is shown that there is a nonsequenceable STS( $v$ ) for all  $v \equiv 1 \pmod 6$ ,  $v > 7$ .

One possible relaxation of the definition of sequenceable STS( $v$ ) would be to require a sequencing of the points so that no  $t$  consecutive points can be partitioned into  $t/3$  blocks, for any  $t \equiv 0 \pmod 3$ ,  $t \leq w$ , where  $w < v$  is some specified integer. Such an STS( $v$ ) could be termed *w-semi-sequenceable*.

A 3-semi-sequenceable STS( $v$ ) has a sequencing of the points so that no three consecutive points form a block. This is identical to a “3-good sequencing.” As noted above, we then generalize this notion to  $\ell$ -good sequencings and we consider the cases  $\ell = 3$  and  $\ell = 4$  in detail.

Although we do not explicitly study  $w$ -semi-sequenceable STS in this paper, we note the following connection between  $w$ -semi-sequenceable STS( $v$ ) and STS( $v$ ) having  $\ell$ -good sequencings.

**Theorem 1.1.** *Let  $u \geq 1$  be an integer. An STS( $v$ ) that has a  $(2u + 1)$ -good sequencing is  $3u$ -semi-sequenceable.*

*Proof.* Let  $\pi$  be a sequencing of the points of an STS( $v$ ) that is not  $3u$ -semi-sequenceable. Then, for some  $t \equiv 0 \pmod 3$ , there are  $t$  consecutive points in  $\pi$  that can be partitioned into  $t/3$  blocks of the STS( $v$ ). Let these  $t$  points be denoted (in order)  $x_1, \dots, x_t$ . Then

$$\{x_1, \dots, x_t\} = \bigcup_{j=1}^{t/3} B_j,$$

where  $B_1, \dots, B_{t/3}$  are blocks in the STS( $v$ ). For  $1 \leq j \leq t/3$ , let

$$m_{lo}(j) = \min\{i : x_i \in B_j\}$$

and let

$$m_{hi}(j) = \max\{i : x_i \in B_j\}.$$

Clearly there is a block  $B_j$  such that  $m_{lo}(j) \geq t/3$ . It also holds that  $m_{hi}(j) \leq t$ . Therefore the block  $B_j \subseteq \{x_{t/3}, \dots, x_t\}$ , which means that the sequencing  $\pi$  is not  $(2t/3 + 1)$ -good.  $\square$

## 2 Existence of 3-good sequencings

In this section, we show that there is a 3-good sequencing for any STS( $v$ ) with  $v \geq 7$ . We prove this in three ways: by a counting argument, by using a greedy algorithm,

and by relabelling the points of the design in a suitable way. The counting argument and greedy algorithm can also be adapted to handle PSTS( $v$ ) with  $v > 3$  ( $v > 5$ , respectively).

### 2.1 A counting argument

Let  $(X, \mathcal{B})$  be an STS( $v$ ) on points  $X = \{1, \dots, v\}$ . For a sequencing  $\pi = [x_1 \ x_2 \ \dots \ x_v]$  of  $X$ , and for any  $i$ ,  $1 \leq i \leq v - 2$ , define  $\pi$  to be *i-forbidden* if  $\{x_i, x_{i+1}, x_{i+2}\} \in \mathcal{B}$ . Let  $\text{forbidden}(i)$  denote the set of *i*-forbidden sequencings. Also, define a sequencing to be *forbidden* if it is *i*-forbidden for at least one value of  $i$  and let  $\text{forbidden}$  denote the set of forbidden sequencings. Clearly, a sequencing is 3-good if and only if it is not forbidden.

**Theorem 2.1.** *Suppose  $v > 3$  and  $(X, \mathcal{B})$  is an STS( $v$ ) on points  $X = \{1, \dots, v\}$ . Then there is a sequencing  $\pi = [x_1 \ x_2 \ \dots \ x_v]$  of  $X$  that is 3-good for  $(X, \mathcal{B})$ .*

*Proof.* Clearly,

$$\text{forbidden} = \bigcup_{i=1}^{v-2} \text{forbidden}(i).$$

For any given value of  $i$ , it holds that  $|\text{forbidden}(i)| = v!/(v-2)$ . This follows because, for any two points,  $x_i$  and  $x_{i+1}$ , the 3-subset  $\{x_i, x_{i+1}, x_{i+2}\} \in \mathcal{B}$  if and only if  $x_{i+2} = \text{third}(x_i, x_{i+1})$ . So given any  $x_i$  and  $x_{i+1}$ , the probability that  $\{x_i, x_{i+1}, x_{i+2}\} \in \mathcal{B}$  is  $1/(v-2)$ .

