6-2024

New examples of self-dual near-extremal ternary codes of length 48 derived from 2-(47,23,11) designs

Sanja Rukavina
Faculty of Mathematics, University of Rijeka

Vladimir Tonchev
Michigan Technological University, tonchev@mtu.edu

Follow this and additional works at: https://digitalcommons.mtu.edu/michigantech-p2
Part of the Mathematics Commons

Examples and Counterexamples 5 (2024) 100130

Contents lists available at ScienceDirect

Examples and Counterexamples

journal homepage: www.elsevier.com/locate/exco

New examples of self-dual near-extremal ternary codes of length 48 derived from 2-$(47, 23, 11)$ designs

Sanja Rukavina [a,*], Vladimir D. Tonchev [b]

[a] Faculty of Mathematics, University of Rijeka, 51000 Rijeka, Croatia

[b] Department of Mathematical Sciences, Michigan Technological University, Houghton, MI 49931, USA

ARTICLE INFO

MSC:
05B05
05B20
94B05

Keywords:
Self-dual code
Near-extremal code
Symmetric 2-design
Automorphism group

ABSTRACT

In a recent paper (Araya and Harada, 2023), Araya and Harada gave examples of self-dual near-extremal ternary codes of length 48 for 145 distinct values of the number A_{12} of codewords of minimum weight 12, and raised the question about the existence of codes for other values of A_{12}. In this note, we use symmetric 2-$(47, 23, 11)$ designs with an automorphism group of order 6 to construct self-dual near-extremal ternary codes of length 48 for 150 new values of A_{12}.

1. Introduction

We assume familiarity with the basic facts and notions from error-correcting codes and combinatorial designs [1–5]. All codes considered in this paper are ternary.

The minimum weight d of a ternary self-dual code of length n divisible by 12 satisfies the upper bound $d \leq n/4+3$ [3, 9.3]. A self-dual ternary code of length n divisible by 12 with minimum weight d is called *extremal* if $d = n/4+3$ [3], and *near-extremal* if $d = n/4$ [6,7]. Any extremal ternary self-dual code supports combinatorial 5-designs by the Assmus–Mattson theorem [8], [3, 8.4]. It was recently proved by Miezaki, Munemasa, and Nakasora [7] that the supports of all codewords of weight $w \leq 6m-3$ in any ternary near-extremal self-dual code of length $n = 12m$ are the blocks of a combinatorial 1-design. Thus, extremal and near-extremal self-dual codes are interesting from both coding and design theoretical point of view.

The classification of extremal ternary self-dual codes of length n divisible by 12 has been completed only for the lengths $n = 12$ and $n = 24$. Up to equivalence, there is one extremal code of length 12, being the extended ternary Golay code, and there are two extremal codes of length 24: the extended quadratic-residue code QR^*_{23} [8] and the Pless symmetry code $C(11)$ [9,10]. The only known extremal code of length $n = 36$ is the Pless symmetry code $C(17)$ [9,10]. It was shown recently that the Pless symmetry code $C(17)$ is equivalent to a code spanned by the incidence matrix of a symmetric 2-$(36, 15, 6)$ design with a trivial full automorphism group [4], as well as by the incidence matrix of a unique 2-$(36, 15, 6)$ design that admits an involution [11]. Two extremal ternary self-dual codes of length 48 are known: the extended quadratic-residue code QR^*_{47} and the Pless symmetry code $C(23)$. Finally, three extremal codes of length $n = 60$ are known: the extended quadratic-residue code QR_{59}, the Pless symmetry code $C(29)$, and a code found by Nebe and Villar [12].

The sparsity of extremal codes has spurred some recent interest in near-extremal codes. Araya and Harada [6] proved that the number A_{3i} of codewords of weight $3i$, $m \leq i \leq 4m$, in a ternary near-extremal self-dual code of length $n = 12$ m, is divisible by 8. In addition, the number A_{12} of codewords of minimum weight 12 in a ternary near-extremal self-dual code of length 48 is $A_{12} = 8\beta$ for some β in the range $1 \leq \beta \leq 4324$ [6, page 1831]. Araya and Harada gave examples of near-extremal ternary codes of length 48 for 145 distinct values of A_{12} (sets $\Gamma_{48,1}$ and $\Gamma_{48,2}$ in [6]), and raised the question for the existence of codes for other values of A_{12} [6, page 1838, Question 1].

