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BIAS-CORRECTED MAXIMUM LIKELIHOOD ESTIMATION OF THE PARAMETERS OF THE WEIGHTED LINDLEY DISTRIBUTION

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Abstract

This report discusses the calculation of analytic second-order bias techniques for the maximum likelihood estimates (for short, MLEs) of the unknown parameters of the distribution in quality and reliability analysis. It is well-known that the MLEs are widely used to estimate the unknown parameters of the probability distributions due to their various desirable properties; for example, the MLEs are asymptotically unbiased, consistent, and asymptotically normal. However, many of these properties depend on an extremely large sample sizes. Those properties, such as unbiasedness, may not be valid for small or even moderate sample sizes, which are more practical in real data applications. Therefore, some bias-corrected techniques for the MLEs are desired in practice, especially when the sample size is small.

Two commonly used popular techniques to reduce the bias of the MLEs , are 'preventive' and 'corrective' approaches. They both can reduce the bias of the MLEs to order $\mathrm{O}(n^{-2})$, whereas the 'preventive' approach does not have an explicit closed-form expression. Consequently, we mainly focus on the 'corrective' approach in this report. To illustrate the importance of the bias-correction in practice, we apply the bias-corrected method to two popular lifetime distributions: the inverse Lindley distribution and the weighted Lindley distribution. Numerical studies based on the

two distributions show that the considered bias-corrected technique is highly recommended over other commonly used estimators without bias-correction. Therefore, special attention should be paid when we estimate the unknown parameters of the probability distributions under the scenario in which the sample size is small or moderate.

Chapter 1

Introduction

In recent years, numerous distributions have been developed in the literature. The main motivation of developing the new distribution is that researchers want to provide a better model to analyze the real data from different research areas. However, in many cases, the poor performance of the distribution is due to inaccurate estimates of the unknown parameters, not its inner properties. It is well-known that the maximum-likelihood estimator (MLE) is the most popular one for estimating the unknown parameter, due to its good properties. However, the MLE is biased in finite sample space. Such bias may significantly affect the fitness of the distribution. This observation motivates us to adopt some bias-corrected technique to reduce the bias of the MLE from order $\mathrm{O}(n^{-1})$ to order $\mathrm{O}(n^{-2})$.

In Chapter 2, we consider the one-parameter inverse Lindley distribution (shortly, IL), which is applicable of modeling the upside-down bathtub shape data. We firstly estimate the unknown parameter based on the MLE. Then we adopt a 'corrective' approach to derive the modified MLE that is bias-free to the second order. As comparison, an alternative bias-correction mechanism based on the parametric bootstrap is considered in this chapter.

In Chapter 3, we focus on the two-parameter weighted Lindley distribution. This distribution is useful for modeling survival data with different shapes, whereas its MLEs are biased in finite samples. This motivates us to construct nearly unbiased estimators for the unknown parameters. We consider a 'corrective' approach to derive modified MLEs that are bias-free to second order. In addition, we adopt an alternative bias-correction mechanism based on the parametric bootstrap. Monte Carlo simulations are conducted to compare the performance between the proposed and two previous methods in the literature. The numerical evidence shows that the bias-corrected estimators are extremely accurate even for very small sample sizes and are superior than the previous estimators in terms of biases and root mean squared errors. Finally, applications to two real data sets are presented for illustrative purposes.

In Chapter 4, We present our conclusions and discuss some future research. Due to the importance of the bias-correction for the MLEs illustrated above, we should pay special attention on estimating the unknown parameters of the lifetime distributions.

It is noteworthy that the considered bias-corrected technique can be easily applied to other commonly used lifetime distributions, such as the weighted exponential distribution and the three-parameter Lindley geometric distribution, which are currently under investigation and will be reported elsewhere.

Chapter 2

The inverse Lindley distribution

2.1 Introduction

The Lindley distribution was originally introduced by Lindley [1] in the context of Batesian statistics as a counter example of fiducial statistics. Its probability density function (pdf) is given by

$$f(t;\theta) = \frac{\theta^2}{\theta+1}(1+t)e^{-\theta t}, \quad t > 0,$$

where the parameter $\theta > 0$. It has been discussed by many authors in different practical cases, such as Bayesian estimation [2] , loading-sharing system mode [3] and stress-strength reliability model [4]. It deserves mentioning that the Lindley distribution provides a flexible shape to model the lifetime data. However, the Lindley distribution may perform poorly for fitting the non-monotone shapes data. This motivates the researchers to develop a modified Lindley distribution discussed as follows.

The inverse Lindley (for short, IL) distribution was originally proposed by [5]. The random variable X is said to follow the IL distribution with the parameter θ, denoted by $X \sim \mathrm{IL}(\theta)$. Its pdf can be written as

$$f(x;\theta) = \frac{\theta^2}{1+\theta}\left(\frac{1+x}{x^3}\right)\exp\left(-\frac{\theta}{x}\right), \quad x > 0, \tag{2.1}$$

where the parameter $\theta > 0$. The corresponding cumulative distribution function (cdf) of the IL distribution is given by

$$F(x;\theta) = \left(1+\frac{\theta}{1+\theta}\frac{1}{x}\right)\exp\left(-\frac{\theta}{x}\right), \quad x > 0. \tag{2.2}$$

Figure 2.1 shows different shapes of the pdf of the IL distribution with different values of θ. It can be seen from figure that the shape of the IL distribution can be upside-down bathtub, right skewed and heavy-tailed. The flexibility of the shape is very useful to model the survival data in practice.

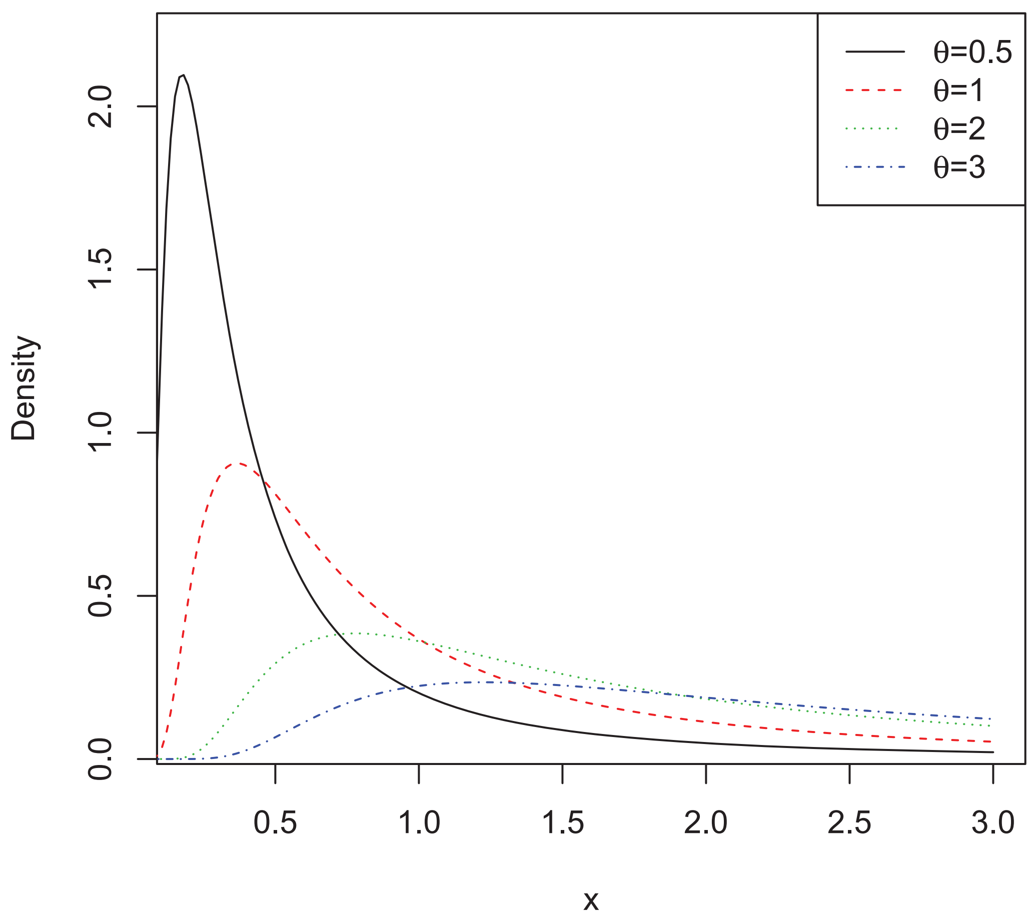

Figure 2.1: Pdf of IL distribution with different values of θ.

2.2 The maximum likelihood estimation

Suppose that $X_1, X_2, \cdots, X_n$ are observations of n independent units taken from the IL distribution. The log likelihood function of θ is given by

$$l(\theta; x) = 2n\log(\theta) - n\log(\theta+1) + \sum_{i=1}^{n}\log(1+x_i) - 3\sum_{i=1}^{n}\log(x_i) - \theta\sum_{i=1}^{n}\frac{1}{x_i}, \quad (2.3)$$

where $x = (x_1, x_2, \cdots, x_n)$. The score function is given by

$$\frac{\partial l}{\partial \theta} = \frac{2n}{\theta} - \frac{n}{1+\theta} - \sum_{i=1}^{n}\frac{1}{x_i}.$$

Define $\bar{x} = \sum_{i=1}^{n} x_i/n$. The MLE of θ, denoted by $\hat{\theta}$, can be easily obtained. Simple algebra shows that,

$$\hat{\theta} = \frac{1 - \bar{x} + \sqrt{1 + 6\bar{x} + \bar{x}^2}}{2\bar{x}}. \quad (2.4)$$

Since the MLE is biased to order $\mathrm{O}(n^{-1})$ in finite samples, we adopt a 'corrective' approach to reduce the bias of MLE to order $\mathrm{O}(n^{-2})$.

