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NOTES AND CORRESPONDENCE

A Random Wave Method for Detecting Phase Imbalance in a Coherent Radar Receiver

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ABSTRACT

The question of monitoring the angle between the in-phase (I) and quadrature (Q) channels of a coherent radar receiver is considered in this paper. Random interference-type scattering (e.g., rain echo) is assumed so that the received signal is a random phasor sum. It is also assumed that I and Q voltages have a $(90 + \phi_0)$ phase difference. The effects of nonzero phase imbalance ϕ_0 on the I and Q statistics are examined. Joint I and Q as well as amplitude and phase probability distributions are derived. It is shown that the cross-correlation coefficient at zero lag between I and Q time series is proportional to ϕ_0, which provides a simple algorithm to detect the phase imbalance. This is verified by Monte Carlo simulations.

1. Introduction

Coherent radars are now common in remote sensing, for example, Doppler radar networks used in monitoring precipitation systems (Doviak and Zrnić 1984), and it is a standard procedure to test the receivers for amplitude and phase imbalances. This is often done by employing deterministic (e.g., sinusoidal) signals as described, for instance, in Churchill et al. (1981). It can, however, be convenient to use the scatterer itself for testing I and Q errors. For example, in the case of weather radar one may want to use the weather echo, which is Rayleigh distributed, to detect phase and amplitude imbalances (E. Mueller 1992, personal communication). One can also use simple noise as a Rayleigh source. Another example is to test a spaceborne radar flying over an ocean surface by the ocean signal itself. The signal is Rayleigh distributed in the case of a rough (compared with the wavelength) surface. The detection of the phase imbalance by using only the "random scatterer" echo is the topic of this paper.

Consider in-phase and quadrature detection (I and Q) of a radar signal due to a randomly moving source. Time series of echoes of such a signal obey well-known statistical properties (Doviak and Zrnić 1984; Goodman 1985), namely, both I and Q have Gaussian-distributed probability density functions (pdfs) with zero means and equal standard deviations. This leads to the Rayleigh pdf of the amplitude and uniform statistics of the relative phase (Goodman 1985; Papoulis 1984). This result applies to a wide variety of signals involving random wave sources, ranging from weather observations (Doviak and Zrnić 1984, 50–51), laser speckle and communications (Dainty 1975; Frieden 1983), to photoelectron statistics (Saleh 1978).

The receivers are, however, not perfect, and amplitude as well as phase imbalances often occur. In this note we deal with the fact that I and Q components are never exactly 90° apart. Let us denote the angle between I and Q as $90 + \phi_0$, where ϕ_0 is the phase imbalance. The question then is, *How does nonzero ϕ_0 affect the statistics of the measured I and Q?* In particular, do I and Q still have uncorrelated Gaussian distributions? If not, can the deviations be used to measure ϕ_0? To answer these questions we find the correlations between I and Q when $\phi_0 \neq 0$ and derive the pdfs of the resultant amplitude and phase.

2. Analytical results

We now briefly review the assumptions in the random phasor sum model (Goodman 1985, p. 45; Papoulis 1984) and then derive necessary pdfs with explicit ϕ_0 dependence.

Let A be the phasor sum of random waves

$$A = I + jQ = ae^{j\phi} = \sum_{k=1}^{N} A_k, \tag{1}$$

where the real part I, and the imaginary part Q, are the in-phase and quadrature voltages, respectively. The

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phase error for each component (assumed constant) is denoted by ϕ_0. Then the in-phase and quadrature components of A are

$$I = \frac{1}{N^{1/2}} \sum_{k=1}^{N} \alpha_k \cos(\phi_k + \phi_0),$$

$$Q = \frac{1}{N^{1/2}} \sum_{k=1}^{N} \alpha_k \sin\phi_k, \tag{2}$$

where $\alpha_k/N^{1/2}$ are normalized random lengths, and ϕ_k are random phases of the phasor components.

The following assumptions are made:

1) The amplitude $\alpha_k/N^{1/2}$ and the phase ϕ_k of the kth component of A are statistically independent of each other and all other phasors.

2) The random variable $\bar{\alpha}_k$ is identically distributed for all k, with mean $\bar{\alpha}$ and second moment $\overline{\alpha^2}$.

3) The phases ϕ_k are uniformly distributed on $(-\pi, \pi)$.

