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IMPROVED PARAMETER ESTIMATION OF THE LOG-LOGISTIC DISTRIBUTION WITH APPLICATIONS

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IMPROVED PARAMETER ESTIMATION OF THE LOG-LOGISTIC DISTRIBUTION WITH APPLICATIONS

By

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Abstract

In this report, we work with parameter estimation of the log-logistic distribution. We first consider one of the most common methods encountered in the literature, the maximum likelihood (ML) method. However, it is widely known that the maximum likelihood estimators (MLEs) are usually biased with a finite sample size. This motivates a study of obtaining unbiased or nearly unbiased estimators for this distribution. Specifically, we consider a certain 'corrective' approach and Efron's bootstrap resampling method, which both can reduce the biases of the MLEs to the second order of magnitude. As a comparison, we also consider the generalized moments (GM) method. Monte Carlo simulation studies are conducted to compare the performances of the various estimators under consideration. Finally, two real-data examples are analyzed to illustrate the potential usefulness of the proposed estimators, especially when the sample size is small or moderate.

Chapter 1

Introduction

The log-logistic distribution is related to the logistic distribution in an identical fashion to how the log-normal and normal distributions are related with each other. A logarithmic transformation on the logistic distribution generates the log-logistic distribution. The probability density function (pdf) of the log-logistic distribution is given by

$$f(x \mid \alpha, \beta) = \frac{(\beta/\alpha)\,(x/\alpha)^{\beta-1}}{\left[1 + (x/\alpha)^{\beta}\right]^{2}}, \quad x > 0, \tag{1.1}$$

where $\alpha > 0$ is the scale parameter, and is the median of this distribution; $\beta > 0$ is the shape parameter, which controls the shape of the distribution, as illustrated in Figure 1.1. We observe that this distribution has radically different shapes, as the distribution can be strictly decreasing, right-skewed, or unimodal. As β increases, this distribution becomes more symmetric.

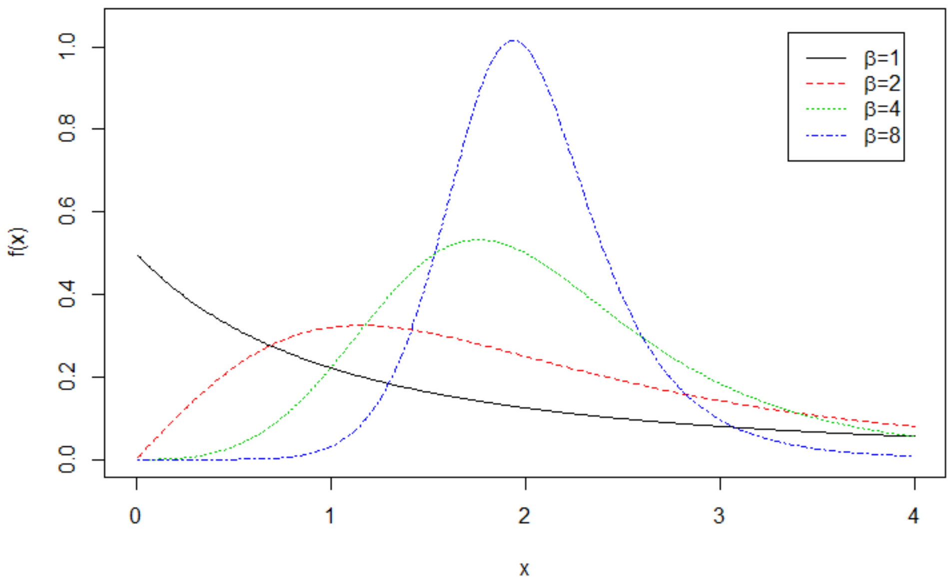

Figure 1.1: The pdf of the log-logistic distributions with $\alpha = 2$ and various values of β.

Because of its flexible shapes, the log-logistic distribution has been illustrated to provide useful fits to data from many different fields, including engineering, economics, hydrology, and survival analysis. For instance, [10] adopted this distribution in modeling economic data. [21] showed its superior performance on fitting precipitation data from various Canadian regions. [3] applied this distribution to maximum annual stream flow data. For further topics related to the log-logistic distribution, we refer the interested reader to [22], and [13].

We are particularly interested in estimating the unknown parameters of the log-logistic distribution. It is well-known the maximum likelihood method is a common choice to estimate the unknown parameters. This is due to its various attractive properties, such as being asymptotically consistent, unbiased, and normal as the sample tends to infinity. However, these attractive properties may not be valid when the sample size of the data is small or moderate, as is encountered in many practical applications. For instance, the maximum likelihood estimators (MLEs) may be severely biased to a certain order for a small sample size; see, for example, [11], [24], among others. It deserves mentioning that [1] recently considered Bayesian estimation of the log-logistic distribution using objective priors. They showed the performances of the Bayesian estimators and the MLEs are quite similar with the various sample sizes, indicating the bias of the Bayesian estimators for small and moderate sample sizes. This motivates a study for obtaining unbiased or nearly unbiased estimators of the unknown parameters for the log-logistic distribution.

In this paper, we first consider a certain 'corrective' approach developed in part by [7], which can correct the bias to the second order of magnitude. The main idea of this 'corrective' approach is to adjust the bias by subtracting it from the original MLEs, and so the obtained estimators are often referred to as bias-corrected MLEs. It is shown that the bias-corrected MLEs of the log-logistic distribution not only have explicit expressions in terms of a convenient matrix notation, but also simultaneously reduce the biases and the root mean square errors (RMSEs) of the parameters.

We then consider Efron's bootstrap resampling method ([8]), which can also reduce the bias to the second order. However, this estimator may accomplish this with an expense of increased variance. As a comparison, we also consider the generalized moments (GM) method, a method commonly used in Hydrology. Monte Carlo simulation studies and real-data applications are provided to compare the performances of the various estimators under consideration. Numerical evidence shows that the proposed bias-corrected MLEs should be recommended for use in practical applications, especially when the sample size is small or moderate.

The rest of this report is organized as follows. Chapter 2 discusses the estimation methods that we will consider. In particular, we discuss in Section 2.1 the MLEs of the parameters for the log-logistic distribution. In Section 2.2, we consider two methods which correct for the bias of the estimators from the ML method. Specifically, Subsection 2.2.1 presents a 'corrective' approach which analytically derives bias-corrected maximum likelihood estimators (MLEs). In Subsection 2.2.2, we discuss Efron's bootstrap resampling method, with which we can derive an alternative bias-correction estimator. In Section 2.3, we discuss another commonly used method in the literature, the generalized moments (GM) method. In Chapter 3, we conduct Monte Carlo simulations to compare the performances of the various estimators. Two real-data examples are analyzed in Chapter 4 for illustrative purposes. Some concluding remarks are provided in Chapter 5 with mathematical derivations given in Appendix A.

Chapter 2

Estimation Methods

2.1 Maximum Likelihood Estimation

Suppose that we have n observations from the log-logistic distribution, denoted by $X_1, \cdots, X_n$. The log-likelihood function of α and β can be written as

$$\log L = n\log(\beta) - n\beta\log(\alpha) + (\beta-1)\sum_{i=1}^{n}\log\left(X_i\right) - 2\sum_{i=1}^{n}\log\left[1+\left(\frac{X_i}{\alpha}\right)^{\beta}\right]. \quad (2.1)$$

Differentiating the above function with respect to α and β, we have

$$\frac{\partial \log L}{\partial \alpha} = -\frac{n\beta}{\alpha} + \frac{2\beta}{\alpha}\sum_{i=1}^{n}\left(\frac{X_i}{\alpha}\right)^{\beta}\left[1+\left(\frac{X_i}{\alpha}\right)^{\beta}\right]^{-1}, \quad (2.2)$$

and

$$\frac{\partial \log L}{\partial \beta} = \frac{n}{\beta} - n\log(\alpha) + \sum_{i=1}^{n} \log\left(X_i\right) - 2\sum_{i=1}^{n} \left(\frac{X_i}{\alpha}\right)^{\beta} \log\left(\frac{X_i}{\alpha}\right) \left[1 + \left(\frac{X_i}{\alpha}\right)^{\beta}\right]^{-1}. \tag{2.3}$$

The MLEs can be obtained by setting the above two equations to zero. Due to the lack of explicit solutions to Equations (2.2) and (2.3), we numerically estimate the MLEs using the *llogisMLE* function from the R *STAR* package, created by [19]. It is well-known that the MLEs are biased with small sample sizes and the bias of an estimator may lead to misleading interpretations of phenomena in practical applications. This motivates a study for obtaining unbiased or nearly unbiased estimators to reduce the bias of the MLEs of the log-logistic distribution.

2.2 Bias-corrected MLEs

In this section, we consider two commonly used bias-correction techniques, both of which can reduce the biases of the MLEs to the second order of magnitude. Specifically, we adopt a 'corrective' analytical approach in Subsection 2.2.1, and then discuss Efron's bootstrap resampling method in Subsection 2.2.2.

2.2.1 A corrective approach

Suppose that based on n randomly selected observations, we are interested in estimating the p unknown parameters, expressed as $\theta = (\theta_1, \cdots, \theta_p)'$. The joint cumulants of the derivatives of the log-likelihood function $L(\theta)$ are given by

$$k_{ij} = \mathbb{E}\left[\frac{\partial^2 L}{\partial\theta_i \partial\theta_j}\right], \quad k_{ijl} = \mathbb{E}\left[\frac{\partial^3 L}{\partial\theta_i \partial\theta_j \partial\theta_l}\right], \quad k_{ij,l} = \mathbb{E}\left[\left(\frac{\partial^2 L}{\partial\theta_i \partial\theta_j}\right)\left(\frac{\partial l}{\partial\theta_l}\right)\right], \tag{2.4}$$

where $i, j, l = 1, 2, \cdots, p$. The derivatives of the joint cumulants are given by

$$k_{ij}^l = \frac{\partial k_{ij}}{\partial\theta_l}. \tag{2.5}$$

Here, we assume that $L(\theta)$ is regular with respect to all derivatives up to the third order, inclusively. We also assume that all expressions in (2.4) and (2.5) are of order O(n).

