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S U P P L E M E N T

OBTAINING THE DROP SIZE DISTRIBUTION

BY ALEXANDER B. KOSTINSKI AND RAYMOND A. SHAW

This document is a supplement to "Fluctuations and Luck in Droplet Growth by Coalescence," by Alexander B. Kostinski and Raymond A. Shaw (*Bull. Amer. Meteor. Soc.*, **86,** 235–244) • ©2005 American Meteorological Society • *Corresponding authors:* Alexander B. Kostinski or Raymond A. Shaw, Dept. of Physics, Michigan Tech, 1400 Townsend Dr., Houghton, MI 49931 • *E-mail:* Kostinski: alex_kostinski@mtu.edu; Shaw: rashaw@mtu.edu • DOI: 10.1175/BAMS-86-2-235.

This paper has established a connection with previous work in stochastic collision–coalescence (e.g., Telford 1955; Robertson 1974) and the broader field related to the Erlang and generalized-Erlang distributions such as statistical signal processing and queueing theory (Evans et al. 2000; Porter and Ogilvie 2000; Syski 1986). In that spirit we illustrate how a droplet size distribution is obtained from the generalized Erlang distributions described in the main paper.

Note that Eq. (SB1.1) is a distribution of times rather than a number distribution of drop sizes. The latter may be useful (e.g., Telford 1955; Robertson 1974) because one may wish to know how the drop size distribution evolves with time. The size of a collector drop is easily calculated if we know the number of collisions it has experienced (ignoring condensation and drop breakup). Therefore, we know the size distribution if we can obtain the distribution of the number of coalescence events for a given time.

We begin with the cumulative distribution, Eq. (SB2.2). This gives us the fraction of drops that have experienced N collisions as a function of time, denoted $p(t|N)$, with a separate curve corresponding to each N. Using the curves for $N = 1 \ldots \infty$ and choosing a single time t, it is possible to find the distribution of the number of collisions given t, $p(N|t)$. A subtlety arises, however, because any drop that has experienced N coalescences also has experienced $N - 1$ coalescences. To find the probability density of a drop experiencing exactly N coalescence events after time

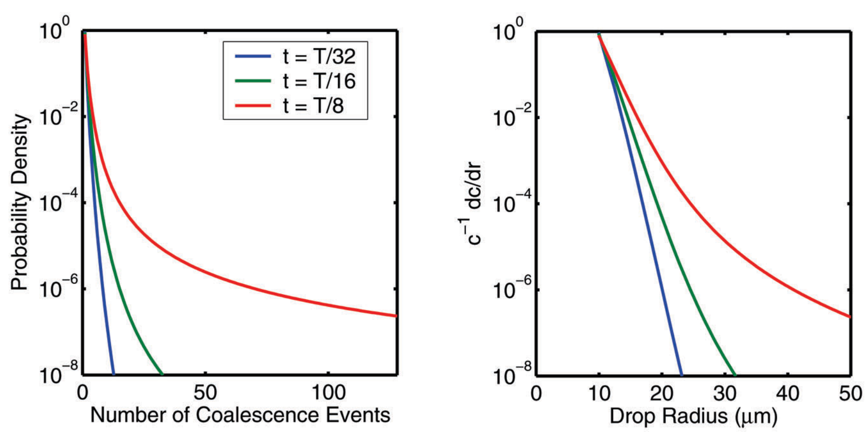

FIG. S1. (left) Each curve is a probability density of *N* coalescence events at a given time (*t* = *T*/32, *t* = *T*/16, and *t* = *T*/8, where $T = \tau\pi^2/6$). This is obtained by taking the difference between successive cumulative pdf's of collision times at a given number of collisions *N*. (right) Drop size distribution at the same times as in the left panel. The distribution is normalized such that it may be interpreted as a probability density of droplet size.

t we must find the difference $p(N|t) = P(t|N) - P(t|N + 1)$. The entire size distribution, then, can be calculated by repeating this for $N = 0 \ldots \infty$. In practice it is necessary only to calculate this difference for the range of N that results in numbers less than the desired resolution in the drop size distribution. This new distribution is properly normalized and can be transformed to a traditional drop size distribution, $\frac{1}{c}\frac{dc}{dr}$, by converting the number of collisions to the drop size via $r_n = r_o(n + 1)^{1/3}$.

An example of $p(N|t)$ is shown in the left panel of Fig. S1, where the mean collision times are assumed to vary as in series (3). The point farthest to the left represents those drops that have experienced $N = 0$ collisions and is obtained by calculating $p(N = 0|t) = 1 - P(t|N = 1)$. The corresponding size distribution is shown in the right panel of Fig. S1. Note the long tails, representing the lucky drops. Finally, we must keep in mind that while the lucky 10^{-6} droplets undergo all 128 coalescence events, about 85% of all droplets undergo no coalescence at all. Thus, the size distributions obtained here remain accurate as long as we restrict ourselves to precipitation initiation.

The restriction to short times relative to the average growth time leads to yet another powerful simplification. Recall that to obtain a size distribution one need only find differences of cumulative generalized Erlang distributions for the relevant N. But calculating these distributions, while straightforward on a computer, does not pass as a "back of the envelope" or even a pocket-calculator operation. For $t \ll \tau$, however, the distribution tails reduce to a much simpler form, as suggested in Fig. S2. Clearly at small t the droplet fraction has a power law dependence. In fact, the result is remarkably simple and is obtained by expanding the exponential functions in Eq. (8) as Nth-order Taylor series, resulting in $P(t|N) \approx N!(t/\tau)^N$. Calculating the droplet fraction and the requisite differences for the size distribution is thereby greatly simplified. Furthermore, this approximate, analytical result may be of broader utility in calculations of precipitation initiation, as discussed in Fig. S2.

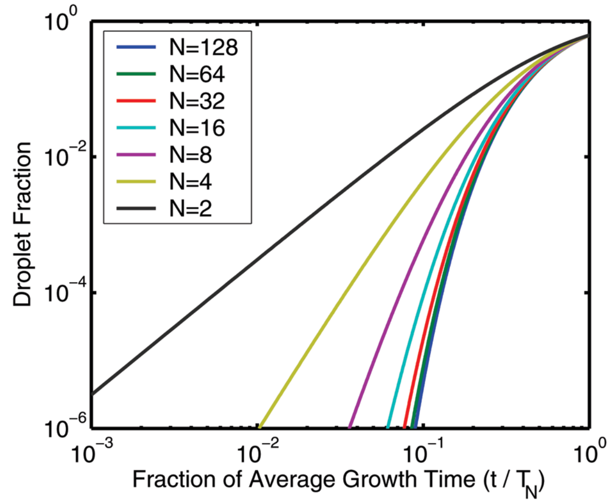

Fig. S2. The fraction of droplets (cumulative generalized Erlang distribution) that have experienced N collisions as a function of time. The figure is expanded and plotted on a log–log scale to emphasize the power law short-time tails of the distribution. One may wish to think of the curves as corresponding to systems with equal liquid water contents, but progressively divided into fewer, larger droplets, such that fewer collisions are necessary to reach the same final drop size. For a final radius of 50 μm, the N = 128 curve corresponds to an initial radius of $r \approx 10$ μm, N = 64 to $r \approx 12.6$ μm, N = 32 to $r \approx 15.9$ μm, and N = 16 to $r = 20$ μm. As in Fig. 2, the abscissa is time relative to the total average growth time, such that the relative importance of the fluctuations may be compared. Clearly, as equal liquid water content is divided into larger (fewer) drops the importance of early luck becomes more pronounced. The simple power law tails at small t allow this luck to be calculated analytically in a straightforward way.