2.3 Bias-corrected MLE

Let $l(\tau)$ be the log-likelihood function with a p-dimensional vector of unknown parameters $\tau = (\tau_1, \cdots, \tau_p)'$ based on a sample of n observations. The joint cumulates of the derivatives of $l(\tau)$ are given by

$$\kappa_{ij} = \mathbb{E}\left[\frac{\partial^2 l}{\partial\tau_i\partial\tau_j}\right], \quad \text{for} \quad i, j = 1, 2, \cdots, p, \tag{2.5}$$

$$\kappa_{ijl} = \mathbb{E}\left[\frac{\partial^3 l}{\partial\tau_i\partial\tau_j\partial\tau_l}\right], \quad \text{for} \quad i, j, l = 1, 2, \cdots, p, \tag{2.6}$$

$$\kappa_{ij,l} = \mathbb{E}\left[\left(\frac{\partial^2 l}{\partial\tau_i\partial\tau_j}\right)\left(\frac{\partial l}{\partial\tau_l}\right)\right], \quad \text{for} \quad i, j, l = 1, 2, \cdots, p, \tag{2.7}$$

$$\kappa_{ij}^{l} = \frac{\partial\kappa_{ij}}{\partial\tau_l}, \quad \text{for} \quad i, j, l = 1, 2, \cdots, p, \tag{2.8}$$

respectively. It is assumed that the log-likelihood function is well behaved and regular with respect to all derivatives up to and including the third order and that all of the four equations given by (2.5)−(2.8) are of order O(n).

Let $K = [-\kappa_{ij}]$ denote the Fisher's information matrix of τ for $i, j = 1, 2, \cdots, p$. [6] show that when the sample data are independent but not necessarily identically distributed, the bias of the sth element of $\hat{\tau}_s$ can be written as

$$\text{Bias}(\hat{\tau}_s) = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{l=1}^{p}\kappa^{si}\kappa^{jl}\left[\frac{1}{2}\kappa_{ijl} + \kappa_{ij,l}\right] + O(n^{-2}), \quad s = 1, 2, \cdots, p, \tag{2.9}$$

where κ^{ij} is the (i,j)th element of the inverse of Fisher's information matrix. Thereafter, [7] show that when all equations in (2.5)$-$(2.8) are of order O(n), equation (2.9) still holds even if observations are not independent. They thus advocate the following convenient form

$$\text{Bias}(\hat{\tau}_s) = \sum_{i=1}^{p} \kappa^{si} \sum_{j=1}^{p} \sum_{l=1}^{p} \left[\kappa_{ij}^{(l)} - \frac{1}{2}\kappa_{ijl} + \kappa_{ij,l} \right] + O(n^{-2}), \quad s = 1, 2, \cdots, p, \quad (2.10)$$

instead of equation (2.9). Define $a_{ij}^{(l)} = \kappa_{ij}^{(l)} - \frac{1}{2}\kappa_{ijl}$ for $i, j, l = 1, 2, \cdots, p$. They also show that the O(n^{-2}) bias expression of $\hat{\tau}$ can be reexpressed as

$$\text{Bias}(\hat{\tau}) = K^{-1} A \cdot \text{vec}(K^{-1}) + O(n^{-2}),$$

where vec is an operator that creates a column vector from a matrix by stacking the column vectors below one another, and

$$A = \left[A^{(1)} \mid A^{(2)} \mid \cdots \mid A^{(p)} \right] \quad \text{with} \quad A^{(l)} = \left[a_{ij}^{(l)} \right].$$

A bias-corrected MLE for τ, denoted by $\hat{\tau}^{\text{CMLE}}$, can thus be constructed as

$$\hat{\tau}^{\text{CMLE}} = \hat{\tau} - \hat{K}^{-1} \hat{A} \cdot \text{vec}(\hat{K}^{-1}),$$

where $\hat{\tau}$ is the MLE of the unknown parameter τ, $\hat{K} = K \mid_{\tau = \hat{\tau}}$, and $\hat{A} = A \mid_{\tau = \hat{\tau}}$. It can be shown that the bias of $\hat{\tau}^{\text{CMLE}}$ will be of order $O(n^{-2})$.

For our problem, we have the case of $p = 1$, that is, $\tau = \theta$. The derivatives of the log-likelihood function of θ can be easily obtained as follows.

$$\begin{aligned} \frac{\partial^2 \ell}{\partial \theta^2} &= -\frac{2n}{\theta^2} + \frac{n}{(1+\theta)^2}, \\ \frac{\partial^3 \ell}{\partial \theta^3} &= \frac{4n}{\theta^3} - \frac{2n}{(1+\theta)^3}. \end{aligned} \tag{2.11}$$

In addition, we have

$$\begin{aligned} K &= \left[\frac{2n}{\theta^2} - \frac{n}{(1+\theta)^2}\right], \\ k_{11}^{(1)} &= k_{111} = \frac{4n}{\theta^3} - \frac{2n}{(1+\theta)^3}, \\ A &= a_{11}^{(1)} = \frac{2n}{\theta^3} - \frac{n}{(1+\theta)^3}. \end{aligned} \tag{2.12}$$

The bias-corrected estimator of the MLE for the IL distribution can be obtained as

$$\hat{\theta}^{\text{CMLE}} = \hat{\theta} - \frac{(\hat{\theta}^3 + 6\hat{\theta}^2 + 6\hat{\theta} + 2)(\hat{\theta}+1)\hat{\theta}}{n(\hat{\theta}^2 + 4\hat{\theta} + 2)^2}. \tag{2.13}$$

Note that, the bias-corrected estimator, $\hat{\theta}^{\text{CMLE}}$ has a simple closed-form expression. So it is easily computed. It should be noted that $\hat{\theta}^{\text{CMLE}}$ is a bias-corrected MLE of θ to order $\mathrm{O}(n^{-1})$ and that its bias is of order $\mathrm{O}(n^{-2})$, because $\mathbb{E}\left[\hat{\theta}^{\text{CMLE}}\right] = \theta + \mathrm{O}(n^{-2})$.

2.4 Simulation studies

In this section, we conduct Monte Carlo simulations to compare the performance of the MLE, bias-corrected MLE, and bootstrap estimator. Let " $\hat{\theta}^{\text{MLE}}$ ", " $\hat{\theta}^{\text{CMLE}}$ ", " $\hat{\theta}^{\text{BOOT}}$ " stand for the MLE, bias-corrected MLE, and bootstrap estimator. Generate the data from IL distribution by the following algorithm:

step 1. Generate U $\sim$ Uniform $(0, 1)$,

step 2. Generate V $\sim$ 1/Exponential (θ),

step 3. Generate W $\sim$ 1/Gamma $(shape = 2, scale = 1/\theta)$,

step 4. If $U \leq \theta/(\theta + 1)$, then $X = V$, otherwise, let $X = W$.

We draw random samples of size $n = \{8, 11, 14, \cdots, 125\}$, with the parameter $\theta = \{0.1, 0.5, 1, 5, 7.5, 15\}$. The replications of simulation studies are based on $M = 20,000$, and the replications of bootstrap are $B = 5,000$. We calculate the average bias of each estimator and its root mean squared error(RMSE), given by

$$\text{Bias}(\hat{\theta}^{\text{est}}) = \frac{1}{M}\sum_{i=1}^{M}\left(\hat{\theta}_i^{\text{est}} - \theta\right) \quad \text{and} \quad \text{RMSE}(\hat{\theta}^{\text{est}}) = \sqrt{\frac{1}{M}\sum_{i=1}^{M}\left(\hat{\theta}_i^{\text{est}} - \theta\right)^2},$$

where $\hat{\theta}^{\text{est}}$ is an estimator of the parameter θ. Figure 2.2 depicts the bias versus the sample size n for a certain value of θ. Figure 2.3 represents the RMSE versus the sample size n for a fixed value of θ. Some conclusions from the two figures can be drawn as follows.

(i) The MLE of θ is positively biased, indicating that the MLE on average overshoots the target parameters θ, and the average of the MLE gets decreasing as the sample size n is increasing .

(ii) The CMLE of θ clearly outperforms the MLE under the same scenario above and these corrected estimators provide substantial bias-correction, especially for the small or moderate sample sizes.

(iii) The bias of the MLE is increasing, when the parameter θ gets larger, as shown in Figure 2.2. When the sample size n gets larger, the bias and RMSE of each estimator decrease and the magnitude of reduction becomes smaller. This is expected because most estimators in statistical theory perform better with increasing n. Therefore, one can only expect that the performances of all the estimators become closer with increasing n in terms of biases and RMSEs.

(iv) The reductions in biases and RMSEs of each estimator are very substantial even for small sample sizes. For instance, when $n = 9$, $\theta = 0.1$, Bias($\hat{\theta}^{\mathrm{CMLE}}$)= 0.000352, Bias($\hat{\theta}^{\mathrm{BOOT}}$)= 0.000692, Bias($\hat{\theta}^{\mathrm{MLE}}$)= 0.006804123, RMSE($\hat{\theta}^{\mathrm{CMLE}}$)= 0.0227, RMSE($\hat{\theta}^{\mathrm{BOOT}}$)= 0.0229, RMSE($\hat{\theta}^{\mathrm{MLE}}$)= 0.0244.