Next, by the union bound,

$$|\text{forbidden}| \leq \sum_{i=1}^{v-2} |\text{forbidden}(i)| = (v-2) \times \frac{v!}{(v-2)} = v! \tag{1}$$

Equality in (1) would be obtained if and only if the sets  $\text{forbidden}(i)$ ,  $1 \leq i \leq v - 2$ , are pairwise disjoint.

We show that equality in (1) is impossible: Consider any two intersecting blocks  $\{a, b, c\}, \{c, d, e\} \in \mathcal{B}$  (here is where we use the assumption that  $v > 3$ ). Then any sequencing in which the first five symbols are  $a \ b \ c \ d \ e$  (in that order) is in  $\text{forbidden}(1) \cap \text{forbidden}(3)$ . Therefore,  $|\text{forbidden}| < v!$  and thus there exists a 3-good sequencing. □

Theorem 2.1 also holds for *partial* STS( $v$ ) when  $v > 3$ .

**Theorem 2.2.** *Suppose  $v > 3$  and  $(X, \mathcal{B})$  is a partial STS( $v$ ) on points  $X = \{1, \dots, v\}$ . Then there is a sequencing  $\pi = [x_1 \ x_2 \ \dots \ x_v]$  of  $X$  that is 3-good for  $(X, \mathcal{B})$ .*

*Proof.* If  $(X, \mathcal{B})$  is an STS( $v$ ), then we are done by Theorem 2.1. Therefore, we can assume there is at least one pair  $\{a, b\}$  that does not appear in any block in  $\mathcal{B}$ . Suppose  $x_i = a$  and  $x_{i+1} = b$ . Then, for every possible  $x_{i+2}$ , we have  $\{x_i, x_{i+1}, x_{i+2}\} \notin \mathcal{B}$ . It then follows that  $|\text{forbidden}(i)| < v!/(v-2)$  for all  $i$ .

Now, when we apply the union bound, we have

$$|\text{forbidden}| \leq \sum_{i=1}^{v-2} |\text{forbidden}(i)| < (v - 2) \times v! / (v - 2) = v!$$

and we are done. □

### 2.2 A greedy algorithm

Theorem 2.1 and a slightly weaker version of Theorem 2.2 can also be proven using a greedy algorithm. First, we consider the case where  $(X, \mathcal{B})$  is an STS( $v$ ), where  $X = \{x_1, \dots, x_v\}$ . We begin by choosing any two distinct values for  $x_1$  and  $x_2$  and then we attempt to define  $x_3, x_4, \dots, x_v$  in turn, in such a way that we end up with a 3-good sequencing. Thus, our strategy is to design a greedy algorithm.

Consider any  $i$  such that  $3 \leq i \leq v - 1$ . We require the following conditions to be satisfied:

1.  $x_i \notin \{x_1, \dots, x_{i-1}\}$ , and
2.  $x_i \neq \text{third}(x_{i-2}, x_{i-1})$ .

Thus, there are at most  $i$  values for  $x_i$  that are ruled out. Because  $i \leq v - 1$ , there is at least one value for  $x_i$  that satisfies the two required conditions.

After choosing  $x_1, x_2, \dots, x_{v-1}$  as described above, there is only one unused value remaining for  $x_v$ . But this might not result in a 3-good sequencing, if it happens that  $\{x_{v-2}, x_{v-1}, x_v\} \in \mathcal{B}$ . However, in this case, it turns out that we can find a slight modification of the sequencing  $[x_1 x_2 \cdots x_v]$  that is 3-good, provided that  $v > 5$ .

Suppose we made sure to select  $x_5$  such that  $\{x_2, x_3, x_5\} \in \mathcal{B}$ , i.e., we define  $x_5 = \text{third}(x_2, x_3)$ . This is an allowable choice for  $x_5$  because

- $\{x_1, x_2, x_3\} \notin \mathcal{B}$  and  $\{x_2, x_3, x_4\} \notin \mathcal{B}$ , which implies that

$$x_5 \notin \{x_1, x_2, x_3, x_4\},$$

and

- $\{x_3, x_4, x_5\} \notin \mathcal{B}$ , because  $\{x_2, x_3, x_5\} \in \mathcal{B}$  and  $x_2 \neq x_4$ .

Now, suppose we have a sequencing  $[x_1 x_2 \cdots x_v]$ , where  $\{x_2, x_3, x_5\} \in \mathcal{B}$ , which fails to be 3-good only because  $\{x_{v-2}, x_{v-1}, x_v\} \in \mathcal{B}$  (which is not allowed). Consider the modified sequencing  $[y_1 y_2 \cdots y_v]$  obtained from  $[x_1 x_2 \cdots x_v]$  by switching  $x_1$  and  $x_v$ . In order to show that  $[y_1 y_2 \cdots y_v]$  is a 3-good sequencing, we need to show that

1.  $\{y_{v-2}, y_{v-1}, y_v\} = \{x_{v-2}, x_{v-1}, x_1\} \notin \mathcal{B}$ , and
2.  $\{y_1, y_2, y_3\} = \{x_v, x_2, x_3\} \notin \mathcal{B}$ .