In the next section, we use symmetric 2-$(47, 23, 11)$ designs admitting an automorphism of order six to find many new examples of self-dual near-extremal ternary codes of length 48. We constructed examples of self-dual near-extremal ternary codes of length 48 with 150 distinct values of A_{12} not covered in [6]. In our computations, in addition to our own computer programs, we used the computer programs of V. Ćepulić [13] for the construction of orbit matrices, and the computer algebra system MAGMA [14] for computing the codes and their weight distributions.

* Corresponding author.
E-mail addresses: sanjar@math.uniri.hr (S. Rukavina), tonchev@mtu.edu (V.D. Tonchev).

https://doi.org/10.1016/j.exco.2023.100130
Received 9 November 2023; Received in revised form 22 November 2023; Accepted 6 December 2023
Available online 15 December 2023

Table 1
2-$(47,23,11)$ designs and their codes.

Orbit matrix	OM1	OM2	OM3	OM4
Non-isomorphic designs	70 400	24 576	63 488	24 576
# Codes with $d=12$	43 338	11 884	22 698	11 884
Inequivalent codes with $d=12$	1662	1073	1200	1073
# Distinct A_{12}	165	152	161	152

Table 2
Γ_{OM_i}, $i=1,2,3,4$.

Γ_{OM_1}	$320, 323, 324, 326, 338, 340, 341, 346, 348, 349, 350, 352, 353, \dots, 357, 359, 360, \dots, 468, 470, 471, \dots, 480, 482, 483, \dots, 486, 489, 490, \dots, 494, 496, 497, \dots, 500, 504, 506, 512, 516, 518, 522, 524, 528, 536, 560$
Γ_{OM_2}	$313, 329, 331, 332, 333, 334, 337, 338, 339, 343, 344, \dots, 349, 351, 352, \dots, 450, 452, 453, \dots, 459, 461, 462, 464, 466, 467, 468, 470, 472, 474, 476, 478, 479, 480, 482, 484, 486, 488, 494, 496, 500, 503, 504, 506, 512, 524, 528, 554, 560$
Γ_{OM_3}	$320, 323, 324, 326, 338, 340, 341, 346, 348, 349, 350, 353, \dots, 357, 359, 360, \dots, 464, 466, 467, 468, 470, 471, \dots, 480, 482, 483, \dots, 486, 488, 489, \dots, 492, 494, 496, 497, \dots, 500, 504, 506, 512, 516, 522, 524, 528, 536, 560$
Γ_{OM_4}	$313, 329, 331, 332, 333, 334, 337, 338, 339, 343, 344, \dots, 349, 351, 352, \dots, 450, 452, 453, \dots, 459, 461, 462, 464, 466, 467, 468, 470, 472, 474, 476, 478, 479, 480, 482, 484, 486, 488, 494, 496, 500, 503, 504, 506, 512, 524, 528, 554, 560$

2. Near-extremal $[48, 24, 12]$ codes derived from symmetric 2-$(47, 23, 11)$ designs

The following statement gives a simple construction of ternary self-dual codes of length 48.

Theorem 2.1. *Let M be a 47×47 $(0,1)$-incidence matrix of a symmetric 2-$(47,23,11)$ design, and let G be the 47×48 matrix obtained by adding to M the all-one column. Then the row space of G over $GF(3)$ is a ternary self-dual code of length 48.*

Proof. Since every row of G has weight $24 \equiv 0 \pmod 3$, and the inner product of every two distinct rows of G is equal to $12 \equiv 0 \pmod 3$, the ternary code L spanned by the rows of G is self-orthogonal. Since 23 is not divisible by 3, and $23-11=12$ is divisible by 3, but not divisible by 9, it follows from [1, Theorem 4.6.2, (b)] that the rank of M over $GF(3)$ is equal to $(47+1)/2 = 24$, hence the code L is self-dual. □

We note that the two known extremal ternary self-dual codes of length 48, QR^*_{47} and $C(23)$, are obtainable via the construction of Theorem 2.1 from symmetric 2-$(47,23,11)$ designs associated with the Paley–Hadamard matrices of type I and II and order 48, respectively [4].

In this paper, we use the method for refinement and indexing of orbit matrices for presumed action of an abelian automorphism group [13,15] to construct 2-$(47,23,11)$ designs invariant under the cyclic group C_6 of order 6 with orbit lengths distribution $(1,2,2,3,3,6,6,6,6,6,6)$. There are 32 orbit matrices for such an action of an automorphism of order six on a 2-$(47,23,11)$ design. Our computations show that only four orbit matrices (given in the Appendix) yield designs which generate near-extremal ternary codes via the construction of Theorem 2.1. The results are summarized in Table 1, where d denotes the minimum weight of a code.