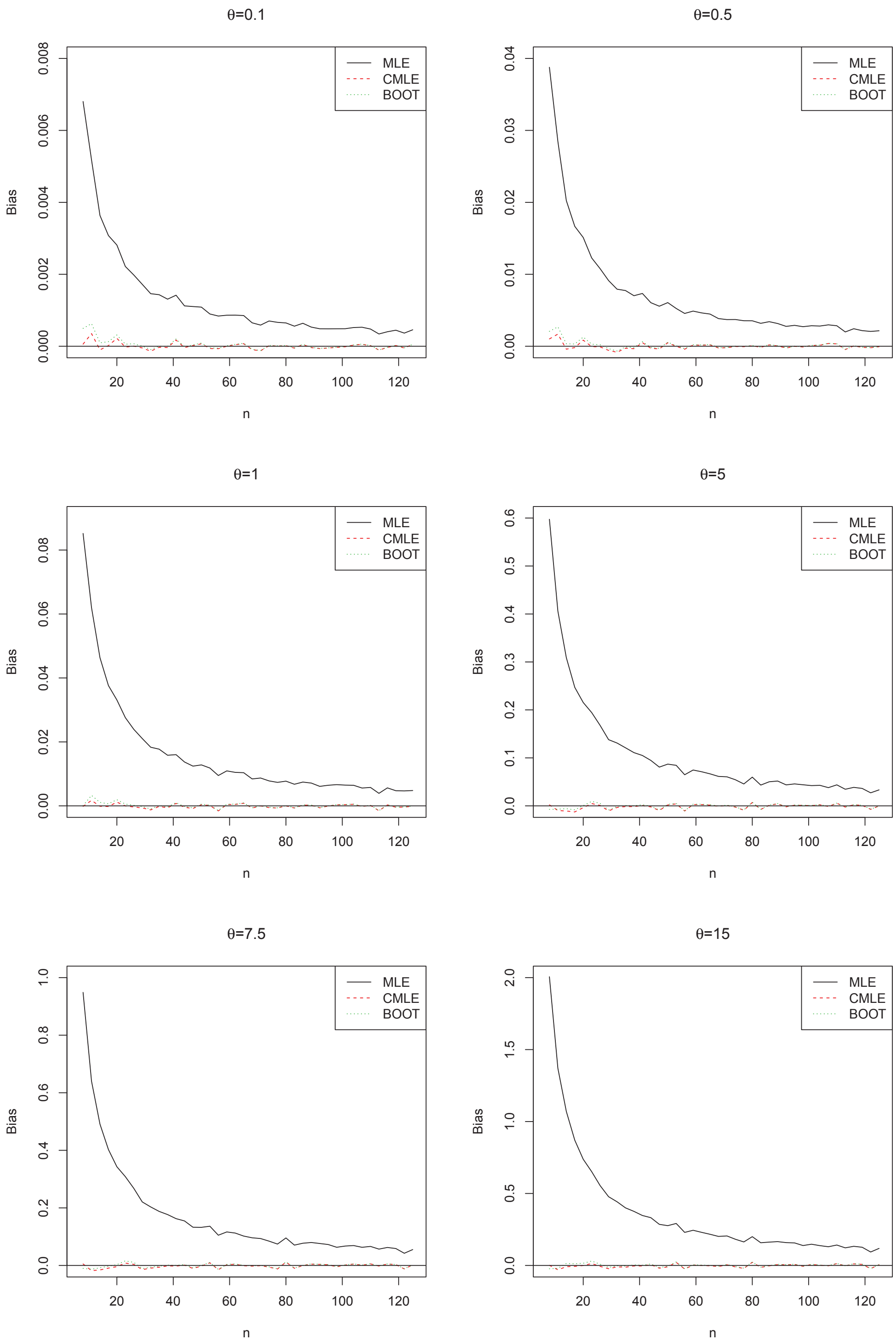

Figure 2.2: Average bias of the considered estimate of θ versus n for $\theta = \{0.1, 0.5, 1, 1.5, 7.5, 15\}$.

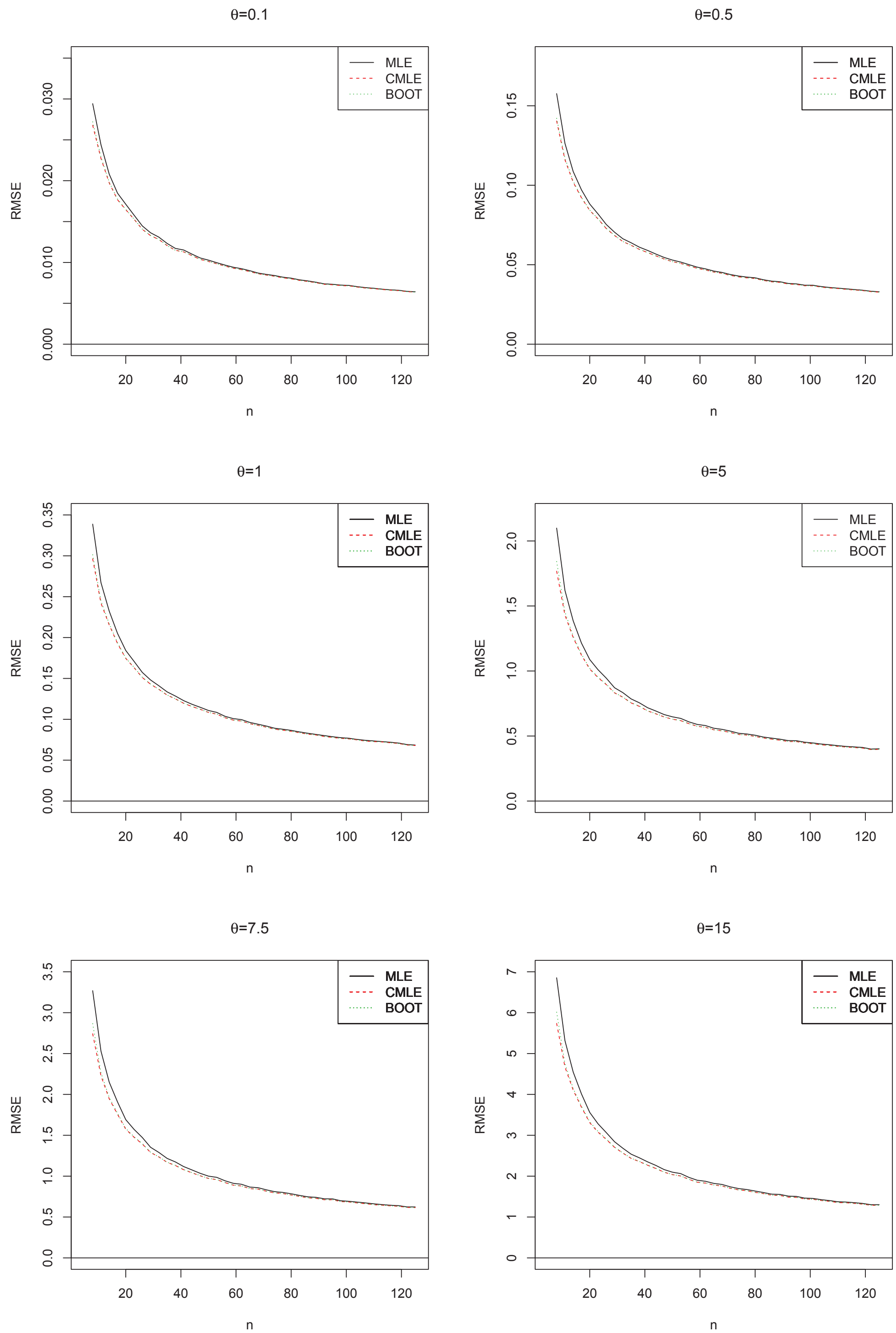

Figure 2.3: RMSE of the considered estimate of $\theta = \{0.1, 0.5, 1, 1.5, 7.5, 15\}$.

2.5 Concluding remarks

In this chapter, we have studied the MLE for the unknown parameter of the inverse IL, which is positively biased in finite samples. We have proposed the bias-corrected estimator, the CMLE, which reduces the bias of the MLE from order $O(n^{-1})$ to order $O(n^{-2})$. Numerical evidence shows that the bias-corrected estimator is strongly recommended over other commonly used estimators without bias-correction, especially when the sample size is small or moderate.

Chapter 3

Bias-corrected maximum likelihood estimation of the parameters of the weighted Lindley distribution

3.1 Introduction

The Lindley distribution was originally introduced by [1] in the context of Batesian statistics as a counterexample of fiducial statistics. Its probability density function (pdf) is given by

$$f(t;\theta) = \frac{\theta^2}{\theta+1}(1+t)e^{-\theta t}, \quad t > 0,$$

where the parameter $\theta > 0$. Since the distribution was proposed, it has been overlooked in the literature partly due to the popularity of the exponential distribution in the context of reliability analysis. Nonetheless, it has recently received considerable attention as a lifetime model to analyze survival data in the competing risks analysis and stress-strength reliability studies; see, for example, [8], [9], [10], [11], [12], among others. [8] provide a nice overview of various statistical properties of the Lindley distribution. Furthermore, they argue that the Lindley distribution could be a better lifetime model than the exponential distribution using a real data set.

