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Game-theoretic view on intermediated exchange

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A GAME-THEORETIC VIEW ON INTERMEDIATED EXCHANGE

By
THOMAS GRASSL

A DISSERTATION
submitted in partial fulfillment of the requirements
for the degree of
DOCTOR OF PHILOSOPHY
(Mathematical Sciences - Applied Mathematics)

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This dissertation, ”A Game-Theoretic View on Intermediated Exchange”, is hereby approved in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in the field of Mathematical Sciences - Applied Mathematics.

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Abstract

Intermediaries permeate modern economic exchange. Most classical models on intermediated exchange are driven by information asymmetry and inventory management. These two factors are of reduced significance in modern economies. This makes it necessary to develop models that correspond more closely to modern financial marketplaces. The goal of this dissertation is to propose and examine such models in a game theoretical context.

The proposed models are driven by asymmetries in the goals of different market participants. Hedging pressure as one of the most critical aspects in the behaviour of commercial entities plays a crucial role.

The first market model shows that no equilibrium solution can exist in a market consisting of a commercial buyer, a commercial seller and a non-commercial intermediary. This indicates a clear economic need for non-commercial trading intermediaries: a direct trade from seller to buyer does not result in an equilibrium solution.

The second market model has two distinct intermediaries between buyer and seller: a spread trader/market maker and a risk-neutral intermediary. In this model a unique, natural equilibrium solution is identified in which the supply-demand surplus is traded by the risk-neutral intermediary, whilst the market maker trades the remainder from seller to buyer. Since the market maker's payoff for trading at the identified equilibrium price is zero, this second model does not provide any motivation for the market maker to enter the market.

The third market model introduces an explicit transaction fee that enables the market maker to secure a positive payoff. Under certain assumptions on this transaction fee the equilibrium solution of the previous model applies and now also provides a financial motivation for the market maker to enter the market. If the transaction fee violates an upper bound that depends on supply, demand and risk-aversity of buyer and seller, the market will be in disequilibrium.

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Thomas Graßl, Houghton, May 2007

Contents

Preface

Activity of intermediaries permeates the modern economics. Market makers, outsourcing companies, headhunters, real estate agents, mortgage brokers are among the most familiar examples.

One might wonder why people on different sides of the transactions would need intermediaries. It turns out that there are a number of critical advantages of intermediated exchange versus direct exchange. Among the gains from intermediated exchange are the centralization of exchange, facilitation of information transfer, reduction of search and bargaining costs, alleviation effects of adverse selection, management of inventory, etc.

A substantial amount of literature deals with these different aspects of an intermediary's activity. An excellent treatment and review of various existing equilibrium theories on intermediation activity can be found in the book by Spulber [Spu99]. The main thesis was summarized by Spulber [Spu99] as follows: "Firms are formed when the gains from intermediated exchange exceed the gains from direct exchange".

The classic theoretical framework for understanding intermediated exchange is based on two major components: information asymmetry, and inventory management. There are many (equilibrium) models of intermediation activity that aim to describe, first of all, the trading of a risky asset in the structural framework of exchanges. In such models, the market is usually assumed to consist of market makers (intermediaries), informed and uninformed traders (often referred to as insiders and liquidity traders respectively). Various models and aspects of exchange trading, based on the distribution of knowledge or information about an asset's true value, are considered by many of authors; for examples, see Dennert [Den93] and Kyle [Kyl85] who describe market makers as uninformed participants. On the other hand, Laffont and Maskin [LM90] presents a model where the market maker is the informed trader whereas other traders are uninformed. Copeland and Galai [CG83] proposes a model in which the bid-ask price spread of market makers depends on the proportion of insiders and liquidity traders. An extension of this demonstrating the effect of adverse selection can be found in Glosten and Milgrom [GM85]. For a detailed review of many other models and aspects of trading at exchanges see Spulber [Spu99].

Realistically speaking, those classical models are applicable to the relatively slow, pit-oriented or specialists' markets/trading before the Internet revolution. Nowadays, however, electronic environments have radically changed the structure of financial trading. The Bloomberg terminal became the standard of information dissemination. Decimalization of prices led to fierce competition between market makers and the elimination of many inefficiencies.

Volumes of trading grew very substantially. Pit trading disappeared on many exchanges in favor of electronic trading, with an ever increasing fraction of algorithmic trading, development of statistical arbitrage models, and 24 hours trading on ECNs. A dramatic increase of professionally managed capital led to the faster movement of "hot money", while the global nature of financial markets made movements of capital across industries/borders much easier.

At the same time, it seems that in the modern economy, the storage and "buffering" functions of intermediaries are declining. Market makers try not to carry a substantial inventory, headhunters do not act themselves on the main area of their customers' activity, real estate agents do not own sizeable properties for their businesses sake, mortgage brokers do not underwrite mortgages, and outsourcing hi-tech companies do not write software.

Overall, nowadays' companies try to carry only the minimally necessary level of inventory, and often, intermediaries do not carry any inventory at all: they serve as pure conduits in the transfer of goods or services.

While the traditional functions of intermediaries are on the decline, intermediation itself increases (see for example Allen and Santomero [AS98]). This makes it necessary to develop a new generation of models that would more closely correspond to the modern structure of the financial marketplace, and to a more information-efficient technological environment. The goal of the present work is to propose such models in a game-theoretical context.

In the proposed models, no information asymmetries between the various parties in the trade will be assumed. Instead, the models will be driven by asymmetries in the goals of the different market participants, and by the "fundamental" differences between various market players. In the considered market models a commercial buyer, a commercial seller and non-commercial intermediaries will trade a certain amount of one commodity. Loosely speaking, this corresponds to the classification of so-called commercials and non-commercials by the Commodity Futures and Trading Commission (CFTC), and to the closely watched long/short commercials/non-commercials numbers released weekly by CFTC (http://www.cftc.gov). As a result the us of the more inflammatory “speculator” term will be avoided.

Hedging pressure as one of the most critical aspects in the behaviour of commercial entities will play a crucial role in the proposed models. it is assumed that trading in the cash market is completely frictionless, and for the sake of simplicity and concise presentation, the models will be presented in the context of financial markets only. However, the general conclusions obtained below are equally applicable to all types of intermediated exchange.

A first model will assume that one intermediary will fulfil a dual function: he will trade price spreads and take risk. The main result of this first model is that, under certain broad assumptions, a naive approach in which buyer and seller fulfil as much of their hedging needs as possible in a direct trade, hence avoiding the costs that would arise by including the intermediary in their trades, will not result in an equilibrium solution. This result shows that there is a clear economic need for non-commercial trading intermediaries. At the same time, the presence of intermediaries does not yet result in establishing an equilibrium price. This total lack of an equilibrium even in such a simple model may provide an additional interpretation of the intrinsic random nature of asset prices.

Since participants in modern day markets tend to specialize their tasks, an extended model will split up the different tasks of the previous intermediary: two separate intermediaries will now be considered, one trading spreads and the other one taking risk. As a main effect, this economic specialization will result in a rather stable market situation. A unique equilibrium price will be identified, a price that will allow for an insightful interpretation on how price negotiations depend on the goals and needs of the different sides in a trade.

This second model will however not sufficiently explain why an intermediary trading spreads will enter the market in the first place. To resolve this issue, a third model will be considered in which the commercial seller and buyer have to pay a fixed transaction fee if they plan to enter a trade with the spread trader. It will be shown that in such a model the equilibrium solution of the previous model will apply as long as the transaction fee satisfies a certain bound. It will be shown that this bound has rather natural properties, and that it agrees with observations that can be made in existing financial markets.

Part I

Introductory remarks

The purpose of this work is to develop a new generation of models describing the modern structure of the financial marketplace. Several models will be examined, based on different assumptions and allowing for different interpretation. However, all models that are to be analysed share one common feature: they are based on a game-theoretic framework. Each model will be presented as a game in strategic form, and the analysis of each of the models will focus on the most prominent of game-theoretic concepts, the so-called Nash equilibrium.
This introductory part will therefore be used to lay the theoretical foundation for the work to come: basic game-theoretic concepts and ideas will be defined and reviewed.

Chapter 1

Basic game-theoretic concepts

The following chapter reviews some of the most prominent game-theoretic concepts: the reader will find definitions of concepts such as games in strategic forms, pure strategies, mixed strategies or Nash equilibria. Furthermore - since the purpose of most of this work is to prove existence or non-existence of a Nash equilibrium solution in the different market models - the most important existence theorems for Nash equilibria will be mentioned and proven.

1.1 Strategic games

In game-theoretic frameworks different types of games can occur. One of the most important types and also the type that will be used in the subsequent market models are games in strategic form:

Definition 1.1.1 (Games in strategic form)
A game in strategic form is a triple

$$\Gamma = (I, S, U), \tag{1.1}$$

where I denotes the set of players, $S = \prod_{i \in I} S^i$ the set of all possible strategy combinations s and S^i the set of strategies s_i of player i. U denotes the payoff function

$$U : S \to \mathcal{E} \tag{1.2}$$

that assigns a result $u \in \mathcal{E}$ to each strategy combination s.[1]

[1] see [Wie02] pg. 110 and [Sch04] pg. 9f.

A game will be played as follows:
Every player i will choose a strategy $s^i \in S^i$ and once all strategies are set, the relevant payoffs will be determined.

Note that one player i can have finitely or infinitely many strategies, i.e. S^i can be a finite set as well as an infinite set. In the finite case with $n_i = |S^i|$ different strategies, player i's strategies will be denoted as

$$S^i = \left\{s^i_1, ..., s^i_n\right\}.$$

The outcome of a single realization of a game is determined by the individual strategy choices of the different players. For $m = |I|$ players, such a combination of individual strategic decisions, short a *strategy combination*, can be written as:

$$s = \left(s^1, s^2, \dots, s^m\right). \tag{1.3}$$

Very often it is necessary to discuss how one single player can effect the outcome of a game given that the other players do not change their strategies. To simplify notation, a strategy combination that does not include player i's strategy will be denoted as s^{-i}. This yields the following simplified notations:

$$s = \left(s^i, s^{-i}\right) \tag{1.4}$$

and

$$S = S^i \times S^{-i} \tag{1.5}$$

where S^{-i} denotes accordingly the set of all strategies of all players except for those of player i. In the above definition, the payoff function U can represent a win, loss, benefit, damage or even something non quantifiable. For the sake of simplicity, it will be assumed that a payoff function is quantifiable, i.e. U will be defined as a map from S to $\mathbf{R}^m$:

$$U : S \to \mathbf{R}^m. \tag{1.6}$$

Using the above short hand notation U can be written as:[2]

$$U = \left(U^1, U^2, \dots, U^m\right) \tag{1.7}$$

$$U^i(s) = U^i\left(s^1, s^2, \dots, s^n\right) = U^i\left(s^i, s^{-i}\right) \text{ for } i \in I. \tag{1.8}$$

[2]see [Sch04] pg. 10f.

1.2 The Nash equilibrium

The purpose of a game-theoretic analysis is not just the formulation of a problem in form of a game, but also its analysis. It is of specific interest to know whether certain strategy combinations will be played more often than others.

1.2.1 Definition

Nash suggests the following equilibrium condition:

Definition 1.2.1 (Nash equilibrium)
In a game $\Gamma = (I, S, U)$ *a strategy combination* $s_* = (s_*^1, ..., s_*^m)$ *is called a Nash equilibrium, if:*[3]

$$U^i\left(s_*^i, s_*^{-i}\right) \geq U^i\left(s^i, s_*^{-i}\right) \ \forall i \in I, \ s^i \in S^i. \tag{1.9}$$

In other words, a Nash equilibrium is a strategy combination s_*, such that no player can singlehandedly improve his payoff. In more detail: no player i is able to obtain a higher payoff just by changing his own strategy s_*^i to a strategy s^i, as long as his opponents keep their strategy combination s_*^{-i} fixed. This means that no player has a reason, to choose a strategy other than his equilibrium strategy.

1.2.2 Famous examples: Prisoner's Dilemma and Rock, Paper, Scissors

The idea of a Nash equilibrium will be demonstrated on two very famous examples: the so called "Prisoner's Dilemma" and the popular game "paper-scissor-stone".

Example 1.2.1 (Prisoner's Dilemma)
After a bank robbery two suspects are interrogated by the police. If both decide not to confess, they remain on remand, but will be set free for lack of evidence after one year. If both of them decide to confess, they will be sentenced to six years in prison. If just one of the two suspects confesses, he will be set free, whilst the other suspect will be sentenced to ten years in prison. A schematic representation of this game may look as follows:
It can easily be seen, that the Nash equilibrium of this game is the strategy combination (confess, confess).[4].

[3] see [Gib92] pg. 8
[4] for a more detailed discussion of this result see [Bit81] pg.250ff.

	s_1^2: **not confess**	s_2^2: **confess**
s_1^1: **not confess**	$(-1, -1)$	$(-10, 0)$
s_2^1: **confess**	$(0, -10)$	$(-6, -6)$

Table 1.1: payoff matrix of Prisoner's Dilemma

Example 1.2.2 (Rock, Paper, Scissors)
A very popular game is the "Rock, Paper, Scissors" game[5]. In this game the payoff function can be represented as follows:

	s_1^2: **rock**	s_2^2: **paper**	s_3^2: **scissors**
s_1^1: **rock**	$(0, 0)$	$(1, -1)$	$(-1, 1)$
s_2^1: **paper**	$(-1, 1)$	$(0, 0)$	$(1, -1)$
s_3^1: **scissors**	$(1, -1)$	$(-1, 1)$	$(0, 0)$

Table 1.2: payoff matrix for Rock, Paper, Scissors

It can be seen, that no matter which strategy combination is played, one player will always be able to improve his payoff by simply changing his own strategy. This means that this game has no Nash equilibrium in pure strategies.

1.2.3 Nash equilibria as fixed points problems

Identifying Nash equilibria reduces to a fixed point problem if the so called *best-response correspondence* is considered.

The principle of the best response is based on the following idea[6]:
If a player i wants to play a game with the greatest possible success, he has to build up expectations on which strategies s^{-i} will be chosen by his contestants. Based on these expectation he can formulate his best possible response strategy. This best response might not be unique: sometimes several possible responses will fit equally well to a strategy combination s^{-i}. It follows that a mapping assigning to each s^{-i} player i's best responses will be a set-valued correspondence:

[5] for more details on this game see [Sch04] pg.17f.
[6] see for this principle also [Sch04] pg. 20f.

Definition 1.2.2 (Best-response correspondence for the player *i*)
Let r^i be the mapping, that yields for any given strategy combination s^{-i} the set of all best responses, i.e. a mapping of the form

$$\begin{aligned} r^i &: S^{-i} \to 2^{S^i} \\ r^i\left(s^{-i}\right) &= \left\{s^i_* \in S^i \;\mid\; U^i\left(s^i_*, s^{-i}\right) \geq U^i\left(s^i, s^{-i}\right)\; \forall s^i \in S^i\right\}. \end{aligned} \tag{1.10}$$

Such a mapping r^i is called best-response correspondence of the player i.

Combining the best-response correspondences for all players i, yields the following mapping:

Definition 1.2.3 (Best-response correspondence)
Let $r^i : S^{-i} \to 2^{S^i}$ be defined as above. Then

$$\begin{aligned} r &: S \to 2^S \\ r(s) &= \prod_{i=1}^{m} r^i(s^{-i}) \end{aligned} \tag{1.11}$$

is called the best-response correspondence of a game Γ.

It is now rather easy to see, that Nash equilibria are in fact nothing else than fixed points of the best-response correspondence.

Theorem 1.2.4 (Nash equilibria as fixed points)
Let $\Gamma = (I, S, U)$ be a game in pure strategies and let r be the best-response correspondence. Then:

$$s_* \textit{ is a Nash equilibrium} \Leftrightarrow s_* \in r\left(s_*\right). \tag{1.12}$$

Proof. Every player i has certain expectations on which strategies s^{-i} the other contestants will play. Since the players are supposed to be rational, these expectations will come true if and only if the expected strategies s^{-i} are best answers on the strategy combination of the other contestants. Therefore if a strategy combination s_* is to be played, it should satisfy

$$s_* \in r\left(s_*\right)$$

This means, that there are mutual best responses if and only if the above relation holds. Hence no player will have an impetus, to change his equilibrium strategy, if and only if $s_* \in r\left(s_*\right)$. But this is just the characterization of a Nash equilibrium and therefore:

$$s_* \text{ is a Nash equilibrium} \Leftrightarrow s_* \in r\left(s_*\right).$$

1.2.4 Occurrence of a Nash equilibrium

By the very definition of a Nash equilibrium, no individual player will have an incentive to play anything else than his Nash equilibrium strategy as long as all other players decide to play their equilibrium strategies. In other words: once all players have agreed to adopt the Nash equilibrium strategy combination, no single player will have an incentive to break this agreement.

However, it is not clear whether or not all players would have adopted such a Nash equilibrium strategy combination in the first place. Only under some rather strict conditions the occurence of a Nash equilibrium strategy combination can be guaranteed. The following Lemma will be stated without further proof:

Lemma 1.2.5 (Occurrence of a Nash equilibrium)
If a game has a unique Nash equilibrium solution, the Nash equilibrium strategy set will be adopted by the players if all players act in a perfectly rational fashion, i.e. if

- *all players try to maximize their expected payoff.*
- *all players pursue their goal without committing mistakes.*
- *all players are intelligent enough to identify the Nash equilibrium solution.*
- *all players know that all other players are perfectly rational.*

It is of course close to impossible to find a perfectly rational player in the real world. However, in the context of an application of game theory to economics, one should expect that all players are at least somewhat close to perfect rationality. Otherwise they would simply be competed out of the market.

1.3 Mixed strategy games

In the above example "Rock, Paper, Scissors" it was not possible to locate a Nash equilibrium. However, it will be possible to identify a Nash equilibrium if only some different kind of strategy concept is considered, the so-called mixed strategies.

1.3.1 Definition of a game in mixed strategies

So far just pure strategies were considered, i.e. the focus of the analysis was just one single realization of a game. But what, if the same game is played over and over again? This leads to the idea of mixed strategies.

Definition 1.3.1 (Mixed strategy)
Let $S^i = \left\{s^i_1, ...s^i_{n_i}\right\}$ *be the strategy set of player i. Then a vector* $\hat{s}^i = \left(\hat{s}^i_1, \hat{s}^i_2, ..., \hat{s}^i_{n_i}\right)^T$ *is called a mixed strategy of i, if:*[7]

$$\sum_{j=1}^{n_i} \hat{s}^i_j = 1, \hat{s}^i_j \geq 0. \tag{1.13}$$

$\hat{S}^i$ *is called the set of all mixed strategies of player i, if:*

$$\hat{S}^i = \left\{\left(\hat{s}^i_1, \ldots, \hat{s}^i_{n_i}\right) \mid \sum_{j=1}^{n_i} \hat{s}^i_j = 1,\ \hat{s}^i_j \geq 0,\ j = 1, \ldots, n_i\right\}. \tag{1.14}$$

The set of all mixed strategies $\hat{S}$ *will then be defined by*

$$\hat{S} = \prod_{i \in I} \hat{S}^i \tag{1.15}$$

An interpretation of the concept of mixed strategies can be given as follows: the i-th player chooses the k-th pure strategy s^i_k with probability $\hat{s}^i_k$. The strategy "I play rock, paper and scissors equally often" could therefore be denoted as the mixed strategy $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.
This interpretation shows, that the set of all pure strategies is a subset of the set of all mixed strategies. Vectors $\hat{s}^i = \left(\hat{s}^i_1, \hat{s}^i_2, ..., \hat{s}^i_{n_i}\right)^T$ where $\hat{s}^i_k = 1$ for one k can therefore be identified with the k-th pure strategy s^i_k. Since $\hat{s}^i_j$ can be understood as the probability of playing the strategy s^i_j, the payoff in mixed strategies can be interpreted as an expected value:

Definition 1.3.2 (Payoff in mixed strategies)
If each player i is playing a mixed strategy, then the payoff $\hat{U}^i$ *to player i is given by:*[8]

$$\hat{U}^i := U^i(\hat{s}) = \sum_{j_1=1}^{n_1} \ldots \sum_{j_m=1}^{n_m} U^i\left(s^1_{j_1}, \ldots, s^m_{j_m}\right) \prod_{k=1}^{m} \hat{s}^k_{j_k}. \tag{1.16}$$

[7] see [Sch04] pg. 30
[8] see [Sch04] pg. 30

Having defined strategies and payoffs for a game in mixed strategies, it is now possible to define a game in mixed strategies:

Definition 1.3.3 (Game in mixed strategies)
Let $\hat{S}$ und $\hat{U}$ be as defined above. Then $\hat{\Gamma} = (I, \hat{S}, \hat{U})$ is called a game in mixed strategies $\Gamma = (I, S, U)$.

A mapping similar to the best-response correspondence in pure strategies can also be defined in mixed strategies:

Definition 1.3.4 (Best-response correspondence in mixed strategies)
For the player i the best-response correspondence in mixed strategies is defined to be the mapping

$$\begin{aligned} \hat{r}^i &: \hat{S}^{-i} \to 2^{\hat{S}^i} \\ \hat{r}^i\left(\hat{s}^{-i}\right) &= \left\{\hat{s}^i_* \in \hat{S}^i \mid \hat{U}^i\left(\hat{s}^i_*, \hat{s}^{-i}\right) \geq \hat{U}^i\left(\hat{s}^i, \hat{s}^{-i}\right) \ \forall \hat{s}^i \in \hat{S}^i\right\}. \end{aligned} \tag{1.17}$$

Combining these correspondence gives the best-response correspondence for the game $\hat{\Gamma}$:

$$\begin{aligned} \hat{r} &: \hat{S} \to 2^{\hat{S}} \\ \hat{r}(\hat{s}) &= \prod_{i=1}^{m} \hat{r}^i\left(\hat{s}^{-i}\right) \end{aligned} \tag{1.18}$$

1.3.2 Nash equilibrium in mixed strategies

Based on the definition of a Nash equilibrium in pure strategies[9] a Nash equlibrium in mixed strategies can be described as follows:

Theorem 1.3.5 (Nash equlibrium in mixed strategies)
A strategy combination $\hat{s}_* = \left(\hat{s}^i_*, \hat{s}^{-i}_*\right) \in \hat{S}$ is a Nash equilibrium if and only if for all pure strategies s^i_j of the i-th player:[10]

$$\hat{U}^i\left(s^i_j, \hat{s}^{-i}_*\right) \leq \hat{U}^i\left(\hat{s}_*\right), \ j = 1, \ldots, n_i, \ i = 1, \ldots, m. \tag{1.19}$$

Proof. The implication

$$\hat{s}_* \text{ is a Nash equilibrium} \Rightarrow \hat{U}^i\left(s^i_j, \hat{s}^{-i}_*\right) \leq \hat{U}^i\left(\hat{s}_*\right), \forall j, \forall i$$

[9] see definition 1.2.1
[10] see for this theorem [Sch04] pg. 38

follows directly from definition 1.2.1.

Suppose now $\hat{U}^i\left(s^i_j, \hat{s}^{-i}_*\right) \leq \hat{U}^i(\hat{s}_*)$ for all pure strategies s^i_j of the i-th player. Then for all mixed strategies:

$$\begin{aligned}\hat{U}^i\left(\hat{s}^i, \hat{s}^{-i}_*\right) &= \sum_{j=1}^{n_i} \hat{U}^i\left(s^i_j, \hat{s}^{-i}_*\right)\hat{s}^i_j \\ &\leq \sum_{j=1}^{n_i} \hat{U}^i(\hat{s}_*)\,\hat{s}^i_j = \hat{U}^i(\hat{s}_*)\end{aligned}$$

Similar to the pure case, Nash equilibria are also fixed points in the case of mixed strategies.

Lemma 1.3.6
Given a game $\hat{\Gamma} = \left(I, \hat{S}, \hat{U}\right)$ and the best-response correspondence $\hat{r}$ the following holds:

$$\hat{s}_* \text{ is a Nash equilibrium} \Leftrightarrow \hat{s}_* \in \hat{r}(\hat{s}_*). \tag{1.20}$$

Proof. See the proof of theorem 1.2.4. □

1.3.3 Properties of mixed strategy games

Mixed strategy games have several key properties that will turn out to be useful when proving the existence of a Nash equilibrium solution.
A first such property is the convexity of the set of mixed strategies $\hat{S}$:

Proposition 1.3.7 (Convexity of $\hat{S}$)
The set of all mixed strategies $\hat{S}$ is convex[11].

Proof. Let $\hat{s}^i, \hat{t}^i \in \hat{S}^i$, $i \in I$. Consider now for $0 \leq \lambda \leq 1$:

$$\hat{q}^i = \lambda\hat{s}^i + (1-\lambda)\hat{t}^i.$$

[11] see [Sch04] pg. 30

Then:

$$\begin{aligned}\sum_{j=1}^{n_i} \hat{q}^i_j &= \sum_{j=1}^{n_i} \left(\lambda \hat{s}^i_j + (1-\lambda)\hat{t}^i_j\right) \\ &= \sum_{j=1}^{n_i} \lambda \hat{s}^i_j + \sum_{j=1}^{n_i}(1-\lambda)\hat{t}^i_j \\ &= \lambda \sum_{j=1}^{n_i} \hat{s}^i_j + (1-\lambda) \sum_{j=1}^{n_i} \hat{t}^i_j \\ &= \lambda + (1-\lambda) \\ &= 1.\end{aligned}$$

This yields $\hat{q}^i \in \hat{S}^i$ and therefore the convexity of $\hat{S}^i$ $\forall\, i \in I$.
Since $\hat{S} = \prod_{i \in I} \hat{S}^i$, also $\hat{S}$ is convex. □

Another important feature of $\hat{S}$ is its compactness:

Proposition 1.3.8 (Compactness of $\hat{S}$)
The set of all mixed strategies $\hat{S}$ ist compact[12].

Proof. To prove that $\hat{S}$ is compact, it is enough to show, that for all $i \in I$:

1. $\hat{S}^i$ is bounded.
2. $\hat{S}^i$ is closed.

The boundedness of $\hat{S}^i$ is trivial (simply by definition 1.3.1).
Consider now a sequence $\hat{s}^i_m \in \hat{S}^i$, that converges to a $\hat{s}^i_0$ for $m \to \infty$.
Then this $\hat{s}^i_0$ satisfies:

$$\begin{aligned}\sum_{j=1}^{n_i} \hat{s}^i_{0,j} &= \lim_{m\to\infty} \sum_{j=1}^{n_i} \hat{s}^i_{m,j} \\ &= \lim_{m\to\infty} 1 \\ &= 1.\end{aligned}$$

Therefore also $\hat{s}^i_0 \in \hat{S}^i$. This yields, that $\hat{S}^i$ is closed.
This shows that all $\hat{S}^i, i \in I$ are compact. For this reason also $\hat{S}$ is compact. □

[12] see [Sch04] pg. 30

Without proof the following remark on the payoff function U will be stated:

Remark 1.3.9
The functions $\hat{U}^i$ are multi-linear, continuous, real-valued functions on the set of all mixed strategy combinations $\prod_{i=1}^{m} \hat{S}^{\,i}$.[13]

An important property of games in mixed strategies follows directly from the definition of mixed strategies and of the payoff in mixed strategies:

Proposition 1.3.10
For every mixed strategy combination $\hat{s}_0$ every player i can choose at least one pure strategy s_k^i such that:[14]

1. $\hat{s}_{0,k}^i > 0$
2. $\hat{U}^i\left(s_k^i, \hat{s}_0^{-i}\right) \leq \hat{U}^i\left(\hat{s}_0\right)$

Proof. Suppose, there exists a player i such that for all k with $\hat{s}_{0,k}^i > 0$:

$$\hat{U}^i\left(s_k^i, \hat{s}_0^{-i}\right) > \hat{U}^i\left(\hat{s}_0\right)$$

Then for all k there holds exactly one of the following relations:

$$\hat{U}^i\left(s_k^i, \hat{s}_0^{-i}\right)\hat{s}_{0,k}^i > \hat{U}^i\left(\hat{s}_0\right)\hat{s}_{0,k}^i$$
$$\hat{U}^i\left(s_k^i, \hat{s}_0^{-i}\right)\hat{s}_{0,k}^i = \hat{U}^i\left(\hat{s}_0\right)\hat{s}_{0,k}^i = 0$$

Summing up over k yields

$$\sum_{j=1}^{n_i} \hat{U}^i\left(s_j^i, \hat{s}_0^{-i}\right)\hat{s}_{0,j}^i > \sum_{j=1}^{n_i} \hat{U}^i\left(\hat{s}_0\right)\hat{s}_{0,j}^i.$$

Together with the definition 1.16 this gives

$$\hat{U}^i\left(\hat{s}_0\right) > \hat{U}^i\left(\hat{s}_0\right).$$

This is a contradiction. Therefore the above assumption is wrong and the proposition is proven. □

[13] see [Sch04] pg. 30
[14] see [Sch04] pg. 37f.

1.4 Fixed point theorems of Brouwer and Kakutani

As was shown in the previous section, Nash equilibria and fixed points of a game are essentially the same. Hence, proving the existence of a Nash equilibrium in a game Γ is nothing else than proving the existence of a fixed point for the relevant best-response correspondence.

Two of the most famous theorems dealing with the existence of fixed points are Brouwer's and Kakutani's fixed point theorems.

The first one of these theorems, Brouwer's fixed point theorem, was developed in 1910[15] and used by John von Neumann in 1928 to prove the existence of a "minimax" solution in two-agent games.[16]

Brouwer's Fixed Point Theorem states the existence of a fixed point for a continuous function r under certain conditions on the domain S:

Theorem 1.4.1 (Brouwer's Fixed Point Theorem)
Let $S \subset \mathbf{R}^n$ convex, compact and non-empty and let $r : S \to S$ be a continuous function. Then there exists an $s_* \in S$ such that $r(s_*) = s_*$.[17]

Proof. The proof of this general form of Brouwer's fixed point theorem is rather long and will hence be omitted. See [Bor85], pg. 28. □

Closely related to Brouwer's fixed point theorem is Kakutani's fixed point theorem. The main difference is that Kakutani's fixed point theorem considers set valued functions, in particular upper semi-continuous multifunctions[18]:

Definition 1.4.2 (Upper Semi-Continuous Multifunction)
A set-valued function (or multifunction) $F : X \to Y$ is called upper semi-continuous at a point $x \in X$, if for each open set $V \in Y$ with $V \subseteq Y$ there is an open set $U \subset X$ containing x such that $F(U) \subseteq V$.

Kakutani's Fixed Point Theorem then states the following:[19]

[15] It is interesting to note, that Brouwer, one of the proponents of intuitonist philosophy, a mathematical approach that disagrees with the classical idea of proving the existence of an object by showing the impossibility of its non-existence, became most famous for this theorem, a theorem, which just proves the existence of a fixed point, but does not tell anything about how to construct this said fixed point.

[16] see [vN28], pg. 295-300 for the original proof

[17] the original version can be found in [Bro11], pg. 161 ff.

[18] for this definition see [Eki03], pg. 229

[19] see for the original version [Kak41], pg. 457-459

Theorem 1.4.3 (Kakutani's Fixed Point Theorem)
Let $S \subset \mathbf{R}^n$ be nonempty, compact and convex. Let $r : S \to 2^S$ be an upper semi-continuous multifunction that assigns to each $s \in S$ a nonempty, closed, convex subset $r(s)$ of S. Then there is a $s_* \in S$, such that $s_* \in r(s_*)$.