The numbers A_{12} of minimum weight codewords in the newly found near-extremal codes are given by

$$A_{12} \in \{8\beta | \beta \in \Gamma_{OM_i}\},$$

where the set Γ_{OM_i}, $i \in \{1,2,3,4\}$, is given in Table 2, and contains all distinct values of β for the near-extremal ternary codes obtained from designs with orbit matrix OM_i. A list of 181 2-$(47,23,11)$ designs that generate codes with different number of codewords of minimum weight 12 is available at
https://www.math.uniri.hr/~sanjar/structures/

All these 181 designs have the cyclic group C_6 as the full automorphism group. The data from Table 2 can be summarized as follows.

Proposition 2.2. *There is a ternary near-extremal self-dual code of length 48 with $A_{12} \in \{8\beta | \beta \in \Gamma\}$, where A_{12} is the number of codewords of weight 12 and* $\Gamma = \{313, 320, 323, 324, 326, 329, 331, 332, 333, 334, 337, \dots, 341, 343, 344, ..., 468, 470, 471, \dots, 480, 482, 483, \dots, 486, 488, 489, \dots, 494, 496, 497, \dots, 500, 503, 504, 506, 512, 516, 518, 522, 524, 528, 536, 554, 560\}$.

Araya and Harada [6] found 86 codes with distinct values $A_{12} \equiv 0 \pmod{48}$ (see $\Gamma_{48,1}$ in [6], p. 1831), plus 59 codes with distinct values A_{12} not divisible by 48 [6, $\Gamma_{48,2}$]. In our list, there are 31 values of A_{12} divisible by 48 that are covered by $\Gamma_{48,1}$ in [6]. Our examples of near-extremal ternary codes for the remaining 150 distinct values of A_{12} were not previously known, since $A_{12}/8 < 282$ for all A_{12} covered by $\Gamma_{48,2}$ in [6]. Therefore, as a result of our construction we found examples of 150 new self-dual near-extremal ternary codes with values of A_{12} which were not previously known, and correspond to the values of β from the set Γ in Proposition 2.2 which are not divisible by six.

Remark 2.3. In [16], fifty-four symmetric 2-$(47,23,11)$ designs admitting a faithful action of a Frobenius group of order 55 were constructed. We computed the ternary codes of these designs and found that fifteen designs yield near-extremal self-dual ternary codes with $A_{12} \in \{1584, 1680, 2640, 3792\}$. Codes with these values of A_{12} were previously found in [6].

Remark 2.4. From Nebe's result [17] it follows that a symmetric 2-$(47,23,11)$ design with an involution cannot yield an extremal ternary self-dual code. This implies that the ternary self-dual codes constructed from the symmetric 2-$(47,23,11)$ designs with an automorphism group of order six cannot be extremal.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The link to the constructed designs is included.

Acknowledgments

The authors would like to thank the reviewers for their careful reading of the manuscript.

Appendix

Orbit matrices for the action of an automorphism of order six on 2-$(47,23,11)$ designs which generate new near-extremal self-dual ternary codes of length 48 via Theorem 2.1:

OM_1	1 2 2 3 3 6 6 6 6 6 6
1	0 2 0 3 0 6 6 6 0 0 0
2	1 2 2 0 0 6 3 0 3 3 3
2	0 2 0 0 3 3 3 3 6 3 0
3	1 2 2 3 3 2 2 2 2 2 2
3	0 2 0 3 0 2 2 2 4 4 4
6	1 1 1 1 1 2 3 4 2 5 2
6	1 1 1 1 1 2 3 4 4 1 4
6	1 0 0 2 2 4 3 2 3 3 3
6	0 0 2 2 1 3 3 3 4 3 2
6	0 1 1 1 2 2 5 2 2 3 4
6	0 1 1 1 2 4 1 4 2 3 4

OM_2	1	2	2	3	3	6	6	6	6	6	6
1	0	2	0	3	0	6	6	6	0	0	0
2	1	2	2	0	0	6	3	0	3	3	3
2	0	2	0	0	3	3	3	3	6	3	0
3	1	2	2	3	3	2	2	2	2	2	2
3	0	2	0	3	0	2	2	2	4	4	4
6	1	1	1	1	1	3	1	5	3	3	3
6	1	1	1	1	1	1	5	3	3	3	3
6	1	0	0	2	2	4	3	2	3	3	3
6	0	0	2	2	1	3	3	3	4	3	2
6	0	1	1	1	2	3	3	3	3	1	5
6	0	1	1	1	2	3	3	3	1	5	3