In a recent paper, [13] introduce the two-parameter Weighted Lindley (shortly LW) distribution as follows. The random variable X is said to follow the WL distribution with parameters θ and c, denoted by $X \sim \mathrm{WL}(\theta,\ c)$, if its pdf can be written as

$$f(x;\theta,c) = \frac{\theta^{c+1}}{(\theta+c)\Gamma(c)} x^{c-1}(1+x)e^{-\theta x}, \quad x > 0, \tag{3.1}$$

where the parameters $\theta > 0$, $c > 0$, and

$$\Gamma(c) = \int_0^\infty y^{c-1} e^{-y}\, dy, \quad c > 0,$$

is the complete gamma function. The WL distribution can be viewed as a mixture of two gamma distributions: one with shape parameter c and scale parameter θ, denoted by Gamma(c, θ), the other with shape parameter $c+1$ and scale parameter θ, Gamma$(c+1, \theta)$. This property can be used to generate random samples from the WL distribution. The corresponding cumulative distribution function (cdf) of the WL distribution is given by

$$F(x; \theta, c) = 1 - \frac{(\theta + c)\Gamma(c, \theta x) + (\theta x)^c e^{-\theta x}}{(\theta + c)\Gamma(c)}, \quad x > 0, \quad \theta,\ c > 0, \tag{3.2}$$

and the hazard rate function of the WL distribution is given by

$$h(x; \theta, c) = \frac{\theta^{c+1} x^{c-1}(1+x)e^{-\theta x}}{(\theta + c)\Gamma(c, \theta x) + (\theta x)^c e^{-\theta x}}, \quad x > 0, \quad \theta,\ c > 0,$$

where

$$\Gamma(a, b) = \int_b^\infty x^{a-1} e^{-x}\, dx, \quad a > 0,\ b \geq 0,$$

is the upper incomplete gamma function.

It is widely known that the maximum likelihood method is often adopted to estimate the unknown parameters of a statistical model because the maximum likelihood estimators (MLEs) have many appealing properties; for example, they are asymptotically unbiased, consistent, and asymptotically normally distributed, etc. It should be noted that most of those properties heavily rely on the large sample size condition, which indicates that they, such as unbiasedness, may not be valid for a small or even moderate sample size; see [14]. As shown by [13], the MLEs of the WL distribution are positively biased on average in finite samples, i.e. the expected value of the estimators exceeds the true value of the parameters. Later on, besides the maximum likelihood method, [15] consider other estimation methods, such as the method of moments estimation (MME), ordinary least-squares estimation (OLSE), and weighted least-squares estimation (WLSE) methods. The numerical evidence they present shows that all of the estimators under consideration are positively biased in finite samples.

For this reason, it has become standard practice to develop nearly unbiased estimators for the WL distribution. To the best of our knowledge, such bias-corrected estimators have not yet been fully explored for the WL distribution in the literature. In this paper, we adopt a 'corrective' approach to derive modified MLEs that are bias-free to second order. Here, the 'corrective' approach means that the bias-correction can be achieved by subtracting the bias (estimated at the MLE of the parameter) from the original MLE. As can be seen in the simulation study, the proposed estimators are extremely accurate even for very small sample sizes and are far superior than the previous estimators in terms of biases and root mean squared errors. Additionally, they have simple closed-form expressions, which means they are quite attractive because they are easy to compute for practitioners. Indeed, such a bias-correction technique has been applied successfully for parameter estimation in other distributions; see, for example, [16], [17], [18], [19], [20], [21], and references cited therein.

As an alternative to the analytically bias-corrected MLEs mentioned above, we consider the bias-corrected MLEs through Efron's bootstrap resampling because the bootstrap estimator is also second-order correct. Note that the bootstrap estimator does not require analytical derivation of the bias function and that the bias-correction is performed numerically. We here refer the interested readers to [22], [23], [24], [25], to name just a few. It deserves mentioning that another analytically bias-corrected MLEs can be developed based on a 'preventive' approach introduced by [26]. This approach can also reduce the bias of the MLEs to order $O(n^{-2})$, whereas it involves modifying the score vector of the log-likelihood function prior to solving for the MLEs, and thus, this approach is not simply attempted in this paper.

The remainder of this paper is organized as follows. In Section 3.2, we briefly discuss point estimation by the maximum likelihood method for the WL distribution. In Section 3.3, we adopt a 'corrective' approach to derive modified MLEs that are bias-free to second order. In addition, an alternative bias-correction mechanism based on Efron's bootstrap resampling is also considered. In Section 3.4, Monte Carlo simulations are conducted to compare the performance between the proposed and two previous methods; MLE and MME. In Section 3.5, applications to two real data sets are presented for illustrative purposes. Finally, Section 3.6 concludes the paper.

3.2 Maximum likelihood estimation

Suppose that $X_1, X_2, \cdots, X_n$ are observations of n independent units taken from the WL distribution. The log-likelihood function of θ and c is given by

$$\begin{aligned} l(\theta, c) = n\big[(c+1)\log(\theta) - \log\big(\Gamma(c)\big) - \log(\theta + c)\big] + (c-1)\sum_{i=1}^{n}\log(x_i) \\ + \sum_{i=1}^{n}\log(1+x_i) - \theta\sum_{i=1}^{n} x_i. \end{aligned} \tag{3.3}$$

The score functions are thus given by

$$\begin{aligned} \frac{\partial l}{\partial \theta}(\theta, c) &= n\left[\frac{c+1}{\theta} - \frac{1}{\theta + c}\right] - \sum_{i=1}^{n} x_i, \\ \frac{\partial l}{\partial c}(\theta, c) &= n\left[\log(\theta) - \frac{1}{\theta + c} - \psi(c)\right] + \sum_{i=1}^{n}\log(x_i), \end{aligned}$$

where $\psi(x) = (d/dc)\log\Gamma(c)$ is the digamma function. The MLEs $\hat{\theta}$ and $\hat{c}$ of the unknown parameters θ and c can be easily obtained by putting the two equations above equal to 0. [13] show that the MLEs of θ and c are, respectively, given by

$$\hat{\theta} = \frac{-\hat{c}(\bar{x}-1) + \sqrt{\big[\hat{c}(\bar{x}-1)\big]^2 + 4\hat{c}(\hat{c}+1)\bar{x}}}{2\bar{x}} \equiv \eta(\hat{c}), \quad \text{say}, \tag{3.4}$$

where $\bar{x}$ is the sample mean and $\hat{c}$ is the solution of the nonlinear equation

$$n\Big[\log\big(\eta(c)\big) - \frac{1}{\eta(c)+c} - \psi(c)\Big] + \sum_{i=1}^{n} \log(x_i) = 0. \tag{3.5}$$

3.3 Bias-corrected MLEs

3.3.1 A corrective approach

For ease of exposition and without loss of generality, let $l(\tau)$ be the log-likelihood function with a p-dimensional vector of unknown parameters $\tau = (\tau_1, \cdots, \tau_p)'$ based on a sample of n observations. The joint cumulants of the derivatives of $l(\tau)$ are given by

$$\kappa_{ij} = \mathbb{E}\Big[\frac{\partial^2 l}{\partial\tau_i \partial\tau_j}\Big], \quad \text{for} \quad i, j = 1, 2, \cdots, p, \tag{3.6}$$

$$\kappa_{ijl} = \mathbb{E}\Big[\frac{\partial^3 l}{\partial\tau_i \partial\tau_j \partial\tau_l}\Big], \quad \text{for} \quad i, j, l = 1, 2, \cdots, p, \tag{3.7}$$

$$\kappa_{ij,l} = \mathbb{E}\Big[\Big(\frac{\partial^2 l}{\partial\tau_i \partial\tau_j}\Big)\Big(\frac{\partial l}{\partial\tau_l}\Big)\Big], \quad \text{for} \quad i, j, l = 1, 2, \cdots, p, \tag{3.8}$$

$$\kappa_{ij}^{l} = \frac{\partial \kappa_{ij}}{\partial\tau_l}, \quad \text{for} \quad i, j, l = 1, 2, \cdots, p, \tag{3.9}$$

respectively. It is assumed that the log-likelihood function is well behaved and regular with respect to all derivatives up to and including the third order and that all of the four equations given by (3.6)−(3.9) are of order O(n).

Let $K = [-\kappa_{ij}]$ denote the Fisher's information matrix of τ for $i, j = 1, 2, \cdots, p$. [6] show that when the sample data are independent but not necessarily identically distributed, the bias of the sth element of $\hat{\tau}_s$ can be written as

$$\text{Bias}(\hat{\tau}_s) = \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{l=1}^{p} \kappa^{si} \kappa^{jl} \left[\frac{1}{2} \kappa_{ijl} + \kappa_{ij,l} \right] + O(n^{-2}), \quad s = 1, 2, \cdots, p, \tag{3.10}$$

where κ^{ij} is the (i, j)th element of the inverse of Fisher's information matrix. Thereafter, [7] show that when all equations in (3.6)−(3.9) are of order O(n), equation (3.10) still holds even if observations are not independent. They thus advocate the following convenient form

$$\text{Bias}(\hat{\tau}_s) = \sum_{i=1}^{p} \kappa^{si} \sum_{j=1}^{p} \sum_{l=1}^{p} \left[\kappa_{ij}^{(l)} - \frac{1}{2} \kappa_{ijl} + \kappa_{ij,l} \right] + O(n^{-2}), \quad s = 1, 2, \cdots, p, \tag{3.11}$$

instead of equation (3.10). Define $a_{ij}^{(l)} = \kappa_{ij}^{(l)} - \frac{1}{2} \kappa_{ijl}$ for $i, j, l = 1, 2, \cdots, p$. They also show that the O(n^{-2}) bias expression of $\hat{\tau}$ can be reexpressed as

$$\text{Bias}(\hat{\tau}) = K^{-1} A \cdot \text{vec}(K^{-1}) + O(n^{-2}),$$

where vec is an operator that creates a column vector from a matrix by stacking the column vectors below one another, and

$$A = \left[A^{(1)} \mid A^{(2)} \mid \cdots \mid A^{(p)}\right] \quad \text{with} \quad A^{(l)} = \left[a_{ij}^{(l)}\right].$$

A bias-corrected MLE for τ, denoted by $\hat{\tau}^{\text{CMLE}}$, can thus be constructed as

$$\hat{\tau}^{\text{CMLE}} = \hat{\tau} - \hat{K}^{-1}\hat{A} \cdot \text{vec}(\hat{K}^{-1}),$$

where $\hat{\tau}$ is the MLE of the unknown parameter τ, $\hat{K} = K \mid_{\tau=\hat{\tau}}$, and $\hat{A} = A \mid_{\tau=\hat{\tau}}$. It can be shown that the bias of $\hat{\tau}^{\text{CMLE}}$ will be of order $O(n^{-2})$.