Proof. For sake of a concise presentation also this proof will be omitted. For a classical proof of this theorem consult [Bor85], pg. 72.

A very elegant proof of this theorem has just recently been suggested by McLennan and Tourky[20]. The reader may recall, that Kakutani's fixed point theorem was introduced in order to prove the existence of Nash equilibria under certain conditions. McLennan and Tourky however constructed games for which the existence of Nash equlibria follows directly out of the games' setup (using the Lemke-Howson algorithm[21]). Using the fact, that equilibrium points exist for this game and the fact, that equilibrium points are nothing else than fixed points for the so called best answer correspondence, McLennan and Tourky were able to conclude that Kakutani's fixed point theorem must hold. □

Kakutani's fixed point theorem is one of the most important theorems in mathematical economics: the noble prize winning works of John Nash, who proved the existence of Nash equilibria, and of Kenneth Arrow and Gerard Debreu, who proved, that the existence of prices that balance supply and demand in a complex economy, both strongly rely on the use of Kakutani's fixed point theorem.

A generalization of Kakutani's fixed point theorem to infinite dimensional normed spaces is due to Fan:[22]

Theorem 1.4.4 (Fan's Fixed Point Theorem)
Let $S \subset X$ be a nonempty, compact and convex subset of a normed linear space X. Let $r : S \to 2^S$ be an upper semi-continuous multifunction that assigns to each $s \in S$ a nonempty, closed, convex subset $r(s)$ of S. Then there is a $s_* \in S$, such that $s_* \in r(s_*)$.

1.5 Existence of a Nash equilibrium

Using the fixed point theorems outlined in the previous section, two important theorems about the existence of Nash equilibria can be proven: one, unfortunately

[20] see [MT05]
[21] see for this algorithm [LH64], pg. 413-423
[22] see [Fan60], pg. 265 ff.

quite restrictive, existence theorem for pure strategy games, and one surprisingly general theorem for games in mixed strategies.

The existence theorem for pure strategies will be valid for quasi-concave utility functions only:

Definition 1.5.1 (Strictly quasi-concave utility function)
Let the strategy set S^i of the player i be convex. A utility function $U^i(s^i, s^{-i})$ is said to be strictly quasi-concave[23]*, if for all s_1^i, $s_2^i \in S^i$, $0 \leq \lambda \leq 1$*

$$U\left(s_1^i, s^{-i}\right) > U\left(s_2^i, s^{-i}\right) \Rightarrow U^i\left(\lambda s_1^i + (1-\lambda)s_2^i, s^{-i}\right) \geq U\left(s_2^i, s^{-i}\right). \tag{1.21}$$

The first existence theorem for Nash equilibria can then be stated as follows:

Theorem 1.5.2 (Existence of a Nash equilibrium in pure strategies)
Let $\Gamma = (I, S, U)$ be a game with the following properties[24]*:*

1. *the strategy set $S^i \subseteq \mathbf{R}^{d_i}$ is compact and convex for all players $i \in I$.*
2. *the payoff function $U^i : S^i \times S^{-i} \to \mathbf{R}$ is continuous and bounded. For fixed $s^{-i} \in S^{-i}$ it is strictly quasi-concave in $s^i \in S^i$.*

Then the game Γ has at least one Nash equilibrium.

Proof. The proof of this existence theorem is based on Brouwer's fixed point theorem. Since Nash equilibria and fixed points of the best-response correspondence r are equivalent it is enough to show, that all assumptions of Brouwer's fixed point theorem are met by r and S.

Since by assumption the strategy sets $S^i \subset R^{d^i}$ are convex and compact, also $S = \prod_{i \in I} S^i$ is convex and compact.

Moreover, it is easy to see, that every player has at least one strategy in his strategy set. Otherwise, a game would be impossible. This means nothing else than that the sets S^i and therefore also S are non-empty.

Consider now the best-response correspondence $r(s)$. r would be defined on the whole strategy set S, if each player i could find a best answer $r^i(s^{-i})$ to each strategy combination s^{-i}. Consider for this purpose that for a fixed s^{-i}, $U^i\left(s^i, s^{-i}\right)$ is a real-valued, continuous and bounded function on the compact set S^i. Hence the

[23] see [Sch04] pg. 190
[24] see [Sch04] pg. 33ff.

Weierstrass extreme value theorem[25] implies that $U^i(s^i, s^{-i})$ takes its maximum for a strategy $s^i \in S^i$.

The reader might have noted that there is one major problem in applying Brouwer's fixed point theorem to the best-response correspondence r: Brouwer's fixed point theorem holds for single valued functions, but r is by definition vector-valued This problem is resolved by considering that the payoff functions U^i are strictly quasi-concave, continuous utility functions. This means nothing else than that for all s^{-i} there is a unique best response $s_0^i = r^i(s^{-i})$, that maximizes the payoff $U^i(s^i, s^{-i})$. Therefore for each player i:

$$r^i\left(s^{-i}\right) = s_0^i \in S^i$$

and for the whole game

$$\begin{aligned} r(s) &= \left(r^1(s^{-1}), \ldots, r^m(s^{-m})\right) \\ &= \left(s_0^1, \ldots, s_0^m\right) \in S. \end{aligned}$$

Therefore r is in fact a single valued mapping from S to S.

It remains to be shown, that r is continuous. Continuity and quasi-concavity of U^i imply that the best response to a strategy combination s^{-i} is just slightly changed by a continuous variation of s^{-i}. Considering the definition of $r^i(s^{-i})$, it is clear that also $r(s)$ has to be continuous.

This shows that all assumptions of Brouwer's fixed point theorem hold for the best answer correspondence r. Therefore there is a fixed point $s_* \in S$ with $r(s_*) = s_*$ and together with theorem 1.2.4 this is equivalent to the existence of a Nash equilibrium s_*. □

A more powerful existence theorem exists for mixed strategy games[26]

Theorem 1.5.3 (Existence of a Nash equilibrium in mixed strategies)
Every finite game in strategic form has a Nash equilibrium in mixed strategies.

Proof. The proof of this theorem will be an application of Kakutani's fixed point theorem. As in the pure strategy case, the claim can be proven by simply showing that all assumptions of Kakutani's theorem are met by $\hat{S}$ and $\hat{r}$.

[25] *Weierstrass extreme value theorem:* a continuous function from a compact space to a subset of real numbers attains its global maximum and minimum on that set.

[26] This theorem has been originally proven by John Nash. See: [Nas51], pg. 288

1. The definition of $\hat{S}^i$ (see definition 1.3.1) implies

$$\hat{S}^i \subset \mathbf{R}^{n_i}.$$

   It was already shown that the sets $\hat{S}^i$ are convex and compact[27]. It is also clear that a strategy set has to be non-empty.

2. $\hat{r}^i(\hat{s}^{-i})$ is clearly non-empty, since there is always a strategy that can be chosen by player i. Therefore also $\hat{r}(\hat{s})$ is non-empty.

3. Let $\hat{p}, \hat{q} \in \hat{r}(\hat{s})$ be best answers to a strategy combination $\hat{s}$. Then for all $i \in I$:

$$\hat{U}^i\left(\hat{p}^i, \hat{s}^{-i}\right) = \hat{U}^i\left(\hat{q}^i, \hat{s}^{-i}\right)$$

   and therefore

$$\hat{U}^i\left(\lambda\hat{p}^i + (1-\lambda)\hat{q}^i, \hat{s}^{-i}\right) \leq \hat{U}^i\left(\hat{p}^i, \hat{s}^{-i}\right),$$

   where $0 \leq \lambda \leq 1$.
   Multilinearity of $\hat{U}^i$ implies:

$$\begin{aligned}\hat{U}^i\left(\lambda\hat{p}^i + (1-\lambda)\hat{q}^i, \hat{s}^{-i}\right) &\geq \min\left\{\hat{U}^i\left(\hat{p}^i, \hat{s}^{-i}\right), \hat{U}^i\left(\hat{q}^i, \hat{s}^{-i}\right)\right\} \\ &= \hat{U}^i\left(\hat{p}^i, \hat{s}^{-i}\right) \\ &= \hat{U}^i\left(\hat{q}^i, \hat{s}^{-i}\right)\end{aligned}$$

   Therefore also $\hat{t}^i := \lambda\hat{p}^i + (1-\lambda)\hat{q}^i \in \hat{r}^i\left(\hat{s}^{-i}\right)$ and $\hat{t} = \left(\hat{t}^1, ..., \hat{t}^n\right) \in \hat{r}\left(\hat{s}\right)$. This means that $\hat{r}\left(\hat{s}\right)$ is convex.

4. Let $\hat{s}_m$ be a sequence of strategy combinations and $\hat{t}_m \in \hat{r}\left(\hat{s}_m\right)$, where for $m \to \infty$

$$\begin{aligned}\hat{s}_m &\to \hat{s}_*, \\ \hat{t}_m &\to \hat{t}_*.\end{aligned}$$

   Then for all $\hat{q}^i \in \hat{S}^i$:

$$\hat{U}^i\left(\hat{t}^i_m, \hat{s}^{-i}_m\right) \geq \hat{U}^i\left(\hat{q}^i, \hat{s}^{-i}_m\right). \tag{1.22}$$

[27] see propositions 1.3.7 and 1.3.8

Continuity of $\hat{U}^i$ implies for all $\hat{q}^i \in \hat{S}^i$:

$$\hat{U}^i\left(\hat{t}_*^i, \hat{s}_*^{-i}\right) \geq \hat{U}^i\left(\hat{q}^i, \hat{s}_*^{-i}\right).$$

This yields for the limit of $\hat{t}_m$:

$$\lim_{m\to\infty} \hat{t}_m \in \hat{r}\left(\hat{s}_*^{-i}\right).$$

Therefore $\hat{r}(\hat{s})$ is closed.

5. Since $\hat{U}$ is continuous, it is clearly upper semi-continuous.

This shows, that all assumptions of Kakutani's fixed point theorem hold. This implies the existence of a fixed point $\hat{s}_* \in \hat{r}(\hat{s}_*)$, or equivalently the existence of a Nash equilibrium at $\hat{s}_*$. □

Part II

Hedging pressure and the need for intermediaries

A first model of intermediated exchange will consider a market in which a commercial buyer/consumer, a commercial seller/producer and a non-commercial intermediary trade a certain commodity. While buyer and seller are entering the market in order to minimize the possible variation in their returns, the intermediary's involvement is based on his goal of earning profit by trading spreads and taking risk. These are key features of an intermediary's activitiy

This fundamental difference between the goals of buyer and seller on the one, and intermediary on the other side will be enough to explain the vital role intermediaries are playing in nowadays markets. A naive expectation on the model might suggest that seller and buyer are trading directly as much as possible, hence avoiding the costs that would arise by including the intermediary. Using a game-theoretic framework, it will however be shown that the intermediary will trade more than just the demand-supply surplus. This corresponds to observations made in daily life where usually almost the complete supply and demand is subject to intermediated exchange.

Chapter 2

Model I: A basic market model

In a two-period model of a market $\mathcal{M}$ a commercial consumer, or buyer B, a commercial producer, or seller S and a non-commercial intermediary I will trade a certain commodity/good G with delivery at time $t = T$. The market participants can either agree on direct trades at $t = 0$ or access the cash market C at time $t = T$.

The assumptions made on the players and the market structure will be described in the following.

2.1 Basic assumptions on the market

1. Accessing the cash market C as well as trading directly with any of the players will be free of transaction costs.

2. There will be no cost of carry and no convenience yield.

3. Trading will take place on a continuous price scale. This means that there is no minimal price step and hence that there is a price p_3 for every two prices $p_1 < p_2$ such that $p_1 < p_3 < p_2$.

4. There will be no bid/ask spread on the cash market.

5. No participant in the cash market C can affect the cash market price $P(t)$ by his trading. Therefore $P(t)$ can be seen as an exogenous parameter: it is a random variable[1].

[1] It might for example be assumed to be lognormal, with μ and σ being the expected mean, and volatility of the corresponding log-return of the prices $P(t)$ respectively.

6. All players know that $P(t)$ is a random variable. Furthermore the expected value of $P(t)$ at time $t = T$,

$$\Lambda = E\left[P(T)\right]$$

is known to all players.

7. The riskless rate of return is known to be r.

8. Supply, or production[2] α of player S and demand, or consumption ν of player B will be deterministic and known to all players at time $t = 0$.

Note that no restrictions on the nature of G have been made: the reader may think of G as being soy beans, stocks, human work force or any other tradeable product.

Comment 2.1.1
Supply α and demand ν are assumed to be deterministic, while the cash market price is assumed to be random. These assumptions might contradict themselves at first glance. However, the considered model depicts only a small part of the complete market action: note that for example none of the three players is able to affect the cash market price by his own trading. Other sellers, buyers and intermediaries act in the market and only a complete knowledge about their combined action will explain the behaviour of the cash market price.

2.2 The roles of the players

The players, seller S, buyer B and intermediary I, enter the market $\mathcal{M}$ playing the roles of a commercial seller and buyer, and a non-commercial intermediary respectively. In more detail, the roles of the players can be described as follows:

- At time $t = 0$
  - B is aware that at time $t = T$ he will need to buy quantity ν of the commodity G.
  - S is aware that at time $t = T$ he will need to sell quantity α of the same commodity G.

[2] Note that the choice of variables for supply and demand relates to the corresponding German expressions "Angebot" (α) and "Nachfrage" (ν)

- The only motivation for both S and B to trade at $t = 0$ is their goal to minimize the possible variation in their returns. It may be described as hedging pressure.
- I enters $\mathcal{M}$ as a risk-neutral intermediary and spread-trader with the goal of gaining an expected profit higher than the risk-free return.

2.3 The possible trades

The supply or demand needs of S and B respectively can be fulfilled in the following ways:

- at time $t = 0$
  - B and S can agree on a direct trade with delivery at $t = T$.
  - B and S can approach a non-commercial intermediary I who will in case of a mutual agreement guarantee that G will be delivered to B or that S's delivery will be taken at $t = T$.
- at time $t = T$, both B and S have access to the cash market C to fulfil their buying or selling needs.
- The non-commercial intermediary I himself will be allowed to enter the cash market C as a drop-off place to transfer his delivery or purchase obligations at $t = T$ to participants of C.

2.4 The goals of the players

Clearly, there are two types of goals that have to be quantified: the goal of I to make profit and the goals of S and B to minimize the variation of their individual returns.

2.4.1 The goals of S and B

By assumption, both S and B, although on opposite sides of the trade, have the same goal, namely minimizing their operatory risk: since the cash market price $P(T)$ is not known in advance, both S and B face some risk concerning their future payment streams. In order to avoid this risk, S and B are willing to pay a certain risk premium at $t = 0$ in exchange for getting an acceptable price for selling or buying G at $t = T$.

Notice that all the players, buyer B, seller S, and intermediary I know Λ, the expected value of $P(t)$ at time $t = T$. In other words, this means that all players have identical information about the marketplace, and hence identical expectations about future prospects.

To be able to define their "acceptable price" at time $t = 0$, both S and B at first need to formulate a goal of what they want to have reached at time $t = T$. One way in which this can be done is the following:

1. S quantifies a minimum price for G that he wants to realize **for every single unit of** G he sells. A bit more formalized, this means the following:

   at time $t = T$, S wants to have sold all α units of G such that the minimal price $p_{\min}$ he realized satisfies

   $$p^S_{\min} \geq (1 - R_S)\Lambda,$$

   where $R_S \in [0, 1)$ denotes the risk premium S is willing to pay for eliminating the price risk.

2. Similarly B quantifies a maximum price for G signalling that he is not willing to pay more than some maximum price **for any single unit of** G he buys. This can be formalized a follows:

   at time $t = T$, B wants to have bought all needed ν units of G such that the maximal price $p_{\max}$ satisfies

   $$p^B_{\max} \leq (1 + R_B)\Lambda,$$

   where $R_B \in [0, \infty)$.

Notice that R_B and R_S may have different interpretations. They may depend on risk aversion of the players, explicitly depend on the price volatility, or incorporate various and different utility functions of the players B and S.

It is now necessary to relate the goals of B and S at $t = T$ with what they need to do at $t = 0$. This can be done by usual discounting arguments; for the sake of clarity of presentation, we omit standard reasoning. The result is that the maximal price for which B can enter into a forward agreement for a unit of G at $t = 0$ is

$$\hat{p}^B_{\max} = (1 + R_B)\Lambda e^{-rT},$$

and that the minimal price for which S can enter into a forward agreement for a unit of G at $t = 0$ is

$$\hat{p}^S_{\min} = (1 - R_S)\Lambda e^{-rT}.$$

This leads to the concept of an acceptable forward offer at time $t = 0$:

Definition 2.4.1 (Acceptable offer)
Under the assumption that goals are formulated as described above

1. *a price $\hat{p}^S$ offered to S at time $t = 0$ is acceptable for S if*

$$\hat{p}^S \geq \hat{p}^S_{\min} = (1 - R_S)\Lambda e^{-rT}, \tag{2.1}$$

2. *Similarly a price $\hat{p}^B$ offered to B at time $t = 0$ is acceptable for B if*

$$\hat{p}^B \leq \hat{p}^B_{\max} = (1 + R_B)\Lambda e^{-rT}. \tag{2.2}$$

At time $t = 0$, both S and B will only agree to offers that are acceptable in the sense defined above. If they are not able to fulfil their supply and demand needs at $t = 0$ then they will buy or sell the remaining units on the cash market C at time $t = T$.

2.4.2 The goal of the non-commercial intermediary I

I is a player whose goal is to make profit by trading spreads, or by taking the risk that B and S face concerning price uncertainty. For simplicity, it is assumed that B and S are fully risk-averse, while I is risk-neutral.

Suppose that at time $t = 0$, I is willing to buy m_S units from S for a price P^S and to sell m_B units of G to B for a price of P^B per unit. Since m_B may not be equal to m_S, the size of the immediate round trade is

$$m_{\min} = \min\{m_B, m_S\}.$$

The immediate profit of I, clear of all obligations, at $t = 0$ equals

$$m_{\min}\left(P^B - P^S\right).$$

Furthermore, at time $t = T$ the intermediary I has to sell the $(m_S - m_{\min})$ or buy the $(m_B - m_{\min})$ remaining units of G on the cash market for a price $P(T)$. Since $E[P(T)] = \Lambda$, I's expected profit/loss $E[U^I]$ from this transaction is

$$(m_S - m_{\min})\left(\Lambda - P^S\right) + (m_B - m_{\min})\left(P^B - \Lambda\right).$$

Note that this profit/loss will be realized at time $t = T$. On the other hand, at time $t = 0$, I's capital changes by

$$m_B P^B - m_S P^S.$$

The riskless interest on this capital change that I can receive during the period from $t = 0$ to $t = T$ is

$$\left(m_B P^B - m_S P^S\right)\left(e^{rT} - 1\right).$$

The overall change of capital for I consists of three parts: immediate profit/loss without further obligations at $t = 0$, the expected profit/loss at $t = T$ and the change in interest payments during the period from $t = 0$ to $t = T$. This means that by entering the market as described, I's capital at time $t = T$ changes by

$$\begin{aligned}
&\underbrace{m_{\min}\left(P^B - P^S\right)}_{\text{immediate profit}} + \underbrace{(m_S - m_{\min})\left(\Lambda - P^S\right) + (m_B - m_{\min})\left(P^B - \Lambda\right)}_{\text{expected profit at } t = T} \\
&+ \underbrace{\left(m_B P^B - m_S P^S\right)\left(e^{rT} - 1\right)}_{\text{change in interest payments}} \\
= &\left(m_B P^B - m_S P^S\right) e^{rT} + (m_S - m_B)\Lambda.
\end{aligned}$$

In other words, the total profit/loss of the trade is equal to the grown initial value of the trade less the cash value of the merchandise dropped on the free market. Hence, I is willing to enter the trade if

$$\left(m_B P^B - m_S P^S\right) e^{rT} + (m_S - m_B)\Lambda > 0.$$

Using a standard discounting argument this goal can be transferred to time $t = 0$. With $\hat{\Lambda} = \Lambda e^{-rT}$ it follows that I will have satisfied his goal if at time $t = 0$

$$\left(m_B P^B - m_S P^S\right) + (m_S - m_B)\hat{\Lambda} > 0.$$

In the special case of $m_S = m_B$ this implies that if only the spread $P^B - P^S$ is positive I is guaranteed a positive immediate payoff without carrying any inventory.

This shows that I is in principle willing to trade G at any price as long as the combination of trading spreads and taking risks promises to generate profit. Hence I might even be willing to trade a negative spread if just the overall revenue promises to be positive.

2.4.3 The quantified goals: rule or strategy?

There are two ways to interpret the goals of the players B and S in the context of a trade.

1. On the one hand these goals can be interpreted as a rule in the game. This means that the players have no influence in actually quantifying their goals and therefore R_S and R_B have to be understood as exogenous parameters.

2. On the other hand these goals can be seen as part of the players' strategies, i.e. when deciding over which strategy to play, a player also has to decide about the relevant parameter in the formalization of his goal.

Both interpretations have their justification. The second alternative for example may describe a situation, in which players B and S constantly revise their goals, and have the freedom to decide on their corresponding risk premia R_B and R_S.

On the other hand, S might for instance represent the sales department of a company producing G. If then the management of the company decides to set a price range for G, the sales department represented by S has to deal with this price range without having had any influence on its formulation and without having the chance of independently revising this price range at a later time.

While this shows that theoretically R_S and R_B can be either a rule or a strategy, the practical interpretation of both possibilities should be rather similar. Consider for this that risk premia are based on mainly two factors:

- the risk that S and B face in a certain situation

- and the risk aversity of the respective player in general.

Hence, a reevaluation of R_S or R_B should correspond to either a substantial change of the risk scenario the respective player is facing or to a change of the player's character. However, on a short time horizon, both factors can be assumed to be rather constant. Since neither the objective riskiness of a trade nor the subjective risk aversity of a player can be changed by a strategic decision, it follows that even when B and S are allowed to decide over R_S and R_B as part of their strategy, their decision will not be strategic. It will simply be a completely predictable response on how much risk they are facing and on how risk-averse they are. Hence, even if S and B are allowed to set R_S and R_B as part of their strategic decision, R_S and R_B can be assumed to be constant external parameters.

2.5 The matchmaking process in a trade

The matchmaking process in a trade will depend on two main assumptions concerning the role of the intermediary I:

- I will make binding price offers $P^{I\to S}$ and $P^{I\to B}$ to S and B. Note that the arrow in the notation shows the direction of the **offer**, not the direction of goods transfer.
- I is willing to accept practically any price, as long as his overall revenue turns out to be positive. This willingness will be reflected in the chance for S and B to enter price negotiations with I with the goal of realizing a price that from their point of view is better than the binding offer made by I.

These two key assumptions can be portrayed in the following matchmaking process[3]:

1. The intermediary I makes a binding offer of an ask price $P^{I\to S}$ and a bid price $P^{I\to B}$ to S and B, respectively.
2. S and B enter negotiations with I by making binding price offers $P^{S\to I}$ and $P^{B\to I}$.
3. I decides whether or not he will accept $P^{S\to I}$ and $P^{B\to I}$. If he accepts a price, all α or ν units respectively will be traded at that price.
4. (a) If *both* S and B have not fulfilled their supply or demand needs yet, they will trade the maximal amount $\min\{\alpha, \nu\}$ directly for a price $P^{S,B} = \Lambda e^{-rT}$.
   (b) If *either* S or B still has open supply or demand needs (because of $\alpha \neq \nu$), the respective player will trade the remaining units with I for

[3] for a similarly structured model see [Spu99] pg. XV

the posted price $P^{I \to S}$ or $P^{I \to B}$ respectively, if this price is acceptable. Otherwise this player will take the risk himself and realize his supply or demand needs at time $t = T$ on the cash market.

It is reasonable to assume, that, if trading G directly, B and S agree on a price of Λe^{-rT}, since both S and B are at the time of the direct negotiation in the same situation: they both can exchange at most N units of G directly and they both are aware that if they cannot agree on a direct trade, they would have to trade all their units with I for $(1 - R_S)\hat{\Lambda}$ or $(1 + R_B)\hat{\Lambda}$ respectively, where

$$\hat{\Lambda} = \Lambda e^{-rT}.$$

2.6 The strategies of the players

The possible strategies of the players follow directly out of the matchmaking process described above.

Consider first that the steps (4a) and (4b) of the matchmaking process were just a consequence of what had happened in the previous steps. Therefore strategic behaviour can only occur in the first three steps of the matchmaking process.

Furthermore, although setting prices $P^{I \to S}$ and $P^{I \to B}$ appears to be a part of I's strategy, in a market with perfect information it is not: I knows that S will only accept a price better than or equal to $(1 - R_S)\hat{\Lambda}$ and that B will only accept a price better than or equal to $(1 + R_B)\hat{\Lambda}$. So clearly $(1 - R_S)\hat{\Lambda}$ and $(1 + R_B)\hat{\Lambda}$ are the best prices I can realize and hence there is no reason to make offers $P^{I \to S} < (1 - R_S)\hat{\Lambda}$ or $P^{I \to B} > (1 + R_B)\hat{\Lambda}$.

On the other hand, I wants to capture a maximal profit, and, if possible, I will not improve his offers to B and S. Hence the only possible binding price offers of I are $P^{I \to S} = (1 - R_S)\hat{\Lambda}$ and $P^{I \to B} = (1 + R_B)\hat{\Lambda}$.

This argument shows that the above model sufficiently describes the concept of hedging pressure:

Intermediary I is under no pressure to enter a trade, since if he does not, all that can happen to him is that he would make no profit. Contrary to that, S and B face the risk of price uncertainty at $t = T$. Hence they have a strong incentive of hedging their positions at time $t = 0$. This results in the fact that they might be forced to accept prices as bad as $(1 - R_S)\hat{\Lambda}$ and $(1 + R_B)\hat{\Lambda}$.

The intermediary I can use this advantage to an extent where one of the players has to trade his whole amount with I for the binding price offer made by I. If I for example wants to buy all α units for $P^{I\to S}$, he only needs to accept B's and decline S's offer in step (3) of the matchmaking process. B would then have already satisfied all his hedging needs with I, and hence only two trading scenarios are left for S: he can either go to the cash market and trade there for an uncertain price or accept I's binding offer. Since S is completely risk-averse, he will accept I's offer as long as it is acceptable for him. But since I will offer an acceptable price, as shown earlier, this implies that S will trade all α units of G for a price of $P^{I\to S}$ per unit, which is exactly what I wanted to achieve.

The above reasoning shows that the only strategic decisions take place in step (2) and (3) of the matchmaking process. Hence the strategies of each player can be defined as follows:

- The strategy of S is well defined by his price offer $P^{S\to I}$.
- The strategy of B is well defined by his price offer $P^{B\to I}$.
- The strategy of I is defined as the set of his responses to $P^{S\to I}$ and $P^{B\to I}$. Each response will either be a price acceptance or rejection.The following scenarios are possible:
  - s^I_B: I accepts B's offer and rejects S's offer. I will therefore sell ν units to B for a price of $P^{B\to I}$ and buy α units from S for a price of $P^{I\to S}$ per unit.
  - s^I_S: I accepts S's offer and rejects B's offer. In this case I will sell ν units to B for a price of $P^{I\to B}$ and buy α units from S for a price of $P^{S\to I}$ per unit.
  - s^I_-: I rejects both offers. S and B will trade $m = \min\{\alpha, \nu\}$ units directly. The remaining $\alpha - m$ or $\nu - m$ units will be bought or sold by I for $P^{I\to S}$ or $P^{I\to B}$ respectively.
  - Theoretically, I could also accept both offers. However, this would only make sense if S and B would offer prices to I that are from his point of view better than his own standing offers $P^{I\to S}$ and $P^{I\to B}$. But clearly, S and B have no motivation to offer such prices. Hence this scenario can be neglected.

Notice that according to the matchmaking process, I decides his strategy after the strategies of S and B are known. Hence I will always be able to formulate the best answer to their strategies. In a "generic case" scenario, where no two options for

I result in the same payoff, I's decision will therefore be completely predictable as soon as S and B have submitted their price offer.

Hence, the game will effectively reduce to a two-person game, where I's decisions are reflected in the respective payoff functions only. For the theoretical analysis this implies that a strategy combination will be a Nash equilibrium solution if the strategies of S and B are best responses to each other. I's strategy will by its very nature be a best response.

2.7 The payoff functions

To construct the payoff functions of the different players consider that - depending on I's decision - in principle three outcome scenarios are possible:

- $s_B = \left(P^{S\to I}, P^{B\to I}, s^I_B\right)$:
  I accepts B's and rejects S's offer. It follows that S will sell his complete supply α for a price of $P^{I\to S}$ to I and that B will buy all ν units he demands for a price of $P^{B\to I}$ from I.
- $s_S = \left(P^{S\to I}, P^{B\to I}, s^I_S\right)$:
  I accepts S's and rejects B's offer. S will then sell α units for a price of $P^{S\to I}$ to I and B will buy ν units for a price of $P^{I\to S}$ from I.
- $s_{DT} = \left(P^{S\to I}, P^{B\to I}, s^I_-\right)$:
  I rejects both offers. It follows that S and B will trade $m = \min\{\alpha, \nu\}$ directly for a price of $\hat{\Lambda} = \Lambda e^{-rT}$ per unit. I will buy/sell the remaining $(\alpha - m)$ or $(\nu - m)$ units for $P^{I\to S}$ or $P^{I\to B}$ respectively.

Notice now that the intermediary is a risk-neutral market participant. His payoff function will therefore not need to take into consideration how much risk I takes on by buying or selling G. It is simply enough to consider I's expected monetary profit at time $t = T$ or equivalently the expected monetary profit at time $t = T$ discounted to time $t = 0$. Suppose a trade results in I buying m_S units from S for P^S and selling m_B units to B for P^B. Then I's monetary, expected payoff (discounted to $t = 0$) is

$$\left(m_B P^B - m_S P^S\right) + (m_S - m_B)\hat{\Lambda}$$

and I's discounted payoff is

$$\begin{aligned} E\left[\hat{U}^I(s_B)\right] &= \nu P^{B\to I} - \alpha P^{I\to S} + (\alpha-\nu)\hat{\Lambda} \\ &= \nu\left(P^{B\to I} - \hat{\Lambda}\right) + \alpha\left(\hat{\Lambda} - P^{I\to S}\right) \end{aligned}$$

$$\begin{aligned} E\left[\hat{U}^I(s_S)\right] &= \nu P^{I\to B} - \alpha P^{S\to I} + (\alpha-\nu)\hat{\Lambda} \\ &= \nu\left(P^{I\to B} - \hat{\Lambda}\right) + \alpha\left(\hat{\Lambda} - P^{S\to I}\right) \end{aligned}$$

$$\begin{aligned} E\left[\hat{U}^I(s_{DT})\right] &= (\nu-m)\,P^{I\to B} - (\alpha-m)\,P^{S\to I} + (\alpha-\nu)\hat{\Lambda} \\ &= (\nu-m)\left(P^{I\to B} - \hat{\Lambda}\right) + (\alpha-m)\left(\hat{\Lambda} - P^{S\to I}\right), \end{aligned}$$

where $m = \min\{\alpha, \nu\}$.

The main goal of S and B is to avoid risk. Both players B and S will transfer all their risk completely at time $t = 0$ no matter which strategy they play: no matter what happened in any of the negotiation steps they will fill all their supply or demand needs for $(1 - R_S)\Lambda e^{-rT}$ and $(1 + R_B)\Lambda e^{-rT}$ respectively in the last step of the matchmaking process.
Therefore it is also in the case of S and B enough to consider only their monetary reward as their payoff function.