Let $K = [-k_{ij}]$ be the Fisher information matrix of θ, where $i, j = 1, 2, \cdots, p$. It can be seen from [7] that if the sample data are independent, the bias of the sth element of $\hat{\theta}$ can be written as

$$\text{Bias}(\hat{\theta}_s) = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{l=1}^{p} k^{si}k^{jl}\left[\frac{1}{2}k_{ijl} + k_{ij,l}\right] + O(n^{-2}), \quad s = 1, 2, \cdots, p, \tag{2.6}$$

where k^{ij} is the (i,j)th element of the inverse of the Fisher information matrix. Later, [6] showed that if the sample data are not identically distributed, Equation (2.6) is still valid for non-independent observations, provided all expressions in (2.4) and (2.5) are of order O(n). More specifically, Equation (2.6) can be reexpressed as

$$\text{Bias}(\hat{\theta}_s) = \sum_{i=1}^{p} k^{si} \sum_{j=1}^{p} \sum_{l=1}^{p} \left[k_{ij}^{(l)} - \frac{1}{2} k_{ijl} \right] k^{jl} + O(n^{-2}), \quad s = 1, 2, \cdots, p. \tag{2.7}$$

Define a matrix $A^{(l)} = a_{ij}^{(l)}$ with its elements given by $a_{ij}^{(l)} = k_{ij}^{(l)} - \frac{1}{2} k_{ijl}$. We have

$$A = \left[A^{(1)} \mid A^{(2)} \mid \cdots \mid A^{(p)} \right] \quad \text{with} \quad A^{(l)} = \left[a_{ij}^{(l)} \right].$$

Accordingly, the bias expression of $\hat{\theta}$ can then be written in matrix form as

$$\text{Bias}(\hat{\theta}) = K^{-1} A \cdot \text{vec}(K^{-1}) + O(n^{-2}).$$

This shows that the bias-corrected MLE of θ, denoted as $\hat{\theta}^{\text{CMLE}}$, is given by

$$\hat{\theta}^{\text{CMLE}} = \hat{\theta} - \hat{K}^{-1} \hat{A} \cdot \text{vec}(\hat{K}^{-1}),$$

where $\hat{\theta}$ is the MLE of θ, $\hat{K} = K \mid_{\theta = \hat{\theta}}$, and $\hat{A} = A \mid_{\theta = \hat{\theta}}$. It should be noted that the bias of $\hat{\theta}^{\text{CMLE}}$ is of the second order.

Since we are working with the log-logistic distribution, we have $p = 2$, with $\theta = (\alpha, \beta)'$.

The joint cumulants of the derivatives of the log-likelihood function are provided below. For further details on the mathematical derivations of these joint cumulants, the reader is invited to refer to Appendix A. The joint cumulants are given as follows

$$
\begin{aligned}
&k_{11} = -\frac{n\beta^2}{3\alpha^2},\ k_{12} = k_{21} = 0,\ k_{22} = -\frac{n\left(1 + (\pi^2 - 6)/9\right)}{\beta^2},\\
&k_{111} = \frac{n\beta^2}{\alpha^3},\ k_{112} = -\frac{n\beta}{2\alpha^2},\ k_{122} = 0,\ k_{222} = \frac{n}{\beta^3}\left(1 + \frac{\pi^2}{6}\right).
\end{aligned}
$$

The corresponding derivatives of the joint cumulants are given by

$$
\begin{aligned}
&k_{11}^{(1)} = \frac{\partial k_{11}}{\partial \alpha} = \frac{2n\beta^2}{3\alpha^3},\ k_{12}^{(1)} = \frac{\partial k_{12}}{\partial \alpha} = 0,\ k_{22}^{(1)} = \frac{\partial k_{22}}{\partial \alpha} = 0,\\
&k_{11}^{(2)} = \frac{\partial k_{11}}{\partial \beta} = -\frac{2n\beta}{3\alpha^2},\ k_{12}^{(2)} = \frac{\partial k_{12}}{\partial \beta} = 0,\ k_{22}^{(2)} = \frac{\partial k_{22}}{\partial \beta} = \frac{2n(3 + \pi^2)}{9\beta^3}.
\end{aligned}
$$

By using $a_{ij}^{(l)} = k_{ij}^{(l)} - \frac{1}{2}k_{ijl}$, simple algebra shows that the matrix of A is found to be

$$
\begin{aligned}
A &= \left[A^{(1)} \mid A^{(2)}\right]\\
&= n\begin{bmatrix} \frac{\beta^2}{6\alpha^3} & \frac{\beta}{4\alpha^2} & \frac{-5\beta}{12\alpha^2} & 0 \\ \frac{\beta}{4\alpha^2} & 0 & 0 & \frac{1}{18\beta^3}\left(3 + \frac{5\pi^2}{2}\right) \end{bmatrix}.
\end{aligned}
\tag{2.8}
$$

We also find that the Fisher information matrix of the log-logistic distribution is given by

$$K = n \begin{bmatrix} \dfrac{\beta^2}{3\alpha^2} & 0 \\ 0 & \dfrac{1 + \frac{1}{9}(-6 + \pi^2)}{\beta^2} \end{bmatrix}. \tag{2.9}$$

Thus, the bias of the MLEs of the log-logistic distribution can be obtained by

$$\text{Bias} \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = K^{-1} A \cdot \text{vec}\big(K^{-1}\big) + O(n^{-2}).$$

Thus, the bias-corrected MLEs of the log-logistic distribution can be constructed as

$$\begin{pmatrix} \hat{\alpha}^{\text{CMLE}} \\ \hat{\beta}^{\text{CMLE}} \end{pmatrix} = \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} - \hat{K}^{-1} \hat{A} \cdot \text{vec}\big(\hat{K}^{-1}\big), \tag{2.10}$$

where $\hat{K} = K \mid_{\alpha=\hat{\alpha},\beta=\hat{\beta}}$ and $\hat{A} = A \mid_{\alpha=\hat{\alpha},\beta=\hat{\beta}}$. It should be noted that the bias-corrected estimators in (2.10) can be computed easily as long as the MLEs are available. Further note that the estimators $\hat{\alpha}^{\text{CMLE}}$and $\hat{\beta}^{\text{CMLE}}$ are unbiased to order $O(n^{-2})$. That is, the expected values of these estimators are $\mathbb{E}\big[\hat{\alpha}^{\text{CMLE}}\big] = \alpha$ + $\text{O}(n^{-2})$ and $\mathbb{E}\big[\hat{\beta}^{\text{CMLE}}\big] = \beta$ + $\text{O}(n^{-2})$.

2.2.2 Correcting Bias Using The Bootstrap

We also consider Efron's bootstrap resampling method, which was introduced by [8]. The main idea of this method is to generate pseudo-samples from the original sample to estimate the bias of the MLEs. We then subtract the estimated bias from the original MLEs to obtain bias-corrected MLEs.

Let $\mathbf{x} = (x_1, \cdots, x_n)'$ be a sample of n randomly selected observations from the random variable X with its cumulative distribution function (cdf) given by F. Let the parameter υ be some function of F, denoted by $\upsilon = t(F)$. Let $\hat{\upsilon}$ be some estimator of υ. We obtain pseudo-samples $\mathbf{x}^* = (\mathbf{x_1^*}, \cdots, \mathbf{x_n^*})'$ from the original sample $\mathbf{x}$ by resampling observations with replacement. We compute the bootstrap replicates of $\hat{\upsilon}$ from these pseudo samples, denoted by $\hat{\upsilon}^* = s(\mathbf{x}^*)$. We use the empirical cdf (ecdf) of $\hat{\upsilon}^*$ to estimate the cdf of $\hat{\upsilon}$, $F_{\hat{\upsilon}}$. We obtain a parametric estimate for F by using a consistent estimator for $F_{\hat{\upsilon}}$, provided F belongs to a parametric family which is known and has a finite dimension, F_{υ}. The bias of the estimator $\hat{\upsilon} = s(\mathbf{x})$ can be estimated by using

$$B_F(\hat{\upsilon}, \upsilon) = \mathbb{E}_F[\hat{\upsilon}] - \upsilon(F). \tag{2.11}$$

Here, we take the expectation with respect to F, as indicated by the subscript of the expectation. Recall that the original sample $\mathbf{x}$ was obtained from F. Furthermore, the bootstrap replicates were obtained from the ecdf of the original sample, $F_{\hat{\upsilon}}$. Thus, the bootstrap bias estimate is obtained by replacing F with $F_{\hat{\upsilon}}$, and we then have the following expression

$$\hat{B}_{F_{\hat{\upsilon}}}(\hat{\upsilon}, \upsilon) = \mathbb{E}_{F_{\hat{\upsilon}}}[\hat{\upsilon}^*] - \hat{\upsilon}.$$

Suppose that we have N bootstrap estimates, denoted by $\left(\hat{\upsilon}^{*(1)}, \cdots, \hat{\upsilon}^{*(N)}\right)$, based on N bootstrap pseudo-samples, which are independently generated from the original sample $\mathbf{x}$. When N is sufficiently large, the expected value of our estimator $\mathbb{E}_{F_{\hat{\upsilon}}}[\hat{\upsilon}^*]$ can be approximated by

$$\hat{\upsilon}^{*(\cdot)} = \frac{1}{N}\sum_{i=1}^{N} \hat{\upsilon}^{*(i)},$$

The bootstrap bias estimate of the parameter becomes $\hat{B}_F(\hat{\upsilon}, \upsilon) = \hat{\upsilon}^{*(\cdot)} - \hat{\upsilon}$, which shows that the bias-corrected estimators based on Efron's bootstrap resampling method are given by

$$\upsilon^B = \hat{\upsilon} - \hat{B}_F(\hat{\upsilon}, \upsilon) = 2\hat{\upsilon} - \hat{\upsilon}^{*(\cdot)}. \tag{2.12}$$

Note that the estimator υ^B is known as a constant bias-corrected MLE since it approximates the function by a constant; see [16]. In our instance, we let $\hat{\upsilon} = \hat{\theta}^{\text{MLE}} = (\hat{\alpha}^{\text{MLE}}, \hat{\beta}^{\text{MLE}})'$. We have υ^B equal to the bootstrap bias-corrected estimate, denoted by $\hat{\theta}^{\text{BOOT}} = (\hat{\alpha}^{\text{BOOT}}, \hat{\beta}^{\text{BOOT}})'$.