1. Choose a block  $\{b, c, e\} \in \mathcal{B}$ , let  $a \neq b, c, e$  and let  $d \neq a, b, c, e$ .
2. Define  $x_1 = a, x_2 = b, x_3 = c, x_4 = d$  and  $x_5 = e$ .
3. **For**  $i = 6$  **to**  $v - 1$  **do** define  $x_i$  to be any element of  $X$  that is distinct from the values  $x_1, \dots, x_{i-1}$  and  $\text{third}(x_{i-2}, x_{i-1})$ .
4. Define  $x_v$  to be the unique value that is distinct from  $x_1, \dots, x_{v-1}$ .
5. **If**  $\{x_{v-2}, x_{v-1}, x_v\} \in \mathcal{B}$  **then** interchange  $x_1$  and  $x_v$ .
6. **Return**  $(\pi = [x_1 \ x_2 \ \cdots \ x_v])$ .

Figure 1: Algorithm to find a 3-good sequencing for a partial STS( $v$ ),  $(X, \mathcal{B})$

To prove 1, we observe that  $\{x_{v-2}, x_{v-1}, x_1\} \notin \mathcal{B}$  because  $\{x_{v-2}, x_{v-1}, x_v\} \in \mathcal{B}$  and  $x_v \neq x_1$ . To prove 2, we observe that  $\{x_2, x_3, x_5\} \in \mathcal{B}$  and  $x_v \neq x_5$  because  $v > 5$ . Thus the sequencing  $[y_1 \ y_2 \ \cdots \ y_v]$  is 3-good.

The above-described process can also be carried out to find a 3-good sequencing for any partial STS( $v$ ) with  $v > 5$ . The resulting algorithm is presented in Figure 1.

From the discussion above, we have the following theorem.

**Theorem 2.3.** *Suppose that  $(X, \mathcal{B})$  is a partial STS( $v$ ) with  $v > 5$ . Then the Algorithm presented in Figure 1 will find a sequencing  $\pi$  that is 3-good for  $(X, \mathcal{B})$ .*

### 2.3 Relabelling points

In this section, we give a short, elegant proof that an STS( $v$ ) with  $v \geq 7$  has a 3-good sequencing. This proof was pointed out to us by Charlie Colbourn (private communication).

Given an STS( $v$ ) with  $v \geq 7$ , say on points  $1, \dots, v$ , relabel the points so that the blocks containing 1 are

$$\{1, 2, v\}, \{1, 3, 4\}, \{1, 5, 6\}, \dots, \{1, v - 2, v - 1\}.$$

Then consider the sequencing  $[1 \ 2 \ \cdots \ v]$ . We observe the following, using the fact that  $v \geq 7$ :

- $\{1, 2, 3\}$  is not a block because  $\{1, 2, v\}$  is a block
- $\{2, 3, 4\}$  is not a block because  $\{1, 3, 4\}$  is a block
- $\{3, 4, 5\}$  is not a block because  $\{1, 3, 4\}$  is a block
- $\{4, 5, 6\}$  is not a block because  $\{1, 5, 6\}$  is a block
- $\{5, 6, 7\}$  is not a block because  $\{1, 5, 6\}$  is a block
- etc.
- $\{v - 3, v - 2, v - 1\}$  is not a block because  $\{1, v - 2, v - 1\}$  is a block
- $\{v - 2, v - 1, v\}$  is not a block because  $\{1, v - 2, v - 1\}$  is a block.

Therefore this sequencing is 3-good.

### 3 Existence of 4-good sequencings

For any integer  $\ell \geq 3$ , it is tempting to conjecture that all “sufficiently large”  $\text{STS}(v)$  have  $\ell$ -good sequencings. In this section, we prove this conjecture for the case  $\ell = 4$ . Then, we present some results on 4-good sequencings of “small”  $\text{STS}(v)$  in Section 3.1.

We might attempt to show the existence of a 4-good sequencing using any of the three methods described in the previous section. It turned out that we were able to do this using a greedy strategy similar to the algorithm presented in Figure 1. In general, when we choose a value for  $x_i$ , it must be distinct from  $x_1, \dots, x_{i-1}$ , of course. It is also required that

$$x_i \notin \{\text{third}(x_{i-3}, x_{i-2}), \text{third}(x_{i-3}, x_{i-1}), \text{third}(x_{i-2}, x_{i-1})\}.$$

There will be a permissible choice for  $x_i$  provided that  $i - 1 + 3 \leq v - 1$ , which is equivalent to the condition  $i \leq v - 3$ . Thus we can define  $x_1, x_2, \dots, x_{v-3}$  in such a way that they satisfy the relevant conditions, and our task would be to somehow fill in the last three positions of the sequencing, after appropriate modifications, to satisfy the desired properties. We describe how to do this now, for sufficiently large values of  $v$ .