Under the above assumptions and Eq. (2), I and Q are Gaussian random variables for large N according to the central limit theorem (Goodman 1985). Note that the mean, variance, and the correlation of I and Q are given by

$$\langle I \rangle = 0,$$

$$\langle Q \rangle = 0, \tag{3}$$

$$\langle I^2 \rangle = \frac{1}{N} \sum_{k=1}^{N} \sum_{n=1}^{N} \langle \alpha_k \alpha_n \rangle \times \langle \cos(\phi_k + \phi_0) \cos(\phi_n + \phi_0) \rangle = \frac{\overline{\alpha^2}}{2} = \sigma^2,$$

$$\langle Q^2 \rangle = \frac{1}{N} \sum_{k=1}^{N} \sum_{n=1}^{N} \langle \alpha_k \alpha_n \rangle \langle \sin\phi_k \sin\phi_n \rangle = \frac{\overline{\alpha^2}}{2} = \sigma^2,$$

$$\langle IQ \rangle = \frac{1}{N} \sum_{k=1}^{N} \sum_{n=1}^{N} \langle \alpha_k \alpha_n \rangle \langle \cos(\phi_k + \phi_0) \sin\phi_n \rangle = -\frac{\overline{\alpha^2}}{2} \sin\phi_0 = -\sigma^2 \sin\phi_0, \tag{4}$$

where $\langle \quad \rangle$ is the expectation of a random variable.

Thus, the probability density functions of I and Q are given by

$$p(I) = \frac{1}{(2\pi)^{1/2}\sigma} \exp\left\{\frac{-I^2}{2\sigma^2}\right\},$$

$$p(Q) = \frac{1}{(2\pi)^{1/2}\sigma} \exp\left\{\frac{-Q^2}{2\sigma^2}\right\}. \tag{5}$$

To specify the correlation of these two random variables, the joint probability density function is considered. Since $\langle I \rangle = 0$ and $\langle Q \rangle = 0$, the correlation coefficient of I and Q is

$$\rho = \frac{\langle IQ \rangle}{\langle I^2 \rangle^{1/2} \langle Q^2 \rangle^{1/2}} = -\sin\phi_0. \tag{6}$$

From Eqs. (5) and (6) the joint probability density function of in-phase and quadrature components is (Goodman 1985)

$$p(IQ) = \frac{1}{2\pi\sigma^2(1-\rho^2)^{1/2}} \times \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{I^2}{\sigma^2} + \frac{Q^2}{\sigma^2} - \frac{2\rho IQ}{\sigma^2}\right]\right\}. \tag{7}$$

Note that when the correlation coefficient is equal to zero, the in-phase I and quadrature Q have independent Gaussian probability densities. The joint pdf of I and Q reduces to

$$p(IQ) = \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2}(I^2 + Q^2)\right\}. \tag{8}$$

To obtain the pdfs of amplitude and phase for the $\phi_0 \neq 0$ case, consider the relations between in-phase and quadrature versus amplitude and phase:

$$\phi = \tan^{-1}\left(\frac{Q}{I}\right),$$

$$a = (I^2 + Q^2)^{1/2}. \tag{9}$$

The Jacobian of this transformation then is

$$\|J\| = \begin{vmatrix} \dfrac{\partial I}{\partial a} & \dfrac{\partial I}{\partial \phi} \\ \dfrac{\partial Q}{\partial a} & \dfrac{\partial Q}{\partial \phi} \end{vmatrix} = a. \tag{10}$$

Therefore the joint pdf of amplitude a and phase ϕ of a random wave with phase imbalance ϕ_0 is

$$p(a, \phi) = p(IQ)\|J\| = \begin{cases} \dfrac{a}{2\pi\sigma^2(1-\rho^2)^{1/2}} \times \exp\left\{\dfrac{-a^2}{2\sigma^2(1-\rho^2)}(1 + \sin\phi_0 \sin 2\phi)\right\} \\ \qquad -\pi < \phi \leqslant \pi \quad \text{and} \quad a \geqslant 0 \\ 0 \quad \text{otherwise.} \end{cases} \tag{11}$$

When $\rho = 0$, this reduces to

$$p(a, \phi) = \begin{cases} \dfrac{a}{2\pi\sigma^2} \exp\left\{-\dfrac{a^2}{2\sigma^2}\right\} \\ \qquad -\pi < \phi \leqslant \pi \quad \text{and} \quad a \geqslant 0 \\ 0 \quad \text{otherwise.} \end{cases} \tag{12}$$