2.3 Generalized Moments

As a comparison, we consider another commonly used method, the generalized moments (GM) method, which utilizes moments of the form $E[X^k] = M_k$, where k can take on a diverse range of values, being positive or negative. In general, the values of k can be chosen to suit the user's needs, and the GM method can thus provide differing weights to the data values. [4] have implemented the GM method for the log-logistic distribution based on similar techniques as are used for the generalized probability weighted moments (GPWM) method, introduced by [12]. For our problem, we consider probability weighted moments (PWMs) of the form

$$\begin{aligned} M_{k,h} = \mathbb{E}[X^k F^h] &= \int_{-\infty}^{\infty} x^k F^h(x) f(x) dx \\ &= \alpha^k B\left(h + 1 + \frac{k}{\beta}, 1 - \frac{k}{\beta}\right), \end{aligned}$$

where $B(\cdot, \cdot)$ is the Beta function. The GM method is obtained by letting $h = 0$: $M_{k,h=0} = \mathbb{E}[X^k F^0] = \mathbb{E}[X^k]$. We then choose two numbers $k = k_1$ and k_2, where the values k_1 and k_2 are positive or negative real numbers satisfying the constraint $-\beta < k < \beta$, which guarantees the existence of the PWM $M_{k,0}$. We then have

$$M_{k_1,0} = \alpha B(1 + \frac{k_1}{\beta}, 1 - \frac{k_1}{\beta}) = \frac{\alpha^{k_1} \pi k_1}{\beta \sin(\pi k_1/\beta)} \tag{2.13}$$

and

$$M_{k_2,0} = \alpha B(1+\frac{k_2}{\beta}, 1-\frac{k_2}{\beta}) = \frac{\alpha^{k_2}\pi k_2}{\beta \sin{(\pi k_2/\beta)}}. \tag{2.14}$$

To obtain the GM estimates of α and β, we first substitute in the sample PWM estimates $\hat{M}_{k_1,0}$ and $\hat{M}_{k_2,0}$ for their respective PWMs and then solve equation (2.13) for α, thus obtaining

$$\alpha = \left[\beta(\pi k_1)^{-1}\hat{M}_{k_1,0} sin\left(\frac{\pi k_1}{\beta}\right)\right]^{\frac{1}{k_1}}. \tag{2.15}$$

After substituting the RHS of equation (2.15) into equation (2.14), we numerically approximate the estimate of β by finding a solution of

$$\hat{M}_{k_2,0} = k_2 k_1^{-\frac{k_2}{k_1}} \left(\frac{\pi}{\beta}\right)^{\frac{k_1-k_2}{k_1}} \left[\hat{M}_{k_1,0}\sin\left(\frac{\pi k_1}{\beta}\right)\right]^{\frac{k_2}{k_1}} \left[\sin\left(\frac{\pi k_2}{\beta}\right)\right]^{-1}. \tag{2.16}$$

We then substitute the estimate of β into equation (2.15) to find the estimate of α. We denote these estimates by $\hat{\alpha}^{\mathrm{GM}}$ and $\hat{\beta}^{\mathrm{GM}}$. It should be noted that a method for determining the optimal values k_1 and k_2 has yet to be identified. Thus, researchers are encouraged to try several pairs of k_1 and k_2 to obtain 'optimal' results; see, for example, [4].

Chapter 3

Simulation Studies

In this chapter, we conduct Monte Carlo simulations to evaluate the performances of the various considered estimators of the log-logistic distribution. The data were simulated using the *rllogis* function in the *STAR* package created by [19].

We draw random samples of size $n = 8, 12, 20, 35, 50,\ 75, 100$ with parameters $\alpha = 1, 1.5, 2$ and $\beta = 1, 1.5, 2$. For the GM methods we only consider $k_1 = 0.50$ and $k_2 = -0.50$. It deserves mentioning that some preliminary studies had shown that different combinations of k_1 and k_2 could result in unresolved computational errors. For each combination of (n, α, β), we used $M = 5,000$ Monte Carlo replications in our simulations. We also used $B = 1,000$ bootstrap replications, meaning that each combination had 25 million replications in total. For an estimator $\hat{\theta}^{est}$ of the parameter θ, we compute the average bias and the root mean square error (RMSE), which are given by

$$\text{Bias}(\hat{\theta}^{\text{est}}) = \frac{1}{M}\sum_{i=1}^{M}\left(\hat{\theta}_i^{\text{est}} - \theta\right), \text{ and } \text{RMSE}(\hat{\theta}^{\text{est}}) = \sqrt{\frac{1}{M}\sum_{i=1}^{M}\left(\hat{\theta}_i^{\text{est}} - \theta\right)^2},$$

respectively. Figures 3.1-3.4 show the average biases of the considered estimates of α and β. Figures 3.5-3.8 depict the RMSEs for α and β. Some conclusions can be drawn as follows.

(i) The MLE and GM estimators of α and β appear to have a positive bias for each simulation under consideration. This indicates that, on average, they overestimate the actual values of the parameters α and β, especially when the sample size is small. Also, the GM estimator had a higher bias and RMSE compared to the MLE in each simulation.

(ii) In each simulation, the CMLE and the BOOT of α and β outperformed the MLEs and GM estimators in terms of bias and RMSE for different sample sizes. Thus, if bias is a concern, then the CMLE and BOOT would be favorable alternatives for estimating α and β.

(iii) As expected, the bias and RMSE of all considered estimators will decrease as n increases. This is mainly because in statistical theory most of the estimators have better performance when the sample size n becomes large.

(iv) As stated above, for small sample sizes, the reduction in bias and RMSE is quite substantial for the bias-corrected estimators. For example, where $n = 8$, $\alpha = 1$, and $\beta = 1.5$, we have Bias($\hat{\alpha}^{\text{MLE}}$) = 0.0875, Bias($\hat{\alpha}^{\text{CMLE}}$) = 0.0010, Bias($\hat{\alpha}^{\text{BOOT}}$) = -0.0215, Bias($\hat{\beta}^{\text{MLE}}$) = 0.3075, Bias($\hat{\beta}^{\text{CMLE}}$) = 0.0192, Bias($\hat{\beta}^{\text{BOOT}}$) = -0.1420, RMSE($\hat{\alpha}^{\text{MLE}}$) = 0.4815, RMSE($\hat{\alpha}^{\text{CMLE}}$) = 0.4317, RMSE($\hat{\alpha}^{\text{BOOT}}$) = 0.4300, MSE($\hat{\beta}^{\text{MLE}}$) =0.7193, RMSE($\hat{\beta}^{\text{CMLE}}$) = 0.5468, RMSE($\hat{\beta}^{\text{BOOT}}$) = 0.5366.

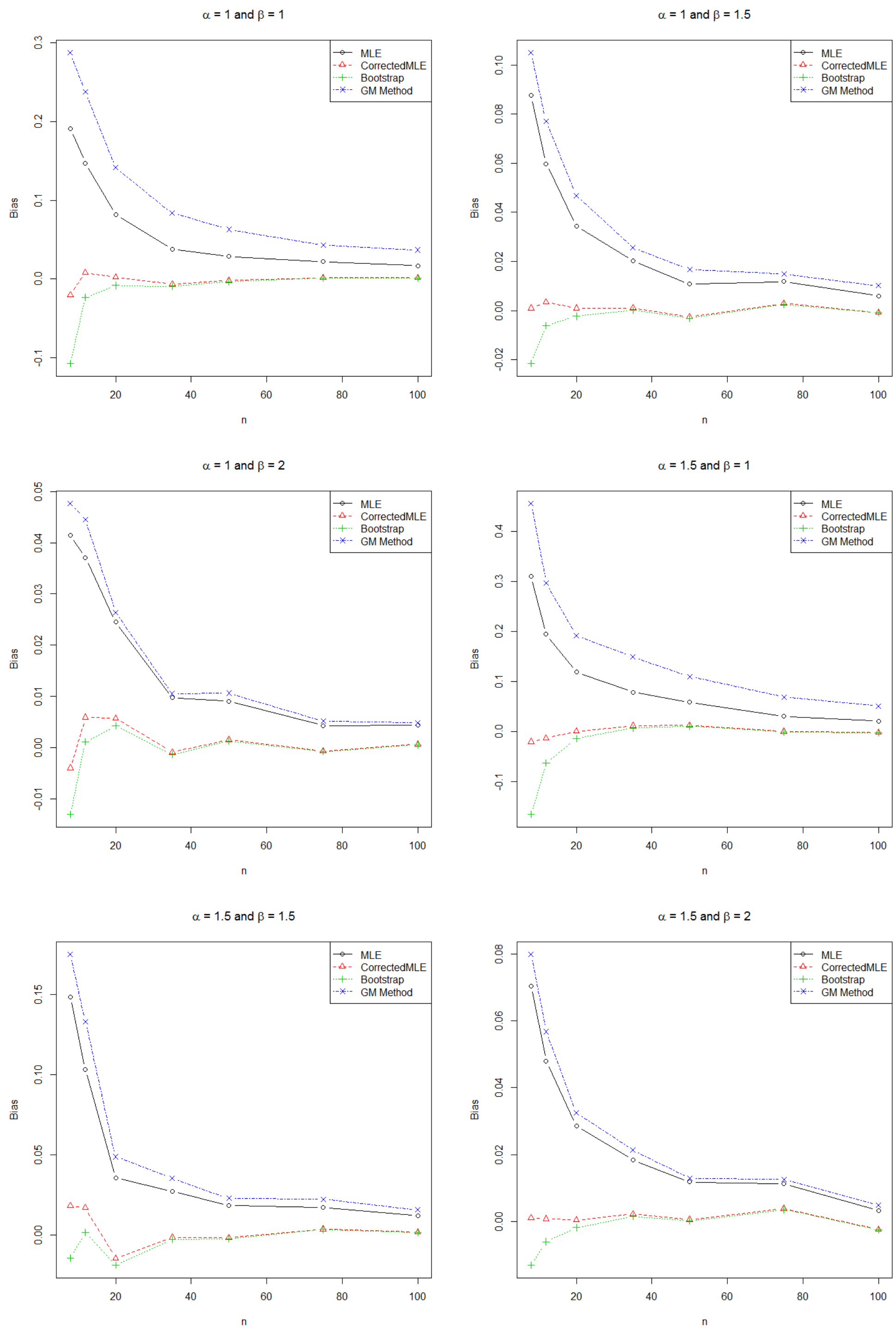

Figure 3.1: Comparison of the average biases of the four different estimation methods for α.