For our problem, we have the case of $p = 2$, i.e., $\tau = (\theta, c)'$. Before adopting the above 'corrective' approach to bias-corrected MLEs, we need the following higher-order derivatives of the log-likelihood function of θ and c in (3.3). Simple algebra shows that

$$\begin{aligned}
\frac{\partial^2 l}{\partial\theta^2} &= -\frac{n(c+1)}{\theta^2} + \frac{n}{(\theta+c)^2} = k_{11}, \\
\frac{\partial^2 l}{\partial\theta\partial c} &= \frac{n}{\theta} + \frac{n}{(\theta+c)^2} = k_{12}, \\
\frac{\partial^2 l}{\partial c^2} &= \frac{n}{(\theta+c)^2} - n\psi'(c) = k_{22}, \\
\frac{\partial^3 l}{\partial\theta^3} &= \frac{2n(c+1)}{\theta^3} - \frac{2n}{(\theta+c)^3} = k_{111}, \\
\frac{\partial^3 l}{\partial\theta^2\partial c} &= -\frac{n}{\theta^2} - \frac{2n}{(\theta+c)^3} = k_{112}, \\
\frac{\partial^3 l}{\partial\theta\partial c^2} &= -\frac{2n}{(\theta+c)^3} = k_{122}, \\
\frac{\partial^3 l}{\partial c^3} &= -\frac{2n}{(\theta+c)^3} - n\psi''(c) = k_{222},
\end{aligned}$$

where $\psi'(c)$ and $\psi''(c)$ are the first and second derivatives of $\psi(c)$, respectively. Of particular note is that the higher-order derivatives do not involve the sample data and thus are equal to their expectations given above. In addition, we have

$$\begin{aligned}
k_{11}^{(1)} &= \frac{\partial k_{11}}{\partial\theta} = k_{111}, \quad k_{12}^{(1)} = \frac{\partial k_{12}}{\partial\theta} = k_{112}, \quad k_{22}^{(1)} = \frac{\partial k_{22}}{\partial\theta} = k_{122}, \\
k_{11}^{(2)} &= \frac{\partial k_{11}}{\partial c} = k_{112}, \quad k_{12}^{(2)} = \frac{\partial k_{12}}{\partial c} = k_{122}, \quad k_{22}^{(2)} = \frac{\partial k_{22}}{\partial c} = k_{222}.
\end{aligned}$$

To implement the 'corrective' approach, we obtain the elements of $A^{(1)}$:

$$a_{11}^{(1)} = k_{11}^{(1)} - \frac{1}{2}k_{111} = \frac{n(c+1)}{\theta^3} - \frac{n}{(\theta+c)^3},$$

$$a_{12}^{(1)} = a_{21}^{(1)} = k_{12}^{(1)} - \frac{1}{2}k_{112} = -\frac{n}{2\theta^2} - \frac{n}{(\theta+c)^3},$$

$$a_{22}^{(1)} = k_{22}^{(1)} - \frac{1}{2}k_{122} = -\frac{n}{(\theta+c)^3}.$$

The elements of $A^{(2)}$ are

$$a_{11}^{(2)} = k_{11}^{(2)} - \frac{1}{2}k_{112} = -\frac{n}{2\theta^2} - \frac{n}{(\theta+c)^3},$$

$$a_{12}^{(2)} = a_{21}^{(2)} = k_{12}^{(2)} - \frac{1}{2}k_{122} = -\frac{n}{(\theta+c)^3},$$

$$a_{22}^{(2)} = k_{22}^{(2)} - \frac{1}{2}k_{222} = -\frac{n}{(\theta+c)^3} - \frac{n}{2}\psi''(c).$$

The matrix of A can thus be written as

$$\begin{aligned} A &= \left[A^{(1)} \mid A^{(2)}\right] \\ &= n\begin{bmatrix} \frac{c+1}{\theta^3} - \frac{1}{(\theta+c)^3} & -\frac{1}{2\theta^2} - \frac{1}{(\theta+c)^3} & -\frac{1}{2\theta^2} - \frac{1}{(\theta+c)^3} & -\frac{1}{(\theta+c)^3} \\ -\frac{1}{2\theta^2} - \frac{1}{(\theta+c)^3} & -\frac{1}{(\theta+c)^3} & -\frac{1}{(\theta+c)^3} & -\frac{1}{(\theta+c)^3} - \frac{1}{2}\psi''(c) \end{bmatrix}. \end{aligned} \tag{3.12}$$

The Fisher information matrix for the WL distribution is given by

$$K = n \begin{bmatrix} \frac{c+1}{\theta^2} - \frac{1}{(\theta+c)^2} & -\frac{1}{\theta} - \frac{1}{(\theta+c)^2} \\ -\frac{1}{\theta} - \frac{1}{(\theta+c)^2} & -\frac{1}{(\theta+c)^2} + \psi'(c) \end{bmatrix}. \tag{3.13}$$

The bias of the MLE of the WL parameters $(\theta,\ c)'$ is given by

$$\text{Bias}\begin{pmatrix} \hat{\theta} \\ \hat{c} \end{pmatrix} = K^{-1}A \cdot \text{vec}\big(K^{-1}\big) + O(n^{-2}).$$

The bias-corrected estimators of the MLEs of the WL distribution can be obtained as

$$\begin{pmatrix} \hat{\theta}^{\text{CMLE}} \\ \hat{c}^{\text{CMLE}} \end{pmatrix} = \begin{pmatrix} \hat{\theta} \\ \hat{c} \end{pmatrix} - \hat{K}^{-1}\hat{A} \cdot \text{vec}\big(\hat{K}^{-1}\big), \tag{3.14}$$

where $\hat{K} = K\ |_{\theta=\hat{\theta},c=\hat{c}}$ and $\hat{A} = A\ |_{\theta=\hat{\theta},c=\hat{c}}$. Note that the bias-corrected estimators in (3.14) have simple closed-form expressions, which means they are quite attractive because they are not computationally burdensome. It should be noted that $(\hat{\theta}^{\text{CMLE}},\ \hat{c}^{\text{CMLE}})'$ is a bias-corrected MLE of $(\theta,\ c)'$ to order $\text{O}(n^{-1})$ and that its bias is of order $\text{O}(n^{-2})$, i.e., $\mathbb{E}\big[\hat{\theta}^{\text{CMLE}}\big] = \theta + \text{O}(n^{-2})$ and $\mathbb{E}\big[\hat{c}^{\text{CMLE}}\big] = c + \text{O}(n^{-2})$. As one would expect, $\hat{\theta}^{\text{CMLE}}$ and $\hat{c}^{\text{CMLE}}$ have superior finite-sample behavior relative to $\hat{\theta}$ and $\hat{c}$, respectively, whose biases are of order $\text{O}(n^{-1})$.

3.3.2 A bootstrap approach

As an alternative to the analytically bias-corrected MLEs mentioned above, we here consider the Efron's [27] bootstrap resampling method for deriving the bias-corrected MLEs. Let $\mathbf{y} = (y_1, \cdots, y_n)'$ be a random sample of size n from the random variable Y with distribution function F. Let $\eta = t(F)$ be a function of F known as a parameter and $\hat{\eta} = s(y)$ be an estimator of η. In Efron's bootstrap resampling, we choose a large number of pseudo-samples $y^* = (y_1^*, \cdots, y_n^*)$ from the sample $\mathbf{y}$ and calculate the corresponding bootstrap replicates of $\hat{\eta}$, say $\hat{\eta}^* = s(y^*)$. Thereafter, the empirical distribution of $\hat{\eta}^*$ is used to estimate the distribution function of $\hat{\eta}$. If F belongs to a parametric family which is known and has finite dimension, F_η, we can then obtain a parametric estimate for F by using a consistent estimator for $F_{\hat{\eta}}$. The bias of the estimator $\hat{\eta} = s(y)$ can be written as

$$B_F(\hat{\eta}, \eta) = \mathbb{E}_F\big[s(y)\big] - \hat{\eta}(F), \tag{3.15}$$

where the subscript F denotes that expectation is taken with respect to F. The bootstrap bias estimate is obtained by replacing F, from which the original sample was obtained, by $F_{\hat{\eta}}$. Hence, the bias can be written as

$$B_{F_{\hat{\eta}}}(\hat{\eta}, \eta) = \mathbb{E}_{F_{\hat{\eta}}}[\hat{\eta}] - \hat{\eta}.$$

For N bootstrap samples generated independently from the original sample $\mathbf{y}$, we calculate the corresponding bootstrap estimates $(\hat{\eta}^{*(1)}, \cdots, \hat{\eta}^{*(n)})$. When N is getting larger, the expected value $E_{F_{\hat{\eta}}}(\hat{\eta})$ can be approximated by

$$\hat{\eta}^{*(\cdot)} = \frac{1}{N} \sum_{i=1}^{N} \hat{\eta}^{*(i)}.$$

The bootstrap bias estimate, obtained from the N replicates of $\hat{\eta}$, is thus $B_{F_{\hat{\eta}}}(\hat{\eta}, \eta) = \hat{\eta}^{*(\cdot)} - \hat{\eta}$. The second-order bias-corrected MLEs of the WL distribution can be obtained as

$$\eta^B = \hat{\eta} - B_{F_{\hat{\eta}}}(\hat{\eta}, \eta) = 2\hat{\eta} - \hat{\eta}^{*(\cdot)}. \tag{3.16}$$

Note that the estimator η^B shall be called the constant bias-corrected MLE since it approximates the function by a constant; see [28].