Note that if S sells a unit of G for a certain price at time $t = 0$ he will give up a commodity with fair value $\hat{\Lambda}$. He will therefore have generated a revenue equal to the difference between the price he realized and $\hat{\Lambda}$. It follows that S's payoff functions for the relevant outcome scenarios are:

$$\begin{aligned} E\left[\hat{U}^S(s_B)\right] &= \alpha\left(P^{I\to S} - \hat{\Lambda}\right) \\ E\left[\hat{U}^S(s_S)\right] &= \alpha\left(P^{S\to I} - \hat{\Lambda}\right) \\ E\left[\hat{U}^S(s_{DT})\right] &= (\alpha-m)\left(P^{I\to S} - \hat{\Lambda}\right). \end{aligned}$$

Similarly if B buys G, he receives a commodity worth $\hat{\Lambda}$ in return for his monetary expense. His payoff functions for each of the possible outcome scenarios are:

$$\begin{aligned} E\left[\hat{U}^B(s_B)\right] &= \nu\left(\hat{\Lambda} - P^{B\to I}\right) \\ E\left[\hat{U}^B(s_S)\right] &= \nu\left(\hat{\Lambda} - P^{I\to B}\right) \\ E\left[\hat{U}^B(s_{DT})\right] &= (\nu-m)\left(\hat{\Lambda} - P^{I\to B}\right). \end{aligned}$$

Comment 2.7.1
The intermediary I is assumed to be a risk-neutral market participant. Since his payoff function is linearly dependent on α and ν he will always trade the maximal possible amount.

This corresponds with observations made in actual markets: most commodity futures exchanges have established position limits for non-commercial entities such as market makers. Large well-capitalized players, e.g. many hedge funds, often take maximally permissible positions in "hot" commodities[4]. When exchanges alter the position limits, these large players who are often fully leveraged, have to rebalance their investment. This causes price shifts and an extensive capital movement.

[4] see for example Fung and Hsieh [FH99]

Chapter 3

Non-existence of a Nash equilibrium solution in Model I

After describing the setup of Model I, it can now be analyzed what results can be expected given that all players act in a rational fashion.

3.1 Impossibility of an equilibrium solution involving direct trade

A naive observer of this market might expect, that S and B would try to minimize the influence of the intermediary I by trading directly, hence trade only the remaining supply-demand surplus with I. However it will be shown that such a behaviour is not a Nash equilibrium solution.

Theorem 3.1.1 (Model I: Impossibility of a direct trade)
Consider a market $\mathcal{M}$ with $\alpha, \nu, R_S, R_B > 0$, i.e. consider a market with nonzero supply in demand in which both S and B are willing to pay for a risk transfer.
Then the intuitive expectation that

1. *S and B trade the maximal possible amount* $\min\{\alpha, \nu\}$ *units directly for a fair price $P^{S,B} = \hat{\Lambda}$ per unit.*

2. *S sells the production surplus* $\max\{\alpha - \nu, 0\}$ *units to I for I's standing offer $P^{I \to S} = (1 - R_S)\hat{\Lambda}$ per unit.*

3. *B buys the consumption deficit* $\max\{\nu - \alpha, 0\}$ *units from I for $P^{I \to B} = (1 + R_B)\hat{\Lambda}$ per unit.*

is not a Nash equilibrium.

Proof. Suppose $\left(P_0^{S\to I}, P_0^{B\to I}\right)$ is a combination of price offers that results in the above naive solution. In other words: $P_0^{S\to I}$ and $P_0^{B\to I}$ are set such that I_R's best answer is the strategy s_-^I, I's only strategy that allows for such a direct trade scenario.

Assume first that the supply exceeds the demand, i.e. that $\alpha > \nu$. Realization of this naive solution

$$s_{0,DT} = \left(P_0^{S\to I}, P_0^{B\to I}, s_-^I\right)$$

would then mean the following:

- S and B trade ν units directly for a price of $\hat{\Lambda}$ per unit.
- I buys the remaining $(\alpha - \nu)$ units from S for $(1 - R_S)\hat{\Lambda}$ per unit.

Hence I's expected payoff would equal

$$\begin{aligned} E\left[\hat{U}^I\left(s_{0,DT}\right)\right] &= (\alpha - \nu)\left(\hat{\Lambda} - (1 - R_S)\hat{\Lambda}\right) \\ &= (\alpha - \nu)R_S\hat{\Lambda}. \end{aligned}$$

The fact that I agreed to such a solution means that he denied both price offers $P_0^{S\to I}$ and $P_0^{B\to I}$ in step 3 of the matchmaking process. He would deny the mentioned offers if his other strategies

- s_S^I: I buys α units from S for $P_0^{S\to I}$ per unit and sells ν units to B for $(1+R_B)\hat{\Lambda}$ per unit
- s_B^I: I sells ν units to B for $P_0^{B\to I}$ per unit and buys α units from S for $(1-R_S)\hat{\Lambda}$ per unit

and, hence, the corresponding strategy combinations

$$s_{0,S} = \left(P_0^{S\to I}, P_0^{B\to I}, s_S^I\right)$$

$$s_{0,B} = \left(P_0^{S\to I}, P_0^{B\to I}, s_B^I\right)$$

would not result in a higher expected payoff than

$$(\alpha - \nu)R_S\hat{\Lambda}.$$

In other words, this implies simultaneously

$$E\left[\hat{U}^I(s_{0,S})\right] \leq E\left[\hat{U}^I(s_{0,DT})\right],$$

$$E\left[\hat{U}^I(s_{0,B})\right] \leq E\left[\hat{U}^I(s_{0,DT})\right].$$

Consider the first equation. Then the expected payoff is

$$\begin{aligned} E\left[\hat{U}^I(s_{0,S})\right] &= \nu\left((1+R_B)\hat{\Lambda} - P_0^{S\to I}\right) + (\alpha-\nu)\left(\hat{\Lambda} - P_0^{S\to I}\right) \\ &= \nu R_B\hat{\Lambda} + \alpha\left(\hat{\Lambda} - P_0^{S\to I}\right). \end{aligned}$$

If this option is declined by I then

$$\nu R_B\hat{\Lambda} + \alpha\left(\hat{\Lambda} - P_0^{S\to I}\right) \leq (\alpha-\nu)R_S\hat{\Lambda}$$

and solving for $P_0^{S\to I}$ yields

$$P_0^{S\to I} \geq \hat{\Lambda}\left((1-R_S) + \frac{\nu}{\alpha}(R_B+R_S)\right).$$

Suppose now that $s_{0,DT}$ is a Nash equilibrium. This implies that $P_0^{S\to I}$ and $P_0^{B\to I}$ are best responses to each other and hence S would not be able to generate a higher payoff by singlehandedly changing his strategy.
Suppose now that S is trying to single-handedly improve his payoff. Consider first that since $P_0^{B\to I}$ is fixed, S can at least single-handedly change I's decision. Clearly if I stays at his current choice of declining both offers or switches to the option s_B^I where he accepts B's offer but declines S's offer, the payoff of S cannot increase.
Hence S needs to choose his strategy so that I will change his decision to accept S's offer, i.e. to the strategy s_S^I. But this means nothing else than that S's new strategy $P_1^{S\to I}$ satisfies

$$P_1^{S\to I} < \hat{\Lambda}\left((1-R_S) + \frac{\nu}{\alpha}(R_B+R_S)\right).$$

Playing $P_1^{S\to I}$ only makes sense if S can also improve his own payoff. Consider that S's previous payoff under the strategy combination $s_{0,DT}$ is given as

$$E\left[\hat{U}^S(s_{0,DT})\right] = -(\alpha-\nu)R_S\hat{\Lambda}.$$

On the other hand the new strategy combination $s_{1,S} = \left(P_1^{S\to I}, P_0^{B\to I}, s_S^I\right)$ results in a payoff of

$$E\left[\hat{U}^S\left(s_{1,S}\right)\right] = \alpha\left(P_1^{S\to I} - \hat{\Lambda}\right).$$

Since S chooses $P_1^{S\to I}$ only if $E\left[\hat{U}^S\left(s_{1,S}\right)\right] > E\left[\hat{U}^S\left(s_{0,DT}\right)\right]$ this yields

$$\alpha\left(P_1^{S\to I} - \hat{\Lambda}\right) > -(\alpha - \nu)R_S\hat{\Lambda}$$

and therefore

$$P_1^{S\to I} > \hat{\Lambda}\left((1 - R_S) + \frac{\nu}{\alpha}R_S\right).$$

Hence $P_1^{S\to I}$ satisfies

$$\hat{\Lambda}\left((1 - R_S) + \frac{\nu}{\alpha}R_S\right) < P_1^{S\to I} < \hat{\Lambda}\left((1 - R_S) + \frac{\nu}{\alpha}(R_B + R_S)\right).$$

A sufficient condition for the existence of such a price $P_1^{S\to I}$ is that

$$\hat{\Lambda}\left((1 - R_S) + \frac{\nu}{\alpha}R_S\right) < \hat{\Lambda}\left((1 - R_S) + \frac{\nu}{\alpha}(R_B + R_S)\right).$$

Therefore the existence of such a price is guaranteed if the condition

$$\frac{\nu}{\alpha}R_B > 0 \tag{3.1}$$

holds. But this is trivially true as long as $R_B > 0$, i.e. as long as B is willing to pay a price for transferring risk.

This argument shows that S can single-handedly change his strategy so that he can guarantee himself a higher payoff. Hence the strategy combination $s_{0,DT}$ cannot be a Nash equilibrium.

Similarly it can be shown that if $\nu > \alpha$ the naive solution with S and B trading as much as possible directly and B selling the remaining units to I cannot be a Nash equilibrium as long as

$$\frac{\alpha}{\nu}R_S > 0. \tag{3.2}$$

and hence as long as $R_S > 0$.

Even in the case of $\alpha = \nu$ the direct trade of all α units of G between S and B will not be a Nash equilibrium: the seller S for instance could singlehandedly improve his payoff by offering I a price $P^{S\to I}$ that satisfies

$$\hat{\Lambda} < P^{S\to I} < (1 + R_B)\hat{\Lambda}.$$

Although this price would be more than the fair discounted value of G, the intermediary I would - given that B's strategy remains unchanged - accept this offer and sell all units for a price of $(1 + R_B)\hat{\Lambda}$ per unit to B. This would result both in a higher payoff for S and I and hence S would have increased his own payoff by single-handedly changing his strategy. Hence the direct exchange cannot be a Nash equilibrium. □

Comment 3.1.2
The existence of an intermediary I results in the fact that a direct trade between S and B will not be an equilibrium solution. The reason for this is that the chance of receiving a higher payoff by trading with I will incline both players to diverge from that strategy. This means that the intermediary will trade more than just the minimal volume $|\alpha - \nu|$. This agrees with observations made in daily life: wherever intermediaries occur, they are not just there for trading the supply-demand surplus, but usually for trading almost the complete supply and demand [1]. Hence the observed phenomenon explains why intermediaries play a vital part in almost all economic activity.

Comment 3.1.3
The described dynamics show that intermediaries often act as a conduits of economic change. Nowadays, "fundamentals" are very often interpreted through the perception of hedge fund managers, who act in the direction of the perceived change of "fundamentals". In this respect, intermediaries are critically important for economic and financial functioning. It is well-known that a successful introduction of new financial products critically depends on attracting the attention of financial intermediaries.

3.2 Non-existence of a Nash equilibrium

The key question that needs to be answered now, is whether or not a Nash equilibrium can be identified at all in this market. In fact, it can be shown, that one such Nash equilibrium exists. However this equilibrium will turn out to be a paradox

[1] see for example Allen and Santomero [AS98], pg. 1470

and a minor change in the rules of the game will prevent this equilibrium from being a legal strategy combination:

Theorem 3.2.1 (Model I: Non-existence of Nash equilibria in the market)
Consider a market $\mathcal{M}$ where $\alpha, \nu, R_S, R_B > 0$.
Then the only existing Nash equilibrium is a strategy combination where $P^{S\to I} = (1 - R_S)\hat{\Lambda}$ and $P^{B\to I} = (1 + R_B)\hat{\Lambda}$.

If the prices offered by S and B to I satisfy

$$P^{S\to I} > (1 - R_S)\hat{\Lambda}$$

and

$$P^{B\to I} > (1 + R_B)\hat{\Lambda}$$

(since S and B want to improve their position by bargaining and the prices $(1 - R_S)\hat{\Lambda}$ and $(1 + R_B)\hat{\Lambda}$ are already standing offers from I), no Nash equilibrium can exist.

Proof. Consider first a combination of offers $\left(P_0^{S\to I}, P_0^{B\to I}\right)$, where $P_0^{S\to I} = (1 - R_S)\hat{\Lambda}$ and $P_0^{B\to I} = (1 + R_B)\hat{\Lambda}$. Then I clearly will accept both offers (strictly speaking he is indifferent between s_S^I and s_B^I, but both strategies will result in the same outcome scenario) and hence the payoffs of S and B will turn out to be

$$E\left[\hat{U}^S\left(s_0, s^I\right)\right] = -\alpha R_S \hat{\Lambda}$$
$$E\left[\hat{U}^B\left(s_0, s^I\right)\right] = -\nu R_B \hat{\Lambda}.$$

Suppose now that S changes his strategy to $P_1^{S\to I}$ while B remains at $P_0^{B\to I}$. Then I will accept B's offer and therefore force S to sell for $(1 - R_S)\hat{\Lambda}$. This means that the change in S's strategy did not influence the outcome of the game.
The same can be shown for a change in B's strategy. But since no player can single-handedly improve his strategy it follows that s_0 actually is a Nash equilibrium.

Suppose now that $P_0^{S\to I} > (1 - R_S)\hat{\Lambda}$ and $P_0^{B\to I} < (1 + R_B)\hat{\Lambda}$. Suppose furthermore that supply exceeds demand, $\alpha > \nu$, and suppose a combination of offers $\left(P_0^{S\to I}, P_0^{B\to I}\right)$ results in the following trades:

- I sells ν units to B for a price of $P_0^{B\to I}$ per unit.

- I buys α units from S for a price of $(1 - R_S)\hat{\Lambda}$ per unit.

In other words: suppose that I's best answer to the strategy combination s_0 is the strategy s^I_B, i.e. suppose that the strategy combination

$$s_{0,B} = \left(P_0^{S\to I}, P_0^{B\to I}, s^I_B\right)$$

is played.

Assume now that $s_{0,B}$ is a Nash equilibrium. Then no player would be able to single-handedly improve his payoff. In particular this would imply that there is no price $P_1^{S\to I}$ such that a combination of offers $\left(P_1^{S\to I}, P_0^{B\to I}\right)$ promises a payoff satisfying

$$E\left[\hat{U}^S\left(s_1, s^I\right)\right] > E\left[\hat{U}^S\left(s_{0,B}\right)\right],$$

where s^I is assumed to be I's best answer to the combination s_1.
Assume now there is such a strategy $P_1^{S\to I}$. Then an increase in S's payoff implies that I must have changed his strategy as well. Otherwise, S would still sell all α units for the cheapest possible price $P^{I\to S} = (1 - R_B)\hat{\Lambda}$. Hence S's new offer $P_1^{S\to I}$ convinced I to change his strategy: he will now either accept S's offer and hence play the strategy s^I_S or reject both offers and therefore opt for a strategy s^I_- leading to a direct trade scenario s_DT.

In order for I to agree to a strategy other than s^I_B it follows that at least one of the following inequalities needs to hold:

$$E\left[\hat{U}^I\left(s_1, s^I_S\right)\right] > E\left[\hat{U}^I\left(s_1, s^I_B\right)\right] \tag{3.3}$$

or

$$E\left[\hat{U}^I\left(s_1, s^I_-\right)\right] > E\left[\hat{U}^I\left(s_1, s^I_B\right)\right] \tag{3.4}$$

As will be shown, S can always find a price such that inequality (3.3) holds no matter how S and B have set their initial offers $P_0^{S\to I}$ and $P_0^{B\to I}$. Hence there is no need to discuss inequality (3.4) as well.

Consider now that

$$\begin{aligned} E\left[\hat{U}^I\left(s_1, s^I_B\right)\right] &= \nu\left(P_0^{B\to I} - \hat{\Lambda}\right) + (\alpha - \nu)\left(\hat{\Lambda} - (1 - R_S)\hat{\Lambda}\right) \\ &= \nu\left(P_0^{B\to I} - \hat{\Lambda}\right) + \alpha R_S\hat{\Lambda} \end{aligned}$$

and that

$$E\left[\hat{U}^I\left(s_1, s_S^I\right)\right] = \nu\left((1+R_B)\hat{\Lambda} - P_1^{S\to I}\right) + (\alpha-\nu)\left(\hat{\Lambda} - P_1^{S\to I}\right) = \nu R_B\hat{\Lambda} + \alpha\left(\hat{\Lambda} - P_1^{S\to I}\right).$$

Inequality (3.3) is satisfied if

$$\nu R_B\hat{\Lambda} + \alpha\left(\hat{\Lambda} - P_1^{S\to I}\right) > \nu\left(P_0^{B\to I} - \hat{\Lambda}\right) + \alpha R_S\hat{\Lambda}.$$

It follows that if

$$P_1^{S\to I} < \frac{\nu}{\alpha}(1+R_B)\hat{\Lambda} + (1-R_S)\hat{\Lambda} - \frac{\nu}{\alpha}P_0^{B\to I},$$

I will prefer the strategy s_S^I over his previous choice s_B^I. It is of course at this point not clear which of the remaining options s_S^I and s_-^I the intermediary I will prefer. However, since S realized a price of $(1-R_S)\hat{\Lambda}$ per unit under the old strategy combination s_0, his payoff will improve as soon as he sells a part of his supply for a price higher than $(1-R_S)\hat{\Lambda}$.

- If I opts for s_-^I, S and B will enter a direct trade, i.e. S will sell ν units to B for a price of $\hat{\Lambda} > (1-R_S)\hat{\Lambda}$ and the remaining $(\alpha-\nu)$ units to I for $(1-R_S)\hat{\Lambda}$ per unit. His overall payoff will therefore have increased.
- If I opts for s_S^I, I will sell all α units for a price of $P_1^{S\to I}$ to S. His payoff will therefore have increased as long as $P_1^{S\to I} > (1-R_S)\hat{\Lambda}$.

S's payoff will therefore have increased no matter which of the strategies s_S^I and s_-^I the intermediary I decides to play if only $P_1^{S\to I} > (1-R_S)\hat{\Lambda}$.

This yields

$$(1-R_S)\hat{\Lambda} < \frac{\nu}{\alpha}(1+R_B)\hat{\Lambda} + (1-R_S)\hat{\Lambda} - \frac{\nu}{\alpha}P_0^{B\to I}$$

and therefore

$$\frac{\nu}{\alpha}(1+R_B)\hat{\Lambda} - \frac{\nu}{\alpha}P_0^{B\to I} > 0.$$

But this implies (as long as $\min\{\alpha,\nu\} > 0$)

$$P_0^{B\to I} < (1+R_B)\hat{\Lambda}.$$

It follows that as long as B is offering a price better than $(1+R_B)\hat{\Lambda}$ to I, S will always find a price $P_1^{S\to I}$ that I will prefer over $P_0^{B\to I}$. But by assumption B will offer such a price and hence s_0 cannot be a Nash equilibrium.
Similarly, if a combination of price offers $\left(P_0^{S\to I}, P_0^{B\to I}\right)$ results in

- I selling ν units to B for a price of $(1 + R_B)\hat{\Lambda}$ per unit
- and I buying α units from S for a price of $P_0^{S\to I}$ per unit

it can be shown that as long as $P_0^{S\to I} > (1 - R_S)\hat{\Lambda}$ is satisfied, B can always find a strategy $P_1^{B\to I}$ such that I prefers selling for $P_1^{B\to I}$ over buying for $P_0^{S\to I}$ and such that B's payoff increases. Therefore also in this case s_0 cannot be a Nash equilibrium.
Equivalent results can be shown for the case $\nu > \alpha$.
Since it has already been shown that no Nash equilibrium can involve a direct trade given that R_S and $R_B > 0$, this shows that no Nash equilibrium can exist at all in this game as long as the rules of the game are restricted to allow only offers made by S and B that satisfy $P^{S\to I} > (1 - R_S)\hat{\Lambda}$ and $P^{B\to I} < (1 + R_B)\hat{\Lambda}$. □

Part III

Market makers and their effect on market behaviour

Market Model I was based on a market with only three participants: a commercial buyer, a commercial seller and a non-commercial intermediary. The intermediary thereby acted both as market maker and risk taker. In modern day markets however, participants tend to specialize their tasks. We will therefore in the following investigate a market model in which the roles of the old intermediary are split between two distinct players: a market maker I_M and a risk-neutral intermediary I_R.

The market maker or spread trader I_M will be an intermediary hoping to make profit on the turn by trading the bid/offer spread between seller and buyer at time $t = 0$. He will enter the trade as soon as the offers made from buyer and seller promise him a nonnegative payoff.

Most financial exchanges are structured in a similar fashion: matchmaking in a trade takes place on a matched bargain basis. If offers from sellers and buyers match up, the exchange's trading system executes the trade. The New York Stock Exchange (NYSE) for example has a single exchange member, the "specialist", as market maker.

In the following market model a market maker will trade every nonnegative price spread, i.e. every price spread that satisfies

$$P^{B\to I} - P^{S\to I} \geq 0,$$

without charging a transaction fee. The existence of this market maker stabilizes the unpredictable market behaviour of the prior model: an equilibrium price exists at which buyer and seller trade the maximal round volume $\min\{\alpha, \nu\}$ with the market maker.

Chapter 4

Model II: A market with market maker

The considered model will be very similar to the one examined in the previous part.
The market participants will now be a commercial buyer B, a commercial seller S and two non-commercial intermediaries: a risk-neutral intermediary I_R and a market maker/spread trader I_M.

The participants will trade a certain commodity/good G with delivery at time $t = T$. This can be done either in form of a direct trade at time $t = 0$ or by accessing the cash market C at time $t = T$.

The assumptions made on the players and the market structure will be described in the following.

4.1 Basic market structure

The same basic assumptions as in Model I will also apply for Model II:

1. Accessing the cash market C as well as trading directly with any of the players will be free of transaction costs.

2. There will be no cost of carry and no convenience yield.

3. Trading will take place on a continuous price scale.

4. There will be no bid/ask spread on the cash market.

5. No participant in the cash market C can affect the cash market price $P(t)$ by his trading. Therefore $P(t)$ can be seen as an exogenous parameter: it is a random variable.

6. All players know that $P(t)$ is a random variable. Furthermore the expected value of $P(t)$ at time $t = T$,

$$\Lambda = E\left[P(T)\right]$$

is known to all players.

7. The rate of return of riskless investments is known to be r.

8. Supply, or production, α and demand, or consumption, ν of G will be deterministic and known to all players at time $t = 0$.

Again no restrictions on the nature of G have been made: it can be any tradeable product.

Furthermore the motives of the market participants will be in analogy to the previous model:

1. S and B are under hedging pressure and want to minimize the possible variation in their returns.

2. I_M enters the market as a market maker trying to realize an immediate, nonnegative profit at time $t = 0$ by trading the bid/offer spread.

3. I_R enters the market as a risk trader with the goal of having gained an expected profit at time $t = T$ higher than the risk-free return.

Note that it is assumed that I_R is only interested in a trade if his expected profit is higher than the risk-free return. On the other hand, I_M is assumed to be interested in trades with immediate, nonnegative profit, i.e. in particular also in trades that promise zero profit. This asymmetry in their motives is due to the fact that I_M's trade is completely riskless (since everything takes place at time $t = 0$) while I_R's trade involves taking risk and might very well result in a loss (even though the expected payoff is positive).

These motives translate directly into quantified goals pursued by S, B and the intermediaries.

1. S wants to realize a price of at least $(1 - R_S)\Lambda$ per unit at $t = T$, where $R_S > 0$. At time $t = 0$ he is only interested in trades with realized price P satisfying

$$P \geq (1 - R_S)\hat{\Lambda},$$

where $\hat{\Lambda} = \Lambda e^{-rT}$.

2. B wants to have paid at most $(1 + R_B)\Lambda$ per unit at $t = T$, where $R_B > 0$. At time $t = 0$ he is only interested in trades in with realized price P satisfying

$$P \leq (1 + R_B)\hat{\Lambda}.$$

3. I_M wants to realize a nonnegative payoff by trading spreads at time $t = 0$.

4. I_R's goal is to realize a profit that - at time $t = T$ - is higher than the risk-free return. But since this cannot be assured at $t = 0$ (he is a risk taker after all) his goal at $t = 0$ is to involve in trades such that his expected payoff at $t = T$ discounted to $t = 0$ is positive.

4.2 The matchmaking process in a trade

In this market model the matchmaking process in a trade will be structured in analogy to the previous model. The only diffference will be that I_M will enter the trading procedure if the commercial buyer offers to pay at least as much as the commercial seller demands, i.e. as long as he can realize a nonnegative payoff. In more detail this means that the trade will be structured as follows:

1. The risk-neutral intermediary I_R makes a binding offer of an ask price $P^{I\to S}$ and a bid price $P^{I\to B}$.

2. S and B enter the market by making binding price offers $P^{S\to I}$ and $P^{B\to I}$, for which either one of the two intermediaries can enter a trade with them.

3. If $P^{S\to I} \leq P^{B\to I}$, I_M will execute the biggest possible trade for these prices, i.e. he will buy $\min\{\alpha, \nu\}$ for $P^{S\to I}$ from the commercial seller and sell the same amount to the buyer B for $P^{B\to I}$.

4. I_R decides whether or not he will accept $P^{S\to I}$ and $P^{B\to I}$. If he accepts a price, all still available units will be traded at that price.

5. (a) If *both* S and B have not fulfilled their supply or demand needs yet, they will trade the maximal amount $\min\{\alpha, \nu\}$ directly for a price $P^{S,B} = \hat{\Lambda}$.

   (b) If *either* S or B still has open supply or demand needs (because of $\alpha \neq \nu$), the respective player will trade the remaining units with I_R for the posted price $P^{I\to S}$ or $P^{I\to B}$ respectively, if these prices are acceptable. Otherwise this player will take the risk himself and realize his supply or demand needs at time $t = T$ on the cash market.

Note that $P^{S,B} = \hat{\Lambda}$ was again assumed to be the natural price for a direct trade between S and B.

4.3 The strategies of the players

Note first that the market maker I_M acts in a completely deterministic fashion once S's and B's price offers are known: if $P^{S\to I} \leq P^{B\to I}$ he will execute as many trades as possible with this nonnegative spread, and if $P^{S\to I} > P^{B\to I}$ he will not interfere at all. This means that I_M's decision on whether or not he will enter a trade will automatically be a best response.

Furthermore - as in the previous model - I_R has no incentive to make binding offers other than $P^{I\to S} = (1 - R_S)\hat{\Lambda}$ and $P^{I\to B} = (1 + R_B)\hat{\Lambda}$ to S and B respectively. These are the best prices he can realize and any other offer would just be purely disadvantageous for him as discussed previously. The only other active interference of I_R takes place in step (4) of the matchmaking process when he decides on whether or not to accept the offers made by the other players.

Steps (5a) and (5b) are simply a consequence of what had happened in the previous steps: at this point S and B have no choice but to trade the remaining units of G in the described fashion.

This reasoning shows that the only decisions that can be subject to a game-theoretic analysis take place in steps (2)-(4) of the matchmaking process. Hence - in analogy to the previous model - the strategies of each player can be defined as follows:

- The strategy of S is well defined by his price offer $P^{S\to I}$.
- The strategy of B is well defined by his price offer $P^{B\to I}$.
- I_M has two possible strategy choices:

- s_+^M: I_M will buy $m = \min\{\alpha, \nu\}$ units for $P^{S\to I}$ from S and sell them to B for $P^{B\to I}$.
- s_-^M: I_M will not enter the market.

- The strategy of I_R is defined as the set of his responses to $P^{S\to I}$ and $P^{B\to I}$, where each response will either be a price acceptance or rejection. The possible strategic action can be
  - s_B^R: I_R accepts B's offer and rejects S's offer.
  - s_S^R: I_R accepts S's offer and rejects B's offer.
  - s_-^R: I_R rejects both offers.
  - Another theoretical possibility could be seen in I_R accepting both offers. However,as in the previous model, such an action would be completely disadvantageous for I_R and can therefore be neglected.

Note that I_M and I_R will already know $P^{B\to I}$ and $P^{S\to I}$ at the time when they have to decide on whether or not to accept these offers. Their strategy choices will therefore automatically be best answers to $P^{B\to I}$ and $P^{S\to I}$. Since all model parameters are known to all market participants, this furthermore implies that I_M's and I_R's action will be predictable once B and S have set their price offers. This means that I_M's and I_R's strategic action can be interpreted as part of the market mechanism, effectively reducing the set of active players to S and B only. In the context of a game-theoretic analysis this implies that a Nash equilibrium solution is identified if B's and S's price offers are best answers to each other.

4.4 Possible outcome scenarios

The introduction of a fourth player allows more variety in outcome scenarios than the previous three-person game.

Consider first the case in which the market maker I_M opts for his strategy s_-^M in order not enter the trade, i.e. the case when $P^{S\to I} > P^{B\to I}$. In such a case the game gets reduced to one resembling the three-person game investigated in the previous part. As discussed earlier the possible outcome scenarios then are as follows:

1. $s_B = \left(P^{S\to I}, P^{B\to I}, s_-^M, s_B^R\right)$:
   I_R accepts B's offer and sells ν units for $P^{B\to I}$. S then sells α units to I_R for a price of $P^{I\to S}$. I_R sells the remaining $|\alpha - \nu|$ units on the cash market for

$P(T)$.
In this case the resulting payoffs are as follows:

$$\begin{aligned}
E\left[\hat{U}^S(s_B)\right] &= \alpha\left(P^{I\to S} - \hat{\Lambda}\right)\\
E\left[\hat{U}^B(s_B)\right] &= \nu\left(\hat{\Lambda} - P^{B\to I}\right)\\
E\left[\hat{U}^{I_M}(s_B)\right] &= 0\\
E\left[\hat{U}^{I_R}(s_B)\right] &= m\left(P^{B\to I} - P^{I\to S}\right) + (\alpha - m)\left(\hat{\Lambda} - P^{I\to S}\right)\\
&\quad + (\nu - m)\left(P^{B\to I} - \hat{\Lambda}\right)\\
&= \nu P^{B\to I} - \alpha P^{I\to S} + (\alpha - \nu)\,\hat{\Lambda},
\end{aligned}$$

where $m = \min\{\alpha, \nu\}$.

2. $s_S = \left(P^{S\to I}, P^{B\to I}, s_-^M, s_S^R\right)$:
I_R accepts S's offer and buys α units for $P^{S\to I}$. B buys ν units from I_R for a price of $P^{I\to B}$. I_R sells/buys the remaining $|\alpha - \nu|$ units on the cash market for $P(T)$.
This results in the following expected payoffs:

$$\begin{aligned}
E\left[\hat{U}^S(s_S)\right] &= \alpha\left(P^{S\to I} - \hat{\Lambda}\right)\\
E\left[\hat{U}^B(s_S)\right] &= \nu\left(\hat{\Lambda} - P^{I\to B}\right)\\
E\left[\hat{U}^{I_M}(s_S)\right] &= 0\\
E\left[\hat{U}^{I_R}(s_S)\right] &= m\left(P^{I\to B} - P^{S\to I}\right) + (\alpha - m)\left(\hat{\Lambda} - P^{S\to I}\right)\\
&\quad + (\nu - m)\left(P^{I\to B} - \hat{\Lambda}\right)\\
&= \nu P^{I\to B} - \alpha P^{S\to I} + (\alpha - \nu)\,\hat{\Lambda}.
\end{aligned}$$

3. $s_{DT} = \left(P^{S\to I}, P^{B\to I}, s_-^M, s_-^R\right)$:
I_R declines both offers. S and B trade $\min\{\alpha, \nu\}$ units directly for $\hat{\Lambda}$ per unit. The remaining $|\alpha - \nu|$ units will be bought/sold by I_R for $P^{I\to S}$ or $P^{I\to B}$ respectively, who in turn will sell/buy them on the cash market for $P(T)$ per unit.