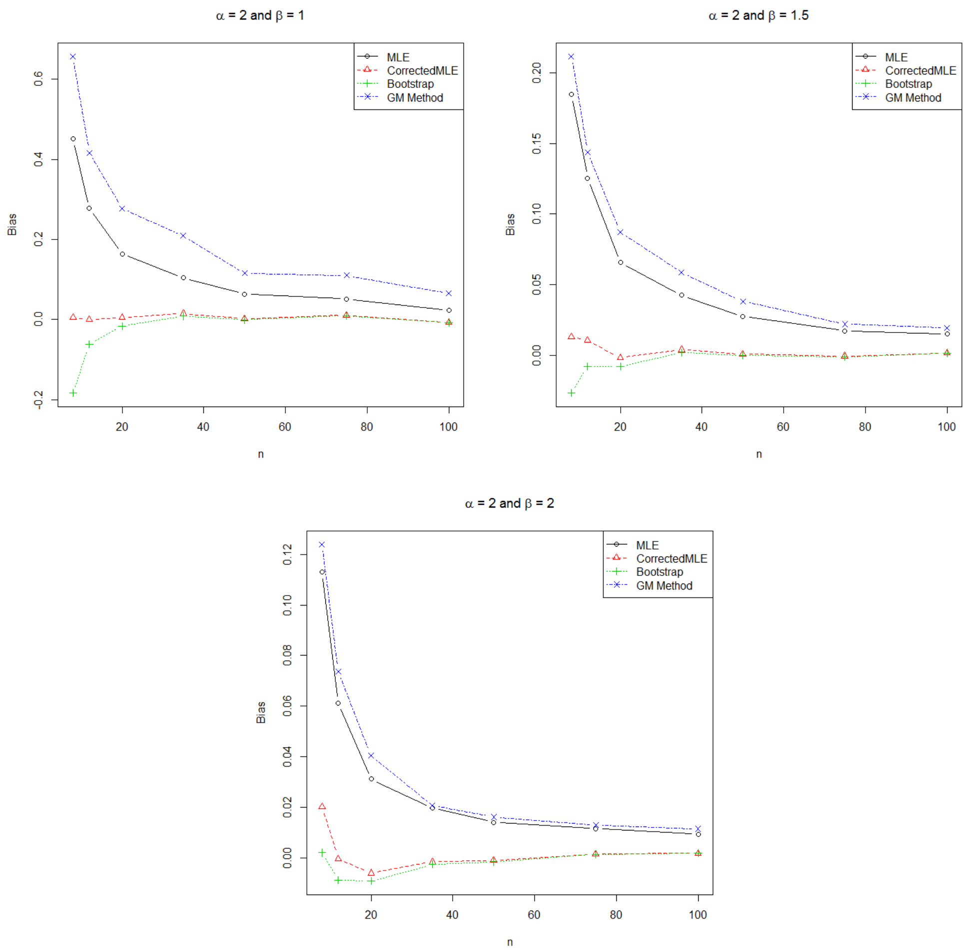

Figure 3.2: Comparison of the average biases of the four different estimation methods for α.

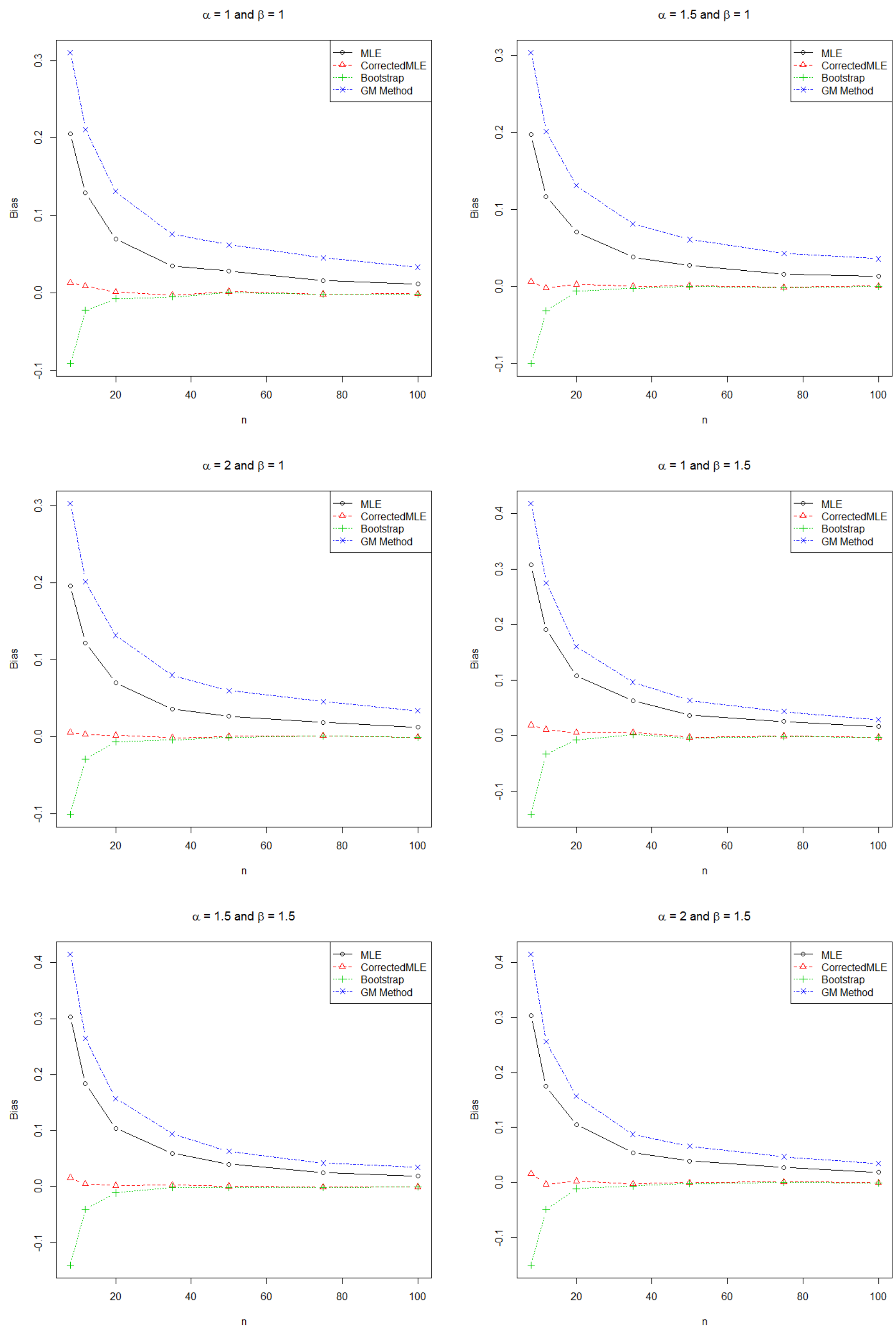

Figure 3.3: Comparison of the average biases of the four different estimation methods for β.

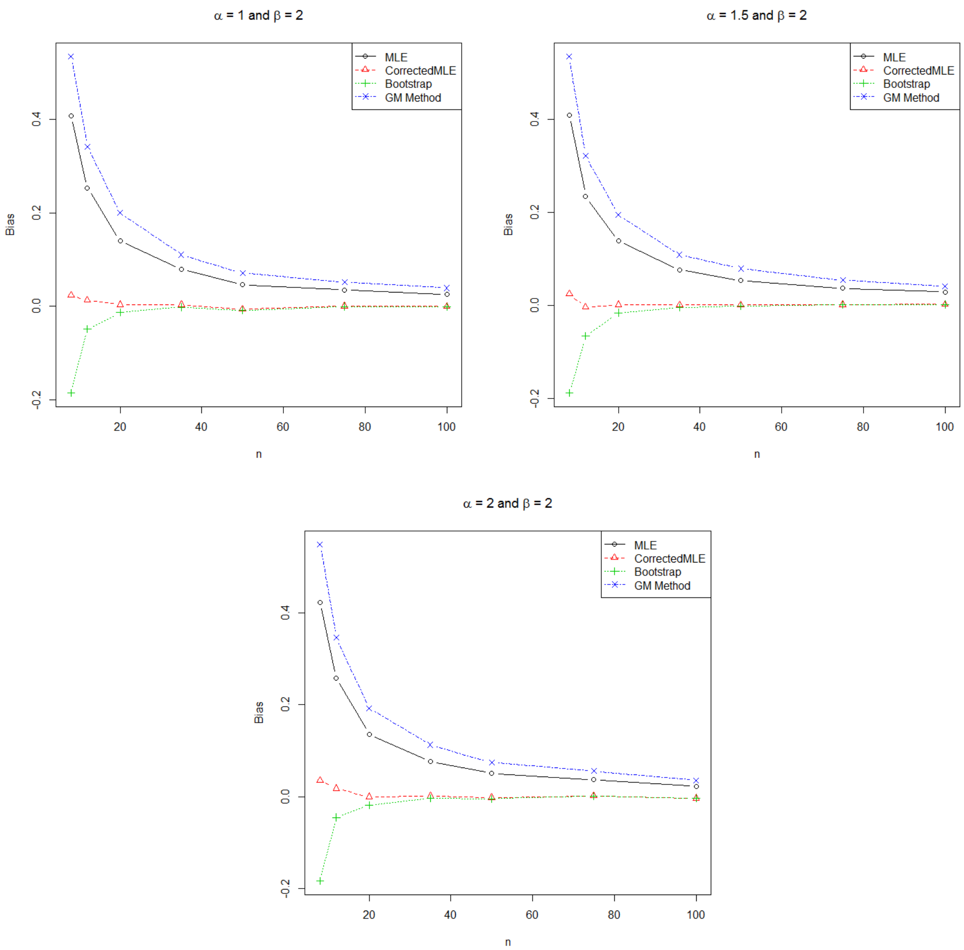

Figure 3.4: Comparison of the average biases of the four different estimation methods for β.