Suppose that  $[x_1 \ x_2 \ \cdots \ x_{v-3}]$  is a 4-good partial sequencing of  $X = \{1, \dots, v\}$  (that is, there is no block contained in any four consecutive points in the sequence  $[x_1 \ x_2 \ \cdots \ x_{v-3}]$ ). Let  $\{\alpha_1, \alpha_2, \alpha_3\} = X \setminus \{x_1, x_2, \dots, x_{v-3}\}$ . Also, let

$$\begin{aligned} \beta_1 &= \text{third}(x_{v-5}, x_{v-4}), \\ \beta_2 &= \text{third}(x_{v-5}, x_{v-3}), \text{ and} \\ \beta_3 &= \text{third}(x_{v-4}, x_{v-3}). \end{aligned}$$

Clearly  $\beta_1, \beta_2$  and  $\beta_3$  are distinct. Observe that  $x_{v-2}$  and  $x_{v-1}$  must be chosen so that  $x_{v-2} \neq \beta_1, \beta_2, \beta_3$  and  $x_{v-1} \neq \beta_3$ .

By permuting  $\alpha_1, \alpha_2, \alpha_3$  if necessary, we can assume the following two conditions hold:

$$\alpha_2 \neq \beta_3 \tag{2}$$

and

$$x_{v-3} \neq \text{third}(\alpha_2, \alpha_3). \tag{3}$$

Now, define the following:

$$\begin{aligned} \gamma &= \text{third}(\alpha_2, x_{v-3}), \\ \delta &= \text{third}(\alpha_2, x_{v-4}), \\ \epsilon &= \text{third}(\alpha_3, x_{v-3}), \text{ and} \\ \eta &= \text{third}(\alpha_2, \alpha_3). \end{aligned}$$

Next, suppose we define  $x_{v-2} = \chi$ ,  $x_{v-1} = \alpha_2$  and  $x_v = \alpha_3$ , where

$$\chi \notin \{x_{v-5}, x_{v-4}, x_{v-3}, \beta_1, \beta_2, \beta_3, \gamma, \delta, \epsilon, \eta\} \tag{4}$$



Table 1: Possible blocks in the last six elements of the sequencing

triple	explanation
$\{x_{v-5}, x_{v-4}, x_{v-3}\}$	greedy algorithm ensures it is not a block
$\{x_{v-5}, x_{v-4}, \chi\}$	$\{x_{v-5}, x_{v-4}, \beta_1\}$ is a block and $\chi \neq \beta_1$
$\{x_{v-5}, x_{v-3}, \chi\}$	$\{x_{v-5}, x_{v-3}, \beta_2\}$ is a block and $\chi \neq \beta_2$
$\{x_{v-4}, x_{v-3}, \chi\}$	$\{x_{v-4}, x_{v-3}, \beta_3\}$ is a block and $\chi \neq \beta_3$
$\{x_{v-4}, x_{v-3}, \alpha_2\}$	$\{x_{v-4}, x_{v-3}, \beta_3\}$ is a block and $\alpha_2 \neq \beta_3$ by (2)
$\{x_{v-4}, \chi, \alpha_2\}$	$\{x_{v-4}, \delta, \alpha_2\}$ is a block and $\chi \neq \delta$
$\{x_{v-3}, \chi, \alpha_2\}$	$\{x_{v-3}, \gamma, \alpha_2\}$ is a block and $\chi \neq \gamma$
$\{x_{v-3}, \chi, \alpha_3\}$	$\{x_{v-3}, \epsilon, \alpha_3\}$ is a block and $\chi \neq \epsilon$
$\{x_{v-3}, \alpha_2, \alpha_3\}$	this is not a block by (3)
$\{\chi, \alpha_2, \alpha_3\}$	$\{\eta, \alpha_2, \alpha_3\}$ is a block and $\chi \neq \eta$ .

is to be determined. Thus, the last six elements of the sequencing will be

$$x_{v-5} \ x_{v-4} \ x_{v-3} \ \chi \ \alpha_2 \ \alpha_3.$$

There should be no block in  $\mathcal{B}$  contained in any four consecutive points chosen from these six points. We enumerate all the relevant triples and verify that none of them are blocks in Table 1.