3.4 Simulation studies

In this section, we carry out Monte Carlo simulations to compare the performance between the proposed and two previous methods in the literature. The WL random variables are generated using the acceptance-rejection algorithm:

Step 1. Generate $u_1, \cdots, u_n$ for Uniform(0, 1);

Step 2. If $u_i \le p = \theta/(\theta+c)(u_i > p)$, generate x_i from Gamma(c, θ) (Gamma$(c+1, \theta)$).

For ease of notation, let "$\hat{\beta}^{\mathrm{CMLE}}$", "$\hat{\beta}^{\mathrm{BOOT}}$", "$\hat{\beta}^{\mathrm{MLE}}$", and "$\hat{\beta}^{\mathrm{MME}}$" stand for the corrective MLE, bootstrap MLE, MLE, and MME of the unknown parameter β for $\beta = \theta, c$, respectively. [9] show that the MMEs of θ and c are given by

$$\hat{\theta}^{\mathrm{MME}} = \frac{-\hat{c}^{\mathrm{MME}}(\bar{x}-1) + \sqrt{\left[\hat{c}^{\mathrm{MME}}(\bar{x}-1)\right]^2 + 4\bar{x}\hat{c}^{\mathrm{MME}}(\hat{c}^{\mathrm{MME}}+1)}}{2\bar{x}}, \tag{3.17}$$

and

$$\hat{c}^{\mathrm{MME}} = \frac{-b(\bar{x}, s^2) + \sqrt{\left[b(\bar{x}, s^2)\right]^2 + 16s^2\left[s^2 + (\bar{x}+1)^2\right]\bar{x}^3}}{2s^2\left[s^2 + (\bar{x}+1)^2\right]}, \tag{3.18}$$

respectively, where $b(\bar{x}, s^2) = s^4 - \bar{x}(\bar{x}^3 + 2\bar{x}^2 + \bar{x} - 4s^2)$ with $\bar{x}$ and s^2 being the sample mean and biased sample variance. Following the similar scenario of [9], we draw random samples of size $n = 10, 20, \cdots, 100$ with parameters $\theta = 0.5, 2$ and $c = 0.5, 1, 2$. The number of Monte Carlo replications is $M = 5,000$ and the number of bootstrap replications is $B = 1,000$ for each combination of (n, θ, c). Hence, each combination entails a total of 50 million replications.

In each simulation, to assess the performance of the methods under consideration, we calculate the average bias and root mean squared error (RMSE) of an estimator β^{est} of the parameter β, which are defined as

$$\mathrm{Bias}(\hat{\beta}^{\mathrm{est}}) = \frac{1}{M}\sum_{i=1}^{M}\left(\hat{\beta}_i^{\mathrm{est}} - \beta\right) \quad \text{and} \quad \mathrm{RMSE}(\hat{\beta}^{\mathrm{est}}) = \sqrt{\frac{1}{M}\sum_{i=1}^{M}\left(\hat{\beta}_i^{\mathrm{est}} - \beta\right)^2},$$

respectively. Figures 3.1 and 3.2 depict the biases of the simulated estimates of θ and c against the sample sizes. The corresponding RMSEs of the simulated estimates of θ and c are also displayed in Figures 3.3 and 3.4, respectively. The four figures reveal important information.

(i) The MLEs and MMEs of θ and c appear positively biased, indicating that the MLEs and MMEs on average overshoot the target parameters θ and c, particularly when the sample size is small. We also observe that in each simulation, the MLE outperforms the MME in terms of bias and RMSE.

(ii) Note that the CMLEs and BOOTs of θ and c clearly perform better than the MLEs and MMEs under the same scenario above and that these corrected estimators provide substantial bias-correction, especially for the small or moderate sample sizes. Consequently, we may treat them as better alternatives of the MLEs and MMEs for θ and c for the case in which bias is a concern.

(iii) When n gets larger, the bias and RMSE of each estimator decrease and the magnitude of reduction becomes smaller. This is expected because most estimators in statistical theory perform better with increasing n. Therefore, one can only expect that the performances of all the estimators become closer with increasing n in terms of biases and RMSEs.

(iv) The reductions in biases and RMSEs of each estimate are very substantial even for small sample sizes. For instance, when $n = 10$, $\theta = 2$, and $c = 1$, Bias($\hat{\theta}^{\text{CMLE}}$)= -0.0322, Bias($\hat{\theta}^{\text{BOOT}}$)= 0.0809, Bias($\hat{\theta}^{\text{MLE}}$)= 0.7397, Bias($\hat{\theta}^{\text{MME}}$)= 1.1015, Bias($\hat{c}^{\text{CMLE}}$)= -0.0247, Bias($\hat{c}^{\text{BOOT}}$)= 0.0733, Bias($\hat{c}^{\text{MLE}}$)= 0.3454, Bias($\hat{c}^{\text{MME}}$)= 0.5595; RMSE($\hat{\theta}^{\text{CMLE}}$)= 1.0652, RMSE($\hat{\theta}^{\text{BOOT}}$)= 1.5704, RMSE($\hat{\theta}^{\text{MLE}}$)= 1.6863, RMSE($\hat{\theta}^{\text{MME}}$)= 2.1653, RMSE($\hat{c}^{\text{CMLE}}$)= 0.4890, RMSE($\hat{c}^{\text{BOOT}}$)= 0.7706, RMSE($\hat{c}^{\text{MLE}}$)= 0.7810, RMSE($\hat{c}^{\text{MME}}$)= 1.0396.

(v) The proposed estimator CMLE consistently outperforms the bootstrap estimator BOOT in terms of bias and RMSE. In particular, the bootstrap bias-correction procedure may lead to an increased RMSE, as shown in Figures 3.3 and 3.4. For example, when $\theta = 2$ and $c = 2$, the RMSE of the bootstrap estimator is larger than that of the MLE for $n \leq 20$, . Hence, the corrected estimators proposed in this paper should be preferred for the WL distribution, instead of the ones via the bootstrap.

3.5 Real data examples

In this section, we illustrate the practical application of the proposed bias-corrected estimators for the WL distribution using two real data sets with one involving a small sample and the other with a moderate sample.

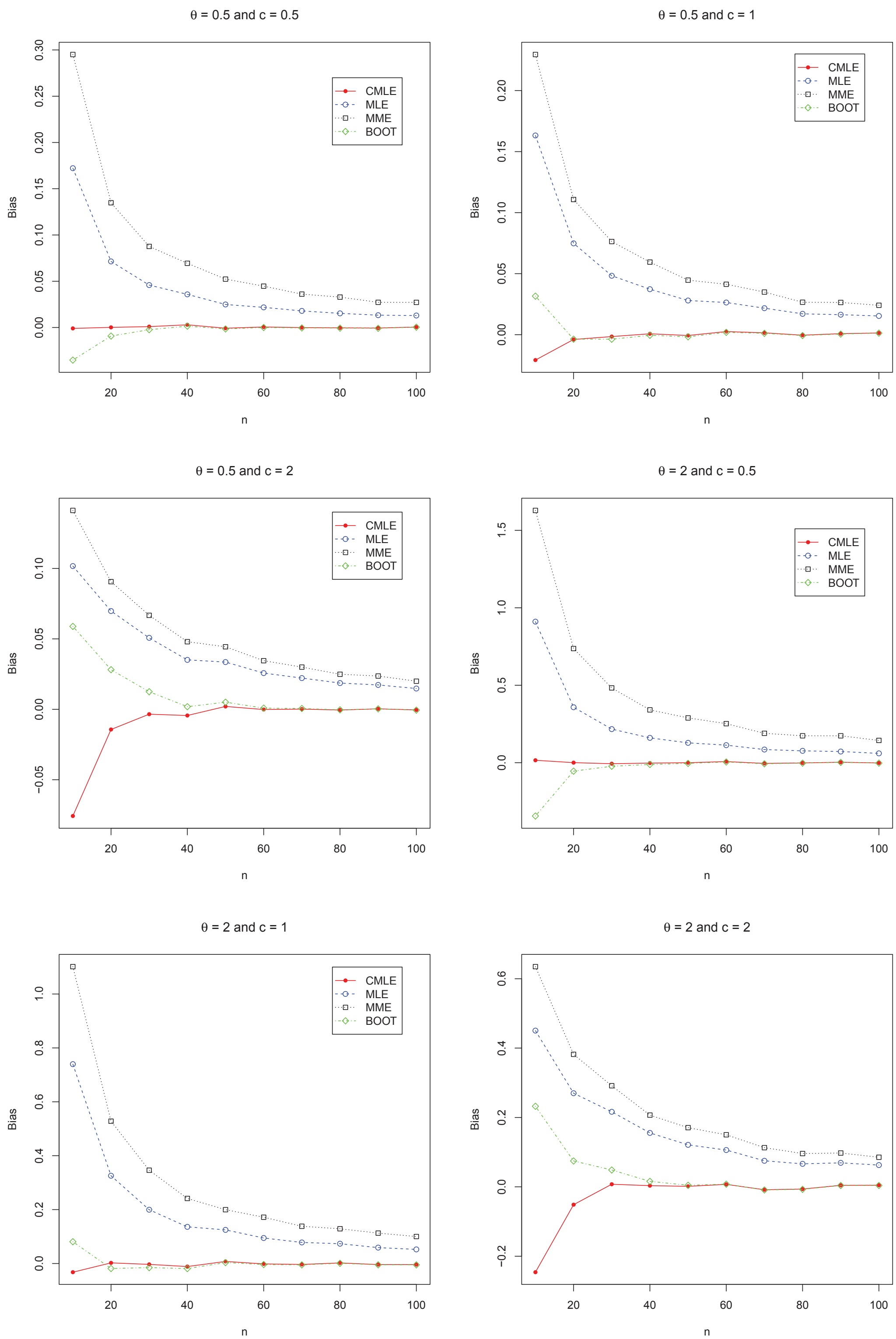

Figure 3.1: Average bias of the considered estimate of θ.