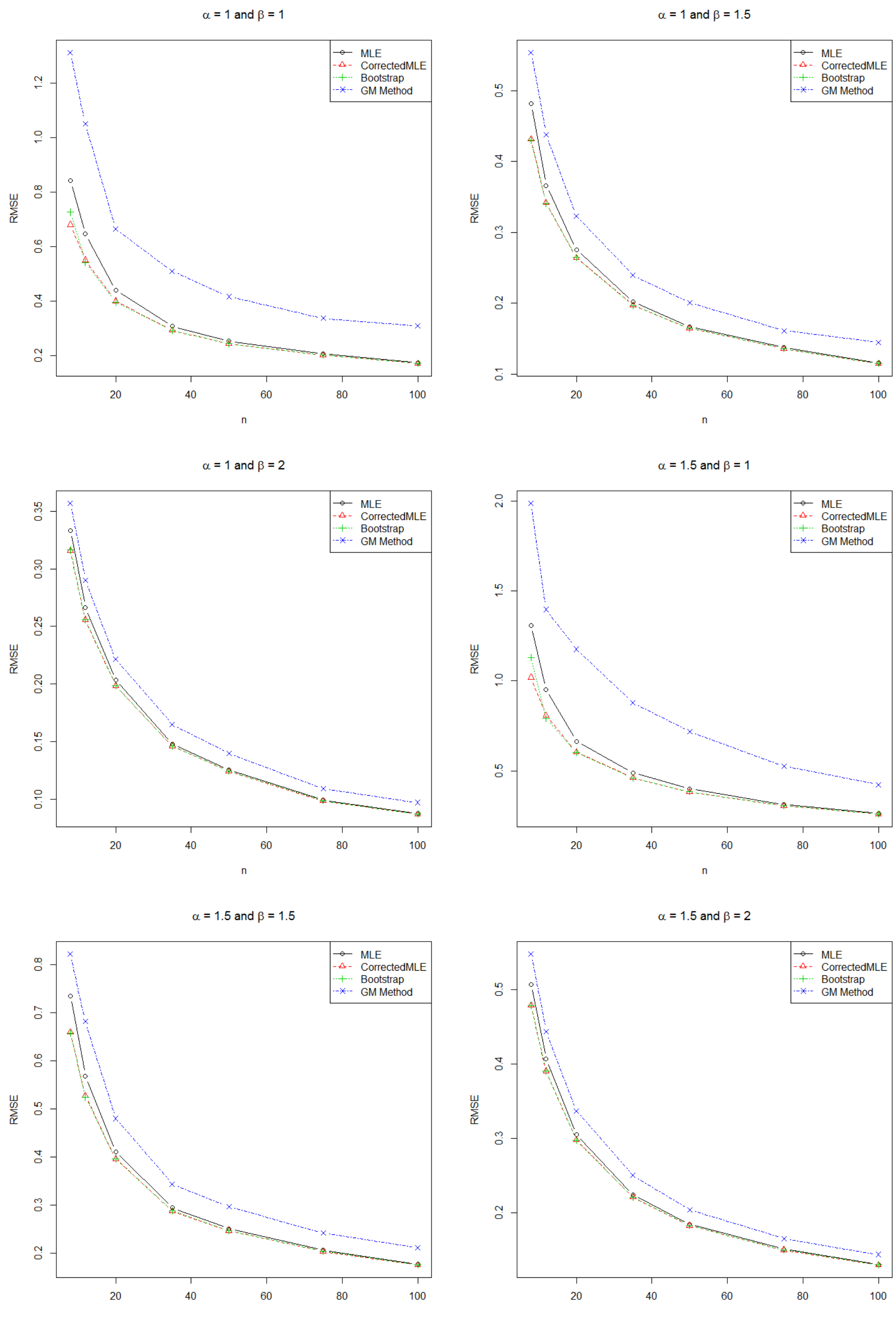

Figure 3.5: Comparison of the RMSEs of the four different estimation methods for α.

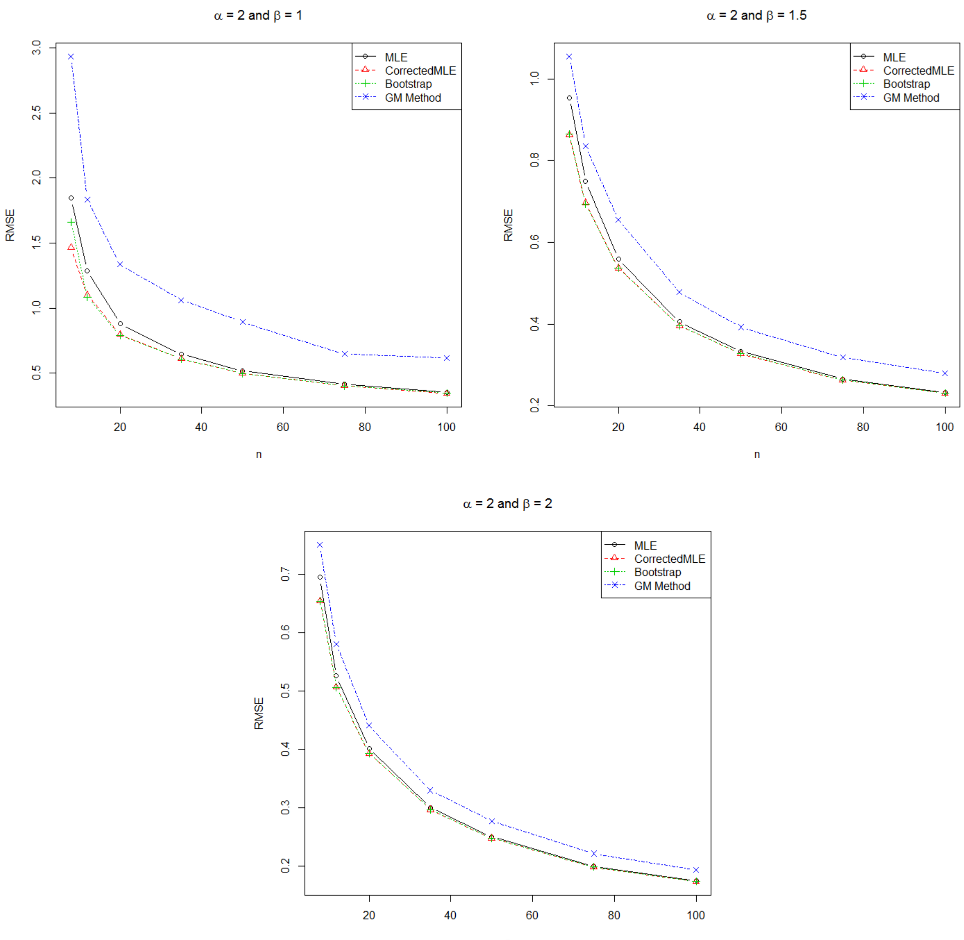

Figure 3.6: Comparison of the RMSEs of the four different estimation methods for α.

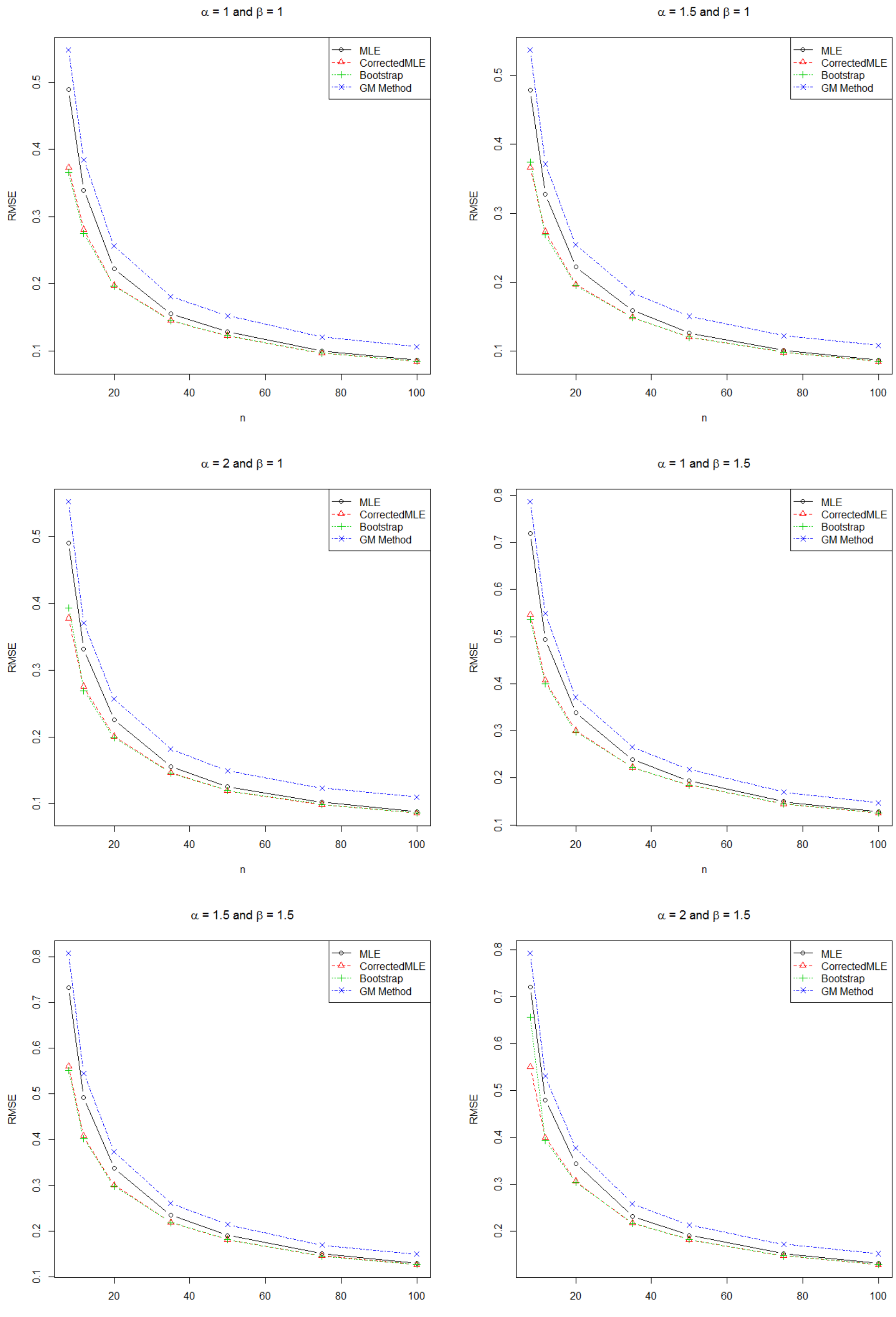

Figure 3.7: Comparison of the RMSEs of the four different estimation methods for β.