The expected payoffs for this scenario can be computed as

$$\begin{aligned} E\left[\hat{U}^S\left(s_{DT}\right)\right] &= (\alpha - m)\left(P^{I\to S} - \hat{\Lambda}\right) \\ E\left[\hat{U}^B\left(s_{DT}\right)\right] &= (\nu - m)\left(\hat{\Lambda} - P^{I\to B}\right) \\ E\left[\hat{U}^{I_M}\left(s_{DT}\right)\right] &= 0 \\ E\left[\hat{U}^{I_R}\left(s_{DT}\right)\right] &= (\alpha - m)\left(\hat{\Lambda} - P^{I\to S}\right) + (\nu - m)\left(P^{I\to B} - \hat{\Lambda}\right). \end{aligned}$$

Consider now an outcome scenario s_M in which the market maker I_M trades the spread, i.e. a strategy combination in which I_M's strategy is given by s_+^M. This happens if $P^{S\to I} \leq P^{B\to I}$. The market maker will use this nonnegative price spread to buy $m = \min\{\alpha, \nu\}$ units from S for a price of $P^{S\to I}$ and sell the same amount for a price of $P^{B\to I}$ to B.

Depending on whether $\alpha > \nu$ or $\alpha < \nu$ either player S or player B will have satisfied all his needs with this deal. I_R now has the choice to accept or decline the remaining player's offer for the last $|\alpha - \nu|$ units.
Suppose first that $\alpha > \nu$. Then - after I_M has traded ν units from S to B - S will still be offering $\alpha - \nu$ units of G. Since it can safely be assumed that S's offer satisfies $P^{S\to I} > P^{I\to S}$ (since otherwise S would have offered to sell G for less or at best the same than what I_R had already offered to S), I_R will certainly decline this offer to use hedging pressure and hence force S into accepting I_R's offer of $P^{I\to S}$. This strategic choice corresponds to either playing s_B^R or s_-^R. Note that the outcome of the game will be exactly the same no matter which of those two strategies I_R opts for.
Similarly, if $\alpha < \nu$, I_R will decline B's offer in order to force B into accepting I_R's standing offer $P^{I\to B}$. This can be done by either playing s_S^R or s_-^R, where again both strategy choices result in exactly the same outcome for all participants. Since in both cases $\alpha > \nu$ and $\nu > \alpha$ the strategy s_-^R is a viable option for I_R, it will without loss of generality be assumed that I_R's response to I_M's strategy s_+^M will always be s_-^R.

The resulting payoffs of a strategy combination $s_M = \left(P^{S\to I}, P^{S\to I}, s_+^M, s_-^R\right)$ can hence be determined as follows:

$$\begin{aligned}
E\left[\hat{U}^{S}\left(s_M\right)\right] &= m\left(P^{S\to I}-\hat{\Lambda}\right)+(\alpha-m)\left(P^{I\to S}-\hat{\Lambda}\right)\\
E\left[\hat{U}^{B}\left(s_M\right)\right] &= m\left(\hat{\Lambda}-P^{B\to I}\right)+(\nu-m)\left(\hat{\Lambda}-P^{I\to B}\right)\\
E\left[\hat{U}^{I_M}\left(s_M\right)\right] &= m\left(P^{B\to I}-P^{S\to I}\right)\\
E\left[\hat{U}^{I_R}\left(s_M\right)\right] &= (\alpha-m)\left(\hat{\Lambda}-P^{I\to S}\right)+(\nu-m)\left(P^{I\to B}-\hat{\Lambda}\right).
\end{aligned}$$

Chapter 5

The existence of a Nash equilibrium solution in Model II

The analysis of the previous, simpler Model I yielded two main results:

1. An outcome scenario in which B and S trade the maximum possible amount of G directly will not be a Nash equilibrium solution. In other words: the intermediary I will not just trade the supply-demand surplus, and will therefore play a vital role in the trade.

2. It was furthermore shown that no Nash equilibrium at all could exist in such a market. This can be understood as the major weakness of the previous result: it provided no insight in how the trades will actually occur.

Splitting up the old intermediary into one trading spreads, I_M, and one trading risk, I_R, will resolve this shortcoming:

1. It will again be shown that the intermediaries do not just trade the supply-demand surplus, hence confirming the result of the simpler model.

2. It will be shown that Model II exhibits exactly one Nash equilibrium solution. This solution will allow for an insightful interpretation on how the considered good will be priced.

To prove these results it will at first be shown that two price offers $\left(P^{S\to I}, P^{B\to I}\right)$ will not be part of a Nash equilibrium solution if they exhibit a negative price spread. In other words, if $s = \left(P^{S\to I}, P^{B\to I}, s^M, s^R\right)$ is to be a Nash equilibrium then

$$P^{B\to I} - P^{S\to I} \geq 0.$$

It will then be shown that of all possible strategy combinations with a nonnegative price spread only one will be a Nash equilibrium solution.

5.1 Non-existence of a Nash equilibrium with negative price spread

Consider a market as described above with S supplying α units, B demanding ν units and with a market maker I_M and a risk trader I_R facilitating the trades. Let

$$m = \min\{\alpha, \nu\}.$$

Suppose that S and B have set their offers $P^{S\to I}$ and $P^{B\to I}$ such that

$$P^{B\to I} < P^{S\to I}.$$

Since the spread $P^{B\to I} - P^{S\to I}$ will be negative, I_M will not enter the trade and opt for strategy s_-^M. Hence, depending on I_R's decision, the following outcome scenarios are possible:

- $s_B = \left(P^{S\to I}, P^{B\to I}, s_-^M, s_B^R\right)$:
  I_R sells ν units to B for $P^{B\to I}$ per unit and buys α units from S for $P^{I\to S}$ per unit
- $s_S = \left(P^{S\to I}, P^{B\to I}, s_-^M, s_S^R\right)$:
  I_R buys α units from S for $P^{S\to I}$ per unit and sells ν units to B for $P^{I\to B}$ per unit
- $s_{DT} = \left(P^{S\to I}, P^{B\to I}, s_-^M, s_-^R\right)$:
  I_R declines both offers. S and B therefore trade m units directly and I_R buys the remaining $(\alpha - m)$ units from S for $P^{I\to S}$ or sells the remaining $(\nu - m)$ units to B for $P^{I\to B}$.

Since none of these scenarios involves a trade with the market maker I_M, this appears to be a very similiar setup than the one discussed in the previous model. In fact, it can be shown that also in this model a trade without active interference of the market maker cannot be a Nash equilibrium.

Consider first a scenario in which S and B set their offers $P^{S\to I}$ and $P^{B\to I}$ such that I_R's decision will result in a strategy combination of the s_B-type. As it will be shown, no such strategy combination can be a Nash equilibrium:

Lemma 5.1.1 (Model II: no Nash equilibrium of the form s_B)
Consider a market as described above with $\alpha, \nu, R_S, R_B > 0$ and with $P^{B\to I} < P^{S\to I}$.
Then a strategy combination $s_B = \left(P^{S\to I}, P^{B\to I}, s_-^M, s_B^R\right)$ where B buys all ν units from I_R for a price of $P^{B\to I}$ is not a Nash equilibrium.

Proof. Suppose a strategy combination $s_{0,B} = \left(P_0^{S\to I}, P_0^{B\to I}, s_-^M, s_B^R\right)$ is a Nash equilibrium and consider that under such a strategy combination S would have to sell all α units he supplies for the worst possible price $P^{I\to S}$.

Since the strategy combination is assumed to be a Nash equilibrium, none of the players should be able to singlehandedly improve his payoff. This implies in particular that also S cannot improve his payoff.

Suppose now that $P_0^{B\to I} > P^{I\to S}$. This means nothing else than that there is $\epsilon > 0$ such that

$$P^{I\to S} < P_0^{B\to I} - \epsilon < P_0^{B\to I}.$$

Suppose now that S changes his offer from $P_0^{S\to I}$ to an offer

$$P_1^{S\to I} = P_0^{B\to I} - \epsilon.$$

Note that such a price $P^{S\to I}$ is a legal price, since a continuous price scale is assumed. It follows that the spread is positive since

$$\begin{aligned} P_0^{B\to I} - P_1^{S\to I} &= P_0^{B\to I} - \left(P_0^{B\to I} - \epsilon\right) \\ &= \epsilon > 0 \end{aligned}$$

and hence that the market maker I_M will enter the trade by playing his strategy s_+^M. I_M will buy $m = \min\{\alpha, \nu\}$ units from S for the suggested price $P_1^{S\to I}$. S will then sell the remaining $\alpha - m$ units to I_R for $P^{I\to S}$. It follows that S's payoff has improved: m units are now sold for a higher price. But this contradicts the assumption of S not being able to singlehandedly improve his payoff. Hence a strategy combination $s_{0,B}$ with $P_0^{B\to I} > P^{I\to S}$ cannot be a Nash equilibrium.

Assume now that $P_0^{B\to I} \leq P^{I\to S}$. In this case, a change to a strategy combination involving the market maker I_M will not increase S's payoff: for I_M to participate in the trade, the spread $P_0^{B\to I} - P_1^{S\to I}$ would need to be nonnegative, i.e.

$$P_1^{S\to I} \leq P_0^{B\to I} \leq P^{I\to S}.$$

But this means that S would sell $m = \min\{\alpha, \nu\}$ units for a price less than or at most equal to $P^{I\to S}$, his payoff would therefore have decreased or at least not increased.

It follows that S can only improve his payoff if he can initiate a change to one of the strategy combinations involving I_R, i.e. if a change from $P_0^{S\to I}$ to $P_1^{S\to I}$ would result in either the strategy combination $s_{1,S}$ or $s_{1,DT}$.

Suppose that S is trying to initiate a change from $s_{0,B}$ to the strategy combination $s_{1,S}$ by changing his offer from $P_0^{S\to I}$ to $P_1^{S\to I}$. I_R would agree to such a change, if $s_{1,S}$ would promise a higher expected payoff than $s_{1,B}$ and $s_{1,DT}$, i.e. if

$$E\left[\hat{U}^{I_R}(s_{1,S})\right] > E\left[\hat{U}^{I_R}(s_{1,B})\right] \tag{5.1}$$

and if

$$E\left[\hat{U}^{I_R}(s_{1,S})\right] > E\left[\hat{U}^{I_R}(s_{1,DT})\right]. \tag{5.2}$$

Note that

$$E\left[\hat{U}^{I_R}(s_{1,B})\right] = \nu P_0^{B\to I} - \alpha P^{I\to S} + (\alpha - \nu)\,\hat{\Lambda}$$

and

$$E\left[\hat{U}^{I_R}(s_{1,DT})\right] = (\alpha - m)\left(\hat{\Lambda} - P^{I\to S}\right) + (\nu - m)\left(P^{I\to B} - \hat{\Lambda}\right)$$

do not depend on S's changed price offer $P_1^{S\to I}$. Hence

$$E\left[\hat{U}^{I_R}(s_{1,B})\right] = E\left[\hat{U}^{I_R}(s_{0,B})\right]$$

and

$$E\left[\hat{U}^{I_R}(s_{1,DT})\right] = E\left[\hat{U}^{I_R}(s_{0,DT})\right].$$

Since the original strategy combination $s_{0,B}$ would have only been played if

$$E\left[\hat{U}^{I_R}(s_{0,B})\right] \geq E\left[\hat{U}^{I_R}(s_{0,DT})\right].$$

it follows that inequality (5.2) will be satisfied as soon as inequality (5.1) holds. With

$$E\left[\hat{U}^{I_R}(s_{1,S})\right] = \nu P^{I\to B} - \alpha P_1^{S\to I} + (\alpha - \nu)\,\hat{\Lambda}$$

inequality (5.1) implies that

$$P_1^{S\to I} < P^{I\to S} + \frac{\nu}{\alpha}\left(P^{I\to B} - P_0^{B\to I}\right). \tag{5.3}$$

Consider now that S would have only changed his strategy in such a way if this would have increased his own payoff. A change from $s_{B,0}$ to $s_{1,S}$ would increase S's payoff if

$$E\left[\hat{U}^{S}(s_{1,S})\right] > E\left[\hat{U}^{S}(s_{0,B})\right], \tag{5.4}$$

i.e. if

$$\alpha\left(P_1^{S\to I} - \hat{\Lambda}\right) > \alpha\left(P^{I\to S} - \hat{\Lambda}\right)$$

It follows that inequality (5.4) holds if

$$P_1^{S\to I} > P^{I\to S}. \tag{5.5}$$

Note that this condition also assures that the price spread is negative (and hence that a strategy combination $s_{1,S}$ not involving the market maker I_M is possible), since

$$\begin{aligned} P_0^{B\to I} - P_1^{S\to I} &< P_0^{B\to I} - P^{I\to S} \\ &\leq 0. \end{aligned}$$

Combining the above bounds (5.3) and (5.5) yields that

$$P^{I\to S} < P_1^{S\to I} < P^{I\to S} + \frac{m}{\alpha}\left(P^{I\to B} - P^{I\to S}\right).$$

But - since pricing is assumed to take place on a continuous price scale - such a price $P_1^{S\to I}$ exists if

$$P^{I\to S} < P^{I\to S} + \frac{m}{\alpha}\left(P^{I\to B} - P^{I\to S}\right)$$

and therefore if

$$0 < \frac{m}{\alpha}\left(P^{I\to B} - P^{I\to S}\right).$$

Since by assumption $P^{I\to B} > P^{I\to S}$ and $m > 0$ this inequality holds. It follows that S can improve his payoff by changing his offer from $P_0^{S\to I}$ to an offer $P_1^{S\to I}$ satisfying the above inequalities.
But this contradicts the assumption of S not being able to singlehandedly improve his payoff. Hence a strategy combination $s_{0,B}$ cannot be a Nash equilibrium. □

Consider now a scenario in which $P^{S\to I}$ and $P^{B\to I}$ are set such that I_M's and I_R's decisions will result in a strategy combination of the form s_S. Also in such a case it can be shown that no Nash equilibrium can exist:

Lemma 5.1.2 (Model II: no Nash equilibrium of the form s_S)
Consider a market as described above with $\alpha, \nu, R_S, R_B > 0$ and with $P^{B\to I} < P^{S\to I}$.
Then a strategy combination $s_S = \left(P^{S\to I}, P^{B\to I}, s_-^M, s_+^R\right)$ where S sells all α units to I_R for a price of $P^{S\to I}$ is not a Nash equilibrium.

Proof. Consider that in the case of a strategy combination

$$s_{0,S} = \left(P_0^{S\to I}, P_0^{B\to I}, s_-^M, s_+^R\right),$$

B would have to buy all ν units he needs for the highest possible price $P^{I\to B}$. Similarly as in the previous case of Lemma 5.1.1, it can be shown, that such a strategy combination cannot be Nash equilibrium:

1. If $P_0^{S\to I} < P^{I\to B}$, B could offer a price

$$P_1^{B\to I} = P_0^{S\to I} + \epsilon < P^{I\to B},$$

where $\epsilon > 0$.
Then I_M would enter the trade since the spread satisfies

$$P_1^{B\to I} - P_0^{S\to I} = \epsilon > 0.$$

It follows that B would now buy $m = \min\{\alpha, \nu\}$ for a price cheaper than $P^{I\to B}$ and hence that B would have improved his payoff.

2. Suppose now that $P_0^{S\to I} \geq P^{I\to B}$ and suppose B wants to induce a switch to a strategy combination $s_{1,B}$. I_R would agree to such a change if

$$E\left[\hat{U}^{I_R}(s_{1,B})\right] > E\left[\hat{U}^{I_R}(s_{1,S})\right] \tag{5.6}$$

and if

$$E\left[\hat{U}^{I_R}(s_{1,B})\right] > E\left[\hat{U}^{I_R}(s_{1,DT})\right]. \tag{5.7}$$

With $E\left[\hat{U}^{I_R}(s_{1,S})\right] = E\left[\hat{U}^{I_R}(s_{0,S})\right]$ and $E\left[\hat{U}^{I_R}(s_{1,DT})\right] = E\left[\hat{U}^{I_R}(s_{0,DT})\right]$ it follows that

$$E\left[\hat{U}^{I_R}(s_{1,S})\right] \geq E\left[\hat{U}^{I_R}(s_{1,DT})\right].$$

Hence inequality (5.7) will be satisfied as soon as inequality (5.6) holds. Using

$$E\left[\hat{U}^{I_R}(s_{1,S})\right] = \nu P^{I\to B} - \alpha P_0^{S\to I} + (\alpha - \nu)\,\hat{\Lambda}$$
$$E\left[\hat{U}^{I_R}(s_{1,B})\right] = \nu P_1^{B\to I} - \alpha P^{I\to S} + (\alpha - \nu)\,\hat{\Lambda}$$

inequality (5.6) implies that

$$P_1^{B\to I} > P^{I\to B} - \frac{\alpha}{\nu}\left(P_0^{S\to I} - P^{I\to S}\right). \tag{5.8}$$

Consider now that B would only agree to a change from $s_{0,S}$ to $s_{1,B}$ if

$$E\left[\hat{U}^{B}(s_{1,B})\right] > E\left[\hat{U}^{B}(s_{0,S})\right], \tag{5.9}$$

i.e. if

$$\nu\left(\hat{\Lambda} - P_1^{B\to I}\right) > \nu\left(\hat{\Lambda} - P^{I\to B}\right).$$

It follows that inequality (5.9) holds if

$$P_1^{B\to I} < P^{I\to B}. \tag{5.10}$$

Combining the inequalities (5.8) and (5.10) yields that B can increase his payoff by inducing a switch to a strategy combination $s_{1,B}$ if his offer satisfies

$$P^{I\to B} > P_1^{B\to I} > P^{I\to B} - \frac{m}{\nu}\left(P^{I\to B} - P^{I\to S}\right).$$

It follows that such an offer exists if

$$P^{I\to B} > P^{I\to B} - \frac{m}{\nu}\left(P^{I\to B} - P^{I\to S}\right),$$

i.e. if

$$0 > -\frac{m}{\nu}\left(P^{I\to B} - P^{I\to S}\right).$$

But since $P^{I\to B} - P^{I\to S} > 0$ and $m > 0$ this inequality is satisfied. It follows that such a price $P_1^{B\to I}$ exists and therefore B would be able to singlehandedly increase his payoff. □

The final case to consider is an outcome scenario s_{DT} in which S and B are trading $m = \min\{\alpha, \nu\}$ units directly for a price of $\hat{\Lambda}$ per unit. But also in this case it can be shown that no Nash equilibrium can exist:

Lemma 5.1.3 (Model II: no Nash equilibrium of the form s_{DT})
Consider a market as described above with $\alpha, \nu, R_S, R_B > 0$ and with $P^{B\to I} < P^{S\to I}$.
Then a strategy combination $s_{DT} = \left(P^{S\to I}, P^{B\to I}, s_-^M, s_-^R\right)$ where B and S trade $m = \min\{\alpha, \nu\}$ directly cannot be a Nash equilibrium.

Proof. For a scenario of the form $s_{0,DT}$ to be possible, the initial offers need to be set such that

$$E\left[\hat{U}^{I_R}(s_{0,DT})\right] \geq E\left[\hat{U}^{I_R}(s_{0,B})\right] \tag{5.11}$$

and

$$E\left[\hat{U}^{I_R}(s_{0,DT})\right] \geq E\left[\hat{U}^{I_R}(s_{0,S})\right]. \tag{5.12}$$

With

$$\begin{aligned} E\left[\hat{U}^{I_R}(s_{0,DT})\right] &= (\alpha - m)\left(\hat{\Lambda} - P^{I\to S}\right) + (\nu - m)\left(P^{I\to B} - \hat{\Lambda}\right) \\ &= (m - \alpha)\, P^{I\to S} + (\nu - m)\, P^{I\to B} + (\alpha - \nu)\,\hat{\Lambda} \end{aligned}$$

and

$$E\left[\hat{U}^{I_R}(s_{0,B})\right] = \nu P_0^{B\to I} - \alpha P^{I\to S} + (\alpha - \nu)\,\hat{\Lambda}$$

inequality (5.11) implies that

$$(m-\alpha)\,P^{I\to S}+(\nu-m)\,P^{I\to B}\geq \nu P_0^{B\to I}-\alpha P^{I\to S}$$

and hence that

$$P_0^{B\to I}\leq P^{I\to B}-\frac{m}{\nu}\left(P^{I\to B}-P^{I\to S}\right). \tag{5.13}$$

Similarly, using

$$E\left[\hat{U}^{I_R}(s_{0,S})\right]=\nu P_B^{I\to I}-\alpha P_0^{S\to I}+(\alpha-\nu)\,\hat{\Lambda}$$

inequality (5.12) yields

$$P_0^{S\to I}\geq P^{I\to S}+\frac{m}{\alpha}\left(P^{I\to B}-P^{I\to S}\right). \tag{5.14}$$

1. Suppose that $m=\alpha$ and that B set his initial offer such that

$$P_0^{B\to I}>\hat{\Lambda}.$$

and that S changes his offer to a price $P_1^{S\to I}$ satisfying

$$\hat{\Lambda}<P_1^{S\to I}\leq P_0^{B\to I}.$$

But since the spread $P_0^{B\to I}-P_1^{S\to I}$ is now nonnegative, I_M will enter the trade and buy all $m=\alpha$ units S supplies for a price of $P_1^{S\to I}>\hat{\Lambda}$ per unit. Under the old strategy combination $s_{0,DT}$, S would have sold all α units directly to B for a price of only $\hat{\Lambda}$. It follows that S has singlehandedly improved his payoff. Hence a strategy combination $s_{0,DT}$ with $m=\alpha$ and $P_0^{B\to I}>\hat{\Lambda}$ cannot be a Nash equilibrium.

2. Suppose now that $m=\alpha$ and that

$$P_0^{B\to I}\leq\hat{\Lambda}.$$

Suppose furthermore that S wants to change from the strategy combination $s_{0,DT}$ to a strategy combination $s_{1,S}$, in which he sells all $m=\alpha$ units to the risk trader I_R for a price of $P_1^{S\to I}$. In order for such a change to be possible the following conditions need to hold:

(a) In order for I_R to play such a strategy the respective payoffs need to satisfy

$$E\left[\hat{U}^{I_R}(s_{1,S})\right] > E\left[\hat{U}^{I_R}(s_{1,B})\right] \tag{5.15}$$

and

$$E\left[\hat{U}^{I_R}(s_{1,S})\right] > E\left[\hat{U}^{I_R}(s_{1,DT})\right]. \tag{5.16}$$

Note that $E\left[\hat{U}^{I_R}(s_{1,B})\right]$ and $E\left[\hat{U}^{I_R}(s_{1,DT})\right]$ do not depend on S's changed offer. Since I_R previously opted for a strategy $s_{0,DT}$ it follows that

$$E\left[\hat{U}^{I_R}(s_{1,DT})\right] = E\left[\hat{U}^{I_R}(s_{0,DT})\right] \geq E\left[\hat{U}^{I_R}(s_{0,B})\right] = E\left[\hat{U}^{I_R}(s_{1,B})\right].$$

This implies that inequality (5.15) will be satisfied as soon as inequality (5.16) holds. Using a version of inequality (5.12) it follows that

$$P_1^{S\to I} < P^{I\to S} + \frac{m}{\alpha}\left(P^{I\to B} - P^{I\to S}\right) = P^{I\to B} \tag{5.17}$$

(b) On the other hand, in order for S to offer such a change in strategy the resulting outcome $s_{1,S}$ also needs to be profitable for S himself, i.e.

$$E\left[\hat{U}^{S}(s_{1,S})\right] > E\left[\hat{U}^{S}(s_{0,DT})\right].$$

With

$$\begin{aligned} E\left[\hat{U}^{S}(s_{0,DT})\right] &= (\alpha - m)\left(P^{I\to S} - \hat{\Lambda}\right) \\ &= 0 \end{aligned}$$

and

$$E\left[\hat{U}^{S}(s_{1,S})\right] = \alpha\left(P_1^{S\to I} - \hat{\Lambda}\right)$$

it follows that

$$P_1^{S\to I} > \hat{\Lambda}. \tag{5.18}$$

(c) Finally, the strategy combination $s_{1,S}$ is only possible if I_M is not involved in the trade. This can be assured if the spread $P_0^{B\to I} - P_1^{S\to I}$ is negative, i.e. if $P_0^{B\to I} < P_1^{S\to I}$. But since by assumption

$$P_0^{B\to I} \leq \hat{\Lambda}$$

the above condition (5.18) already guarantees that

$$P_0^{B\to I} \leq \hat{\Lambda} < P_1^{S\to I}.$$

This shows that S can profitably switch to a strategy combination $s_{1,S}$ if his new offer $P_1^{S\to I}$ satisifies the bounds (5.17) and (5.18), i.e. if

$$\hat{\Lambda} < P_1^{S\to I} < P^{I\to B}.$$

Such a price exists if

$$\hat{\Lambda} < P^{I\to B},$$

but since B is willing to pay a premium $R_B > 0$ for transferring his risk to I_R, it follows that this condition holds:

$$P^{I\to B} = (1 + R_B)\hat{\Lambda} > \hat{\Lambda}.$$

It follows that S can singlehandedly induce a profitable change from the strategy combination $s_{0,DT}$ to a strategy combination $s_{1,S}$ if $m = \alpha$ and $P_0^{B\to I} \leq \hat{\Lambda}$. The old outcome $s_{0,DT}$ can therefore not be a Nash equilibrium under these assumptions.

Combining the results for $P_0^{B\to I} > \hat{\Lambda}$ and for $P_0^{B\to I} \leq \hat{\Lambda}$ it follows that a strategy combination $s_{0,DT}$ cannot be a Nash equilibrium if $m = \alpha$.

Similiarly it can be shown that if $m = \nu$, B can always find a way to singlehandedly increase his payoff. Hence, also in this case a strategy combination $s_{0,DT}$ cannot be a Nash equilibrium. □

The previous three Lemmata imply that no strategy combination that does not involve the market maker I_M can be a Nash equilibrium:

Theorem 5.1.4 (Model II: non-existence of Nash equilibria without I_M) *Consider a market as described above with $\alpha, \nu, R_S, R_B > 0$ and with $P^{B\to I} < P^{S\to I}$.*
Then a strategy combination $s = \left(P^{S\to I}, P^{B\to I}, s_-^M, s^R\right)$ with $P^{B\to I} < P^{S\to I}$, i.e. a strategy combination not involving the market maker I_M is not a Nash equilibrium.

Proof. Consider a strategy combination $s = \left(P^{S\to I}, P^{B\to I}, s_-^M, s^R\right)$ with

$$P^{B\to I} < P^{S\to I}.$$

Since

$$P^{B\to I} - P^{S\to I} < 0,$$

the market maker I_M will not enter the trader and opt for strategy s_-^M. The trade will therefore be either of the form s_B, s_S or s_{DT} depending on I_R's decision. But by Lemmata 5.1.1, 5.1.2 and 5.1.3 no such outcome scenario can be a Nash equilibrium. This proves the claim. □

5.2 A Nash equilibrium involving the market maker

Consider a market as described above with S supplying α units, B demanding ν units and a market maker I_M and a risk trader I_R facilitating the trades. Let

$$m = \min\{\alpha, \nu\}$$

and let

$$\left(P_0^{S\to I}, P_0^{B\to I}\right)$$

be a combination of price offers with

$$P^{I\to S} < P_0^{S\to I} \leq P_0^{B\to I} < P^{I\to B}.$$

Since I_M can generate profit by trading the spread, he will opt for his strategy s_+^M. With I_R's response being the strategy s_-^R, the resulting strategy combination will be

$$s_{0,M} = \left(P_0^{S\to I}, P_0^{B\to I}, s_+^M, s_-^R\right).$$

The trades will then occur in the following fashion:

1. I_M buys m units from S for $P_0^{S\to I}$ and sells m units to B for $P_0^{B\to I}$.
2. I_R buys $\alpha - m$ units from S for $P^{I\to B}$ or sells $m - \nu$ units to B for $P^{I\to B}$.
3. I_R evens out his position at $t = T$ for $P(T)$.

The expected payoffs of this strategy combination can hence be found as:

$$\begin{aligned}
E\left[\hat{U}^B(s_{0,M})\right] &= m\left(\hat{\Lambda} - P_0^{B\to I}\right) + (\nu - m)\left(\hat{\Lambda} - P^{I\to B}\right)\\
E\left[\hat{U}^S(s_{0,M})\right] &= m\left(P_0^{S\to I} - \hat{\Lambda}\right) + (\alpha - m)\left(P^{I\to S} - \hat{\Lambda}\right)\\
E\left[\hat{U}^{I_R}(s_{0,M})\right] &= (\alpha - m)\left(\hat{\Lambda} - P^{I\to S}\right) + (\nu - m)\left(P^{I\to B} - \hat{\Lambda}\right)\\
&= \alpha\left(\hat{\Lambda} - P^{I\to S}\right) + \nu\left(P^{I\to B} - \hat{\Lambda}\right) + m\left(P^{I\to S} - P^{I\to B}\right)\\
E\left[\hat{U}^{I_M}(s_{0,M})\right] &= m\left(P_0^{B\to I} - P_0^{S\to I}\right).
\end{aligned}$$

Suppose now that this strategy combination is a Nash equilibrium. This means that neither S nor B can improve their payoff by singlehandedly changing their strategy.

5.2.1 An upper bound on S's price offer

Assuming that the above strategy combination is a Nash equilibrium implies in particular that the prices $P_0^{S\to I}$ and $P_0^{B\to I}$ are set in a way such that the commercial buyer B cannot singlehandedly change his strategy in a profitable way.

Note first that B cannot improve his payoff with an offer $P_1^{B\to I} > P_0^{B\to I}$. Since $P_0^{S\to I} \leq P_0^{B\to I} < P_1^{B\to I}$, B would still buy m units from the market maker I_M and $\nu - m$ units from the risk trader I_R with the only difference that the trades with I_M occur at a higher price $P_1^{B\to I} > P_0^{B\to I}$.

A potentially profitable change can therefore ccur in only two ways:

If $P_0^{S\to I} < P_0^{B\to I}$, B could set a new offer $P_1^{B\to I} < P_0^{B\to I}$ that still satisfies a non-negative price spread $P_0^{S\to I} \leq P_1^{B\to I}$. I_M would still be involved and since B would now buy m units for a lower price $P_1^{B\to I}$ he would certainly have singlehandedly improved his payoff. This case will be discussed in more detail later. For now the focus will be put on the second possible case:

Suppose B changes his strategy from $P_0^{B\to I}$ to $P_1^{B\to I}$ where $P_1^{B\to I} < P_0^{S\to I} \leq P_0^{B\to I}$. This means that I_M cannot trade a nonnegative spread anymore and hence that the nature of the trade will depend on how I_R sets his strategy. I_R has the choice between the following strategies:

- s_B^R: I_R sells ν units to B for $P_1^{B\to I}$ per unit and buys α units from S for $P^{I\to S}$ per unit
- s_S^R: I_R buys α units from S for $P_0^{S\to I}$ per unit and sells ν units to B for $P^{I\to B}$ per unit
- s_{DT}^R: I_R declines both offers. S and B therefore trade m units directly and I_R buys the remaining $(\alpha - m)$ units from S for $P^{I\to S}$ or sells the remaining $(\nu - m)$ units to B for $P^{I\to B}$.