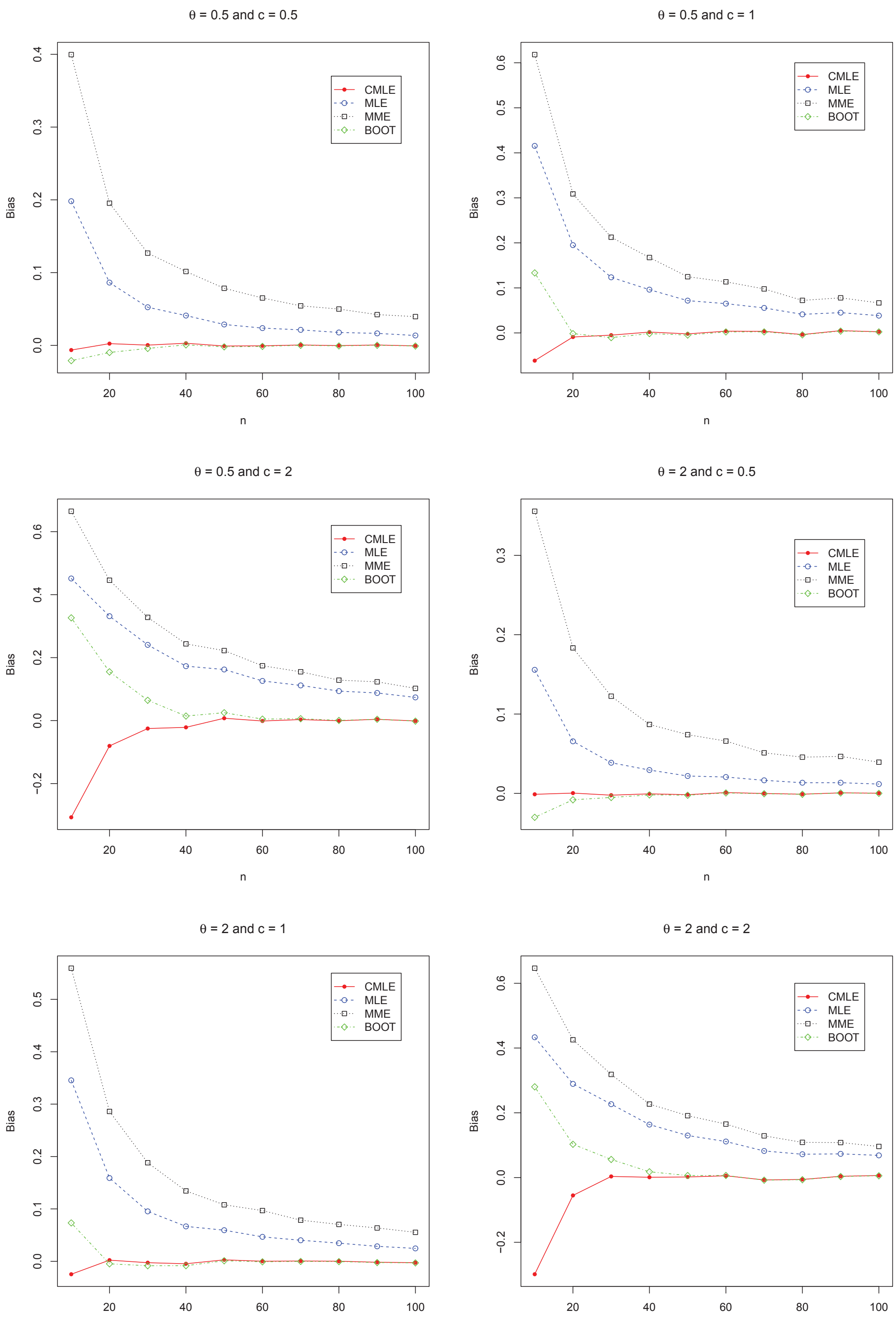

Figure 3.2: Average bias of the considered estimate of c.

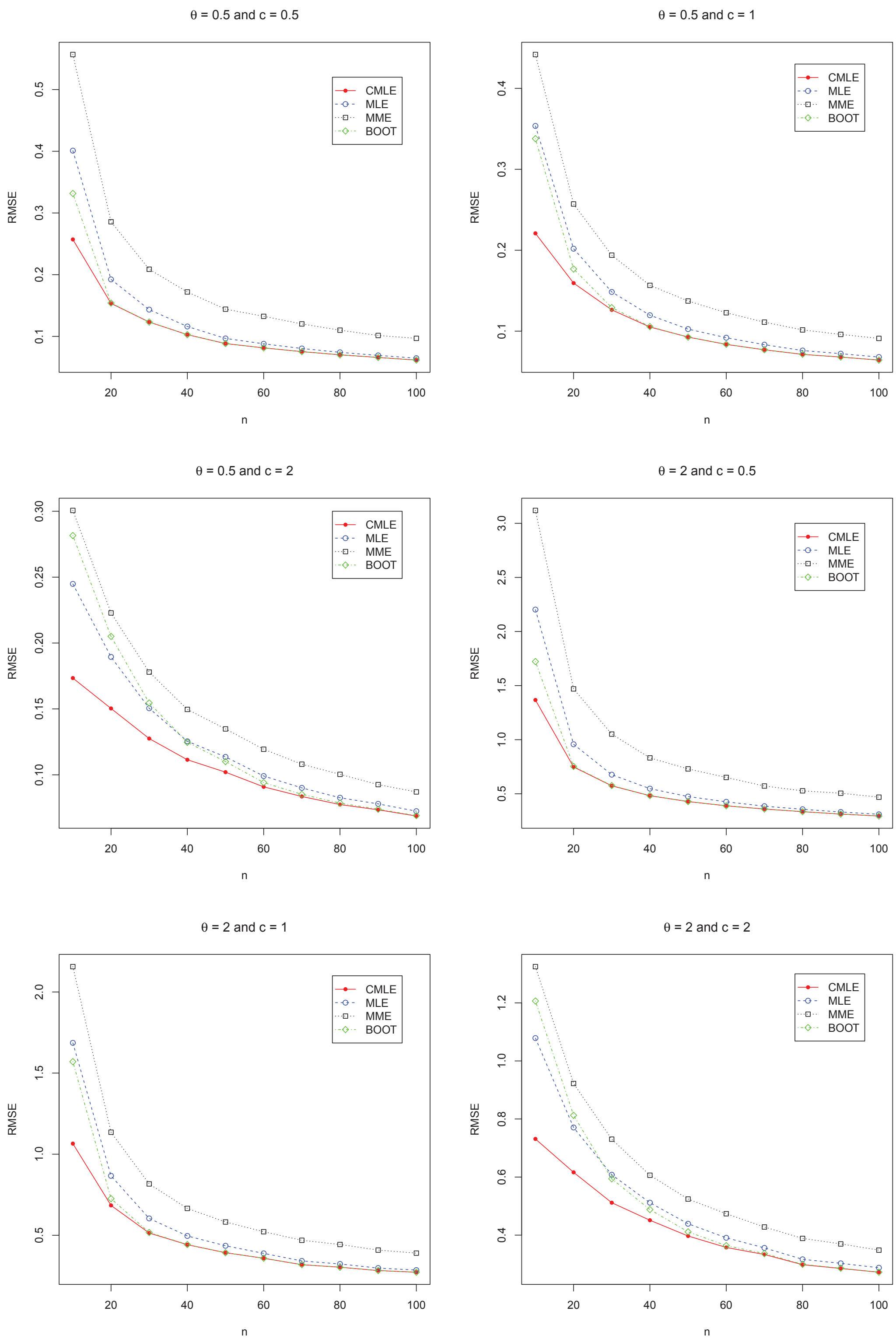

Figure 3.3: RMSE of the considered estimate of θ.