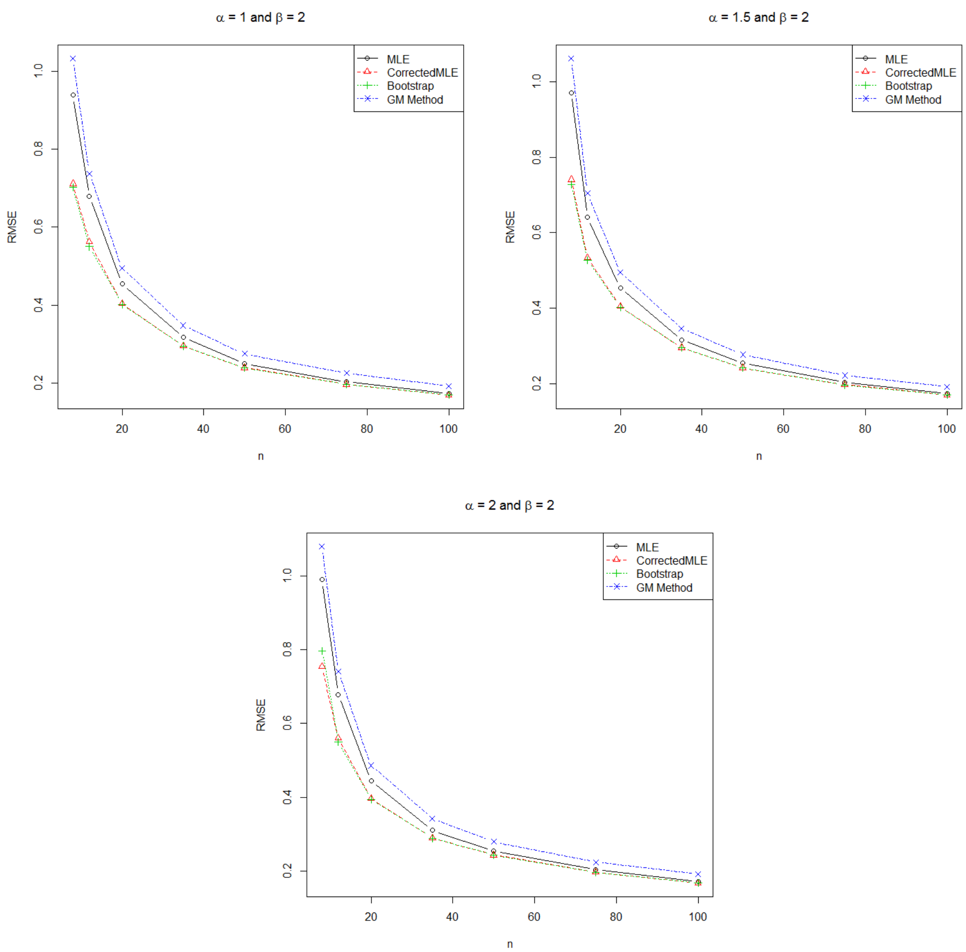

Figure 3.8: Comparison of the RMSEs of the four different estimation methods for β.

Chapter 4

Application to Real Data Examples

In this chapter, we compare the performances of the estimators under consideration through two real data sets, as illustrated in Examples 6.1 and 6.2.

Example 6.1 This data set focuses on the time to breakdown of an insulating fluid between electrodes at a voltage of 34 kV. These data are originally from [18] and were later used by [1]. The data are presented in Table 4.1.

0.96	4.15	0.19	0.78	8.01	31.75	7.35	6.50	8.27	33.91
32.52	3.16	4.85	2.78	4.67	1.31	12.06	36.71	72.89	

Table 4.1
The time to breakdown of an insulating fluid

We choose $k_1 = 0.75$ and $k_2 = 0.35$ for the GM method. Table 4.2 lists the estimated values of the unknown parameters of the log-logistic distribution. We observe that the MLEs and GM estimates are both larger than the bias-corrected estimates of α and β, which shows that estimation by the ML and GM methods are overestimating both parameters, especially α. Figure 4.1 depicts the pdf and cdf of the log-logistic distribution evaluated with the different values of the estimates of α and β in Table 4.2. It can be seen from the figure that the density shapes based on the MLE and GM methods may be misleading, and thus we recommend the use of bias-corrected MLEs for this data set.

Estimate	α	β
MLE	6.253730	1.173462
GM	6.585568	1.499451
CMLE	5.895189	1.094631
BOOT	5.936168	1.094750

Table 4.2
Point estimates of α and β for the insulating fluid data

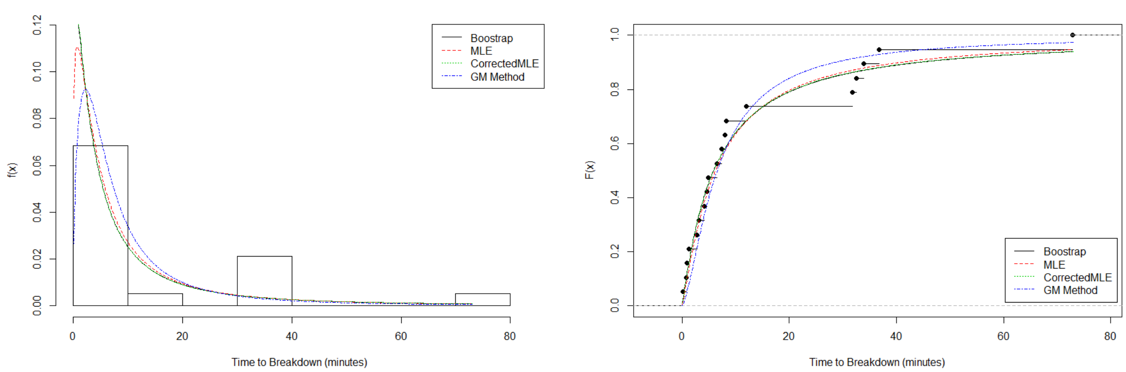

Figure 4.1: The pdf and cdf of the log-logistic distribution fitted to the time to breakdown of an insulating fluid data using the various estimates of α and β.

Example 6.2 This second data set focuses on the time to failure of an electronic device studied by [23]. The data set can be found in Table 4.3.

5	11	21	31	46	75	98	122	145
65	196	224	245	293	321	330	350	420

Table 4.3
The time to failure of an electronic device

Estimate	α	β
MLE	122.263427	1.419075
GM	111.970504	1.947895
CMLE	117.203965	1.318449
BOOT	119.582026	1.292038

Table 4.4
Point estimates of α and β for the electronic device failure data.

We choose $k_1 = 0.75$ and $k_2 = 0.35$ for the GM method. Table 4.4 contains the estimated values for α and β based on the considered estimation methods. We observe that the MLEs are larger than the corresponding bias-corrected estimates of α and β. Figure 4.2 displays the estimated pdfs and cdfs of the log-logistic distribution calculated with the various estimated parameter values in Table 4.4. Once again, the fitted densities and cdfs are quite different for the bias-corrected estimators versus the densities fitted using the ML and GM methods. Again, like in Example 6.1, the MLEs and GM estimates may be misleading. Thus, we again have a preference for the bias-corrected MLEs of the log-logistic distribution, especially since the sample size is small.

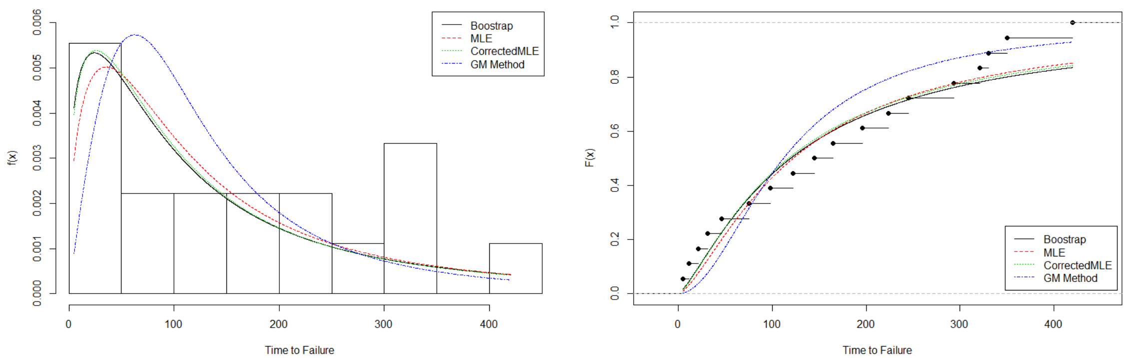

Figure 4.2: The pdf and cdf of the log-logistic distribution fitted to the time to failure of an insulating fluid data using the various estimates of α and β.

Chapter 5

Concluding Remarks

We have derived the second-order bias-corrected MLEs based on a 'corrective' method developed in part by [7]. The derived bias-corrected MLEs not only have explicit expressions in terms of a convenient matrix notation, but also simultaneously reduce the bias and the root mean square errors (RMSEs) of the parameters of the log-logistic distribution. As a comparison, we have also considered Efron's bootstrap resampling method and the GM method. Numerical results from both simulation studies and real-data applications strongly suggest that the bias-corrected MLEs should be recommended for use in practical applications, especially when the sample size is small or even moderate.

References

[1] Abbas, K. and Tang, Y. (2015). Objective Bayesian analysis for log-logistic distribution. *Communications in Statistics - Simulation and Computation*, DOI:10.1080/03610918.2014.925925.

[2] Ali, M.M. (1987). On order statistics from the log-logistic distribution. *Journal of Statistical Planning and Inference* **17**, 103-108.

[3] Ashkar, F., and Mahdi, S. (2003). Comparison of two fitting methods for the log-logistic distribution. *Water Resources Research* **39**, 12-17.

[4] Ashkar, F. , and Mahdi, S. (2006). Fitting the log-logistic distribution by generalized moments. *Journal of Hydrology* **328**, 694-703.

[5] Burr, I. W. (1942). Cumulative frequency functions. *Annals of Mathematical Statistics* **13**, 215-232.

[6] Cordeiro, G.M. and Klein, R. (1994). Bias correction in ARMA models. *Statistics and probability Letters* **19**, 169-176.

[7] Cox, D.R. and Snell, E.J. (1968). A general definition of residuals. *Journal of the Royal Statistical Society. Series B. (Methodological)* **30**, 248-275.

[8] Efron, B. (1979) Bootstrap methods: another look at the Jackknife. *The Annals of Statistics* **7**, 1-26.

[9] Efron, B. and Tibshirani, R. (1986) Bootstrap methods for standard errors, confidence intervals, and other measures of statistical accuracy. *Statistical Science* **1**, 54-75.

[10] Fisk, P. R. (1961). The Graduation of Income Distributions *Econometrica* **29**, 171-185.

[11] Giles, D. E., Feng, H., and Godwin, R. T. (2013). On the bias of the maximum likelihood estimator for the two-parameter Lomax distribution. *Communications in Statistics - Theory and Methods* **42**, 1934-1950.

[12] Greenwood, J.A., Landwehr, J.M., Matalas, N.C., and Wallis, J.R. (1979). Probability weighted moments: definition and relation to paramters of several distributions expressible in inverse form. *Water Resources Research* **15**, 1049-1054.

[13] Gupta, R.C., Akman, O., and Lvin, S. (1999). A study of log-logistic model in survival analysis. *Biometrical Journal* **41**, 431-443.

[14] Haldane, J. B. S. and Smith, S. M. (1956). The Sampling Distribution of a Maximum-Likelihood Estimate. *Biometrika* **43**, 96-103.