Depending on I_R's behaviour the new strategy combination

$$s_1 = \left(P_0^{S\to I}, P_1^{B\to I}, s_-^M, s^R\right)$$

can lead to three outcomes:

1. a strategy combination $s_{1,B} = \left(P_0^{S\to I}, P^{B\to I}, s_-^M, s_B^R\right)$, i.e. a scenario where I_R accepts B's offer.

The relevant payoffs of B and I_R are

$$E\left[\hat{U}^B(s_{1,B})\right] = \nu\left(\hat{\Lambda} - P_1^{B\to I}\right)$$
$$E\left[\hat{U}^{I_R}(s_{1,B})\right] = \nu\left(P_1^{B\to I} - \hat{\Lambda}\right) + \alpha\left(\hat{\Lambda} - P^{I\to S}\right).$$

2. a strategy combination $s_{1,S} = \left(P_0^{S\to I}, P^{B\to I}, s_-^M, s_S^R\right)$, i.e. a scenario where I_R accepts S's offer.
   The relevant payoffs of B and I_R are

$$E\left[\hat{U}^B(s_{1,S})\right] = \nu\left(\hat{\Lambda} - P^{I\to B}\right)$$
$$E\left[\hat{U}^{I_R}(s_{1,S})\right] = \nu\left(P^{I\to B} - \hat{\Lambda}\right) + \alpha\left(\hat{\Lambda} - P_0^{S\to I}\right).$$

3. a strategy combination $s_{DT} = \left(P_0^{S\to I}, P^{B\to I}, s_-^M, s_-^R\right)$, i.e. a scenario where I_R declines both offers.
   The relevant payoffs of B and I_R are in this case

$$E\left[\hat{U}^B(s_{1,DT})\right] = (\nu - m)\left(\hat{\Lambda} - P^{I\to B}\right)$$
$$E\left[\hat{U}^{I_R}(s_{1,DT})\right] = (\alpha - m)\left(\hat{\Lambda} - P^{I\to S}\right) + (\nu - m)\left(P^{I\to B} - \hat{\Lambda}\right).$$

Consider now that B would only change his strategy in such a way if the resulting scenario appears to be more profitable than the previous strategy combination $s_{0,M}$. However - as will be shown below - a clever choice of $P_0^{S\to I}$ will prevent B from changing his strategy.

Suppose first that B's change in strategy results in the strategy combination $s_{1,B}$, i.e. in a strategy combination in which I_R would accept B's offer. Such a change in strategy is profitable for B if the strategy combination $s_{1,B}$ promises him a higher payoff than the strategy combination $s_{0,M}$, i.e. if

$$E\left[\hat{U}^B(s_{1,B})\right] > E\left[\hat{U}^B(s_{0,M})\right].$$

With

$$E\left[\hat{U}^B(s_{0,M})\right] = m\left(\hat{\Lambda} - P_0^{B\to I}\right) + (\nu - m)\left(\hat{\Lambda} - P^{I\to B}\right)$$
$$= m\left(P^{I\to B} - P_0^{B\to I}\right) + \nu\left(\hat{\Lambda} - P^{I\to B}\right)$$

it follows that

$$\nu\left(\hat{\Lambda} - P_1^{B\to I}\right) > m\left(P^{I\to B} - P_0^{B\to I}\right) + \nu\left(\hat{\Lambda} - P^{I\to B}\right)$$

and hence that

$$P_1^{B\to I} < \frac{m}{\nu} P_0^{B\to I} + \left(1 - \frac{m}{\nu}\right) P^{I\to B}.$$

Consider now that with $P_0^{B\to I} < P^{I\to B}$ and $m \leq \nu$ it follows that

$$\begin{aligned}\frac{m}{\nu} P_0^{B\to I} + \left(1 - \frac{m}{\nu}\right) P^{I\to B} &\geq \frac{m}{\nu} P_0^{B\to I} + \left(1 - \frac{m}{\nu}\right) P_0^{B\to I} \\ &= P_0^{B\to I}.\end{aligned}$$

But since $P_1^{B\to I} < P_0^{B\to I} \leq \frac{m}{\nu} P_0^{B\to I} + \left(1 - \frac{m}{\nu}\right) P^{I\to B}$ the inequality

$$P_1^{B\to I} < \frac{m}{\nu} P_0^{B\to I} + \left(1 - \frac{m}{\nu}\right) P^{I\to B}.$$

is automatically satisfied. The commercial buyer B would therefore certainly prefer such an outcome over the original strategy combination $s_{0,M}$.

Suppose now that B's change in strategy results in a strategy combination $s_{1,DT}$ involving direct trade between B and S. Such an outcome would be profitable for B if

$$E\left[\hat{U}^B(s_{1,DT})\right] > E\left[\hat{U}^B(s_{0,M})\right],$$

i.e. if

$$(\nu - m)\left(\hat{\Lambda} - P^{I\to B}\right) > m\left(\hat{\Lambda} - P_0^{B\to I}\right) + (\nu - m)\left(\hat{\Lambda} - P^{I\to B}\right).$$

This implies that

$$0 > m\left(\hat{\Lambda} - P_0^{B\to I}\right)$$

and hence that $P_0^{B\to I} > \hat{\Lambda}$. This means that such a change is profitable for B, if his initial offer was higher than the discounted expected value of $P(T)$. Hence, under the proper circumstances, also this outcome scenario might be advantageous for B.

The remaining possibility is that B's change in strategy leads to an outcome scenario in which the intermediary I_R decides to accept S's offer. Since this forces B to accept I_R's offer of $P^{I\to B}$ this appears to be the worst possible case for B and in fact it is. Such a scenario would increase B's payoff if

$$E\left[\hat{U}^B(s_{1,S})\right] > E\left[\hat{U}^B(s_{0,M})\right]$$

and therefore if

$$\nu\left(\hat{\Lambda} - P^{I\to B}\right) > m\left(\hat{\Lambda} - P_0^{B\to I}\right) + (\nu - m)\left(\hat{\Lambda} - P^{I\to B}\right).$$

It follows that

$$m\left(\hat{\Lambda} - P^{I\to B}\right) > m\left(\hat{\Lambda} - P_0^{B\to I}\right)$$

and therefore that $P_0^{B\to I} > P^{I\to B}$. In other words this means that such an outcome would only be more favourable than s_0 if B would have initially offered a price $P_0^{B\to I}$ that was higer (and therefore from his point of view worse) than I_R's offer $P^{I\to B}$. But offering such a price is purely disadvantageous since I_R had already offered a better price. Hence, B will never be interested in a change from $s_{0,M}$ to $s_{1,S}$.

The previous results can be summarized as follows:

Lemma 5.2.1 (Model II: *B*'s view on different outcome scenarios)
Consider a combination of price offers $\left(P_0^{S\to I}, P_0^{B\to I}\right)$ with

$$P^{I\to S} < P_0^{S\to I} \le P_0^{B\to I} < P^{I\to B},$$

i.e. a strategy combination $s_{0,M} = \left(P_0^{S\to I}, P_0^{B\to I}, s_+^M, s_-^R\right)$ in which the market maker I_M actively takes part in the trade.

If B is then changing his offer to an offer satisfying $P_1^{B\to I} < P_0^{S\to I} \le P_0^{B\to I}$, the following can be said about whether or not B will prefer the resulting strategy combination $s_{1,S}$, $s_{1,B}$ or $s_{1,DT}$ over the original strategy combination $s_{0,M}$:

1. *No matter how B's original offer $P_0^{B\to I}$ was set, B will always prefer $s_{0,M}$ over $s_{1,S}$.*
2. *If $P_0^{B\to I}$ satisfies $P_0^{B\to I} > \hat{\Lambda}$, a change from $s_{0,M}$ to $s_{1,DT}$ will be profitable for B. Otherwise such a change will be disadvantageous for B.*
3. *A change from $s_{0,M}$ to $s_{1,B}$ will always be profitable for B.*

It follows that if it could be made sure that I_R does not opt for the strategy s_B^R and in the case of $P_0^{B\to I} > \hat{\Lambda}$ also not for the strategy s_-^R, then B will not be able to increase his profit by setting an offer $P_1^{B\to I} < P_0^{S\to I} \le P_0^{B\to I}$. This means that B would certainly decide not to change his offer in such a way and rather stick to his old offer $P_0^{B\to I}$.

As it will be shown, S is able to guarantee such a behaviour if he only sets his initial offer $P_0^{S\to I}$ in a clever way:

Proposition 5.2.2 (Model II: An upper bound on $P^{S\to I}$)
Suppose that $\alpha, \nu, R_S, R_B > 0$.
Consider a combination of offers $\left(P_0^{S\to I}, P_0^{B\to I}\right)$ with

$$P^{I\to S} < P_0^{S\to I} \leq P_0^{B\to I} < P^{I\to B},$$

i.e. a strategy combination

$$s_{0,M} = \left(P_0^{S\to I}, P_0^{B\to I}, s_+^M, s_-^R\right)$$

that results in the market maker I_M being involved in the trade.
If S's offer $P_0^{S\to I}$ satisfies

$$P_0^{S\to I} \leq \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}, \tag{5.19}$$

B will not be able to singlehandedly generate a higher payoff by changing his offer $P_0^{B\to I}$ to an offer $P_1^{B\to I}$ satisfying $P_1^{B\to I} < P_0^{S\to I} \leq P_0^{B\to I}$. This means that B will not be able to profitably switch to an outcome scenario that does not involve the market maker I_M.

Proof. Consider first the case $P_0^{B\to I} \leq \hat{\Lambda}$. By Lemma 5.2.1 only a switch to the strategy combination $s_{1,B}$ appears to be profitable for B in this case. This implies that just one of the following inequalities has to hold in order to assure that a strategy change $P_1^{B\to I} < P_0^{S\to I} \leq P_0^{B\to I}$ cannot be profitable for B:

$$E\left[\hat{U}^{I_R}(s_{1,S})\right] > E\left[\hat{U}^{I_R}(s_{1,B})\right] \tag{5.20}$$

or

$$E\left[\hat{U}^{I_R}(s_{1,DT})\right] > E\left[\hat{U}^{I_R}(s_{1,B})\right]. \tag{5.21}$$

Inequality (5.20) implies the following:

$$\nu\left(P^{I\to B} - \hat{\Lambda}\right) + \alpha\left(\hat{\Lambda} - P_0^{S\to I}\right) > \nu\left(P_1^{B\to I} - \hat{\Lambda}\right) + \alpha\left(\hat{\Lambda} - P^{I\to S}\right).$$

Simplifying yields

$$\nu P^{I\to B} - \alpha P_0^{S\to I} > \nu P_1^{B\to I} - \alpha P^{I\to S}.$$

It follows that

$$P_1^{B\to I} < P^{I\to B} - \frac{\alpha}{\nu}\left(P_0^{S\to I} - P^{I\to S}\right).$$

Since by assumption $P_1^{B\to I} < P_0^{S\to I}$, S can make sure that the above inequality holds by simply setting his primary offer in the following way:

$$P_0^{S\to I} \leq P^{I\to B} - \frac{\alpha}{\nu}\left(P_0^{S\to I} - P^{I\to S}\right).$$

It follows that

$$P_0^{S\to I} \leq \frac{P^{I\to B} + \frac{\alpha}{\nu}P^{I\to S}}{1 + \frac{\alpha}{\nu}}$$

and hence that

$$P_0^{S\to I} \leq \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}. \tag{5.22}$$

Consider now that

$$E\left[\hat{U}^{I_R}(s_{1,DT})\right] = \alpha\left(\hat{\Lambda} - P^{I\to S}\right) + \nu\left(P^{I\to B} - \hat{\Lambda}\right) + m\left(P^{I\to S} - P^{I\to B}\right)$$

and that

$$E\left[\hat{U}^{I_R}(s_{1,B})\right] = \nu\left(P_1^{B\to I} - \hat{\Lambda}\right) + \alpha\left(\hat{\Lambda} - P^{I\to S}\right).$$

Inequality (5.21) will therefore hold if

$$\nu\left(P^{I\to B} - \hat{\Lambda}\right) + m\left(P^{I\to S} - P^{I\to B}\right) > \nu\left(P_1^{B\to I} - \hat{\Lambda}\right).$$

This yields

$$m\left(P^{I\to S} - P^{I\to B}\right) > \nu\left(P_1^{B\to I} - P^{I\to B}\right).$$

It follows that I_R will prefer $s_{1,DT}$ over $s_{1,B}$ if B's offer $P_1^{B\to I}$ satisfies

$$P_1^{B\to I} < P^{I\to B} + \frac{m}{\nu}\left(P^{I\to S} - P^{I\to B}\right).$$

Since by assumption $P_1^{B\to I} < P_0^{S\to I}$ S could assure that this inequality holds by offering a price $P_0^{S\to I}$ satisfying

$$P_0^{S\to I} \leq P^{I\to B} + \frac{m}{\nu}\left(P^{I\to S} - P^{I\to B}\right). \tag{5.23}$$

The results (5.22) and (5.23) suggest that S can make sure that one of the inequalities (5.20) and (5.21) holds by setting

$$P_0^{S\to I} \le \max\left\{\frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}, P^{I\to B} + \frac{m}{\nu}\left(P^{I\to S} - P^{I\to B}\right)\right\}.$$

Note that if $m = \nu$

$$P^{I\to B} + \frac{m}{\nu}\left(P^{I\to S} - P^{I\to B}\right) = P^{I\to S}.$$

The assumption $P^{I\to S} < P^{I\to B}$ then trivially implies

$$P^{I\to B} + \frac{m}{\nu}\left(P^{I\to S} - P^{I\to B}\right) = P^{I\to S} < \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}.$$

If on the other hand $m = \alpha$, it follows that

$$\begin{aligned}
P^{I\to B} + \frac{m}{\nu}\left(P^{I\to S} - P^{I\to B}\right) &= P^{I\to B} + \frac{\alpha}{\nu}\left(P^{I\to S} - P^{I\to B}\right) \\
&= \frac{(\alpha+\nu)\, P^{I\to B} + \left(\frac{\alpha^2}{\nu} + \alpha\right)\left(P^{I\to S} - P^{I\to B}\right)}{\alpha+\nu} \\
&= \frac{\nu P^{I\to B} + \alpha P^{I\to S} - \frac{\alpha^2}{\nu}\left(P^{I\to B} - P^{I\to S}\right)}{\alpha+\nu} \\
&< \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}.
\end{aligned}$$

Since for both the case $m = \nu$ and the case $m = \alpha$ it was shown that

$$P^{I\to B} + \frac{m}{\nu}\left(P^{I\to S} - P^{I\to B}\right) < \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}.$$

it follows that

$$\max\left\{\frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}, P^{I\to B} + \frac{m}{\nu}\left(P^{I\to S} - P^{I\to B}\right)\right\} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}.$$

This means that one of the inequalities (5.20) and (5.21) will hold if S sets his initial offer such that

$$P_0^{S\to I} \le \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu} \tag{5.24}$$

is satisfied. Such a choice of $P_0^{S\to I}$ will then prevent B from profitably changing his strategy to a scenario not involving the market maker in the case $P_0^{B\to I} \leq \hat{\Lambda}$.

Suppose now that B's original offer satisfied $P_0^{B\to I} > \hat{\Lambda}$. In this case, Lemma 5.2.1 implies that both $s_{1,B}$ and $s_{1,DT}$ appear to be profitable scenarios for B. This means that a change of the form $P_1^{B\to I} < P_0^{S\to I} \leq P_0^{B\to I}$ can only be avoided, if I_R would opt for $s_{1,S}$ no matter how B sets his offer. This means that the following two inequalities have to hold simultaneously:

$$E\left[\hat{U}^{I_R}(s_{1,S})\right] > E\left[\hat{U}^{I_R}(s_{1,B})\right] \tag{5.25}$$

and

$$E\left[\hat{U}^{I_R}(s_{1,S})\right] > E\left[\hat{U}^{I_R}(s_{1,DT})\right]. \tag{5.26}$$

Consider first that inequality (5.25) is the same as inequality (5.20) which has already been discussed earlier. By the previous result, it can therefore be assured that inequality (5.25) holds if S's original offer is set according to

$$P_0^{S\to I} \leq \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}. \tag{5.27}$$

Considering that

$$E\left[\hat{U}^{I_R}(s_{1,S})\right] = \nu\left(P^{I\to B} - \hat{\Lambda}\right) + \alpha\left(\hat{\Lambda} - P_0^{S\to I}\right)$$

and that

$$E\left[\hat{U}^{I_R}(s_{1,DT})\right] = \alpha\left(\hat{\Lambda} - P^{I\to S}\right) + \nu\left(P^{I\to B} - \hat{\Lambda}\right) + m\left(P^{I\to S} - P^{I\to B}\right)$$

the second inequality (5.26) yields that

$$\alpha\left(\hat{\Lambda} - P_0^{S\to I}\right) > \alpha\left(\hat{\Lambda} - P^{I\to S}\right) + m\left(P^{I\to S} - P^{I\to B}\right).$$

Simplifying yields

$$\alpha\left(P^{I\to S} - P_0^{S\to I}\right) > m\left(P^{I\to S} - P^{I\to B}\right)$$

and it follows that

$$P_0^{S\to I} < P^{I\to S} + \frac{m}{\alpha}\left(P^{I\to B} - P^{I\to S}\right). \tag{5.28}$$

In order for both inequalities (5.27) and (5.28) to be satisfied, S needs to set $P_0^{S\to I}$ such that

$$P_0^{S\to I} \leq \min\left\{\frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}, P^{I\to S} + \frac{m}{\alpha}\left(P^{I\to B} - P^{I\to S}\right)\right\},$$

with equality only possible if

$$\min\left\{\frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}, P^{I\to S} + \frac{m}{\alpha}\left(P^{I\to B} - P^{I\to S}\right)\right\} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}.$$

Consider now that if $m = \alpha$

$$P^{I\to S} + \frac{m}{\alpha}\left(P^{I\to B} - P^{I\to S}\right) = P^{I\to B}.$$

The assumption $P^{I\to S} < P^{I\to B}$ then trivially implies

$$\frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu} < P^{I\to B} = P^{I\to S} + \frac{m}{\alpha}\left(P^{I\to B} - P^{I\to S}\right).$$

If on the other hand $m = \nu$ then $P^{I\to B} - P^{I\to S} > 0$ implies

$$\begin{aligned}
P^{I\to S} + \frac{m}{\alpha}\left(P^{I\to B} - P^{I\to S}\right) &= P^{I\to S} + \frac{\nu}{\alpha}\left(P^{I\to B} - P^{I\to S}\right)\\
&= \frac{(\alpha+\nu)\left(P^{I\to S} + \frac{\nu}{\alpha}\left(P^{I\to B} - P^{I\to S}\right)\right)}{\alpha+\nu}\\
&= \frac{\alpha P^{I\to S} + \nu P^{I\to B} + \frac{\nu^2}{\alpha}\left(P^{I\to B} - P^{I\to S}\right)}{\alpha+\nu}\\
&> \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}.
\end{aligned}$$

Hence it has been shown that for $m = \nu$ and $m = \alpha$

$$\min\left\{\frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}, P^{I\to S} + \frac{m}{\alpha}\left(P^{I\to B} - P^{I\to S}\right)\right\} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}.$$

It follows that in the case $P_0^{B\to I} > \hat{\Lambda}$ the commercial seller S can prevent B from switching from a scenario involving the market maker I_M into one not involving I_M by choosing any price satisfying

$$P_0^{S\to I} \leq \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}. \tag{5.29}$$

Note that (5.29) is the same bound as the bound (5.24) that was derived for the case $P_0^{B\to I} \leq \hat{\Lambda}$. It follows that - no matter how $P_0^{B\to I}$ was set - if $P_0^{S\to I}$ satisfies the above inequality, an offer $P_1^{B\to I} < P_0^{S\to I} \leq P_0^{B\to I}$ will not improve B's payoff. □

The following corollary shows that if $P_0^{S\to I}$ does not satisfy the bound (5.19), B will in fact be able to profitably switch to a scenario not involving I_M:

Corollary 5.2.3
Suppose that $\alpha, \nu, R_S, R_B > 0$.
Consider a strategy combination $s_{0,M} = \left(P_0^{S\to I}, P_0^{B\to I}, s_+^M, s_-^R\right)$ with

$$P_0^{S\to I} > \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}.$$

Then B will be able to singlehandedly initiate a profitable change to a scenario not involving I_M.

Proof. Suppose $\frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu} < P_0^{S\to I} \leq P_0^{B\to I}$, i.e. suppose there is a $d > 0$ such that

$$P_0^{S\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B} + d}{\alpha + \nu}.$$

Suppose furthermore that B changes his offer $P_0^{B\to I}$ to an offer $P_1^{B\to I}$ satisfying

$$P_1^{B\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}.$$

Then since $P_1^{B\to I} < P_0^{S\to I}$, the spread $P_1^{B\to I} - P_0^{S\to I}$ will be negative. This implies that the market maker I_M will not enter the trade. The resulting outcome scenario will therefore depend on the decision of I_R. I_R's payoff for a strategy combination $s_{1,B}$ can be found as

$$\begin{aligned}
E\left[\hat{U}^{I_R}(s_{1,B})\right] &= \nu\left(P_1^{B\to I} - \hat{\Lambda}\right) + \alpha\left(\hat{\Lambda} - P^{I\to S}\right) \\
&= \nu\left(\frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu} - \hat{\Lambda}\right) + \alpha\left(\hat{\Lambda} - P^{I\to S}\right) \\
&= \frac{\left(\alpha\nu - \alpha^2 - \alpha\nu\right) P^{I\to S} + \nu^2 P^{I\to B} + \left(\alpha^2 - \nu^2\right)\hat{\Lambda}}{\alpha + \nu} \\
&= \frac{\nu^2 P^{I\to B} + \left(\alpha^2 - \nu^2\right)\hat{\Lambda} - \alpha^2 P^{I\to S}}{\alpha + \nu}.
\end{aligned}$$

For a strategy combination $s_{1,S}$ it follows that

$$\begin{aligned}
E\left[\hat{U}^{I_R}(s_{1,S})\right] &= \nu\left(P^{I\to B} - \hat{\Lambda}\right) + \alpha\left(\hat{\Lambda} - P_0^{S\to I}\right) \\
&= \nu\left(P^{I\to B} - \hat{\Lambda}\right) + \alpha\left(\hat{\Lambda} - \frac{\alpha P^{I\to S} + \nu P^{I\to B} + d}{\alpha+\nu}\right) \\
&= \frac{\left(\alpha\nu + \nu^2 - \alpha\nu\right) P^{I\to B} + \left(\alpha^2 - \nu^2\right)\hat{\Lambda} - \alpha^2 P^{I\to S} - \alpha d}{\alpha+\nu} \\
&= \frac{\nu^2 P^{I\to B} + \left(\alpha^2 - \nu^2\right)\hat{\Lambda} - \alpha^2 P^{I\to S} - \alpha d}{\alpha+\nu} \\
&= E\left[\hat{U}^{I_R}(s_{1,B})\right] - \frac{\alpha d}{\alpha+\nu} < E\left[\hat{U}^{I_R}(s_{1,B})\right].
\end{aligned}$$

A case distinction for the strategy combination $s_{1,DT}$ yields

$$\begin{aligned}
E\left[\hat{U}^{I_R}(s_{1,DT})\right] &= (\alpha - m)\left(\hat{\Lambda} - P^{I\to S}\right) + (\nu - m)\left(P^{I\to B} - \hat{\Lambda}\right) \\
&= \begin{cases} (\alpha-\nu)\left(\hat{\Lambda} - P^{I\to S}\right) & \text{if } m = \nu \\ (\nu-\alpha)\left(P^{I\to B} - \hat{\Lambda}\right) & \text{if } m = \alpha \end{cases} \\
&= \begin{cases} \frac{\nu^2 P^{I\to S} + \left(\alpha^2-\nu^2\right)\hat{\Lambda} - \alpha^2 P^{I\to S}}{\alpha+\nu} & \text{if } m = \nu \\ \frac{\nu^2 P^{I\to B} + \left(\alpha^2-\nu^2\right)\hat{\Lambda} - \alpha^2 P^{I\to B}}{\alpha+\nu} & \text{if } m = \alpha \end{cases} \\
&< \frac{\nu^2 P^{I\to B} + \left(\alpha^2 - \nu^2\right)\hat{\Lambda} - \alpha^2 P^{I\to S}}{\alpha+\nu} \\
&= E\left[\hat{U}^{I_R}(s_{1,B})\right].
\end{aligned}$$

This shows that I_R can expect the highest payoff if the strategy combination $s_{1,B}$ is played. He will therefore play the strategy s_B^R. This means that B's changed price offer has induced a switch from the strategy combination $s_{0,M}$ to the combination $s_{1,B}$. But by Lemma 5.2.1 a switch from $s_{0,M}$ to $s_{1,B}$ will be profitable for B. It follows that B has singlehandedly improved his payoff by setting the offer

$$P_1^{B\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}.$$

5.2.2 A lower bound on B's price offer

The above result shows that if a combination of offers $\left(P_0^{S\to I}, P_0^{B\to I}\right)$ induces the market maker I_M to enter the trade (i.e. if $P_0^{S\to I} \leq P_0^{B\to I}$), the commercial buyer

B will not be able to profitably change to an outcome scenario not involving I_M if only $P_0^{S\to I}$ is set in a clever way. The purpose of this section is to show that also S will not be able to make such a strategy change if $P_0^{B\to I}$ satisfies a very similar condition.

Consider first that S cannot increase his payoff by lowering his offer to $P_1^{S\to I} < P_0^{S\to I}$. Since $P_1^{S\to I} < P_0^{S\to I} \leq P_0^{B\to I}$, the market maker would still be involved and would this time buy m units from S for a lower price. The remaining $\alpha - m$ units would still be sold to I_R for $P^{I\to S}$. The overall payoff of S would therefore have decreased.

This means that as in the previous case there are two possibilities how such a profitable change could occur:

1. If $P_0^{S\to I} < P_0^{B\to I}$, S could offer a price $P_1^{S\to I}$ such that $P_0^{S\to I} < P_1^{S\to I} \leq P_0^{B\to I}$. I_M would still be involved and S would sell to I_M for a higher price. This would certainly increase S's payoff.
2. S could offer a price $P_1^{S\to I}$ satisfying $P_0^{S\to I} \leq P_0^{B\to I} < P_1^{S\to I}$. Now the spread is negative and hence I_M would not take part in the trade. Depending on I_R's decision such a choice of strategy might prove to be profitable for S.

As in the previous discussion the focus will be put on the second possibility. The first case will be discussed later.

Suppose now that S changed his strategy to a price offer $P_1^{S\to I}$ satisfying $P_0^{S\to I} \leq P_0^{B\to I} < P_1^{S\to I}$. Since the spread is negative, I_M will not take part in the trade and the outcome will depend on I_R's decision.

Depending on I_R's behaviour the new strategy combination

$$s_1 = \left(P_1^{S\to I}, P_0^{B\to I}, s_-^M, s^R\right)$$

can lead to three outcome scenarios:

1. a strategy combination $s_{1,S} = \left(P_1^{S\to I}, P_0^{B\to I}, s_-^M, s_S^R\right)$, i.e. a scenario where I_R accepts S's offer.
   The relevant payoffs of S and I_R are

$$E\left[\hat{U}^S(s_{1,S})\right] = \alpha\left(P_1^{S\to I} - \hat{\Lambda}\right)$$
$$E\left[\hat{U}^{I_R}(s_{1,S})\right] = \nu\left(P^{I\to B} - \hat{\Lambda}\right) + \alpha\left(\hat{\Lambda} - P_1^{S\to I}\right).$$

2. a strategy combination $s_{1,B} = \left(P_1^{S\to I}, P_0^{B\to I}, s_-^M, s_B^R\right)$, i.e. a scenario where I_R accepts B's offer.
   The relevant payoffs of S and I_R are
$$E\left[\hat{U}^S(s_{1,B})\right] = \alpha\left(P^{I\to S} - \hat{\Lambda}\right)$$
$$E\left[\hat{U}^{I_R}(s_{1,B})\right] = \nu\left(P_0^{B\to I} - \hat{\Lambda}\right) + \alpha\left(\hat{\Lambda} - P^{I\to S}\right).$$

3. a strategy combination $s_{1,DT} = \left(P_1^{S\to I}, P_0^{B\to I}, s_-^M, s_-^R\right)$, i.e. a scenario where I_R declines both offers.
   The relevant payoffs of S and I_R are in this case
$$E\left[\hat{U}^S(s_{1,DT})\right] = (\alpha - m)\left(P^{I\to S} - \hat{\Lambda}\right)$$
$$E\left[\hat{U}^{I_R}(s_{1,DT})\right] = (\alpha - m)\left(\hat{\Lambda} - P^{I\to S}\right) + (\nu - m)\left(P^{I\to B} - \hat{\Lambda}\right).$$

Consider now that for S a switch from $s_{0,M}$ to $s_{1,S}$ would be profitable if
$$E\left[\hat{U}^S(s_{1,S})\right] > E\left[\hat{U}^S(s_{0,M})\right]$$
and hence if
$$\alpha\left(P_1^{S\to I} - \hat{\Lambda}\right) > m\left(P_0^{S\to I} - \hat{\Lambda}\right) + (\alpha - m)\left(P^{I\to S} - \hat{\Lambda}\right).$$
It follows that
$$P_1^{S\to I} > P^{I\to S} + \frac{m}{\alpha}\left(P_0^{S\to I} - P^{I\to S}\right)$$
or by rearranging terms that
$$P^{S\to I} > \frac{m}{\alpha}P_0^{S\to I} + \left(1 - \frac{m}{\alpha}\right)P^{I\to S}.$$
With $P_0^{S\to I} > P^{I\to S}$ and $m \leq \alpha$ it follows that
$$\frac{m}{\alpha}P_0^{S\to I} + \left(1 - \frac{m}{\alpha}\right)P^{I\to S} \leq \frac{m}{\alpha}P_0^{S\to I} + \left(1 - \frac{m}{\alpha}\right)P_0^{S\to I} = P_0^{S\to I}.$$
This implies
$$P_1^{S\to I} > P_0^{S\to I} \geq \frac{m}{\alpha}P_0^{S\to I} + \left(1 - \frac{m}{\alpha}\right)P^{I\to S}.$$

It would therefore always be advantageous for the commercial seller to switch to the strategy combination $s_{1,S}$.

Suppose now that S's change in strategy results in a scenario S and B trade m units directly. Such an outcome would improve S's payoff if

$$E\left[\hat{U}^S(s_{1,DT})\right] > E\left[\hat{U}^S(s_{0,M})\right],$$

i.e. if

$$(\alpha - m)\left(P^{I\to S} - \hat{\Lambda}\right) > m\left(P_0^{S\to I} - \hat{\Lambda}\right) + (\alpha - m)\left(P^{I\to S} - \hat{\Lambda}\right).$$

It follows that

$$0 > m\left(P_0^{S\to I} - \hat{\Lambda}\right).$$

This implies that S would show interest into changing into a direct trade scenario if $P_0^{S\to I} < \hat{\Lambda}$, i.e. if his initial offer was to sell the good G for less than the discounted expected value.