OM_3	1	2	2	3	3	6	6	6	6	6	6
1	0	2	0	3	0	6	6	6	0	0	0
2	1	2	2	0	0	3	3	3	6	3	0
2	0	2	0	0	3	6	3	0	3	3	3
3	1	2	2	3	3	2	2	2	2	2	2
3	0	2	0	3	0	2	2	2	4	4	4
6	1	1	1	1	1	2	5	2	2	3	4
6	1	1	1	1	1	4	1	4	2	3	4
6	1	0	0	2	2	3	3	3	4	3	2
6	0	0	2	2	1	4	3	2	3	3	3
6	0	1	1	1	2	2	3	4	2	5	2
6	0	1	1	1	2	2	3	4	4	1	4

OM_4	1	2	2	3	3	6	6	6	6	6	6
1	0	2	0	3	0	6	6	6	0	0	0
2	1	2	2	0	0	3	3	3	6	3	0
2	0	2	0	0	3	6	3	0	3	3	3
3	1	2	2	3	3	2	2	2	2	2	2
3	0	2	0	3	0	2	2	2	4	4	4
6	1	1	1	1	1	3	3	3	3	1	5
6	1	1	1	1	1	3	3	3	1	5	3
6	1	0	0	2	2	3	3	3	4	3	2
6	0	0	2	2	1	4	3	2	3	3	3
6	0	1	1	1	2	3	1	5	3	3	3
6	0	1	1	1	2	1	5	3	3	3	3

References

[1] E.F. Assmus Jr., J.D. Key, Designs and their Codes, Cambridge University Press, Cambridge, 1992.

[2] T. Beth, D. Jungnickel, H. Lenz, Design Theory, second ed., Cambridge University Press, 1999.

[3] W.C. Huffman, V. Pless, Fundamentals of Error-Correcting Codes, Cambridge University Press, Cambridge, 2003.

[4] V.D. Tonchev, On Pless symmetry codes, ternary QR codes, and related Hadamard matrices and designs, Des. Codes Cryptogr. 90 (2022) 2753–2762.

[5] V.D. Tonchev, Combinatorial Configurations, John Wiley & Sons, Inc., New York, 1988.

[6] M. Araya, M. Harada, Some restrictions on the weight enumerators of near-extremal ternary self-dual codes and quaternary hermitian self-dual codes, Des. Codes Cryptogr. 91 (2023) 1813–1843.

[7] T. Miezaki, A. Munemasa, H. Nakasora, A note on Assmus–Mattson type theorems, Des. Codes Cryptogr. 89 (2021) 843–858.

[8] E.F. Assmus Jr., H.F. Mattson Jr., New 5-designs, J. Combin. Theory Ser. A 6 (1969) 122–151.

[9] V. Pless, On a new family of symmetry codes and related new five-designs, Bull. Amer. Math. Soc. 75 (6) (1969) 1339–1342.

[10] V. Pless, Symmetry codes over $GF(3)$ and new five-designs, J. Combin. Theory, Ser. A 12 (1972) 119–142.

[11] S. Rukavina, V.D. Tonchev, Extremal ternary self-dual codes of length 36 and symmetric $2\text{-}(36, 15, 6)$ designs with an automorphism of order 2, J. Algebraic Combin. 57 (2023) 905–913.

[12] G. Nebe, D. Villar, An analogue of the Pless symmetry codes, in: Seventh International Workshop on Optimal Codes and Related Topics, Albena, Bulgaria, 2013, pp. 158–163.

[13] V. Ćepulić, On symmetric block designs (40, 13, 4) with automorphisms of order 5, Discrete Math. 128 (1–3) (1994) 45–60.

[14] W. Bosma, J. Cannon, Handbook of magma functions, in: Department of Mathematics, University of Sydney, 1994, available at http://magma.maths.usyd.edu.au/magma.

[15] D. Crnković, S. Rukavina, Construction of block designs admitting an abelian automorphism group, Metrika 62 (2–3) (2005) 175–183.

[16] D. Crnković, S. Rukavina, Some symmetric (47, 23, 11) designs, Glas. Mat. 38 (58) (2003) 1–9.

[17] G. Nebe, On extremal self-dual ternary codes of length 48, Int. J. Comb. 2012 (2012) 154281.