[15] Kus, C.S. and Kaya, M.F. (2006). Estimation of parameters of the log-logistic distribution based on progressive censoring using the EM algorithm. *Hacettepe Journal of Mathematics and Statistics* **35**, 203-211.

[16] MacKinnon, J. and Smith, A. (1998). Approximate bias correction in econometrics. *Journal of Econometrics* **85**, 205-230.

[17] Mielke, P.W. and Johnson, E.S. (1973). Three parameter Kappa distribution maximum likelihood estimates and likelihood ratio tests. *Monthly Weather Review* **101**, 701-709.

[18] Nelson, N.L. (1982). Applied Life Data Analysis. *Wiley*, New York.

[19] Pouzat, C. (2012). STAR: Spike Train Analysis with R. R package version 0.3-7. *http://CRAN.R-project.org/package=STAR*.

[20] R Core Team (2015). R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. *http://www.R-project.org/*.

[21] Shoukri, M. M., Mian, I. U. H., and Tracy, D. S. (1988). Sampling properties of estimators of the log-logistic distribution with application to Canadian precipitation data. *The Canadian Journal of Statistics* **16**, 223-236.

[22] Tiku, M. L. and Suresh, R. P. (1992). A new method of estimation for location and scale parameters. *Journal of Statistical Planning and Inference* **30**, 281-292, North Holland.

[23] Wang, F. (2000). A new model with bathtub-shaped failure rate using an additive Burr XII distribution. *Reliability Engineering and System Safety* **70**, 305-312.

[24] Wang, M. and Wang, W. (2015). Bias-corrected maximum likelihood estimation of the parameters of the weighted Lindley distribution. *Communications in Statistics - Simulation and Computation*, DOI:10.1080/03610918.2014.970696.

Appendix A

Mathematical Derivations of the Joint Cumulants

To obtain the bias-corrected MLEs developed in part by [7], we need to calculate higher-order derivatives of the log-likelihood function of the log-logistic distribution. These derivatives, taken with respect to α and β, are given as follows

$$\frac{\partial^2 \log(L)}{\partial \alpha^2} = \frac{n\beta}{\alpha^2} - \frac{2\beta(\beta+1)}{\alpha^2} \sum_{i=1}^{n} \left(\frac{X_i}{\alpha}\right)^{\beta} \left[1 + \left(\frac{X_i}{\alpha}\right)^{\beta}\right]^{-1} + \frac{2\beta^2}{\alpha^2} \sum_{i=1}^{n} \left(\frac{X_i}{\alpha}\right)^{2\beta} \left[1 + \left(\frac{X_i}{\alpha}\right)^{\beta}\right]^{-2},$$

$$\frac{\partial^2 logL}{\partial\beta^2} = -\frac{n}{\beta^2} - 2\sum_{i=1}^{n}\left[\log\left(\frac{X_i}{\alpha}\right)\right]^2\left(\frac{X_i}{\alpha}\right)^{\beta}\left[1+\left(\frac{X_i}{\alpha}\right)^{\beta}\right]^{-2},$$

$$\begin{aligned}\frac{\partial^2 log(L)}{\partial\alpha\partial\beta} &= -\frac{n}{\alpha} + \frac{2}{\alpha}\sum_{i=1}^{n}\left[\beta\left(\frac{X_i}{\alpha}\right)^{\beta}\left[\log\left(\frac{X_i}{\alpha}\right)\right]\left[1+\left(\frac{X_i}{\alpha}\right)^{\beta}\right]^{-1}\right] \\ &+ \sum_{i=1}^{n}\left[\left(\frac{X_i}{\alpha}\right)^{\beta}\left[1+\left(\frac{X_i}{\alpha}\right)^{\beta}\right]^{-1}\right] \\ &- \sum_{i=1}^{n}\left[\beta\left(\frac{X_i}{\alpha}\right)^{2\beta}\left[\log\left(\frac{X_i}{\alpha}\right)\right]\left[1+\left(\frac{X_i}{\alpha}\right)^{\beta}\right]^{-2}\right],\end{aligned}$$

$$\begin{aligned}\frac{\partial^3 log(L)}{\partial\alpha^3} &= -\frac{2n\beta}{\alpha^3} + \frac{4\beta^3}{\alpha^3}\sum_{i=1}^{n}\left[\left(\frac{X_i}{\alpha}\right)^{3\beta}\left[1+\left(\frac{X_i}{\alpha}\right)^{\beta}\right]^{-3}\right] \\ &- \frac{4\beta^2}{\alpha^3}\sum_{i=1}^{n}\left[\left(\frac{X_i}{\alpha}\right)^{2\beta}\left[1+\left(\frac{X_i}{\alpha}\right)^{\beta}\right]^{-2}\right] \\ &- \frac{4\beta^3}{\alpha^3}\sum_{i=1}^{n}\left[\left(\frac{X_i}{\alpha}\right)^{2\beta}\left[1+\left(\frac{X_i}{\alpha}\right)^{\beta}\right]^{-2}\right] \\ &- \frac{2\beta^2}{\alpha^3}\sum_{i=1}^{n}\left[(1+\beta)\left(\frac{X_i}{\alpha}\right)^{2\beta}\left[1+(\frac{X_i}{\alpha})^{\beta}\right]^{-2}\right] \\ &+ \frac{2\beta^2}{\alpha^3}(1+\beta)\sum_{i=1}^{n}\left(\frac{X_i}{\alpha}\right)^{\beta}\left[1+\left(\frac{X_i}{\alpha}\right)^{\beta}\right]^{-1} \\ &+ \frac{4\beta\,(1+\beta)}{\alpha^3}\sum_{i=1}^{n}\left(\frac{X_i}{\alpha}\right)^{\beta}\left[1+\left(\frac{X_i}{\alpha}\right)^{\beta}\right]^{-1},\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^3 log(L)}{\partial\alpha^2\partial\beta} &= \frac{n}{\alpha^2} + \frac{4\beta}{\alpha^2}\sum_{i=1}^{n}\left[\left(\frac{X_i}{\alpha}\right)^{2\beta}\left[1+\left(\frac{X_i}{\alpha}\right)^{\beta}\right]^{-2}\right] \\
&- \frac{2\beta}{\alpha^2}\sum_{i=1}^{n}\left[\left(\frac{X_i}{\alpha}\right)^{\beta}\left[1+\left(\frac{X_i}{\alpha}\right)^{\beta}\right]^{-1}\right] \\
&- \frac{2\,(1+\beta)}{\alpha^2}\sum_{i=1}^{n}\left[\left(\frac{X_i}{\alpha}\right)^{\beta}\left[1+\left(\frac{X_i}{\alpha}\right)^{\beta}\right]^{-1}\right] \\
&- \frac{4\beta^2}{\alpha^2}\sum_{i=1}^{n}\left[\left(\frac{X_i}{\alpha}\right)^{3\beta}\left[1+\left(\frac{X_i}{\alpha}\right)^{\beta}\right]^{-3}\log\left(\frac{X_i}{\alpha}\right)\right] \\
&+ \frac{4\beta^2}{\alpha^2}\sum_{i=1}^{n}\left[\left(\frac{X_i}{\alpha}\right)^{2\beta}\left[1+\left(\frac{X_i}{\alpha}\right)^{\beta}\right]^{-2}\log\left(\frac{X_i}{\alpha}\right)\right] \\
&+ \frac{2\beta(1+\beta)}{\alpha^2}\sum_{i=1}^{n}\left[\left(\frac{X_i}{\alpha}\right)^{2\beta}\left[1+\left(\frac{X_i}{\alpha}\right)^{\beta}\right]^{-2}\log\left(\frac{X_i}{\alpha}\right)\right] \\
&- \frac{2\beta(1+\beta)}{\alpha^2}\sum_{i=1}^{n}\left[\left(\frac{X_i}{\alpha}\right)^{\beta}\left[1+\left(\frac{X_i}{\alpha}\right)^{\beta}\right]^{-1}\log\left(\frac{X_i}{\alpha}\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3 log(L)}{\partial\alpha\partial\beta^2} &= -\frac{4}{\alpha}\sum_{i=1}^{n}\left[\left(\frac{X_i}{\alpha}\right)^{2\beta}\log\left(\frac{X_i}{\alpha}\right)\left[1+\left(\frac{X_i}{\alpha}\right)^{\beta}\right]^{-2}\right] \\
&+ \frac{4}{\alpha}\sum_{i=1}^{n}\left[\left(\frac{X_i}{\alpha}\right)^{\beta}\log\left(\frac{X_i}{\alpha}\right)\left[1+\left(\frac{X_i}{\alpha}\right)^{\beta}\right]^{-1}\right] \\
&+ \frac{4\beta}{\alpha}\sum_{i=1}^{n}\left[\left(\frac{X_i}{\alpha}\right)^{3\beta}\left[\log\left(\frac{X_i}{\alpha}\right)\right]^2\left[1+\left(\frac{X_i}{\alpha}\right)^{\beta}\right]^{-3}\right] \\
&- \frac{6\beta}{\alpha}\sum_{i=1}^{n}\left[\left(\frac{X_i}{\alpha}\right)^{2\beta}\left[\log\left(\frac{X_i}{\alpha}\right)\right]^2\left[1+\left(\frac{X_i}{\alpha}\right)^{\beta}\right]^{-2}\right] \\
&+ \frac{2\beta}{\alpha}\sum_{i=1}^{n}\left[\left(\frac{X_i}{\alpha}\right)^{\beta}\left[\log\left(\frac{X_i}{\alpha}\right)\right]^2\left[1+\left(\frac{X_i}{\alpha}\right)^{\beta}\right]^{-1}\right],
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3 log(L)}{\partial\beta^3} &= \frac{2n}{\beta^3} + 4\sum_{i=1}^{n}\left[\left(\frac{X_i}{\alpha}\right)^{2\beta}\left[\log\left(\frac{X_i}{\alpha}\right)\right]^3\left[1+\frac{X_i}{\alpha}\right]^{-3}\right] \\
&- 2\sum_{i=1}^{n}\left[\left(\frac{X_i}{\alpha}\right)^{\beta}\left[\log\left(\frac{X_i}{\alpha}\right)\right]^3\left[1+\frac{X_i}{\alpha}\right]^{-2}\right].
\end{aligned}
$$