The last alternative, the strategy combination $s_{1,B}$ appears to be purely disadvantageous for S: with $P_0^{S\to I} > P^{I\to S}$ it follows that

$$\begin{aligned} E\left[\hat{U}^S(s_{1,B})\right] &= \alpha\left(P^{I\to S} - \hat{\Lambda}\right) \\ &= m\left(P^{I\to S} - \hat{\Lambda}\right) + (\alpha - m)\left(P^{I\to S} - \hat{\Lambda}\right) \\ &< m\left(P^{S\to I} - \hat{\Lambda}\right) + (\alpha - m)\left(P^{I\to S} - \hat{\Lambda}\right) \\ &= E\left[\hat{U}^S(s_{0,M})\right]. \end{aligned}$$

This shows that S would always prefer an outcome $s_{0,M}$ over an outcome $s_{1,B}$.

The previous results can be summarized as follows:

Lemma 5.2.4 (Model II: *S*'s view on different outcome scenarios)
Consider a strategy combination $s_{0,M} = \left(P_0^{S\to I}, P_0^{B\to I}, s_+^M, s_-^R\right)$ *with*

$$P^{I\to S} < P_0^{S\to I} \leq P_0^{B\to I} < P^{I\to B},$$

i.e. a strategy combination that results in the market maker I_M being involved in the trade. If S is changing his offer to an offer satisfying $P_0^{S\to I} \leq P_0^{B\to I} < P_1^{S\to I}$, the following can be said about possible outcome scenarios:

1. *S will always prefer $s_{0,M}$ over $s_{1,B}$.*

2. *A change from $s_{0,M}$ to $s_{1,S}$ will always be profitable for S.*

3. *If $P_0^{S\to I} < \hat{\Lambda}$, a change from $s_{0,M}$ to $s_{1,DT}$ will be profitable. Otherwise such a change will not increase S's payoff.*

This means in other words that S would not opt to change his old offer $P_0^{S\to I}$ to an offer $P_1^{S\to I}$ satisfying $P_0^{S\to I} \leq P_0^{B\to I} < P_1^{S\to I}$ if such a change would result in the outcome scenario $s_{1,B}$ (or, if $P_0^{S\to I} \geq \hat{\Lambda}$ also $s_{1,DT}$), i.e. if I_R would prefer $s_{1,B}$ (or $s_{1,DT}$) no matter what S is offering.

I_R will behave in such a way if B is setting his initial offer such that it satisfies a similar bound than the one derived in the previous section:

Proposition 5.2.5 (Model II: A lower bound on $P^{B\to I}$)
Suppose that $\alpha, \nu, R_S, R_B > 0$.
Consider a strategy combination $s_{0,M} = \left(P_0^{S\to I}, P_0^{B\to I}, s_+^M, s_-^R\right)$ with

$$P^{I\to S} < P_0^{S\to I} \leq P_0^{B\to I} < P^{I\to B},$$

i.e. a strategy combination that results in the market maker I_M being involved in the trade.

If B's offer $P_0^{B\to I}$ satisfies

$$P_0^{B\to I} \geq \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}, \tag{5.30}$$

then S will not be able to singlehandedly generate a higher payoff by changing his offer $P_0^{S\to I}$ to an offer $P_1^{S\to I}$ satisfying $P_0^{S\to I} \leq P_0^{B\to I} < P_1^{S\to I}$. This means that S will not be able to profitably switch to an outcome scenario that does not involve the market maker I_M.

Proof. Suppose first that $P_0^{S\to I} \geq \hat{\Lambda}$. In this case, Lemma 5.2.4 implies that only a change from $s_{0,M}$ to $s_{1,S}$ would be profitable for S. Such a change can be prevented if I_R would prefer $s_{1,B}$ or $s_{1,DT}$ no matter what price S is offering. This means that one of the following inequalities has to hold:

$$E\left[\hat{U}^{I_R}(s_{1,B})\right] > E\left[\hat{U}^{I_R}(s_{1,S})\right] \tag{5.31}$$

or

$$E\left[\hat{U}^{I_R}(s_{1,DT})\right] > E\left[\hat{U}^{I_R}(s_{1,S})\right]. \tag{5.32}$$

Using that

$$\begin{aligned} E\left[\hat{U}^{I_R}(s_{1,B})\right] &= \nu\left(P_0^{B\to I} - \hat{\Lambda}\right) + \alpha\left(\hat{\Lambda} - P^{I\to S}\right) \\ &= \nu P_0^{B\to I} - \alpha P^{I\to S} + (\alpha - \nu)\hat{\Lambda} \end{aligned}$$

and

$$\begin{aligned} E\left[\hat{U}^{I_R}(s_{1,S})\right] &= \nu\left(P^{I\to B} - \hat{\Lambda}\right) + \alpha\left(\hat{\Lambda} - P_1^{S\to I}\right) \\ &= \nu P^{I\to B} - \alpha P_1^{S\to I} + (\alpha - \nu)\hat{\Lambda}, \end{aligned}$$

inequality (5.31) implies that

$$\nu P_0^{B\to I} - \alpha P^{I\to S} > \nu P^{I\to B} - \alpha P_1^{S\to I}.$$

This yields that

$$P_1^{S\to I} > P^{I\to S} + \frac{\nu}{\alpha}\left(P^{I\to B} - P_0^{B\to I}\right).$$

Since by assumption $P_1^{S\to I} > P_0^{B\to I}$, B can make sure that this inequality holds by setting $P_0^{B\to I}$ such that it satisfies

$$P_0^{B\to I} \geq P^{I\to S} + \frac{\nu}{\alpha}\left(P^{I\to B} - P_0^{B\to I}\right).$$

It follows that

$$P_0^{B\to I} \geq \frac{P^{I\to S} + \frac{\nu}{\alpha}P^{I\to B}}{1 + \frac{\nu}{\alpha}}$$

and hence that

$$P_0^{B\to I} \geq \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}. \tag{5.33}$$

With

$$\begin{aligned} E\left[\hat{U}^{I_R}(s_{1,DT})\right] &= \alpha\left(\hat{\Lambda} - P^{I\to S}\right) + \nu\left(P^{I\to B} - \hat{\Lambda}\right) + m\left(P^{I\to S} - P^{I\to B}\right) \\ &= \nu P^{I\to B} - \alpha P^{I\to S} + (\alpha - \nu)\hat{\Lambda} + m\left(P^{I\to S} - P^{I\to B}\right) \end{aligned}$$

it follows that inequality (5.32)

$$E\left[\hat{U}^{I_R}(s_{1,DT})\right] > E\left[\hat{U}^{I_R}(s_{1,S})\right]$$

holds if

$$m\left(P^{I\to S} - P^{I\to B}\right) > \alpha\left(P^{I\to S} - P_1^{S\to I}\right).$$

This implies

$$P_1^{S\to I} > P^{I\to S} - \frac{m}{\alpha}\left(P^{I\to S} - P^{I\to B}\right).$$

Since by assumption $P_1^{S\to I} > P_0^{B\to I}$, B could assure that this inequality holds by offering a price $P_0^{B\to I}$ satisfying

$$P_0^{B\to I} \geq P^{I\to S} - \frac{m}{\alpha}\left(P^{I\to S} - P^{I\to B}\right). \tag{5.34}$$

The results (5.33) and (5.34) imply that B can make sure that one of the inequalities (5.31) and (5.32) holds by setting

$$P_0^{B\to I} \geq \min\left\{\frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}, P^{I\to S} - \frac{m}{\alpha}\left(P^{I\to S} - P^{I\to B}\right)\right\}.$$

Note that if $m = \alpha$

$$P^{I\to S} - \frac{m}{\alpha}\left(P^{I\to S} - P^{I\to B}\right) = P^{I\to B}.$$

Since $\alpha, \nu > 0$ and $P^{I\to S} < P^{I\to B}$ it follows that

$$P^{I\to S} - \frac{m}{\nu}\left(P^{I\to S} - P^{I\to B}\right) = P^{I\to B} \geq \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}.$$

If on the other hand $m = \nu$, it follows that

$$\begin{aligned}
P^{I\to S} - \frac{m}{\alpha}\left(P^{I\to S} - P^{I\to B}\right) &= P^{I\to S} - \frac{\nu}{\alpha}\left(P^{I\to S} - P^{I\to B}\right) \\
&= \frac{(\alpha+\nu)\, P^{I\to S} - \left(\frac{\nu^2}{\alpha} + \nu\right)\left(P^{I\to S} - P^{I\to B}\right)}{\alpha+\nu} \\
&= \frac{\nu P^{I\to B} + \alpha P^{I\to S} + \frac{\nu^2}{\alpha}\left(P^{I\to B} - P^{I\to S}\right)}{\alpha+\nu} \\
&> \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}.
\end{aligned}$$

Since for both the case $m = \nu$ and the case $m = \alpha$ it was shown that

$$P^{I\to S} - \frac{m}{\alpha}\left(P^{I\to S} - P^{I\to B}\right) > \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}.$$

it follows that

$$\min\left\{\frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}, P^{I\to S} - \frac{m}{\alpha}\left(P^{I\to S} - P^{I\to B}\right)\right\} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}.$$

This means that at least one of the inequalities (5.31) and (5.32) will hold if B sets his initial offer such that

$$P_0^{B\to I} \geq \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu} \tag{5.35}$$

is satisfied. Such a choice of $P_0^{B\to I}$ will then prevent S from profitably changing his strategy to a scenario not involving the market maker in the case $P_0^{S\to I} \geq \hat{\Lambda}$.

Suppose now that $P_0^{S\to I} < \hat{\Lambda}$. In this case both a change to $s_{1,S}$ and to $s_{1,DT}$ will be of interest for S. A change in strategy can therefore only be avoided, if I_R would prefer $s_{1,B}$ over both $s_{1,S}$ and $s_{1,DT}$. This means that the inequalities

$$E\left[\hat{U}^{I_R}(s_{1,B})\right] > E\left[\hat{U}^{I_R}(s_{1,S})\right] \tag{5.36}$$

and

$$E\left[\hat{U}^{I_R}(s_{1,B})\right] > E\left[\hat{U}^{I_R}(s_{1,DT})\right]. \tag{5.37}$$

have to hold simultaneously.

Note that the discussion of the case $P_0^{S\to I} \geq \hat{\Lambda}$ implies that inequality (5.36) is satisfied if

$$P_0^{B\to I} \geq \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}. \tag{5.38}$$

Inequality (5.36) will be satisfied if

$$\nu\left(P_0^{B\to I} - \hat{\Lambda}\right) + \alpha\left(\hat{\Lambda} - P^{I\to S}\right) > \alpha\left(\hat{\Lambda} - P^{I\to S}\right) + \nu\left(P^{I\to B} - \hat{\Lambda}\right) + m\left(P^{I\to S} - P^{I\to B}\right).$$

It follows that

$$\nu\left(P_0^{B\to I} - \hat{\Lambda}\right) > \nu\left(P^{I\to B} - \hat{\Lambda}\right) + m\left(P^{I\to S} - P^{I\to B}\right)$$

and hence that

$$\nu\left(P_0^{B\to I} - P^{I\to B}\right) > m\left(P^{I\to S} - P^{I\to B}\right)$$

This yields that

$$P_0^{B\to I} > P^{I\to B} - \frac{m}{\nu}\left(P^{I\to B} - P^{I\to S}\right). \tag{5.39}$$

In order for both inequalities (5.38) and (5.39) to be satisfied at the same time, B needs to set $P_0^{B\to I}$ such that

$$P_0^{B\to I} \geq \max\left\{\frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}, P^{I\to B} - \frac{m}{\nu}\left(P^{I\to B} - P^{I\to S}\right)\right\},$$

with equality only possible if

$$\max\left\{\frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}, P^{I\to B} - \frac{m}{\alpha}\left(P^{I\to B} - P^{I\to S}\right)\right\} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}.$$

Consider now that if $m = \nu$

$$P^{I\to B} - \frac{m}{\nu}\left(P^{I\to B} - P^{I\to S}\right) = P^{I\to S}.$$

Since by assumption $\alpha, \nu > 0$ and $P^{I\to S} < P^{I\to B}$ it follows that

$$\frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu} > P^{I\to S} = P^{I\to B} - \frac{m}{\nu}\left(P^{I\to B} - P^{I\to S}\right).$$

If on the other hand $m = \alpha$ then $P^{I\to B} - P^{I\to S} > 0$ implies

$$\begin{aligned}
P^{I\to B} - \frac{m}{\nu}\left(P^{I\to B} - P^{I\to S}\right) &= P^{I\to B} - \frac{\alpha}{\nu}\left(P^{I\to B} - P^{I\to S}\right) \\
&= \frac{(\alpha+\nu)\left(P^{I\to B} - \frac{\alpha}{\nu}\left(P^{I\to B} - P^{I\to S}\right)\right)}{\alpha+\nu} \\
&= \frac{\alpha P^{I\to S} + \nu P^{I\to B} - \frac{\alpha^2}{\nu}\left(P^{I\to B} - P^{I\to S}\right)}{\alpha+\nu} \\
&< \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}.
\end{aligned}$$

Hence it follows that for both $m = \nu$ and $m = \alpha$

$$\max\left\{\frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}, P^{I\to B} - \frac{m}{\nu}\left(P^{I\to B} - P^{I\to S}\right)\right\} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}.$$

This implies that in the case $P_0^{S\to I} < \hat{\Lambda}$ the commercial buyer B can prevent S from switching from a scenario involving the market maker I_M into one not involving I_M by choosing any price satisfying

$$P_0^{B\to I} \geq \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}. \tag{5.40}$$

Note that the bound (5.40) derived for the case $P_0^{S\to I} < \hat{\Lambda}$ and the bound (5.35) derived for the case $P_0^{S\to I} \geq \hat{\Lambda}$ are the same. It follows that - no matter how $P_0^{S\to I}$ was set - if $P_0^{B\to I}$ satisfies the above inequality, an offer $P_1^{S\to I}$ with $P_0^{S\to I} \leq P_0^{B\to I} < P_1^{S\to I}$ will not improve S's payoff. □

Similarly as with the bound for $P_0^{S\to I}$ it can also be shown for this bound that - as soon as inequality (5.30) is not satisfied - S will be able to profitably switch to a scenario not involving I_M:

Corollary 5.2.6
Suppose that $\alpha, \nu, R_S, R_B > 0$.
Consider a strategy combination $s_{0,M} = \left(P_0^{S\to I}, P_0^{B\to I}, s_+^M, s_-^R\right)$ with

$$P_0^{B\to I} < \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}.$$

Then S will be able to singlehandedly initiate a profitable change to a scenario not involving I_M.

Proof. Suppose $P_0^{S\to I} \leq P_0^{B\to I} < \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}$, i.e. suppose there is a $d > 0$ such that

$$P_0^{B\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B} - d}{\alpha + \nu}.$$

Suppose furthermore that S changes his offer $P_0^{S\to I}$ to an offer $P_1^{S\to I}$ satisfying

$$P_1^{S\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}.$$

Then the spread $P_0^{B\to I} - P_1^{S\to I} = -\frac{d}{\alpha+\nu}$ will be negative. This implies that the market maker I_M will not enter the trade. The resulting outcome scenario will therefore depend on the decision of I_R. If I_R opts for the strategy s_S his payoff will be

$$\begin{aligned}
E\left[\hat{U}^{I_R}(s_{1,S})\right] &= \nu\left(P^{I\to B} - \hat{\Lambda}\right) + \alpha\left(\hat{\Lambda} - P_1^{S\to I}\right) \\
&= \nu\left(P^{I\to B} - \hat{\Lambda}\right) + \alpha\left(\hat{\Lambda} - \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}\right) \\
&= \frac{\left(\alpha\nu + \nu^2 - \alpha\nu\right) P^{I\to B} + \left(\alpha^2 - \nu^2\right)\hat{\Lambda} - \alpha^2 P^{I\to S}}{\alpha+\nu} \\
&= \frac{\nu^2 P^{I\to B} + \left(\alpha^2 - \nu^2\right)\hat{\Lambda} - \alpha^2 P^{I\to S}}{\alpha+\nu}.
\end{aligned}$$

If the strategy combination $s_{1,B}$ is played, I_R's payoff turns out to be

$$\begin{aligned}
E\left[\hat{U}^{I_R}(s_{1,B})\right] &= \nu\left(P_0^{B\to I} - \hat{\Lambda}\right) + \alpha\left(\hat{\Lambda} - P^{I\to S}\right) \\
&= \nu\left(\frac{\alpha P^{I\to S} + \nu P^{I\to B} - d}{\alpha+\nu} - \hat{\Lambda}\right) + \alpha\left(\hat{\Lambda} - P^{I\to S}\right) \\
&= \frac{\left(\alpha\nu - \alpha^2 + \alpha\nu\right) P^{I\to S} + \nu^2 P^{I\to B} + \left(\alpha^2 - \nu^2\right)\hat{\Lambda} - \nu d}{\alpha+\nu} \\
&= \frac{\nu^2 P^{I\to B} + \left(\alpha^2 - \nu^2\right)\hat{\Lambda} - \alpha^2 P^{I\to S}}{\alpha+\nu} - \frac{\nu d}{\alpha+\nu} \\
&= E\left[\hat{U}^{I_R}(s_{1,S})\right] - \frac{\nu d}{\alpha+\nu} < E\left[\hat{U}^{I_R}(s_{1,S})\right].
\end{aligned}$$

Finally, if the strategy combination $s_{1,DT}$ will be played, I_R's payoff can be computed as

$$\begin{aligned}
E\left[\hat{U}^{I_R}(s_{DT})\right] &= (\alpha - m)\left(\hat{\Lambda} - P^{I\to S}\right) + (\nu - m)\left(P^{I\to B} - \hat{\Lambda}\right) \\
&= \begin{cases} (\alpha-\nu)\left(\hat{\Lambda} - P^{I\to S}\right) & \text{if } m = \nu \\ (\nu-\alpha)\left(P^{I\to B} - \hat{\Lambda}\right) & \text{if } m = \alpha \end{cases} \\
&= \begin{cases} \frac{\nu^2 P^{I\to S} + \left(\alpha^2-\nu^2\right)\hat{\Lambda} - \alpha^2 P^{I\to S}}{\alpha+\nu} & \text{if } m = \nu \\ \frac{\nu^2 P^{I\to B} + \left(\alpha^2-\nu^2\right)\hat{\Lambda} - \alpha^2 P^{I\to B}}{\alpha+\nu} & \text{if } m = \alpha \end{cases} \\
&< \frac{\nu^2 P^{I\to B} + \left(\alpha^2 - \nu^2\right)\hat{\Lambda} - \alpha^2 P^{I\to S}}{\alpha+\nu} \\
&= E\left[\hat{U}^{I_R}(s_{1,S})\right].
\end{aligned}$$

This shows that I_R will play the strategy s_S^R since the strategy combination $s_{1,S}$ is promising him the highest expected payoff. But by Lemma 5.2.4 a switch from $s_{0,M}$ to $s_{1,S}$ will be profitable for S. It follows that S has singlehandedly improved his payoff by setting the offer

$$P_1^{S\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}.$$

5.2.3 A Nash equilibrium solution

In the previous two sections it has been shown that if

$$P^{I\to S} < P_0^{S\to I} \leq P_0^{B\to I} < P^{I\to B},$$

a clever choice of $P_0^{S\to I}$ and $P_0^{B\to I}$ can prevent B and S from profitably changing to a scenario that does not involve the market maker. What has not been discussed yet is whether or not S and B can still profitably change their strategy as long as the resulting strategy combination does still involve the market maker I_M. If for offers $\left(P_0^{S\to I}, P_0^{B\to I}\right)$ no such change is possible, a Nash equilibrium has been found. In fact, it can be shown that one such combination of offers exists:

Theorem 5.2.7 (Model II: A Nash equilibrium solution involving I_M)
Suppose that $\alpha, \nu, R_S, R_B > 0$.
Consider a strategy combination $s_{0,M} = \left(P_0^{S\to I}, P_0^{B\to I}, s_+^M, s_-^R\right)$ with

$$P^{I\to S} < P_0^{S\to I} \leq P_0^{B\to I} < P^{I\to B},$$

i.e. a strategy combination that results in the market maker I_M being involved in the trade.

Then $s_{0,M}$ is a Nash equilibrium if and only if $P_0^{B\to I}$ and $P_0^{S\to I}$ satisfy

$$P_0^{S\to I} = P_0^{B\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}. \tag{5.41}$$

Proof. Consider first a strategy combination $s_{0,M} = \left(P_0^{S\to I}, P_0^{B\to I}, s_+^M, s_-^R\right)$ with

$$P_0^{S\to I} = P_0^{B\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}.$$

Proposition 5.2.2 then implies that B will not be able to singlehandedly increase his profit by offering a price $P_1^{B\to I}$ satisfying $P_1^{B\to I} < P_0^{S\to I} = P_0^{B\to I}$. Furthermore, also an offer satisfying $P_0^{S\to I} = P_0^{B\to I} < P_1^{B\to I}$ will not increase B's payoff: since the spread $P_1^{B\to I} - P_0^{S\to I}$ is positive, I_M will still be involved, only this time B will buy the needed units for a higher price. Hence B will not be able to increase his payoff by singlehandedly changing his strategy.

Similarly, also S will not be able to singlehandedly improve his payoff: on the one hand an offer $P_1^{S\to I}$ with $P_0^{S\to I} = P_0^{B\to I} < P_1^{S\to I}$ will not increase S's payoff according to Proposition 5.2.5; on the other hand an offer $P_1^{S\to I}$ satisfying $P_1^{S\to I} < P_0^{S\to I} = P_0^{B\to I}$ will only result in S selling v units for a lower price to I_M.

Since neither S nor B can singlehandedly improve their payoff this shows that $s_{0,M}$ is in fact a Nash equilibrium.

Suppose now that a strategy combination $s_{0,M} = \left(P_0^{S\to I}, P_0^{B\to I}, s_+^M, s_-^R\right)$ with

$$P^{I\to S} < P_0^{S\to I} \leq P_0^{B\to I} < P^{I\to B}$$

is a Nash equilibrium. Then neither S nor B should be able to singlehandedly improve their payoff.

Corollary 5.2.3 implies that B could profitably change to a scenario not involving the market maker if

$$P_0^{S\to I} > \frac{\alpha P^{I\to S} + v P^{I\to B}}{\alpha + v}.$$

This implies that $s_{0,M}$ can only be a Nash equlibrium if

$$P_0^{S\to I} \leq \frac{\alpha P^{I\to S} + v P^{I\to B}}{\alpha + v}.$$

Similarly, Corollary 5.2.6 implies that S can only be prevented from switching to a scenario without I_M if

$$P_0^{B\to I} \geq \frac{\alpha P^{I\to S} + v P^{I\to B}}{\alpha + v}.$$

Suppose now that

$$P_0^{S\to I} < \frac{\alpha P^{I\to S} + v P^{I\to B}}{\alpha + v} \leq P_0^{B\to I}.$$

Then S could change his offer $P_0^{S\to I}$ to an offer $P_1^{S\to I}$ satisfying

$$P_0^{S\to I} < P_1^{S\to I} \leq \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}.$$

In such a case the spread $P_0^{B\to I} - P_1^{S\to I}$ would still be nonnegative and hence I_M would still be involved in the trade. This implies that I_M would now buy $m = \min\{\alpha, \nu\}$ units for a price $P_1^{S\to I} > P_0^{S\to I}$, i.e. S would sell m units for a higher price. Since the remaining $\alpha - m$ would still be sold to I_R for $P^{I\to S}$, this means that S would have singlehandedly increased his payoff. It follows that a strategy combination with

$$P_0^{S\to I} < \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}$$

cannot be a Nash equilibrium. This implies that for $s_{0,M}$ to be a Nash equilibirum S's offer $P_0^{S\to I}$ has to satisfy

$$P_0^{S\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}.$$

Similarly, if

$$P_0^{S\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu} < P_0^{B\to I}$$

B could singlehandedly improve his payoff by offering a price $P_1^{B\to I}$ with

$$P_0^{B\to I} > P_1^{B\to I} \geq \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}$$

since B would then buy the needed $m = \min\{\alpha, \nu\}$ units for a lower price from I_M. Hence for $s_{0,M}$ to be a Nash equilibrium $P_0^{B\to I}$ has to satisfy

$$P_0^{B\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}.$$

But this shows that if a strategy combination $s_{0,M}$ with

$$P^{I\to S} < P_0^{S\to I} \leq P_0^{B\to I} < P^{I\to B},$$

is a Nash equilibrium, then

$$P_0^{S\to I} = P_0^{B\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}.$$

This proves the claim. □

5.3 Interpretation of the Nash equilibrium solution

In the previous two sections it was shown that

1. no strategy combination with a negative price spread can be a Nash equilibrium.
2. exactly one strategy combination with a nonnegative price spread will be a Nash equilibrium.

Combining these results, the following can be said about a market with a commercial buyer B, a commercial seller S and two intermediaries I_R and I_M:

Theorem 5.3.1 (Model II: Existence & Uniqueness of a Nash equilibrium) *Consider a market as described above with $\alpha, \nu, R_S, R_B > 0$.*

Then a strategy combination $s_0 = \left(P_0^{S\to I}, P_0^{B\to I}, s^M, s^R\right)$ is a Nash equilibrium if and only if $P_0^{B\to I}$ and $P_0^{S\to I}$ satisfy

$$P_0^{S\to I} = P_0^{B\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}. \tag{5.42}$$

Proof. The theorem follows directly from theorems 5.1.4 and 5.2.7. □

Since the strategy combination $s_{0,M} = \left(P_0^{S\to I}, P_0^{B\to I}, s_+^M, s_-^R\right)$ with

$$P_0^{S\to I} = P_0^{B\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}.$$

is a unique Nash equilibrium solution, trades will occur in a completely predictable fashion (assuming that all players act perfectly rational[1]):

1. S will sell $m = \min\{\alpha, \nu\}$ units to I_M for

$$P_0^{S\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}.$$

2. B will buy $m = \min\{\alpha, \nu\}$ units from I_M for

$$P_0^{B\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}.$$

3. I_R will buy/sell the remaining $|\alpha - \nu|$ units for $P^{I\to S}$ or $P^{I\to B}$ respectively.

[1] see section 1.2.4

5.3.1 A fair forward price

Theorem 5.3.1 suggests that $\min\{\alpha, \nu\}$ units of G will be traded at a price of

$$F = \frac{\alpha P^{I \to S} + \nu P^{I \to B}}{\alpha + \nu}.$$

Considering that $P^{I \to S} = (1 - R_S)\,\hat{\Lambda}$ and $P^{I \to B} = (1 + R_B)\,\hat{\Lambda}$ it follows that

$$\begin{aligned} F &= \frac{\alpha P^{I \to S} + \nu P^{I \to B}}{\alpha + \nu} \\ &= \left(\frac{\alpha\,(1 - R_S) + \nu\,(1 + R_B)}{\alpha + \nu} \right) \hat{\Lambda} \\ &= \left(1 - \left(\frac{\alpha}{\alpha + \nu} \right) R_S + \left(\frac{\nu}{\alpha + \nu} \right) R_B \right) \hat{\Lambda}. \end{aligned}$$

It can be seen that this price depends on all parameters that were assumed to be relevant for this market model in quite a natural fashion:

1. $\hat{\Lambda}$: the expected price of G at time $t = T$ discounted to time $t = 0$ is the base price. If $\hat{\Lambda}$ changes, also F will change correspondingly.

2. R_S: the risk premium S is willing to pay determines a lower limit for F since

$$\begin{aligned} F &= \left(1 - \left(\frac{\alpha}{\alpha + \nu} \right) R_S + \left(\frac{\nu}{\alpha + \nu} \right) R_B \right) \hat{\Lambda} \\ &\geq \left(1 - \left(\frac{\alpha}{\alpha + \nu} \right) R_S \right) \hat{\Lambda} \\ &\geq (1 - R_S)\,\hat{\Lambda}. \end{aligned}$$

   Furthermore, if everything else is assumed to be constant, an increased risk premium R_S will result in F selling $\min\{\alpha, \nu\}$ units of G for a lower price, since

$$\frac{\partial F}{\partial R_S} = -\left(\frac{\alpha}{\alpha + \nu} \right) \hat{\Lambda} < 0.$$

   In other words: if S is willing to pay more for transferring his risk, he will sell G for a cheaper price.

3. R_B: similarly to R_S also B's risk premium influences F. An upper limit of F can be found as

$$\begin{aligned} F &= \left(1 - \left(\frac{\alpha}{\alpha+\nu}\right) R_S + \left(\frac{\nu}{\alpha+\nu}\right) R_B\right) \hat{\Lambda} \\ &\leq \left(1 + \left(\frac{\nu}{\alpha+\nu}\right) R_B\right) \hat{\Lambda} \\ &\leq (1 + R_B)\, \hat{\Lambda}. \end{aligned}$$

   Assuming that all other parameters remain constant, increasing R_B will increase F:

$$\frac{\partial F}{\partial R_B} = \left(\frac{\nu}{\alpha+\nu}\right) \hat{\Lambda} > 0.$$

   This means that if B is willing to pay more for a risk transfer, buying G will be more expensive.

4. α, ν: supply α and demand ν will determine how close F will be to the extreme values $P^{I\to S} = (1 - R_S)\, \hat{\Lambda}$ and $P^{I\to B} = (1 + R_B)\, \hat{\Lambda}$.

   If the supply α is increased while everything else remains constant, F will decrease since

$$\begin{aligned} \frac{\partial F}{\partial \alpha} &= -\frac{\nu}{(\alpha+\nu)^2} R_S \hat{\Lambda} - \frac{\nu}{(\alpha+\nu)^2} R_B \hat{\Lambda} \\ &= -\frac{\nu}{(\alpha+\nu)^2} (R_S + R_B)\, \hat{\Lambda} \\ &< 0. \end{aligned}$$

   This result is not surprising: basic economic theory suggests that an increase in supply that is not accompanied by an increase in demand should result in a lower price.

   In the case of a huge supply surplus, i.e. if $\alpha >> \nu$, S will be under a lot more pressure than B to hedge his risk. This pressure is reflected in the forward price F:

$$\begin{aligned} F &= \left(1 - \underbrace{\left(\frac{\alpha}{\alpha+\nu}\right)}_{\approx 1} R_S + \underbrace{\left(\frac{\nu}{\alpha+\nu}\right)}_{\approx 0} R_B\right) \hat{\Lambda} \\ &\approx (1 - R_S)\, \hat{\Lambda}. \end{aligned}$$

If on the other hand ν is increased while α is kept constant, F will increase as well:

$$\begin{aligned}\frac{\partial F}{\partial \nu} &= \frac{\alpha}{(\alpha+\nu)^2} R_S \hat{\Lambda} + \frac{\alpha}{(\alpha+\nu)^2} R_B \hat{\Lambda} \\ &= \frac{\alpha}{(\alpha+\nu)^2} (R_S + R_B) \hat{\Lambda} \\ &> 0.\end{aligned}$$

In a market characterized by an extreme demand surplus $\nu >> \alpha$, most of the hedging pressure will be on B's side of the trade. Hence

$$\begin{aligned}F &= \Bigg(1 - \underbrace{\left(\frac{\alpha}{\alpha+\nu}\right)}_{\approx 0} R_S + \underbrace{\left(\frac{\nu}{\alpha+\nu}\right)}_{\approx 1} R_B\Bigg) \hat{\Lambda} \\ &\approx (1 + R_B) \hat{\Lambda}.\end{aligned}$$

Finally, if $\alpha \approx \nu$ similar pressure will apply to both S and B. The price F will therefore mainly depend on how R_S and R_B are set:

$$\begin{aligned}F &= \Bigg(1 - \underbrace{\left(\frac{\alpha}{\alpha+\nu}\right)}_{\approx \frac{1}{2}} R_S + \underbrace{\left(\frac{\nu}{\alpha+\nu}\right)}_{\approx \frac{1}{2}} R_B\Bigg) \hat{\Lambda} \\ &\approx \left(1 + \frac{1}{2}(R_B - R_S)\right) \hat{\Lambda}.\end{aligned}$$

If furthermore both S and B are willing to pay roughly the same risk premium $R_S \approx R_B$ then

$$F \approx \hat{\Lambda},$$

i.e. the lion's share of the trade will take place for approximately the discounted price $\hat{\Lambda}$.