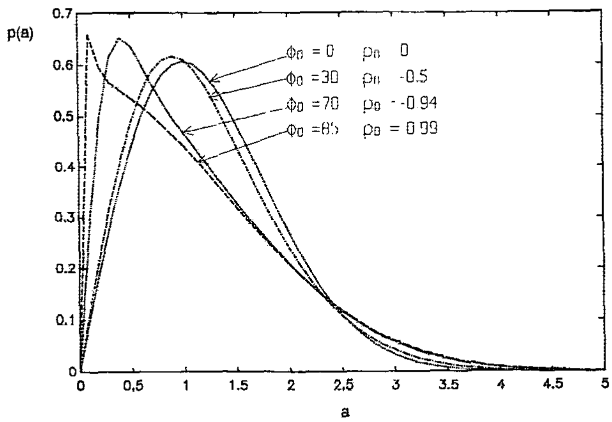

FIG. 1. Amplitude pdfs for various values of phase imbalance.

The pdf of amplitude a can be obtained from $p(a, \phi)$ by

$$p(a) = \int_{-\pi}^{\pi} p(a, \phi) d\phi. \tag{13}$$

Some examples of the amplitude pdf with $\sigma^2 = 1$ are shown in Fig. 1.

When the correlation coefficient is zero, I and Q have independent Gaussian probability densities, and the amplitude pdf is

$$p(a) = \begin{cases} \dfrac{a}{\sigma^2} \exp\left(-\dfrac{a^2}{2\sigma^2}\right), & a \geqslant 0 \\ 0, & a < 0, \end{cases} \tag{14}$$

that is, the amplitude is a Rayleigh-distributed variable.[1]

The probability density function of the phase ϕ can be calculated as

$$p(\phi) = \begin{cases} \displaystyle\int_0^{\infty} p(a, \phi) da = \frac{\cos\phi_0}{2\pi[1 + \sin\phi_0 \sin 2\phi]} \\ \qquad\qquad -\pi < \phi \leqslant \pi \\ 0 \quad \text{otherwise}. \end{cases} \tag{15}$$

The results of the phase pdf calculations are shown in Fig. 2. When the correlation coefficient is zero, the phase pdf reduces to a uniform distribution.[2]

[1] On the other hand, if $\rho = \pm 1$, the amplitude pdf becomes Gaussian distribution for $a \geqslant 0$.

[2] If the correlation coefficient is equal to ± 1, the pdf for phase is

$$p(\phi) = \frac{1}{2}\left[\delta\left(\phi \pm \frac{3}{4}\pi\right) + \delta\left(\phi \mp \frac{1}{4}\pi\right)\right].$$

3. Simulation results

To test the theoretical results obtained in the previous section, a Monte Carlo simulation was performed. According to Eq. (2), I and Q pairs of the phasor sum A were created from $N = 10\,000$ random terms with $\phi_0 = 3°$, $\phi_0 = 30°$, and $\phi_0 = 80°$, respectively.

The correlation coefficient between I and Q and the pdfs of I and Q were then calculated based on 20 000 simulated pairs of I and Q (each pair produced from a sum of 10 000 terms). The estimated values ρ are -0.046 (vs theoretical value -0.052), -0.4995 (vs -0.5), and -0.9712 (vs -0.985) when $\phi_0 = 3°, 30°$, and $80°$.

The phase histograms were then calculated according to Eq. (9) from 20 000 simulated values of I and Q. These histograms are shown in Figs. 3–5 where the emergence of peaks at $-\pi/4$, $+3\pi/4$ confirms analytical results.

4. Discussion

It was assumed from the outset that the variances of the I and Q channels are equal. This is often not so. Therefore effects of amplitude imbalance must also be considered. In this regard it is sufficient to note that multiplying either I or Q by a constant (to simulate gain mismatch) does not affect the I and Q correlation coefficient. Thus estimation of phase imbalance via Eq. (6) is not affected by the amplitude imbalance.