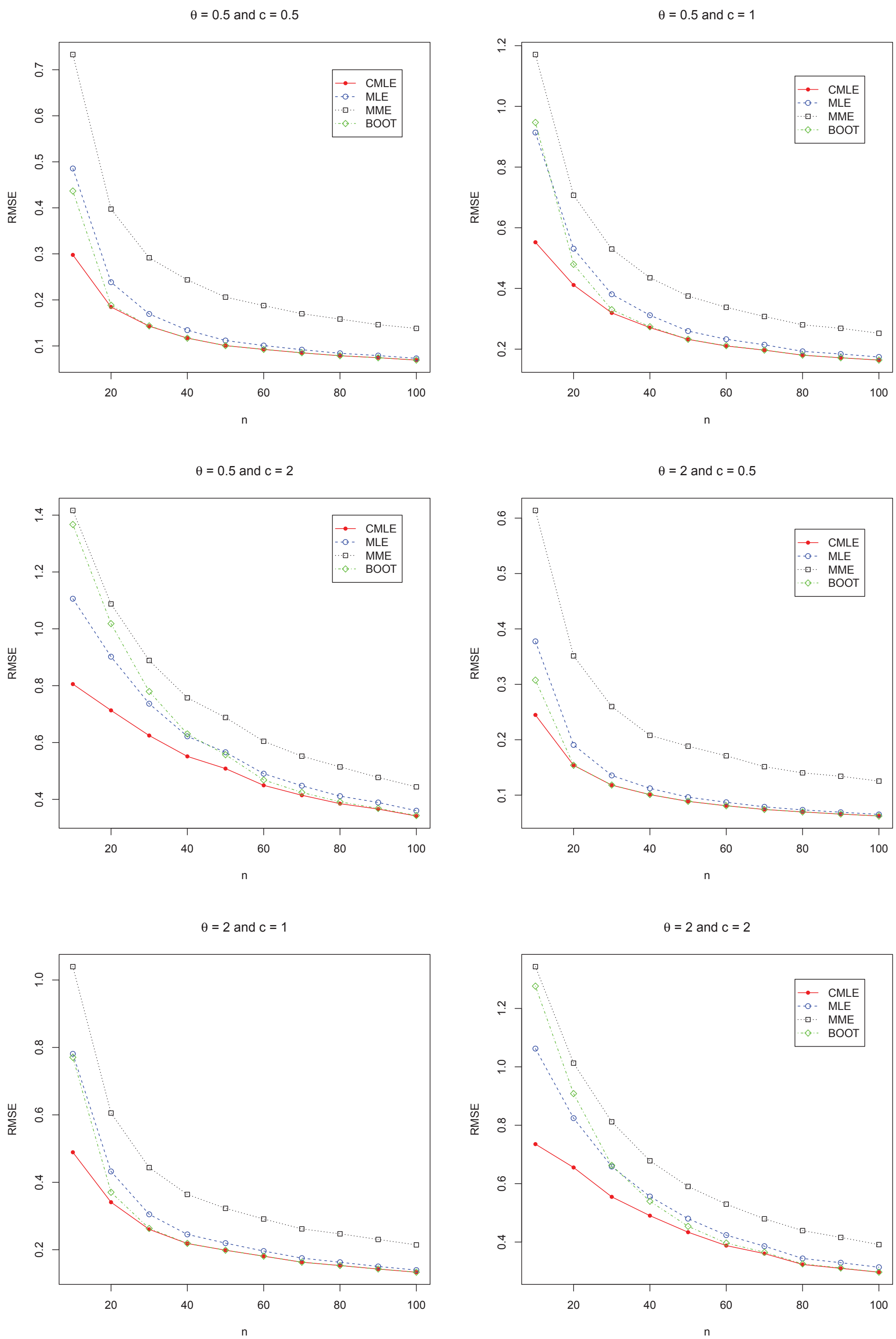

Figure 3.4: RMSE of the considered estimate of c.

Example 3.1 We shall now analyze a data set on the lifetime failure of an electronic device. The data were used by [29] as an illustration of the additive Burr XII distribution. Later on, the data were further analyzed by [9] for comparing different estimation methods for the WL distribution. The data are given in Table 3.1.

5	11	21	31	46	75	98	122	145
165	196	224	245	293	321	330	350	420

Table 3.1
The time to failure of 18 electronic devices

The point estimates for the WL distribution are provided in Table 3.2. Note that the bias-corrected estimates of θ and c are smaller than the MLEs and MMEs, especially for estimating c. This would justify that estimation by the maximum likelihood and method of moments are overestimating both θ and c. Figure 3.5 depicts the WL density given by (3.1) evaluated at different estimates of θ and c in Table 3.2. It can be seen from Figure 3.5 that given the small sample size, the shape of densities based on the MLE and MME may be misleading and that correction for bias in the estimation for the WL distribution should be extremely important in real data analysis.

Estimate	θ	c
MLE	0.00726	0.27681
MME	0.01060	0.83755
CMLE	0.00554	0.03536
BOOT	0.00621	0.11136

Table 3.2
Point estimates of θ and c for Example 3.1.

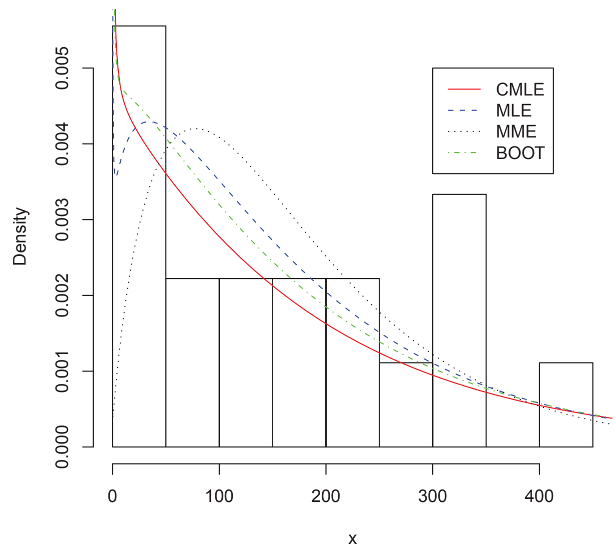

Figure 3.5: Estimated fitted density functions of the lifetime failure of an electronic device for Example 3.1.

Example 3.2 The data set is given by [30] on the failure stresses (in GPa) of 65 single carbon fibers of length 50mm. The data were recently used as an illustrative example for the WL distribution by [9]. The data are presented in Table 3.3.

1.339	1.434	1.549	1.574	1.589	1.613	1.746	1.753	1.764	1.807
1.812	1.840	1.852	1.852	1.862	1.864	1.931	1.952	1.974	2.019
2.051	2.055	2.058	2.088	2.125	2.162	2.171	2.172	2.180	2.194
2.211	2.270	2.272	2.280	2.299	2.308	2.335	2.349	2.356	2.386
2.390	2.410	2.430	2.431	2.458	2.471	2.497	2.514	2.558	2.577
2.593	2.601	2.604	2.620	2.633	2.670	2.682	2.699	2.705	2.735
2.785	3.020	3.042	3.116	3.174					

Table 3.3
The failure stresses (in GPa) of 65 single carbon fibers of length 50mm.

The point estimates of θ and c obtained by all the considered methods are summarized in Table 3.4. It is worth pointing out that all the estimations are obviously different, which indicates that even when the sample size is moderate, the bias correction is still necessary because it contains useful information. Figure 3.6 contains the WL density given by (3.1) evaluated at the point estimates of θ and c in Table 3.4. Note that the estimated density obtained from the MLE is too peaked and the CMLE and BOOT density estimates are almost overlapping and less peaked than the two previous estimates in the literature.

Estimate	θ	c
MLE	12.82271	28.08819
MME	13.28780	29.13167
CMLE	12.23304	26.78033
BOOT	12.28989	26.88698

Table 3.4
Point estimates of θ and c for Example 3.2.

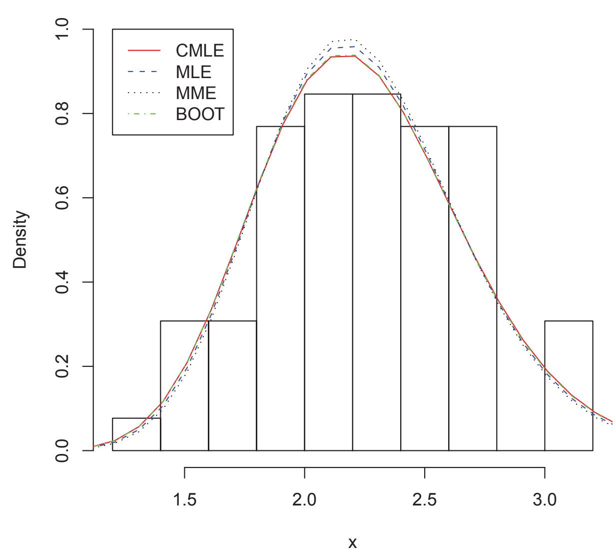

Figure 3.6: Estimated fitted density functions of the failure stresses (in GPa) of 65 single carbon fibers of length 50mm for Example 3.2.

3.6 Concluding remarks

In this chapter, we have adopted a 'corrective' approach to derive simple closed-form expressions for the second order biases of the MLEs of the parameters that index the weighted Lindley distribution. The biases of the proposed estimators are of order $\mathrm{O}(n^{-2})$, whereas for the MLEs they are of order $\mathrm{O}(n^{-1})$, indicating that the newly proposed estimators converge to their true value considerably faster than those of the MLEs. In addition, we have also considered an alternative bias-correction mechanism through Efron's bootstrap resampling. The numerical evidence shows that the proposed estimators are quite attractive because they outperform those of the MLE and MME in terms of bias and RMSE. It deserves mentioning that unlike the bias-corrected MLEs via the bootstrap, the proposed estimators are available in closed form and are thus easy to compute without requiring data resampling. Consequently, the proposed bias-corrected estimators are strongly recommended over other estimators without bias-correction, especially when the sample size is small or moderate, which is often encountered in the context of reliability analysis.

Chapter 4

Concluding remarks and future work

The main goal of this report is to illustrate the importance of the bias-correction of the MLEs of the probability distributions, especially when the sample size is small or moderate. It has been shown that the fitted distributions based on the MLEs and bias-corrected MLEs can be significantly different for both the one-parameter inverse Lindley distribution and the two-parameter weighted Lindley distribution. We thus have a preference of the considered bias-corrected technique, because it reduces the bias of the MLE from order $\mathrm{O}(n^{-1})$ to order $\mathrm{O}(n^{-2})$, indicating that the bias-corrected estimator converges to the true value faster than the one based on the MLE. Moreover, the considered technique can be easily implemented in practical situations as long as the MLE of the unknown parameter is available.

Recently, numerous distributions have been developed in the literature. The main motivation of these new distributions is that researchers want to provide a better fit for the real-data applications. In this report, we have shown that the poor performance of a distribution maybe due to inaccurate estimators of the unknown parameters, rather than the inner properties of the distribution. Consequently, special attention should be paid when we apply a distribution to analyze the real data in practice. In an on-going work, we study the application of the bias-corrected technique to some other commonly used lifetime distributions, such as weighted exponential distribution [31] and the three-parameter Lindley geometric distribution[32], which are currently under investigation and will be reported elsewhere in the near future.

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