Note that if X has the log-logistic distribution with parameters α and β, then $Y = \left(\frac{X}{\alpha}\right)^{\beta} \sim f(y) = \dfrac{1}{(1+y^2)}, y > 0$. To find the joint cumulants of the log-logistic distribution, we follow the results of [1], and obtain the following expectations

$$
\mathbb{E}\left\{\left(\frac{X}{\alpha}\right)^{\beta}\left[1+\left(\frac{X}{\alpha}\right)^{\beta}\right]^{-1}\right\} = \int_0^{\infty} y[1+y]^{-3}dy = \frac{1}{2},
$$

$$\mathbb{E}\left\{\left(\frac{X}{\alpha}\right)^{2\beta}\left[1+\left(\frac{X}{\alpha}\right)^{\beta}\right]^{-2}\right\} = \int_0^\infty y^2[1+y]^{-4}dy = \frac{1}{3},$$

$$\beta\mathbb{E}\left\{\left(\frac{X}{\alpha}\right)^{\beta}\left[\log\left(\frac{X}{\alpha}\right)\right]\left[1+\left(\frac{X}{\alpha}\right)^{\beta}\right]^{-1}\right\} = \int_0^\infty y[\log(y)][1+y]^{-3}dy = \frac{1}{2},$$

$$\beta^2\mathbb{E}\left\{\left(\frac{X}{\alpha}\right)^{\beta}\left[\log\left(\frac{X}{\alpha}\right)\right]^2\left[1+\left(\frac{X}{\alpha}\right)^{\beta}\right]^{-2}\right\}$$
$$= \int_0^\infty y[\log(y)]^2[1+y]^{-4}dy = \frac{1}{18}(-6+\pi^2),$$

$$\beta\mathbb{E}\left\{\left(\frac{X}{\alpha}\right)^{2\beta}\left[\log\left(\frac{X}{\alpha}\right)\right]\left[1+\left(\frac{X}{\alpha}\right)^{\beta}\right]^{-2}\right\} = \int_0^\infty y^2[\log(y)][1+y]^{-4}dy = \frac{1}{2}.$$

In addition, we have also computed the following expectations

$$\mathbb{E}\left\{\left(\frac{X}{\alpha}\right)^{3\beta}\left[1+\left(\frac{X}{\alpha}\right)^{\beta}\right]\right\} = \int_0^\infty y^3[1+y]^{-3}dy = \frac{1}{4},$$

$$\beta\mathbb{E}\left\{\left(\frac{X}{\alpha}\right)^{3\beta}\left[\log\left(\frac{X}{\alpha}\right)\right]\left[1+\left(\frac{X}{\alpha}\right)^{\beta}\right]^{-3}\right\} = \int_0^\infty y^3[\log(y)][1+y]^{-5}dy = \frac{11}{24},$$

$$\beta^2 \mathbb{E}\left\{ \left(\frac{X}{\alpha}\right)^{3\beta} \left[\log\left(\frac{X}{\alpha}\right)\right]^2 \left[1 + \left(\frac{X}{\alpha}\right)^{\beta}\right]^{-3} \right\}$$
$$= \int_0^\infty y^3 [\log(y)]^2 [1+y]^{-5} dy = \frac{1}{12}(6+\pi^2),$$

$$\beta^2 \mathbb{E}\left\{ \left(\frac{X}{\alpha}\right)^{2\beta} \left[\log\left(\frac{X}{\alpha}\right)\right]^2 \left[1 + \left(\frac{X}{\alpha}\right)^{\beta}\right]^{-2} \right\}$$
$$= \int_0^\infty y^2 [\log(y)]^2 [1+y]^{-4} dy = \frac{1}{9}(3+\pi^2),$$

$$\beta^2 \mathbb{E}\left\{ \left(\frac{X}{\alpha}\right)^{\beta} \left[\log\left(\frac{X}{\alpha}\right)\right]^2 \left[1 + \left(\frac{X}{\alpha}\right)^{\beta}\right]^{-1} \right\} = \int_0^\infty y [\log(y)]^2 [1+y]^{-3} dy = \frac{\pi^2}{6},$$

$$\beta^3 \mathbb{E}\left\{ \left(\frac{X}{\alpha}\right)^{2\beta} \left[\log\left(\frac{X}{\alpha}\right)\right]^3 \left[1 + \left(\frac{X}{\alpha}\right)^{\beta}\right]^{-3} \right\}$$
$$= \int_0^\infty y^2 [\log(y)]^3 [1+y]^{-5} dy = \frac{1}{24}(-6+\pi^2),$$

and

$$\beta^3 \mathbb{E}\left\{ \left(\frac{X}{\alpha}\right)^{\beta} \left[\log\left(\frac{X}{\alpha}\right)\right]^3 \left[1 + \left(\frac{X}{\alpha}\right)^{\beta}\right]^{-2} \right\} = \int_0^\infty y [\log(y)]^3 [1+y]^{-4} dy = 0.$$

Appendix B

Code for Bias-Corrected Estimates

B.1 LLtabularResultsCode.r

```
##Bias Corrected MLE function
BMLE = function(alphaHat, betaHat, n){

  k11 <- (-1*betaHat^2)/(3*alphaHat^2)
  k12 <- 0
  k21 <- 0
  k22 <- -1*(1+((1/9)*(-6+pi^2)))/(betaHat^2)

  k111 <- (betaHat^2)/(alphaHat^3)
  k112 <- (-1*betaHat/(2*alphaHat^2))
  k122 <- 0
  k222 <- (1+(pi*pi/6))/(betaHat^3)
```

```
  k11one <- (2*betaHat^2)/(3*alphaHat^3)
  k12one <- 0
  k22one <- 0
  k11two <- (-2*betaHat)/(3*alphaHat^2)
  k12two <- 0
  k22two <- (2*(3+pi^2))/(9*betaHat^3)

  a11one <- betaHat^2/(6*alphaHat^3)

  a12one <- betaHat/(4*alphaHat^2)

  a21one <- a12one
  a22one <- 0
  a11two <- (-5*betaHat)/(12*alphaHat^2)

  a12two <- 0
  a21two <- 0
  a22two <- (3+(5*pi*pi/2))/(18*betaHat^3)


  kMatrix <- matrix(c(-1*k11, -1*k21, -1*k12, -1*k22), ↩
     nrow=2, ncol=2, byrow=FALSE)

  aMatrix <- matrix(c(a11one, a21one, a12one, a22one, ↩
     a11two, a21two, a12two, a22two), nrow=2, ncol=4, ↩
     byrow=FALSE)

  Inv = solve(kMatrix)
  bias <- Inv%*%aMatrix%*%vec(Inv)/n
  alphaHatStar <- alphaHat - bias[1,1]
  betaHatStar <- betaHat - bias[2,1]
  c(alphaHatStar, betaHatStar)
}
```

```
##Generalized Moments Function
GM = function(data, l1, l2, n)
{
  Ml1hat <- mean(data^l1)
  Ml2hat <- mean(data^l2)
}

##uniroot function for GM Method
betaGMfunc=function(data, l1, l2)
{
  Ml1hat <- mean(data^l1)
  Ml2hat <- mean(data^l2)
  betaGMfunct <- numeric(0)
  betaGMfunct <- uniroot(function(betaGM) (l2*l1^(-1*l2/↩
     l1)*(pi/betaGM)^((l1-l2)/l1)*(Ml1hat*sin(pi*l1/↩
     betaGM))^(l2/l1)*sin(pi*l2/betaGM)^-1 - Ml2hat), ↩
     interval=c(1,5000))
  return(betaGMfunct$root)
}
##function for alphaGM
alphaGMfunc=function(data, l1, l2, betaGM)
{
  Ml1hat <- mean(data^l1)
  Ml2hat <- mean(data^l2)
  alphaGMfunct <- numeric(0)
  alphaGMfunct <- (betaGM*(pi*l1)^-1 * Ml1hat * sin(pi*↩
     l1/betaGM))^(1/l1)
  return(alphaGMfunct)
}

##Bootstrap function
BOOTFunc = function(data, aHat, bHat)
{
```

```
  set.seed(1)
  alphaHatBootSum <- betaHatBootSum <-numeric(0)
  for(i in 1:10000)
  {
    index<- sample(1:n, n, replace = TRUE)
    MLE <- llogisMLE(data[index])
    alphaHatBootSum[i] <- exp(MLE$estimate[1])
    betaHatBootSum[i] <- 1/(MLE$estimate[2])
  }
  alphaHatBootMean <- mean(alphaHatBootSum)
  betaHatBootMean <- mean(betaHatBootSum)

  aHatBoot <- 2*aHat-alphaHatBootMean
  bHatBoot <- 2*bHat-betaHatBootMean
  return(c(aHatBoot,bHatBoot))
}


LLestFunc = function(data, n, l1, l2)
{
  MLE <- llogisMLE(data)
  location <- MLE$estimate[1]
  scale <- MLE$estimate[2]
  location <- as.numeric(location)
  scale <- as.numeric(scale)
  (alphaHat <- exp(location))                          #MLE for ↩
     alpha
  (betaHat <- 1/scale)                                 #MLE for ↩
     beta
  estimators <- BMLE(alphaHat, betaHat, n)
  alphaHatStar <- estimators[1]                        #CMLE for↩
      alpha
  betaHatStar <- estimators[2]                         #CMLE for↩
      beta
```

```
  bootEst <- BOOTFunc(data, aHat = alphaHat, bHat = ↩
     betaHat)
  alphaHatBoot <- bootEst[1]
  betaHatBoot <- bootEst[2]
  betaGM <- numeric(0)
  betaGM <- betaGMfunc(data, l1, l2)           #GM ↩
     estimator for beta
  betaGM
  alphaGM <- numeric(0)
  alphaGM <- alphaGMfunc(data, l1, l2, betaGM) #GM ↩
     estimator for alpha

  # the lines below make an easy to read table
  results <- cbind(rbind(alphaHat,alphaHatStar,↩
     alphaHatBoot,alphaGM),
                  rbind(betaHat,betaHatStar,betaHatBoot↩
                     ,betaGM))
  rownames(results) <- c("MLE", "CMLE", "BOOT", "GM")
  rownames(results)
  colnames(results) <- c("Alpha", "Beta")
  results
  return(t(results))
}

##################################################
## Functions are above                          ##
## Below is an example of how to implement      ##
## the above functions                          ##
##################################################

library(STAR)
data <- c(5,11,21,31,46,75,98,122,145,
          165,196,224,245,293,321,330,350,420)
data
```

```
n <-length(data)
l1 <- 0.75
l2 <- 0.35
LLestimates <- LLestFunc(data=data, n=n, l1=l1, l2=l2)
LLestimates #Table of the results for alpha and beta
```