Suppose  $v \geq 14$ . Our strategy is to define  $\chi$  to be one of  $x_1, x_2, \dots, x_8$ , in such a way that (4) is satisfied. Note that  $v - 5 \geq 9$ , so we are guaranteed that  $\chi \neq x_{v-5}, x_{v-4}, x_{v-3}$ . We can choose  $\chi \in \{x_1, x_2, \dots, x_8\}$  because at least one of these eight values is not in the set  $\{\beta_1, \beta_2, \beta_3, \gamma, \delta, \epsilon, \eta\}$ , which has size 7. Suppose we take  $\chi = x_\kappa$ , where  $\kappa \in \{1, 2, \dots, 8\}$ . Then we redefine  $x_\kappa = \alpha_1$ . Another way to describe this process is to temporarily define  $x_{v-2} = \alpha_1$  and then interchange  $x_{v-2}$  with  $x_\kappa$ .

Now, when we initially choose  $x_1, x_2, x_3, \dots$ , we have no idea which value  $\alpha_1$  we will be interchanging with  $x_\kappa$ . So it is necessary to ensure that any value we “swap in” will not result in a block being contained in four successive points of the sequencing. Clearly we only have to worry about the first  $8 + 3 = 11$  points,  $x_1, x_2, x_3, \dots, x_{11}$ .

Define

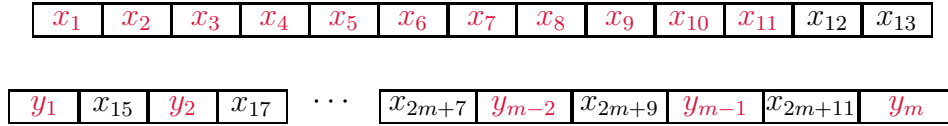
$$Y = \{\text{third}(x_i, x_j) : 1 \leq i < j \leq 11, |i - j| \leq 3\} \setminus \{x_1, \dots, x_{11}\}.$$

(Note, in the definition of  $Y$ , that we do not care about pairs of points that are more than three positions apart.) Denote the points in  $Y$  as  $y_1, \dots, y_m$ . It is not hard to verify that  $m \leq 27$ , because there are ten pairs  $x_i, x_j$  in  $\{x_1, \dots, x_{11}\}$  with  $j - i = 1$ , nine pairs with  $j - i = 2$  and eight pairs with  $j - i = 3$ .

Having already chosen  $x_1, \dots, x_{11}$ , we want to “pre-specify” some of the next points. Due to the changes that are introduced, this part of the algorithm will be referred to as the “modified greedy approach.” To be specific, we define  $x_{14} = y_1$ ,

$x_{16} = y_2, \dots, x_{2m+12} = y_m$ . Note that no three of the  $y_i$ 's are contained in four consecutive points of the sequencing, from  $x_{12}$  to  $x_{2m+12}$ .

The following diagram might be helpful in the subsequent discussion:



In this diagram, the red values have been defined and we need to determine the black values. Let us consider how the greedy algorithm must be modified in order to accomplish this. We have the following additional restrictions “looking ahead” when choosing values for  $x_{12}, x_{13}, x_{15}, \dots, x_{2m+11}$ :

- each of  $x_{12}, x_{13}, x_{15}, \dots, x_{2m+11}$  must be distinct from  $y_1, \dots, y_m$ ;
- we require that  $\{x_{11}, x_{12}, y_1\} \notin \mathcal{B}$ , so we must define

$$x_{12} \neq \text{third}(x_{11}, y_1);$$

- we require that

$$\{x_{11}, x_{13}, y_1\}, \{x_{12}, x_{13}, y_1\}, \{x_{13}, y_1, y_2\} \notin \mathcal{B},$$

so we must define

$$x_{13} \neq \text{third}(x_{11}, y_1), \text{third}(x_{12}, y_1), \text{third}(y_1, y_2);$$

- in the “general” case, for  $i = 2, \dots, m - 1$ , we require that

$$\{x_{2i+9}, x_{2i+11}, y_i\}, \{y_{i-1}, x_{2i+11}, y_i\}, \{x_{2i+11}, y_i, y_{i+1}\} \notin \mathcal{B},$$

so we must define

$$x_{2i+11} \neq \text{third}(x_{2i+9}, y_i), \text{third}(y_{i-1}, y_i), \text{third}(y_i, y_{i+1});$$

- finally, we require that

$$\{x_{2m+9}, x_{2m+11}, y_m\}, \{y_{m-1}, x_{2m+11}, y_m\} \notin \mathcal{B},$$

so we must define

$$x_{2m+11} \neq \text{third}(x_{2m+9}, y_m), \text{third}(y_{m-1}, y_m).$$

Of course, we need to ensure that the greedy algorithm can choose values for all these  $x_i$ 's.