5.3.2 The role of the intermediaries

In the previous model (Model I) with only one intermediary I trading both price spreads and risk, it was shown that I will usually be involved in more than just the trade of the supply-demand surplus.

A similiar observation can be made here:

1. The risk trader I_R will only trade the supply-demand surplus:
   - if $\alpha > \nu$, I_R will buy $\alpha - \nu$ units at time $t = 0$ from S for $P^{I \to S}$ (i.e. the cheapest possible price) and sell them at time $t = T$ for the cash market price $P(T)$.
   - if $\nu > \alpha$, I_R will sell $\nu - \alpha$ units at time $t = 0$ to B for $P^{I \to B}$ (i.e. the highest possible price) and buy them at time $t = T$ for the cash market price $P(T)$.
2. The market maker I_M will be involved in all remaining trades. For a price of

$$\frac{\alpha P^{I \to S} + \nu P^{I \to B}}{\alpha + \nu}.$$

   he will
   - buy $\min\{\alpha, \nu\}$ units from S
   - and sell the same amount to B.

   Both trades will occur at time $t = 0$.

This shows that I_R and I_M will in fact be involved in all occuring trades. Their pure existence will prevent S and B from involving into any kind of direct trade. This agrees with observations made in daily life: the presence of intermediaries in markets usually leads to them trading almost the complete supply and demand.

On the other hand, the motivation for I_M to enter this trade is not completely clear. Since he is selling and buying for the same price, it follows that his payoff satisfies

$$\begin{aligned} E\left[\hat{U}^{I_M}(s_{0,M})\right] &= \min\{\alpha, \nu\}\left(P_0^{B \to I} - P_0^{S \to I}\right) \\ &= 0. \end{aligned}$$

While this does at least not constitute a monetary loss for I_M, it does however mean that I_M would work for free. He would invest his time and attention to carry out a trade without the prospect of a monetary return.

This outcome is due to the assumption of I_M trading every nonnegative price spread, i.e. in particular also a zero price spread. Simply excluding a zero price spread (and hence allowing I_R to enter the trade in the case of price offers $P_0^{S \to I} = P_0^{B \to I}$) wouldn't really solve this problem:

1. The reader may convince himself that in a market where I_M is only trading positive spreads, a Nash equilibrium $s_{0,M} = \left(P_0^{S\to I}, P_0^{B\to I}, s_+^M, s_-^R\right)$ would have to satisfy[2]
$$P_0^{S\to I} < \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu} < P_0^{B\to I}.$$

2. But if continuous pricing is allowed, any offer $P_0^{S\to I}$ satisfying $P_0^{S\to I} < \frac{\alpha P^{I\to S}+\nu P^{I\to B}}{\alpha+\nu}$ would allow for another offer $P_1^{S\to I}$ with
$$P_0^{S\to I} < P_1^{S\to I} < \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}.$$
Since this new offer $P_1^{S\to I}$ would result in a higher payoff for S it follows that $P_0^{S\to I}$ couldn't have been part of a Nash equilibrium solution. Analogous arguments apply to $P_0^{B\to I}$. Hence no Nash equilibrium can exist.

3. If on the other hand trading takes place on a discrete price scale, a Nash equilibrium will exist. $P_0^{S\to I}$ and $P_0^{B\to I}$ will be set such that
$$P_0^{S\to I} < \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu} < P_0^{B\to I}$$
is satisfied and such that $P_0^{S\to I}$ and $P_0^{B\to I}$ are as close to each other as possible. The latter means that depending on whether or not
$$F = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}$$
is a legal price on a discrete price scale with minimal tick size τ, the price spread will satisfy
$$P_0^{B\to I} - P_0^{S\to I} = \begin{cases} \tau & \text{if } F \text{ is not legal.} \\ 2\tau & \text{if } F \text{ is legal.} \end{cases}$$
But even this Nash equilibrium would only give a very weak explanation for I_M's participation in the trade: I_M's payoff would now be positive, but only to the smallest possible extent, since
$$\begin{aligned} E\left[\hat{U}^{I_M}\left(s_{0,M}\right)\right] &= \min\{\alpha,\nu\}\left(P_0^{B\to I} - P_0^{S\to I}\right) \\ &= \begin{cases} \tau\min\{\alpha,\nu\} & \text{if } F \text{ is not legal.} \\ 2\tau\min\{\alpha,\nu\} & \text{if } F \text{ is legal.} \end{cases} \end{aligned}$$

[2] as a version of Theorem 5.2.7

> Considering that the tick size is usually only a very small fraction of the actual price of a traded good, it becomes apparent, that simply receiving one tick per unit might not be enough to justify the amount of work that I_M has to invest in order to trade as much as $\min\{\alpha, \nu\}$ units.

In an analysis of the effective transaction fee received by market makers, Locke and Venkatesh (see [LV91]) show that while for some futures contracts market makers are willing to trade for an average revenue of a tick or even less per trade, some contracts show a significantly higher transaction cost (for example currency futures). Hence, an extension of the previous model needs to take into account that the previous assumption of I_M entering the trade whenever the price spread is nonnegative (or even positive) does not sufficiently describe I_M's motivation to participate in the market. I_M needs an additional monetary incentive to enter a trade: a fixed transaction fee.

Part IV

Transaction costs and the existence of an equilibrium price

On most financial exchanges, the matchmaking process between potential buyers and sellers is operated on a matched bargain or order driven basis: when offers by a buyer and a seller match up, the trade will be executed by the exchange's matching system. In the previous model a market maker was introduced to simulate this type of matchmaking.

It was shown that the existence of such a market maker leads to a completely predictable equilibrium solution. This solution can however not explain why a market maker would enter a trade in the first place. To explain a market maker's participation an additional monetary incentive is necessary: a fixed transaction fee.

This corresponds to the structure of real-world financial exchanges: a financial exchange supplies services to its participants in order to facilitate trades. The existence of for example the specialist on the NYSE can be seen as such a service. In return however, the exchange will charge its participants for the services it supplies.

In the following model, a market maker I_M will execute a trade as soon as a commercial buyer B and a commercial seller S offer prices with a nonnegative price spread

$$P^{B \to I} - P^{S \to I} \geq 0.$$

In return for executing the trade, I_M will charge a transaction fee T_M that buyer B and seller S have to pay per traded unit of G.

It will be shown that the introduction of such a transaction fee will result in an equilibrium solution analogous to the one observed in the previous model. However, depending on how the transaction fee T_M was set such an equilibrium solution will not always exist. Hence the market participants will under certain conditions again behave in an unpredictable manner.

Chapter 6

Model III: A market with transaction costs

In the following model, we will consider a market structured in complete analogy to the previous market model. Only one of the previous assumptions will be dropped: trading with the market maker I_M will now involve the payment of a transaction fee T_M.

6.1 Basic market structure

Four participants - a commercial buyer B, a commercial seller S, a market maker I_M and a risk-neutral intermediary I_R - will trade a good G on a market $\mathcal{M}$. The four participants can fulfil their needs by either trading with each other directly at time $t = 0$ or by accessing the cash market C at time $t = T$.

The following assumptions are made on the market structure:

1. Accessing the cash market C as well as trading directly with S, B or I_R will be free of transaction costs.

2. The spread trader/market maker I_M will charge a transaction fee of T_M per unit of G that is traded with him.

3. There will be no cost of carry and no convenience yield.

4. Trading will take place on a continuous price scale.

5. There will be no bid/ask spread on the cash market.

6. No participant in the cash market C can affect the cash market price $P(t)$ by his trading. Therefore $P(t)$ can be seen as an exogenous parameter: it is a random variable.

7. All players know that $P(t)$ is a random variable. Furthermore the expected value of $P(t)$ at time $t = T$,

$$\Lambda = E\left[P(T)\right]$$

is known to all players.

8. The return of riskless investments is known to be r.

9. Supply, or production, α and demand, or consumption, ν of G will be deterministic and known to all players at time $t = 0$.

Each of the four players enters the market with a certain motivation:
S and B are under hedging pressure and want to minimize the possible variation in their returns. They are willing to pay certain risk premia R_S and R_B for transferring this type of risk at time $t = 0$ to one of the other players. This means that at time $t = T$ S wants to have sold all his supply for at least $(1 - R_S)\Lambda$ per unit while B wants have bought all he demands for not more than $(1 + R_B)\Lambda$ per unit.

I_R is a risk-neutral intermediary, entering the market with the goal of earning an expected profit higher than the risk-free return. The market maker I_M is trading on the turn, i.e. he is trying to make a risk-free profit by trading a nonnegative price spread.

I_M will furthermore generate income by charging a transaction fee T_M for his services. It will be assumed that this transaction fee is a fair reflection of the amount of work that I_M has to invest in order to execute a trade. In other words: while I_M could still realize a monetary profit by trading a negative price spread

$$P^{B\to I} - P^{S\to I} < 0$$

with

$$|P^{B\to I} - P^{S\to I}| < T_M,$$

this profit would be too small to warrant his participation in a trade.

The matchmaking process in a trade will be just like in the previous case:

1. I_R will make binding offers $P^{I\to B} = (1 + R_B)\,\hat{\Lambda}$ and $P^{I\to S} = (1 - R_S)\,\hat{\Lambda}$, on which S and B will react with their respective price offers $P^{S\to I}$ and $P^{B\to I}$.

2. If the spread satisfies

$$P^{B\to I} - P^{S\to I} \geq 0,$$

the market maker I_M will enter the trade: he will buy $m = \min\{\alpha, \nu\}$ units of G from S for $P^{S\to I}$ and sell them to B for $P^{B\to I}$ per unit.

3. I_R will decide whether or not he will accept $P^{S\to I}$ and $P^{B\to I}$. If he accepts a price, all still available units will be traded at that price.

4. (a) If *both* S and B have not fulfilled their supply or demand needs yet, they will trade the maximal amount $\min\{\alpha, \nu\}$ directly for a price $P^{S,B} = \hat{\Lambda}$.

   (b) If *either* S or B still has open supply or demand needs (because of $\alpha \neq \nu$), the respective player will trade the remaining units with I_R for the posted price $P^{I\to S}$ or $P^{I\to B}$ respectively, if these prices are acceptable. Otherwise this player will take the risk himself and realize his supply or demand needs at time $t = T$ on the cash market.

6.2 Strategies and payoffs

As discussed in more detail previously, the only strategic decision made by player S is his price offer $P^{S\to I}$, B's only strategic action his price offer $P^{B\to I}$. I_R has the choice between

- s^R_B: I_R accepts B's and rejects S's price offer
- s^R_S: I_R accepts S's and rejects B's price offer
- and s^R_-: I_R rejects both price offers.

The market maker can either enter the market or not, hence he has the choice between

- s^M_+: I_M trades $m = \min\{\alpha, \nu\}$ from S to B by accepting their respective price offers

- and s_-^M: I_M does not enter the trade.

One might however wonder, whether or not I_M should also be allowed to set the transaction costs T_M as part of his strategy. As it will be shown later, I_M's participation in a trade and hence his payoff will depend on how T_M relates to α, ν, R_B, R_S and Λ. If I_M would therefore be allowed to adjust T_M in a strategic decision, he could make sure that he will be involved in the resulting equilibrium solution.

Allowing I_M to adjust T_M would however contradict the very idea of these transaction costs: it was assumed earlier that the transaction costs T_M represent the fair monetary value of the work and attention I_M has to invest in a trade. But this value should be rather constant on a short time horizon. Furthermore, it can be observed in daily life that transaction costs are quite stable and are changed only on rare occasions. Key features for a financial market place to attract its customers are price transparency and stability. Such features would however be missing if I_M would be allowed to adjust the transaction costs T_M in every single round. T_M will therefore be assumed to be only an exogenous parameter and not a part of I_M's strategy.

Note that the structure of the matchmaking process again implies that I_M's and I_R's strategy choices will automatically be best responses to S's and B's price offers, effectively reducing the game to a two-person game between S and B. Since the transaction costs T_M are not part of I_M's strategy, It follows furthermore that the strategy setup is the same as in the previous model. Hence also in this model four possible outcome scenarios can be identified:

- $s_B = \left(P^{S\to I}, P^{B\to I}, s_-^M, s_B^R\right)$:
  in this scenario, I_M is not involved in the trade and I_R accepts B's and rejects S's offer. The relevant payoffs are

$$\begin{aligned}
E\left[\hat{U}^S\left(s_B\right)\right] &= \alpha\left(P^{I\to S} - \hat{\Lambda}\right)\\
E\left[\hat{U}^B\left(s_B\right)\right] &= \nu\left(\hat{\Lambda} - P^{B\to I}\right)\\
E\left[\hat{U}^{I_M}\left(s_B\right)\right] &= 0\\
E\left[\hat{U}^{I_R}\left(s_B\right)\right] &= \nu P^{B\to I} - \alpha P^{I\to S} + (\alpha - \nu)\,\hat{\Lambda}.
\end{aligned}$$

- $s_S = \left(P^{S\to I}, P^{B\to I}, s_-^M, s_S^R\right)$:
  I_M is not involved in the trade and I_R accepts S's and rejects B's offer. The

relevant payoffs are

$$\begin{aligned} E\left[\hat{U}^S(s_S)\right] &= \alpha\left(P^{S\to I} - \hat{\Lambda}\right) \\ E\left[\hat{U}^B(s_S)\right] &= \nu\left(\hat{\Lambda} - P^{I\to B}\right) \\ E\left[\hat{U}^{I_M}(s_S)\right] &= 0 \\ E\left[\hat{U}^{I_R}(s_S)\right] &= \nu P^{I\to B} - \alpha P^{S\to I} + (\alpha - \nu)\hat{\Lambda}. \end{aligned}$$

- $s_{DT} = \left(P^{S\to I}, P^{B\to I}, s_-^M, s_-^R\right)$:
  I_M is not involved in the trade and I_R rejects both S's and B's offer. The relevant payoffs are

$$\begin{aligned} E\left[\hat{U}^S(s_{DT})\right] &= (\alpha - m)\left(P^{I\to S} - \hat{\Lambda}\right) \\ E\left[\hat{U}^B(s_{DT})\right] &= (\nu - m)\left(\hat{\Lambda} - P^{I\to B}\right) \\ E\left[\hat{U}^{I_M}(s_{DT})\right] &= 0 \\ E\left[\hat{U}^{I_R}(s_{DT})\right] &= (\alpha - m)\left(\hat{\Lambda} - P^{I\to S}\right) + (\nu - m)\left(P^{I\to B} - \hat{\Lambda}\right). \end{aligned}$$

- $s_M = \left(P^{S\to I}, P^{B\to I}, s_+^M, s_-^R\right)$:
  because of a nonnegative price spread I_M will trade $m = \min\{\alpha, \nu\}$ units of G from S to B by accepting their respective price offers. He will charge a transaction fee of T_M per unit from each S and B.

$$\begin{aligned} E\left[\hat{U}^S(s_M)\right] &= m\left(P^{S\to I} - \hat{\Lambda}\right) + (\alpha - m)\left(P^{I\to S} - \hat{\Lambda}\right) - m\,T_M \\ E\left[\hat{U}^B(s_M)\right] &= m\left(\hat{\Lambda} - P^{B\to I}\right) + (\nu - m)\left(\hat{\Lambda} - P^{I\to B}\right) - m\,T_M \\ E\left[\hat{U}^{I_M}(s_M)\right] &= m\left(P^{B\to I} - P^{S\to I}\right) + 2m\,T_M \\ E\left[\hat{U}^{I_R}(s_M)\right] &= (\alpha - m)\left(\hat{\Lambda} - P^{I\to S}\right) + (\nu - m)\left(P^{I\to B} - \hat{\Lambda}\right). \end{aligned}$$

6.3 Feasibility of strategy combinations

One of the key assumptions in the discussion of the previous two models (Model I and Model II) was the complete risk aversity of B and S. Their primary goal is to realize a price corresponding to their respective risk evaluation: by entering the market at time $t = 0$ rather than at time $t = T$, S wants to make sure that all his supply will be sold at time $t = T$ for a price of at least $(1 - R_S)\Lambda$, while B's goal is that whatever he demands at time $t = T$ will have been bought at a price not exceeding $(1 + R_B)\Lambda$. The setup of Model I and Model II guaranteed both S and B the realization of their goals: it was in I_R's very own interest to place standing offers $P^{I\to S} = (1 - R_S)\hat{\Lambda}$ and $P^{I\to B} = (1 + R_B)\hat{\Lambda}$ that just narrowly satisfied these condiitons. As long as S offered a price $P^{S\to I} \geq P^{I\to S}$ and B a price $P^{B\to I} \leq P^{I\to B}$ the worst that could happen to S or B was a trade at $P^{I\to S}$ or $P^{I\to B}$.

With the primary goal of S and B already satisfied, the focus of the game-theoretic analysis was then put on their secondary goals, namely maximizing their respective payoffs. The key questions that were examined were how and at what prices the goods transfer will occur given that every market participant is trying to maximize his respective monetary payoff in a rational fashion. For this purpose every combination of offers $\left(P^{S\to I}, P^{B\to I}\right)$ satisfying $P^{S\to I} \geq P^{I\to S}$ and $P^{B\to I} \leq P^{I\to B}$ was examined for whether or not it could be part of an equilibrium solution. If an equilbrium solution was found it certainly was a feasible solution since no matter how this equilibrium solution was assembled, it at least satisfied S's and B's goals for transferring their operatory risk for a risk premium not exceeding R_S or R_B.

In the discussion of this model however not all combinations of price offers with $P^{S\to I} \geq P^{I\to S}$ and $P^{B\to I} \leq P^{I\to B}$ will be feasible. Clearly, if S's offer $P^{S\to I}$ satisfies

$$P^{I\to S} + T_M \leq P^{S\to I},$$

the resulting outcome would certainly be a legal solution from S's point of view. In a trade with I_R he would at least receive a payment of $P^{I\to S} = (1 - R_S)\hat{\Lambda}$ per unit, and if I_M would enter the trade, S would receive his posted price $P^{S\to I}$ minus the transaction fee T_M that S has to pay for a trade with I_M. But since

$$P^{I\to S} \leq P^{S\to I} - T_M$$

it follows that S would have received at least the desired price of $P^{I\to S}$ per unit. If S's offer however satisfies

$$P^{I\to S} \leq P^{S\to I} < P^{I\to S} + T_M,$$

the feasibility of the resulting outcome scenario will not be guaranteed. If

$$P^{S\to I} \leq P^{B\to I},$$

then S would have failed to meet his primary goal: because of the nonnegative price spread I_M would enter the market and buy $m = \min\{\alpha, \nu\}$ units from S for a price of $P^{S\to I}$. Additionally he would charge a transaction fee of T_M per unit. S would therefore have realized an effective price of

$$P^{S\to I} - T_M < P^{I\to S} = (1 - R_S)\hat{\Lambda}$$

per unit. But this means that S would have failed to realize his primary pricing goal for m of the units he sells and hence that he would have failed his goal as defined in Definition 2.4.1 as a whole.

If on the other hand

$$P^{S\to I} > P^{B\to I},$$

the strategy combination would be perfectly feasible. The trade would now take place without the market maker I_M and hence - since no transaction fees need to be paid - the worst that could happen to S would be a price of $P^{I\to S}$ per unit.

Similarly a strategy combination is not feasible from B's point of view if the price offers satisfy

$$P^{I\to B} - T_M < P^{B\to I} \leq P^{I\to B}$$

and

$$P^{S\to I} \leq P^{B\to I}.$$

All other possible combinations of price offers will be feasible.

The previous discussion suggests the following proposition:

Proposition 6.3.1 (Model III: Non-feasibility of a strategy combination) *Consider a market as described above where I_M charges a transaction fee T_M per traded unit.*
Then a combination of price offers $s = \left(P^{S\to I}, P^{B\to I}\right)$ with $P^{S\to I} \leq P^{B\to I}$ is not feasible if

$$P^{I\to S} \leq P^{S\to I} < P^{I\to S} + T_M$$

or if

$$P^{I\to B} - T_M < P^{B\to I} \leq P^{I\to B}.$$

All other combinations of price offers are feasible as long as $P^{I\to S} \leq P^{S\to I}$ and $P^{B\to I} \leq P^{I\to B}$ are satisfied.

Chapter 7

The existence of an equilibrium solution in Model III

In Model II no transaction fees were charged by the market maker/spread trader I_M. It was shown that then only one equilibrium solution could exist: $m = \min\{\alpha, \nu\}$ units of G were traded from S via I_M to B for a price of

$$F = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu},$$

the remaining $\alpha - m$ or $\nu - m$ were traded from S to I_R or from I_R to B for I_R's posted offers of $P^{I\to S}$ or $P^{I\to B}$ respectively. This equilibrium solution will under certain conditions also be an equilibrium solution in Model III. However, the introduction of transaction fees T_M will under certain conditions reintroduce randomness into the market behaviour.

7.1 Equilibrium analysis for trades with I_M

Suppose first that S and B have set their offers such that I_M will decide to enter the market, i.e. suppose that

$$P^{B\to I} - P^{S\to I} \geq 0.$$

Note that Proposition 6.3.1 implies that such a combination of offers is only feasible if $P^{B\to I}$ and $P^{S\to I}$ satisfy

$$P^{I\to S} + T_M \leq P^{S\to I} \leq P^{B\to I} \leq P^{I\to B} - T_M.$$

Other than for this feasibility restriction it will be shown that a trade with I_M results in the same equilibrium solution as the one identified in the previous model:

Theorem 7.1.1 (Model III: an equilibrium involving I_M)
Consider a market with transaction fee T_M as described above and suppose that $\alpha, \nu, R_S, R_B > 0$.
Consider a strategy combination $s_{0,M} = \left(P_0^{S\to I}, P_0^{B\to I}, s_+^M, s_-^R\right)$ with

$$P^{I\to S} + T_M \le P_0^{S\to I} \le P_0^{B\to I} \le P^{I\to B} - T_M,$$

i.e. a feasible strategy combination that results in the market maker I_M being involved in the trade.

Then $s_{0,M}$ is a Nash equilibrium if and only if $P_0^{B\to I}$ and $P_0^{S\to I}$ satisfy

$$P_0^{S\to I} = P_0^{B\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}. \tag{7.1}$$

To prove this theorem it will be necessary to check whether one of the players can singlehandedly improve his payoff. One way in which an improvement might be possible is a switch to an outcome scenario not involving I_M, i.e. a scenario where M plays his strategy s_-^M. Such a switch will either result in a strategy combination of the type s_B, s_S or s_{DT}. For player S the following can be said about his preferences concerning these three outcome scenarios:

Lemma 7.1.2 (Model III: S's view on a switch to s_-^M)
Suppose that $\alpha, \nu, R_S, R_B > 0$.
Consider a combination of price offers $\left(P_0^{S\to I}, P_0^{B\to I}\right)$ with

$$P^{I\to S} + T_M \le P_0^{S\to I} \le P_0^{B\to I} \le P^{I\to B} - T_M,$$

i.e. a combination of price offers resulting in a strategy combination $s_{0,M}$ involving the market maker I_M:

$$s_{0,M} = \left(P_0^{S\to I}, P_0^{B\to I}, s_+^M, s_-^R\right).$$

If S is changing his offer to an offer satisfying $P_0^{S\to I} \le P_0^{B\to I} < P_1^{S\to I}$, the following can be said about possible outcome scenarios:

1. *S will always prefer $s_{0,M}$ over $s_{1,B} = \left(P_1^{S\to I}, P_0^{B\to I}, s_-^M, s_B^R\right)$.*

2. *A change from* $s_{0,M}$ *to* $s_{1,S} = \left(P_1^{S\to I}, P_0^{B\to I}, s_-^M, s_S^R\right)$ *will always be profitable for* S.

3. *If* $P_0^{S\to I} < \hat{\Lambda} + T_M$*, a change from* $s_{0,M}$ *to* $s_{1,DT} = \left(P_1^{S\to I}, P_0^{B\to I}, s_-^M, s_-^R\right)$ *will be advantageous for* S*. Otherwise such a change will not be profitable for* S.

Proof. Suppose S wants to change from a strategy combination

$$s_{0,M} = \left(P_0^{S\to I}, P_0^{B\to I}, s_+^M, s_-^R\right),$$

where

$$P^{I\to S} + T_M \le P_0^{S\to I} \le P_0^{B\to I} \le P^{I\to B} - T_M,$$

to a strategy combination not involving I_M. He can do so by offering a price $P_1^{S\to I}$ satisfying

$$P_1^{S\to I} > P_0^{B\to I}.$$

Since now I_M's strategy will then be s_-^M, only outcome scenarios of the type s_B, s_S and s_{DT} are possible after such a switch. To discuss the profitability of the possible changes consider first that S's payoff under the old strategy combination $s_{0,M}$ was given by

$$E\left[\hat{U}^S\left(s_{0,M}\right)\right] = m\left(P_0^{S\to I} - \hat{\Lambda}\right) + (\alpha - m)\left(P^{I\to S} - \hat{\Lambda}\right) - m\,T_M.$$

This payoff can now be compared to the payoffs of other possible outcome scenarios as follows:

1. $s_{1,B} = \left(P_1^{S\to I}, P_0^{B\to I}, s_-^M, s_B^R\right)$:
   in this scenario I_R will accept B's offer and S's payoff will be given by

   $$E\left[\hat{U}^S\left(s_{1,B}\right)\right] = \alpha\left(P^{I\to S} - \hat{\Lambda}\right).$$

   Since S will only be interested in such an outcome if

   $$E\left[\hat{U}^S\left(s_{1,B}\right)\right] > E\left[\hat{U}^S\left(s_{0,M}\right)\right],$$

   it follows that

   $$\alpha\left(P^{I\to S} - \hat{\Lambda}\right) > m\left(P_0^{S\to I} - \hat{\Lambda}\right) + (\alpha - m)\left(P^{I\to S} - \hat{\Lambda}\right) - m\,T_M$$

and hence that

$$m\left(P_0^{S\to I} - P^{I\to S} - T_M\right) < 0.$$

This implies that such a change would only be profitable for S if

$$P_0^{S\to I} < P^{I\to S} + T_M.$$

The feasibility of the strategy combination $s_{0,M}$ however implies that

$$P_0^{S\to I} \geq P^{I\to S} + T_M.$$

It follows that a switch from $s_{0,M}$ to $s_{1,B}$ will never be profitable for S.

2. $s_{1,S} = \left(P_1^{S\to I}, P_0^{B\to I}, s_-^M, s_S^R\right)$:
in this scenario, I_R is accepting S's price offer and buys α units of G from S for a price of $P_1^{S\to I}$ per unit. A change from $s_{0,M}$ to $s_{1,S}$ will be profitable for S if

$$E\left[\hat{U}^S\left(s_{1,S}\right)\right] > E\left[\hat{U}^S\left(s_{0,M}\right)\right].$$

Since

$$E\left[\hat{U}^S\left(s_{1,S}\right)\right] = \alpha\left(P_1^{S\to I} - \hat{\Lambda}\right)$$

it follows that

$$\alpha\left(P_1^{S\to I} - \hat{\Lambda}\right) > m\left(P_0^{S\to I} - \hat{\Lambda}\right) + (\alpha - m)\left(P^{I\to S} - \hat{\Lambda}\right) - m\, T_M$$

and hence that

$$P_1^{S\to I} > mP_0^{S\to I} + (\alpha - m)P^{I\to S} - m\, T_M.$$

This implies that a change from $s_{0,M}$ to $s_{1,S}$ will be profitable for S if $P_1^{S\to I}$ satisfies

$$\alpha P_1^{S\to I} > \frac{m}{\alpha}P_0^{S\to I} + \left(1 - \frac{m}{\alpha}\right)P^{I\to S} - \frac{m}{\alpha}\, T_M.$$

With $T_M > 0$, $P_1^{S\to I} > P_0^{S\to I} > P^{I\to S}$ and $0 < m \leq \alpha$ it follows that

$$\begin{aligned}\frac{m}{\alpha}P_0^{S\to I} + \left(1 - \frac{m}{\alpha}\right)P^{I\to S} - \frac{m}{\alpha}\, T_M &< \frac{m}{\alpha}P_0^{S\to I} + \left(1 - \frac{m}{\alpha}\right)P^{I\to S} \\ &\leq \frac{m}{\alpha}P_0^{S\to I} + \left(1 - \frac{m}{\alpha}\right)P_0^{S\to I} \\ &\leq P_0^{S\to I} \\ &< P_1^{S\to I}.\end{aligned}$$

The inequality

$$E\left[\hat{U}^S(s_{1,S})\right] > E\left[\hat{U}^S(s_{0,M})\right]$$

is therefore satisfied for every price offer $P_1^{S\to I} > P_0^{S\to I}$ and it follows that a switch from $s_{0,M}$ to $s_{1,S}$ will always be advantageous for player S.