The results obtained thus far are expressions for the I and Q correlation coefficient, joint I and Q pdf as well as marginal pdfs of amplitude and phase—all as functions of the phase imbalance ϕ_0. But which one of these expressions is the most suitable for detecting small ϕ_0? We suggest the correlation coefficient between I and Q series as a possible "signature" of ϕ_0. For small angles,

$$|\rho| \sim \phi_0, \tag{16}$$

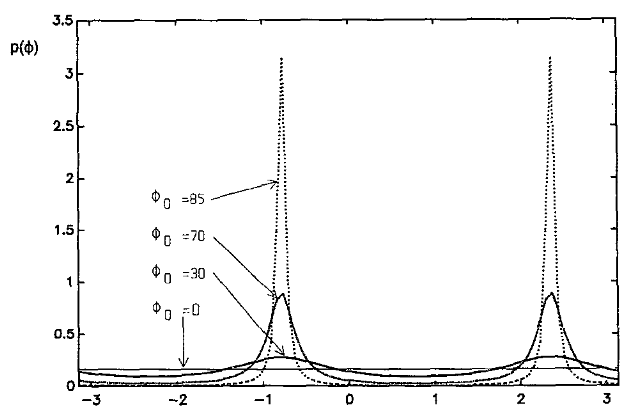

FIG. 2. Phase pdfs for various values of phase imbalance.

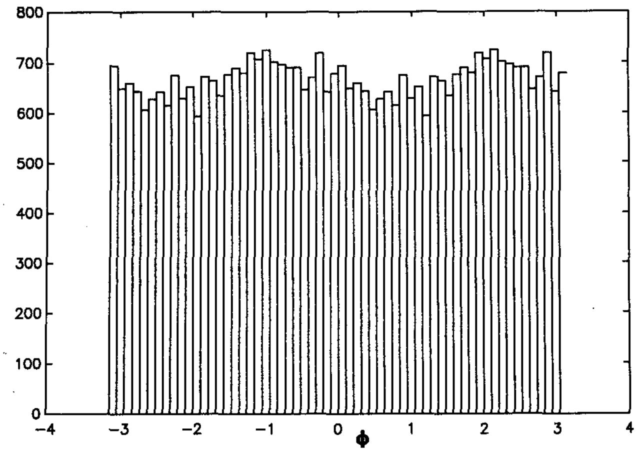

FIG. 3. Histogram of phase with phase imbalance $\phi_0 = 3°$.

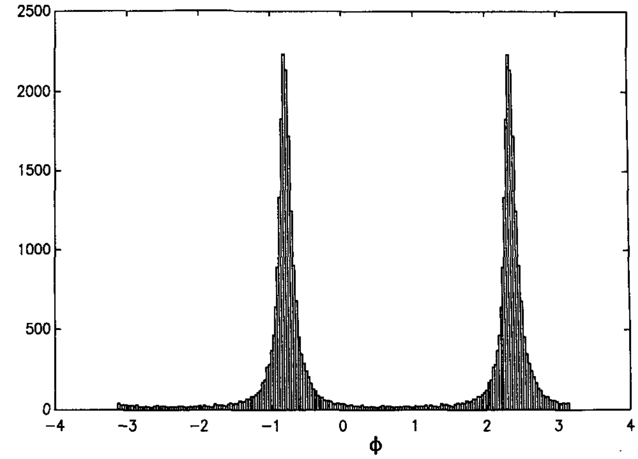

FIG. 5. Histogram of phase with phase imbalance $\phi_0 = 80°$.

therefore, a small phase imbalance implies a small correlation coefficient. Consider $\phi_0 = 3°$, for example. The corresponding coefficient $|\rho| \approx 0.05$.

Thus we must make a *statistically reliable* measurement of $|\rho| = 0.05$ in order to detect $\phi_0 = 3°$. How many samples does one need to "distinguish" between $|\rho| = 0.00$ and $|\rho| = 0.05$? The answer can be computed using the methods found in the statistical estimation literature. For our purposes here it will suffice to point out that the variance of cross-correlation coefficient at zero lag between two Gaussian sequences obeys the following equation (Shumway 1988, 22–28):

$$\mathrm{var}[\rho_{IQ}] \sim \frac{1}{N}, \qquad (17)$$

where N is a number of samples and var denotes the variance. Thus, one can simply demand that $(\mathrm{var}[\rho_{IQ}])^{1/2} \ll |\rho| = 0.05 = 1/20$. This implies that