Now consider what happens when we swap  $x_\kappa$  with  $\alpha_1$ . The value  $\alpha_1 \notin Y$ , so  $\alpha_1$  cannot form a block with any two of the points  $x_1, \dots, x_{11}$ . Because  $\kappa \leq 8$ , there are no blocks contained in any four consecutive points chosen from the first 11

1. Determine  $x_1, \dots, x_{11}$  using the greedy approach.
2. Fill in the values  $y_1, \dots, y_m$  and then determine the remaining values  $x_{12}, \dots, x_{2m+11}$  using the modified greedy approach.
3. Determine  $x_{2m+13}, \dots, x_{v-3}$  using the greedy approach.
4. Define the values  $x_{v-2} = \alpha_1, x_{v-1} = \alpha_2, x_v = \alpha_3$  as described in the text, and then swap  $x_{v-2}$  with  $x_{v-3}$ .
5. **Return**  $(\pi = [x_1 \ x_2 \ \dots \ x_v])$ .

Figure 2: Algorithm to find a 4-good sequencing for an STS( $v$ ),  $(X, \mathcal{B})$

points of the sequencing. At the opposite end, we have guaranteed that there are no blocks contained in any four consecutive points chosen from the last six points of the sequencing, because of the way that  $x_{v-3}$  was chosen.

Summarizing, the resulting algorithm has the high-level structure described in Figure 2.

All the above steps can be carried out if we ensure that the first  $2m + 12$  elements of the sequencing do not overlap with the last six elements of the sequencing. Because  $m \leq 27$ , this condition is guaranteed to hold if  $v - 5 \geq 2 \times 27 + 12 + 1$ , or  $v \geq 72$ . So we have proven the following.

**Theorem 3.1.** *Suppose  $v > 71$  and  $(X, \mathcal{B})$  is an STS( $v$ ) on points  $X = \{1, \dots, v\}$ . Then there is a sequencing  $\pi = [x_1 \ x_2 \ \dots \ x_v]$  of  $X$  that is 4-good for  $(X, \mathcal{B})$ .*

A similar result can also be proven for PSTS( $v$ ) using this technique.

### 3.1 Results on 4-good sequencings of STS( $v$ ) for $v \leq 15$

We have shown in Section 1 that there is no 4-good sequencing for the unique STS(7). Here, we use a counting method to establish the same result, as well as an analogous result for the unique STS(9).

Suppose we take the points of an STS( $v$ ) to be  $1, \dots, v$ . Suppose, by relabelling points if necessary, that  $[1 \ 2 \ 3 \ 4 \ \dots \ v]$  a 4-good sequencing of an STS( $v$ ). We say that a block  $B$  is of *type  $i$*  if  $|B \cap \{1, 2, 3, 4\}| = i$ . Clearly, we must have  $i \in \{0, 1, 2\}$ .

For  $i = 0, 1, 2$ , let  $b_i$  denote the number of blocks of type  $i$ . Simple counting allows us to determine the values  $b_0, b_1$  and  $b_2$ . First, because the sequencing is 4-good, we have  $b_2 = \binom{4}{2} = 6$ . Next, because each point appears in  $(v - 1)/2$  blocks, we have  $b_1 = 4((v - 1)/2 - 3) = 2v - 14$ . Finally, because the total number of blocks is  $v(v - 1)/6$ , we have  $b_0 = v(v - 1)/6 - (2v - 14) - 6 = v(v - 1)/6 - 2v + 8$ .

Now, if  $v = 7$ , we obtain  $b_2 = 6, b_1 = 0$  and  $b_0 = 1$ . The block of type 0 must be  $\{5, 6, 7\}$ . Since these are the last three points of the sequencing, the sequencing is not even 3-good. Therefore there is no 4-good sequencing of the STS(7).

If  $v = 9$ , we obtain  $b_2 = 6, b_1 = 4$  and  $b_0 = 2$ . If the sequencing is 4-good, then any block  $B$  of type 0 must contain both 5 and 9 (if not, then  $B \subseteq \{5, 6, 7, 8\}$  or

$B \subseteq \{6, 7, 8, 9\}$ , neither of which can occur if the sequencing is 4-good). But there is at most one block in the  $\text{STS}(v)$  that contains the pair  $\{5, 9\}$ , so  $b_0 = 2$  is impossible. Therefore there is no 4-good sequencing of the  $\text{STS}(9)$ .

However, for  $v = 13, 15$ , we quickly found 4-good sequencings of all the nonisomorphic  $\text{STS}(v)$  by computer, by using a simple backtracking algorithm. We have found such sequencings for the two nonisomorphic  $\text{STS}(13)$  and the 80 nonisomorphic  $\text{STS}(15)$ ; these are presented in the Appendix.

### 4 Conclusion

For any integer  $\ell \geq 3$ , let  $n(\ell)$  denote the smallest integer such that the following property is satisfied:

$$\text{any STS}(v) \text{ with } v > n(\ell) \text{ has an } \ell\text{-good sequencing.} \tag{5}$$

Also, define  $n(\ell) = \infty$  if no integer satisfying (5) exists.

We conjecture that  $n(\ell)$  is finite for every integer  $\ell \geq 3$ . Further, based on the computational results from Section 3.1, we conjecture that  $n(4) = 9$ .

### Acknowledgement

We thank Charlie Colbourn for providing the construction presented in Section 2.3.

### Appendix: 4-good sequencings for $\text{STS}(13)$ and $\text{STS}(15)$

We present 4-good sequencings for the two nonisomorphic  $\text{STS}(13)$  and the 80 nonisomorphic  $\text{STS}(15)$ . These designs are listed in the same order as in the *Handbook of Combinatorial Designs* [2, Table 1.27 and 1.28].

Table 2: 4-good sequencings for the  $\text{STS}(13)$

- 1:  $[0\ 1\ 3\ 6\ 2\ 4\ 7\ 8\ 5\ 9\ b\ a\ c]$       2:  $[0\ 1\ 3\ 6\ 2\ 4\ 7\ 8\ 5\ 9\ a\ b\ c]$

Table 3: 4-good sequencings for the  $\text{STS}(15)$

- 1:  $[0\ 1\ 3\ 6\ 7\ 4\ 2\ 8\ a\ 5\ 9\ d\ c\ b\ e]$     2:  $[0\ 1\ 3\ 6\ 7\ 4\ 2\ 8\ a\ 5\ 9\ d\ b\ c\ e]$   
 3:  $[0\ 1\ 3\ 6\ 7\ 4\ 2\ 8\ a\ 5\ 9\ d\ b\ c\ e]$     4:  $[0\ 1\ 3\ 6\ 7\ 4\ 2\ 8\ a\ 5\ 9\ d\ b\ c\ e]$   
 5:  $[0\ 1\ 3\ 6\ 7\ 4\ 2\ 8\ a\ 5\ 9\ b\ c\ d\ e]$     6:  $[0\ 1\ 3\ 6\ 7\ 4\ 2\ 8\ a\ 5\ 9\ c\ d\ b\ e]$   
 7:  $[0\ 1\ 3\ 6\ 7\ 4\ 2\ 8\ a\ 5\ 9\ c\ b\ d\ e]$     8:  $[0\ 1\ 3\ 6\ 7\ 4\ 2\ 8\ 9\ 5\ a\ d\ b\ c\ e]$   
 9:  $[0\ 1\ 3\ 6\ 7\ 4\ 2\ 8\ 9\ 5\ a\ d\ c\ b\ e]$     10:  $[0\ 1\ 3\ 6\ 7\ 4\ 2\ 8\ 9\ 5\ a\ c\ b\ d\ e]$   
 11:  $[0\ 1\ 3\ 6\ 7\ 4\ 2\ 8\ 9\ a\ b\ 5\ d\ c\ e]$     12:  $[0\ 1\ 3\ 6\ 7\ 4\ 2\ 8\ 9\ 5\ a\ b\ c\ d\ e]$   
 13:  $[0\ 1\ 3\ 6\ 7\ 4\ 2\ 8\ 9\ a\ b\ 5\ d\ c\ e]$     14:  $[0\ 1\ 3\ 6\ 7\ 4\ 2\ 8\ a\ 5\ 9\ d\ b\ c\ e]$   
 15:  $[0\ 1\ 3\ 6\ 7\ 4\ 2\ 8\ 9\ 5\ a\ b\ c\ d\ e]$     16:  $[0\ 1\ 3\ 6\ 7\ 4\ 2\ 8\ a\ 5\ 9\ d\ c\ b\ e]$   
 17:  $[0\ 1\ 3\ 6\ 7\ 4\ 2\ 8\ 9\ a\ b\ 5\ d\ c\ e]$     18:  $[0\ 1\ 3\ 6\ 7\ 4\ 2\ 8\ 9\ 5\ a\ d\ b\ c\ e]$   
 19:  $[0\ 1\ 3\ 6\ 7\ 4\ 2\ 8\ a\ 5\ 9\ c\ e\ b\ d]$     20:  $[0\ 1\ 3\ 6\ 7\ 4\ 2\ 8\ 9\ 5\ a\ b\ c\ d\ e]$   
 21:  $[0\ 1\ 3\ 6\ 7\ 4\ 2\ 8\ 9\ a\ b\ 5\ d\ c\ e]$     22:  $[0\ 1\ 3\ 6\ 7\ 4\ 2\ 8\ 9\ 5\ a\ c\ d\ b\ e]$