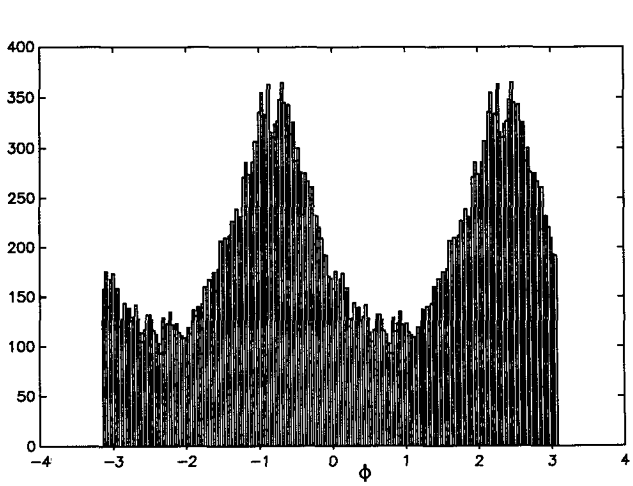

FIG. 4. Histogram of phase with phase imbalance $\phi_0 = 30°$.

$$\frac{1}{N^{1/2}} \ll \frac{1}{20} \quad \text{or} \quad N \gg 20^2 = 400. \qquad (18)$$

In general, $N \gg |\rho|^{-2}$. We see that even for $\phi_0 = 0.5°$, N must satisfy $N \gg 100^2 = 10\,000$, which is quite feasible.

Indeed, a typical radar pulse is on the order of a microsecond (10^{-6} s), so that 10^6–10^8 samples, along with the correlation coefficient, can be obtained in a few seconds (Mueller 1992, personal communication). It seems reasonable therefore to expect that the measurement of the IQ cross-correlation coefficient at zero lag can provide an estimate of the phase imbalance ϕ_0 as small as a fraction of a degree. Figure 6 shows typical estimates of ρ as a function of number of samples. Although just over 10^4 samples are shown, convergence is achieved in all three cases. The figure also demonstrates that for larger values of phase imbalance the estimate of ρ converges with fewer samples.

Finally let us examine briefly the impact of the phase imbalance on Doppler velocity estimation, which is of interest in the weather radar field. One can start with the definition $\omega_d = d\gamma/dt = -4\pi v_r/\lambda$, where ω_d is the Doppler shift, v_r is the radial component of velocity, and γ is the absolute phase in I and Q components (Doviak and Zrnić 1984). According to our assumptions, γ in one of the channels is corrupted by a constant ϕ_0. In so far as ϕ_0 remains constant, Doppler measurements are not affected by the phase imbalance. When ϕ_0 fluctuates in time, one needs to know the pdf of such "phase jitter" (Doviak and Zrnić 1984, p. 163). A Gaussian pdf for the fluctuating part is often a reasonable assumption (Passarelli and Zrnić 1989), and, in that case, one can still use the phase histogram to estimate ϕ_0. Indeed, Gaussian phase jitter produces a peak at zero while the phase imbalance causes peaks at $\pm\pi/4$, $\pm3\pi/4$, and the two effects can be separated.

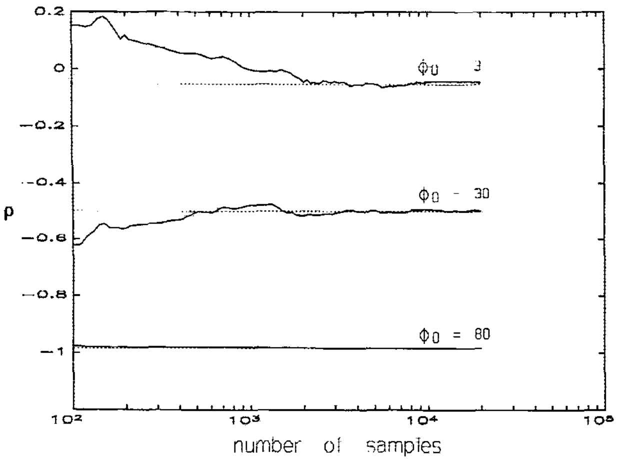

FIG. 6. Typical estimates of ρ as a function of number of samples.

5. Summary

In this note we have considered the effects of the phase imbalance ϕ_0 on the I and Q statistics. A relation between the correlation coefficient and the phase imbalance was derived [Eq. (6)] and it was shown with statistical arguments and through the numerical examples that the IQ cross-correlation coefficient measurement may provide a practical way to estimate ϕ_0. Expressions for the joint I and Q pdf as well as marginal amplitude and phase pdfs have also been obtained as functions of the phase imbalance ϕ_0 and it was shown that the phase pdf develops peaks at $\pm\pi/4$, $\pm3\pi/4$ as the correlation coefficient increases. Numerical illustrations and Monte Carlo simulations have also been given.

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