23: [0 1 3 6 7 4 5 8 2 9 a b c d e]	24: [0 1 3 6 7 4 5 8 2 9 a b c d e]
25: [0 1 3 6 7 4 5 8 2 9 a b c d e]	26: [0 1 3 6 7 4 5 8 2 a b 9 c d e]
27: [0 1 3 6 7 4 5 8 2 9 a b c d e]	28: [0 1 3 6 7 4 5 8 2 9 a b c d e]
29: [0 1 3 6 7 4 5 8 2 9 a b c d e]	30: [0 1 3 6 7 4 5 8 2 9 a b c d e]
31: [0 1 3 6 7 4 5 8 2 9 a b c d e]	32: [0 1 3 6 7 4 5 8 2 9 a b c d e]
33: [0 1 3 6 7 4 5 8 2 9 a b c d e]	34: [0 1 3 6 7 4 5 8 2 9 a b c d e]
35: [0 1 3 6 2 4 5 7 8 a 9 b c d e]	36: [0 1 3 6 2 4 5 7 8 a 9 b c d e]
37: [0 1 3 6 2 4 5 7 8 9 a b c d e]	38: [0 1 3 6 2 4 5 7 8 9 a c b d e]
39: [0 1 3 6 2 4 5 7 8 9 a c b d e]	40: [0 1 3 6 2 4 5 7 8 a 9 b c d e]
41: [0 1 3 6 2 4 5 7 8 9 a b c d e]	42: [0 1 3 6 2 4 5 7 8 9 a b c d e]
43: [0 1 3 6 2 4 5 7 8 9 a c b d e]	44: [0 1 3 6 2 4 5 7 8 a 9 b c d e]
45: [0 1 3 6 2 4 5 7 8 9 a b c d e]	46: [0 1 3 6 2 4 5 7 8 9 a c b d e]
47: [0 1 3 6 2 4 5 7 8 9 a c b d e]	48: [0 1 3 6 2 4 5 7 8 9 a c b d e]
49: [0 1 3 6 2 4 5 7 8 9 a b c d e]	50: [0 1 3 6 2 4 5 7 8 9 a b c d e]
51: [0 1 3 6 2 4 5 7 9 8 a b c d e]	52: [0 1 3 6 2 4 5 7 9 8 a b c d e]
53: [0 1 3 6 2 4 5 7 9 8 a b c d e]	54: [0 1 3 6 2 4 5 7 8 9 a b c d e]
55: [0 1 3 6 2 4 5 7 8 9 a b c d e]	56: [0 1 3 6 2 4 5 7 9 8 a b c d e]
57: [0 1 3 6 2 4 5 7 9 8 a b c d e]	58: [0 1 3 6 2 4 5 7 9 8 a b c d e]
59: [0 1 3 6 2 4 5 7 8 9 a b c d e]	60: [0 1 3 6 2 4 5 7 8 9 a b c d e]
61: [0 1 3 6 7 4 2 8 9 a b 5 d c e]	62: [0 1 3 6 7 4 5 8 2 9 a b c d e]
63: [0 1 3 6 7 4 5 8 2 9 a b c d e]	64: [0 1 3 6 7 4 5 8 2 a 9 c b d e]
65: [0 1 3 6 2 4 5 7 8 9 a b c d e]	66: [0 1 3 6 2 4 5 7 9 8 a b c d e]
67: [0 1 3 6 2 4 5 7 8 9 a b c d e]	68: [0 1 3 6 2 4 5 7 8 9 a b c d e]
69: [0 1 3 6 2 4 5 7 8 9 a b c d e]	70: [0 1 3 6 2 4 5 7 8 9 a b c d e]
71: [0 1 3 6 2 4 5 7 8 9 a b c d e]	72: [0 1 3 6 2 4 5 7 8 9 a b c d e]
73: [0 1 3 6 2 4 5 7 8 9 a b c d e]	74: [0 1 3 6 2 4 5 7 8 9 a b c d e]
75: [0 1 3 6 2 4 5 7 8 9 a b c d e]	76: [0 1 3 6 2 4 5 7 8 9 a b c d e]
77: [0 1 3 6 2 4 5 7 8 9 a b c d e]	78: [0 1 3 6 2 4 5 7 8 9 a b c d e]
79: [0 1 3 6 2 4 5 7 9 8 a b c d e]	80: [0 1 3 6 2 5 7 4 8 a d 9 b c e]

## References

- [1] B. Alspach, D. L. Kreher and A. Pastine, Sequenceable triple systems, Preprint.
- [2] C. J. Colbourn and J. H. Dinitz, *Handbook of Combinatorial Designs, Second Edition*, Chapman & Hall/CRC, 2006.
- [3] D. L. Kreher and D. R. Stinson, Nonsequenceable Steiner triple systems, *Bull. ICA* **86** (2019), 64–68.
- [4] C. J. Colbourn and A. Rosa, *Triple Systems*, Oxford University Press, 1999.

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