3. $s_{1,DT} = \left(P_1^{S\to I}, P_0^{B\to I}, s_-^M, s_-^R\right)$:
I_R is rejecting both offers and S and B will involve in a direct trade for a price of $\hat{\Lambda}$ per unit. In such a case, S's payoff can be determined as

$$E\left[\hat{U}^S(s_{1,DT})\right] = (\alpha - m)\left(P^{I\to S} - \hat{\Lambda}\right)$$

This payoff will be an improvement for S if

$$E\left[\hat{U}^S(s_{1,DT})\right] > E\left[\hat{U}^S(s_{0,M})\right]$$

and hence if

$$(\alpha - m)\left(P^{I\to S} - \hat{\Lambda}\right) > m\left(P_0^{S\to I} - \hat{\Lambda}\right) + (\alpha - m)\left(P^{I\to S} - \hat{\Lambda}\right) - m\,T_M.$$

It follows that a switch from $s_{0,M}$ to $s_{1,DT}$ will be advantageous for S if

$$0 > m\left(P_0^{S\to I} - \hat{\Lambda}\right) - m\,T_M.$$

or in other words if

$$P_0^{S\to I} < \hat{\Lambda} + T_M.$$

□

Similarly, if B wants to change from a scenario with I_M to one not involving I_M the following can be said about his preference concerning the possible outcomes:

Lemma 7.1.3 (Model III: *B*'s view on a switch to s_-^M)
Suppose that $\alpha, \nu, R_S, R_B > 0$.
Consider a combination of price offers $\left(P_0^{S\to I}, P_0^{B\to I}\right)$ with

$$P^{I\to S} + T_M \le P_0^{S\to I} \le P_0^{B\to I} \le P^{I\to B} - T_M,$$

i.e. a combination of price offers resulting in a strategy combination $s_{0,M}$ involving the market maker I_M:

$$s_{0,M} = \left(P_0^{S\to I}, P_0^{B\to I}, s_+^M, s_-^R\right).$$

If B is changing his offer to an offer satisfying $P_1^{B\to I} < P_0^{S\to I} \le P_0^{B\to I}$, the following can be said about possible outcome scenarios:

1. *B will always prefer $s_{0,M}$ over $s_{1,S} = \left(P_0^{S\to I}, P_1^{B\to I}, s_-^M, s_S^R\right)$.*

2. *A change from $s_{0,M}$ to $s_{1,B} = \left(P_0^{S\to I}, P_1^{B\to I}, s_-^M, s_B^R\right)$ will always be profitable for B.*

3. *If $P_0^{B\to I} > \hat{\Lambda} - T_M$, a change from $s_{0,M}$ to $s_{1,DT} = \left(P_0^{S\to I}, P_1^{B\to I}, s_-^M, s_-^R\right)$ will be advantageous for B. Otherwise such a change will not be profitable for B.*

Proof. Suppose B wants to initiate a change from a strategy combination

$$s_{0,M} = \left(P_0^{S\to I}, P_0^{B\to I}, s_+^M, s_-^R\right).$$

to one not involving the market maker I_M. He can do so by offering a price satisfying

$$P_1^{B\to I} < P_0^{S\to I} \leq P_0^{B\to I}.$$

With I_M then being forced into playing s_-^M, the remaining possible outcome scenarios are of the type s_S, s_B and s_{DT}. With

$$E\left[\hat{U}^B(s_M)\right] = m\left(\hat{\Lambda} - P_0^{B\to I}\right) + (\nu - m)\left(\hat{\Lambda} - P^{I\to B}\right) - m\,T_M$$

being B's payoff under the old strategy combination $s_{0,M}$, the following can be said about the profitability of a switch:

1. $s_{1,S} = \left(P_0^{S\to I}, P_1^{B\to I}, s_-^M, s_S^R\right)$:
   in such a scenario I_R will accept S's and reject B's offer. B's payoff will therefore be given by

   $$E\left[\hat{U}^B(s_{1,S})\right] = \nu\left(\hat{\Lambda} - P^{I\to B}\right).$$

   B will only be interested in such a switch if he can thereby increase his payoff, i.e. if

   $$E\left[\hat{U}^B(s_{1,S})\right] > E\left[\hat{U}^B(s_{0,M})\right].$$

   This means that

   $$\nu\left(\hat{\Lambda} - P^{I\to B}\right) > m\left(\hat{\Lambda} - P_0^{B\to I}\right) + (\nu - m)\left(\hat{\Lambda} - P^{I\to B}\right) - m\,T_M$$

and hence that

$$m\left(P^{I\to B} - P_0^{B\to I} - T_M\right) < 0.$$

This implies that such a change would only be profitable for S if

$$P_0^{B\to I} > P^{I\to B} - T_M.$$

Since this is in contradiction to the feasibility condition

$$P_0^{B\to I} \leq P^{I\to B} - T_M.$$

it follows that a switch from $s_{0,M}$ to $s_{1,S}$ will never be profitable for B.

2. $s_{1,B} = \left(P_0^{S\to I}, P_1^{B\to I}, s_-^M, s_B^R\right)$:
this strategy combination results in I_R buying ν units from B for B's posted price $P_1^{B\to I}$. The resulting payoff will be

$$E\left[\hat{U}^B\left(s_{1,B}\right)\right] = \nu\left(\hat{\Lambda} - P_1^{B\to I}\right).$$

A change from $s_{0,M}$ to $s_{1,B}$ will be profitable for B if

$$E\left[\hat{U}^B\left(s_{1,B}\right)\right] > E\left[\hat{U}^B\left(s_{0,M}\right)\right].$$

This implies that a change would be profitable if

$$\nu\left(\hat{\Lambda} - P_1^{B\to I}\right) > m\left(\hat{\Lambda} - P_0^{B\to I}\right) + (\nu - m)\left(\hat{\Lambda} - P^{I\to B}\right) - m\,T_M$$

and hence if

$$\nu P_1^{B\to I} < m P_0^{B\to I} + (\nu - m) P^{I\to B} + m\,T_M.$$

It follows that B would like to change from $s_{0,M}$ to $s_{1,B}$ if $P_1^{B\to I}$ satisfies

$$P_1^{B\to I} < \frac{m}{\nu} P_0^{B\to I} + \left(1 - \frac{m}{\nu}\right) P^{I\to B} + \frac{m}{\nu}\,T_M.$$

With $T_M > 0$, $P_1^{B\to I} < P_0^{B\to I} < P^{I\to B}$ and $0 < m \leq \nu$ it follows that

$$\begin{aligned}\frac{m}{\nu} P_0^{B\to I} + \left(1 - \frac{m}{\nu}\right) P^{I\to B} + \frac{m}{\nu}\,T_M &> \frac{m}{\nu} P_0^{B\to I} + \left(1 - \frac{m}{\nu}\right) P^{I\to B} \\ &\geq \frac{m}{\nu} P_0^{B\to I} + \left(1 - \frac{m}{\nu}\right) P_0^{B\to I} \\ &\geq P_0^{B\to I} \\ &> P_1^{B\to I}.\end{aligned}$$

Hence,

$$E\left[\hat{U}^B(s_{1,B})\right] > E\left[\hat{U}^B(s_{0,M})\right]$$

will be satisfied for every price offer $P_1^{B\to I} < P_0^{B\to I}$. A switch from $s_{0,M}$ to $s_{1,B}$ will therefore always be advantageous for player B.

3. $s_{1,DT} = \left(P_0^{S\to I}, P_1^{B\to I}, s_-^M, s_-^R\right)$:
in this scenario, I_R is rejecting both offers and S and B will involve in a direct trade for a price of $\hat{\Lambda}$ per unit. In such a case, B's payoff can be determined as

$$E\left[\hat{U}^B(s_{1,DT})\right] = (v-m)\left(\hat{\Lambda} - P^{I\to B}\right).$$

B's payoff will have improved if

$$E\left[\hat{U}^B(s_{1,DT})\right] > E\left[\hat{U}^B(s_{0,M})\right]$$

and hence if

$$(v-m)\left(\hat{\Lambda} - P^{I\to B}\right) > m\left(\hat{\Lambda} - P_0^{B\to I}\right) + (v-m)\left(\hat{\Lambda} - P^{I\to B}\right) - m\,T_M.$$

Simplifying yields

$$0 > m\left(\hat{\Lambda} - P_0^{B\to I}\right) - m\,T_M.$$

or in other words if

$$P_0^{B\to I} > \hat{\Lambda} - T_M.$$

□

Having discussed B's and S's preferences, the main theorem of this section can now be proven:

Proof of Theorem 7.1.1. Consider a set of price offers $\left(P_0^{S\to I}, P_0^{B\to I}\right)$ with

$$P^{I\to S} + T_M \leq P_0^{S\to I} \leq P_0^{B\to I} \leq P^{I\to B} - T_M.$$

and the resulting strategy combination

$$s_{0,M} = \left(P_0^{S\to I}, P_0^{B\to I}, s_+^M, s_-^R\right).$$

Suppose that $s_{0,M}$ is a Nash equilibrium. Then no player should be able to singlehandedly improve his payoff.

Suppose first that $P_0^{S\to I} < P_0^{B\to I}$. Then S could singlehandedly receive a higher payoff by simply offering a price $P_1^{S\to I} = P_0^{B\to I}$, since S would then sell $m = \{\alpha, \nu\}$ units of G to I_M for a price of $P_1^{S\to I} > P_0^{S\to I}$. Similarly, also B could singlehandedly improve his payoff by offering $P_1^{B\to I} = P_0^{S\to I} < P_0^{B\to I}$. It follows that if $s_{0,M}$ is to be a Nash equilibrium solution, $P_0^{B\to I}$ and $P_0^{S\to I}$ have to satisfy

$$P_0^{S\to I} = P_0^{B\to I}. \tag{7.2}$$

Suppose now that $P_0^{S\to I} = P_0^{B\to I}$ and suppose that S wants to improve his payoff. Consider for this purpose that S's payoff under the old strategy combination $s_{0,M}$ was given by

$$E\left[\hat{U}^S\left(s_{0,M}\right)\right] = m\left(P_0^{S\to I} - \hat{\Lambda}\right) + (\alpha - m)\left(P^{I\to S} - \hat{\Lambda}\right) - m\, T_M$$

Since decreasing his offer to a price $P_1^{S\to I} < P_0^{S\to I}$ would mean that S would still trade with I_M, but this time for a lower price, an increase in S's payoff can only be realized if S offers a price $P_1^{S\to I} > P_0^{S\to I}$. Since the price spread then satisfies

$$P_0^{B\to I} - P_1^{S\to I} = P_0^{S\to I} - P_1^{S\to I} > 0,$$

it follows that this change in strategy will lead to an outcome scenario not involving the market maker I_M, i.e. a scenario of the type s_B, s_S or s_{DT}.

Lemma 7.1.2 implies that a change that is profitable for S could only occur with a switch to a strategy combination $s_{1,S}$, or in the case of

$$P_0^{S\to I} < \hat{\Lambda} + T_M$$

also to a strategy combination $s_{1,DT}$. Hence, for S to be unable to profitably initiate a change, I_R would need to decline playing his strategy s_S^R and if necessary also his strategy s_-^R.

In the case $P_0^{S\to I} \geq \hat{\Lambda} + T_M$, only $s_{1,S}$ would be profitable for S and hence - in order to prevent S from changing profitably - I_R's payoffs would need to satisfy

$$E\left[\hat{U}^{I_R}\left(s_{1,DT}\right)\right] > E\left[\hat{U}^{I_R}\left(s_{1,S}\right)\right]$$

or

$$E\left[\hat{U}^{I_R}\left(s_{1,B}\right)\right] > E\left[\hat{U}^{I_R}\left(s_{1,S}\right)\right].$$

If $P_0^{S\to I} < \hat{\Lambda} + T_M$, a profitable change of S away from $s_{0,M}$ can only be avoided if I_R would opt for s_B^R, i.e. if

$$E\left[\hat{U}^{I_R}\left(s_{1,B}\right)\right] > E\left[\hat{U}^{I_R}\left(s_{1,S}\right)\right]$$

and

$$E\left[\hat{U}^{I_R}\left(s_{1,B}\right)\right] > E\left[\hat{U}^{I_R}\left(s_{1,DT}\right)\right].$$

Note now that in the Model II I_R's payoffs were arranged in an almost completely identical setup in the proof of Proposition 5.2.5 (see inequalities (5.33), (5.34), (5.38) and (5.39)). The only difference was that instead of a case distinction at $P_0^{S\to I} = \hat{\Lambda} + T_M$, the two cases were separated along $P_0^{S\to I} = \hat{\Lambda}$. However, the exact value at which this distinction took place did not influence the analysis of Model II. Since the introduction of a transaction fee T_M did not change I_R's payoff functions for the different outcome scenarios, it follows that the analysis from the proof of Proposition 5.2.5 can be directly applied to this set of inequalities. Using a result from the proof of Proposition 5.2.5 it follows that these inequalities will be satisfied if[1]

$$P_0^{B\to I} \geq \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}. \tag{7.3}$$

Suppose now that B wants to single-handedly improve his payoff. Since increasing his price offer would only result in I_M still being involved but this time selling G to B for a higher price, B can only improve by decreasing his price offer to an offer $P_1^{B\to I} < P_0^{B\to I}$. Since such an offer would result in a negative price spread, I_M would opt to play his strategy s_-^M. The possible outcome scenarios of such a strategy switch are $s_{1,B}$, $s_{1,S}$ and $s_{1,DT}$.

By Lemma 7.1.3 only a change to $s_{1,B}$, and in the case of

$$P_0^{B\to I} > \hat{\Lambda} - T_M$$

[1] for the equivalent result in the Model II see inequality (5.40).

also to $s_{1,DT}$ would be profitable for B. Hence, in order not to prevent B from profitably changing his strategy, I_R's relevant payoffs for the three strategy combinations $s_{1,B}$, $s_{1,S}$ and $s_{1,DT}$ need to satisfy

$$E\left[\hat{U}^{I_R}(s_{1,DT})\right] > E\left[\hat{U}^{I_R}(s_{1,B})\right]$$

and/or (depending on whether $P_0^{B\to I} > \hat{\Lambda} - T_M$ or $P_0^{B\to I} \leq \hat{\Lambda} - T_M$)

$$E\left[\hat{U}^{I_R}(s_{1,S})\right] > E\left[\hat{U}^{I_R}(s_{1,B})\right].$$

The same setup can be found in the proof of Proposition 5.2.2 (see inequalities (5.22),(5.23),(5.27) and (5.28)). The resulting bound (5.29) can therefore also be applied in this case. It follows that the above inequalities will be satisfied in the described manner if

$$P_0^{S\to I} \leq \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}. \tag{7.4}$$

Combining inequalities (7.2), (7.3) and (7.4) yields that $s_{0,M}$ is a Nash equilibrium only if

$$P_0^{S\to I} = P_0^{B\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}. \tag{7.5}$$

Suppose now that a strategy combination $s_{0,M}$ satisfies

$$P_0^{S\to I} = P_0^{B\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}.$$

If S would change his strategy to an offer $P_1^{S\to I} < P_0^{S\to I}$, I_M would still be involved. S would then sell $m = \min\{\alpha, \nu\}$ units of G for a cheaper price. Changing his offer in this way would therefore decrease S's payoff. Similarly, B's payoff would decrease if B would offer a price $P_1^{B\to I} > P_0^{B\to I}$.

Suppose now that S is trying to improve his payoff by offering a price $P_1^{S\to I}$ with $P_0^{S\to I} < P_1^{S\to I}$. With the spread $P_0^{B\to I} - P_1^{S\to I}$ being negative, I_M's response to this strategy change will be a switch to strategy s_-^M. Since $P_0^{B\to I}$ satisfies

$$P_0^{B\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu},$$

previous reasoning[2] implies that I_R will opt for s_B^R: I_R will accept B's offer and buy α units from S for the standing offer $P^{I\to S}$. But by Lemma 7.1.2 such a change will not improve S's payoff. It follows that there is no way for S to profitably leave a strategy combination $s_{0,M}$ with

$$P_0^{S\to I} = P_0^{B\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}.$$

Similarly, also B isn't able to increase his payoff by offering a price satisfying $P_1^{B\to I} < P_0^{B\to I}$: I_R would then opt for strategy s_S^R and the resulting strategy combination $s_{1,S}$ would not increase B's payoff according to Lemma 7.1.3. Hence also B cannot profitably change away from $s_{0,M}$.

It follows that if in a strategy combination $s_{0,M}$ the price offers are set according to

$$P_0^{S\to I} = P_0^{B\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu},$$

neither S nor B can profitably change their strategy. Since I_M's and I_R's strategies are (by the very setup of the matchmaking process) best answers to S's and B's offers, it follows that $s_{0,M}$ is a Nash equilibrium. This proves the claim. □

An immediate consequence of Theorem 7.1.1 is that such an equilibrium solution will not always exist:

Corollary 7.1.4 (Model III: Existence of an equilibrium involving I_M)
Suppose that $\alpha, \nu, R_S, R_B > 0$.
A necessary and sufficient condition for the existence of a Nash equilibrium involving the market maker I_M is that the transaction fee T_M satisfies

$$T_M \leq \frac{m}{\alpha + \nu}\left(P^{I\to B} - P^{I\to S}\right), \tag{7.6}$$

where $m = \min\{\alpha, \nu\}$.

Proof. Suppose first that

$$T_M \leq \frac{m}{\alpha + \nu}\left(P^{I\to B} - P^{I\to S}\right).$$

[2] see the proof of Corollary 5.2.3.

This implies in particular that

$$T_M \le \frac{\nu}{\alpha+\nu}\left(P^{I\to B} - P^{I\to S}\right)$$

and hence that

$$\begin{aligned} P^{I\to S} + T_M &\le P^{I\to S} + \frac{\nu}{\alpha+\nu}\left(P^{I\to B} - P^{I\to S}\right) \\ &= \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}. \end{aligned}$$

T_M also satisfies

$$T_M \le \frac{\alpha}{\alpha+\nu}\left(P^{I\to B} - P^{I\to S}\right).$$

This implies that

$$\begin{aligned} P^{I\to B} - T_M &\ge P^{I\to B} - \frac{\alpha}{\alpha+\nu}\left(P^{I\to B} - P^{I\to S}\right) \\ &= \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}. \end{aligned}$$

Combining these two results it follows that

$$P^{I\to S} + T_M \le \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu} \le P^{I\to B} - T_M.$$

Hence, offers $P_0^{B\to I}$ and $P_0^{S\to I}$ with

$$P_0^{B\to I} = P_0^{S\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}$$

satisfy the feasibility condition

$$P^{I\to S} + T_M \le P_0^{S\to I} \le P_0^{B\to I} \le P^{I\to B} - T_M.$$

Since Theorem 7.1.1 then implies that such a set of offers results in a Nash equilibrium solution of the $s_{0,M}$-type, the existence of a Nash equilibrium has been proven.

Suppose now that

$$T_M > \frac{m}{\alpha+\nu}\left(P^{I\to B} - P^{I\to S}\right)$$

and assume furthermore that $m = \nu$. Then

$$\begin{aligned} P^{I\to S} + T_M &> P^{I\to S} + \frac{\nu}{\alpha+\nu}\left(P^{I\to B} - P^{I\to S}\right) \\ &= \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}. \end{aligned}$$

This shows that a combination of offers with

$$P_0^{B\to I} = P_0^{S\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}$$

violates the feasibility condition

$$P^{I\to S} + T_M \leq P_0^{S\to I}.$$

Theorem 7.1.1 now implies that no equilibrium solution can exist.

Similarly, if $m = \alpha$ it follows that

$$P^{I\to B} - T_M < \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}$$

and therefore that a combination of such offers is not feasible. By Theorem 7.1.1 it follows that no Nash equilibrium of the type $s_{0,M}$ exists. It can be concluded that if

$$T_M > \frac{m}{\alpha+\nu}\left(P^{I\to B} - P^{I\to S}\right)$$

no equilibrium solution of the form $s_{0,M}$ will exist, no matter whether $m = \alpha$ or $m = \nu$. This proves the claim. □

7.2 Equilibrium analysis for trades without I_M

Suppose now that a combination of price offers $\left(P_0^{S\to I}, P_0^{B\to I}\right)$ satisfies

$$P_0^{S\to I} > P_0^{B\to I}.$$

In this situation I_M's reponse will be his strategy s_-^M, i.e. the negative price spread will push I_M out of the market. Depending on I_R's response there are three possible outcome scenarios: a direct trade scenario of the form $s_{0,DT}$, a scenario $s_{0,B}$ where B's offer is accepted and a scenario $s_{0,S}$ where S's offer is accepted. It will be shown that none of these scenarios will result in a Nash equilibrium solution.

Lemma 7.2.1 (Model III: non-existence of a s_{DT}-type equilibrium)
Consider a market with transaction fee T_M as described above and suppose that $\alpha, \nu, R_S, R_B > 0$. Then no strategy combination of the form

$$s_{0,DT} = \left(P_0^{S\to I}, P_0^{B\to I}, s_-^M, s_-^R\right),$$

i.e. no strategy combination resulting in a direct trade scenario, will be a Nash equilibrium solution.

Proof. Let $\left(P_0^{S\to I}, P_0^{B\to I}\right)$ with

$$P_0^{S\to I} > P_0^{B\to I}$$

be a combination of price offers resulting in a direct trade scenario. For I_R to opt for a direct trade the following conditions need to hold:

$$E\left[\hat{U}^{I_R}\left(s_{0,DT}\right)\right] \geq E\left[\hat{U}^{I_R}\left(s_{0,B}\right)\right] \tag{7.7}$$

and

$$E\left[\hat{U}^{I_R}\left(s_{0,DT}\right)\right] \geq E\left[\hat{U}^{I_R}\left(s_{0,S}\right)\right]. \tag{7.8}$$

From inequality (7.7) it follows that

$$(\alpha - m)\left(\hat{\Lambda} - P^{I\to S}\right) + (\nu - m)\left(P^{I\to B} - \hat{\Lambda}\right) \geq \nu P_0^{B\to I} - \alpha P^{I\to S} + (\alpha - \nu)\,\hat{\Lambda}$$

and hence that

$$P_0^{B\to I} \leq P^{I\to B} - \frac{m}{\nu}\left(P^{I\to B} - P^{I\to S}\right). \tag{7.9}$$

Inequality (7.8) on the other hand implies that

$$(\alpha - m)\left(\hat{\Lambda} - P^{I\to S}\right) + (\nu - m)\left(P^{I\to B} - \hat{\Lambda}\right) \geq \nu P^{I\to B} - \alpha P_0^{S\to I} + (\alpha - \nu)\,\hat{\Lambda}.$$

This yields

$$P_0^{S\to I} \geq P^{I\to S} + \frac{m}{\alpha}\left(P^{I\to B} - P^{I\to S}\right). \tag{7.10}$$

Suppose now that S wants to improve his payoff by offering a price $P_1^{S\to I}$. A change to a scenario $s_{1,S}$ would be profitable for S if

$$E\left[\hat{U}^{S}\left(s_{1,S}\right)\right] > E\left[\hat{U}^{S}\left(s_{0,DT}\right)\right],$$

i.e. if

$$\alpha\left(P_1^{S\to I}-\hat{\Lambda}\right)>(\alpha-m)\left(P^{I\to S}-\hat{\Lambda}\right).$$

It follows that a change from $s_{0,DT}$ to $s_{1,S}$ would be profitable for S if

$$P_1^{S\to I}>P^{I\to S}+\frac{m}{\alpha}\left(\hat{\Lambda}-P^{I\to S}\right). \tag{7.11}$$

Note now that I_R would agree to such a change if

$$E\left[\hat{U}^{I_R}(s_{1,S})\right]>E\left[\hat{U}^{I_R}(s_{1,DT})\right] \tag{7.12}$$

and

$$E\left[\hat{U}^{I_R}(s_{1,S})\right]>E\left[\hat{U}^{I_R}(s_{1,B})\right]. \tag{7.13}$$

Note first that inequality (7.13) will hold automatically as long as inequality (7.12) is satisfied:
$E\left[\hat{U}^{I_R}(s_{1,DT})\right]$ and $E\left[\hat{U}^{I_R}(s_{1,B})\right]$ do not depend on S's price offer. A change from $P_0^{S\to I}$ to $P_1^{S\to I}$ will therefore not have changed those payoffs. Hence

$$E\left[\hat{U}^{I_R}(s_{1,DT})\right]=E\left[\hat{U}^{I_R}(s_{0,DT})\right]$$

and

$$E\left[\hat{U}^{I_R}(s_{1,B})\right]=E\left[\hat{U}^{I_R}(s_{0,B})\right].$$

For a strategy combination $s_{0,DT}$ to be played a necessary condition was

$$E\left[\hat{U}^{I_R}(s_{0,DT})\right]\geq E\left[\hat{U}^{I_R}(s_{0,B})\right].$$

Hence also

$$E\left[\hat{U}^{I_R}(s_{1,DT})\right]\geq E\left[\hat{U}^{I_R}(s_{1,B})\right].$$

If therefore

$$E\left[\hat{U}^{I_R}(s_{1,S})\right]>E\left[\hat{U}^{I_R}(s_{1,DT})\right]$$

holds, it follows that

$$E\left[\hat{U}^{I_R}(s_{1,S})\right]>E\left[\hat{U}^{I_R}(s_{1,DT})\right]\geq E\left[\hat{U}^{I_R}(s_{1,B})\right]$$

and hence that

$$E\left[\hat{U}^{I_R}(s_{1,S})\right] > E\left[\hat{U}^{I_R}(s_{1,B})\right].$$

Inequality (7.12) is satisfied as long as

$$P_1^{S\to I} < P^{I\to S} + \frac{m}{\alpha}\left(P^{I\to B} - P^{I\to S}\right) \tag{7.14}$$

(using a version of inequality (7.10)).

Finally, a change to $s_{1,S} = \left(P_1^{S\to I}, P_0^{B\to I}, s_-^M, s_S^R\right)$ is only possible if I_M opts for his strategy s_-^M, i.e. if

$$P_1^{S\to I} > P_0^{B\to I}.$$

Inequality (7.9) implies that this can be assured as long as $P_1^{S\to I}$ satisfies

$$P_1^{S\to I} > P^{I\to B} - \frac{m}{\nu}\left(P^{I\to B} - P^{I\to S}\right). \tag{7.15}$$

Combining inequalities (7.11), (7.14) and (7.15) yields that S will be able to profitably switch from $s_{0,DT}$ to $s_{1,S}$ if his offer $P_1^{S\to I}$ satisfies:

$$\begin{aligned} &\max\left\{P^{I\to B} - \frac{m}{\nu}\left(P^{I\to B} - P^{I\to S}\right), P^{I\to S} + \frac{m}{\alpha}\left(\hat{\Lambda} - P^{I\to S}\right)\right\} \\ < \quad & P_1^{S\to I} < P^{I\to S} + \frac{m}{\alpha}\left(P^{I\to B} - P^{I\to S}\right). \end{aligned} \tag{7.16}$$

It follows that such a price $P_1^{S\to I}$ will exist if

$$P^{I\to B} - \frac{m}{\nu}\left(P^{I\to B} - P^{I\to S}\right) < P^{I\to S} + \frac{m}{\alpha}\left(P^{I\to B} - P^{I\to S}\right) \tag{7.17}$$

and if

$$P^{I\to S} + \frac{m}{\alpha}\left(\hat{\Lambda} - P^{I\to S}\right) < P^{I\to S} + \frac{m}{\alpha}\left(P^{I\to B} - P^{I\to S}\right). \tag{7.18}$$

Inequality (7.17) implies that

$$\left(1 - \frac{m}{\nu} - \frac{m}{\alpha}\right)\left(P^{I\to B} - P^{I\to S}\right) < 0.$$

With $P^{I\to B} > P^{I\to S}$ it follows that

$$1 - \frac{m}{\nu} - \frac{m}{\alpha} < 0.$$

But this inequality will always be satisfied since

$$1 - \frac{m}{\nu} - \frac{m}{\alpha} = \left\{ \begin{array}{ll} -\frac{\nu}{\alpha} & \text{if } m = \nu \\ -\frac{\alpha}{\nu} & \text{if } m = \alpha \end{array} \right\} < 0$$

Inequality (7.18) will hold if

$$\frac{m}{\alpha}\left(\hat{\Lambda} - P^{I\to S}\right) < \frac{m}{\alpha}\left(P^{I\to B} - P^{I\to S}\right)$$

and hence if

$$\hat{\Lambda} < P^{I\to B}.$$

But since B is assumed to be willing to pay a price for transferring his risk, this inequality will automatically be satisfied.

It follows that both inequalities (7.17) and (7.18) will always hold and hence that a price offer $P_1^{S\to I}$ satisfying the bounds (7.16) will exist. But this means that S is able to single-handedly induce a profitable change to a strategy combination $s_{1,S}$. Similarly it could also have been shown that B will always be able to single-handedly switch to a strategy combination $s_{1,B}$ in a profitable manner. This implies that $s_{0,DT}$ could not have been a Nash equilibrium. □

A similar result can also be shown for a strategy combination of the s_B type:

Lemma 7.2.2 (Model III: non-existence of a s_B-type equilibrium)
Consider a market with transaction fee T_M as described above and suppose that $\alpha, \nu, R_S, R_B > 0$. Then no strategy combination of the form

$$s_{0,B} = \left(P_0^{S\to I}, P_0^{B\to I}, s_-^M, s_B^R\right),$$

i.e. no strategy combination in which I_R sells ν units to B for $P_0^{B\to I}$ per unit, will be a Nash equilibrium solution.

Proof. Suppose that S and B set their offers $P_0^{S\to I}$ and $P_0^{B\to I}$ such that I_M will not enter the market by playing his strategy s_-^M and such that I_R will opt for s_B^R. The resulting strategy combination

$$s_{0,B} = \left(P_0^{S\to I}, P_0^{B\to I}, s_-^M, s_B^R\right)$$

will result in I_R selling ν units to B for $P_0^{B\to I}$ per unit. S on the other hand will sell all α units to I_R for a price of $P^{I\to S}$ per unit, i.e. for the cheapest possible price.

Suppose furthermore that $s_{0,B}$ is a Nash equilibrium.

Suppose now that S wants to induce a change to a strategy combination that is promising him a higher payoff. There are essentially two ways for S to do this: he can offer a price $P_1^{S\to I} > P_0^{B\to I}$ or a price $P_2^{S\to I} \leq P_0^{B\to I}$. In the first case all trades will take place without I_M's participation while in the second case I_M will trade $\min\{\alpha, \nu\}$ units from S to B at their posted price offers.

Suppose first that S is offering a price $P_1^{S\to I} > P_0^{B\to I}$. Note that S cannot induce a change to a direct trade scenario: S's changed price offer will only affect I_R's payoff if I_R opts to play s_S^R, i.e.

$$E\left[\hat{U}^{I_R}(s_{0,B})\right] = E\left[\hat{U}^{I_R}(s_{1,B})\right]$$

and

$$E\left[\hat{U}^{I_R}(s_{0,DT})\right] = E\left[\hat{U}^{I_R}(s_{1,DT})\right]$$

Since under the old price offers $P_0^{S\to I}$ and $P_0^{B\to I}$ I_R decided to play his strategy s_B^R, he preferred an outcome scenario $s_{0,B}$ over a direct trade scenario $s_{0,DT}$. Hence

$$E\left[\hat{U}^{I_R}(s_{0,B})\right] \geq E\left[\hat{U}^{I_R}(s_{0,DT})\right]$$

and therefore also

$$E\left[\hat{U}^{I_R}(s_{1,B})\right] \geq E\left[\hat{U}^{I_R}(s_{1,DT})\right].$$

Since an outcome scenario $s_{1,B}$ will not improve S's payoff his only option for a profitable change will hence be a strategy combination $s_{1,S}$. I_R will opt for such a change if

$$E\left[\hat{U}^{I_R}(s_{1,S})\right] \geq E\left[\hat{U}^{I_R}(s_{1,B})\right],$$

i.e. if

$$\nu P^{I\to B} - \alpha P_1^{S\to I} + (\alpha - \nu)\hat{\Lambda} > \nu P_0^{B\to I} - \alpha P^{I\to S} + (\alpha - \nu)\hat{\Lambda}.$$

It follows that

$$\nu P^{I\to B} - \alpha P_1^{S\to I} > \nu P_0^{B\to I} - \alpha P^{I\to S}$$

and hence that

$$P_1^{S\to I} < P^{I\to S} + \frac{\nu}{\alpha}\left(P^{I\to B} - P_0^{B\to I}\right). \tag{7.19}$$

Consider now that $s_{0,B}$ was assumed to be a Nash equilibrium. This means that no player can profitably change his strategy. Since $P_1^{S\to I} > P_0^{B\to I}$, S will not be able to find a price offer $P_1^{S\to I}$ satisfying inequality (7.19) if $P_0^{B\to I}$ is set such that

$$P_0^{B\to I} \geq P^{I\to S} + \frac{\nu}{\alpha}\left(P^{I\to B} - P_0^{B\to I}\right).$$

It follows that for $s_{0,B}$ to be a Nash equilibrium

$$P_0^{B\to I} \geq \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}. \tag{7.20}$$

Suppose now that S is trying to induce a profitable change by offering a price $P_2^{S\to I} \leq P_0^{B\to I}$. In such a case I_M would enter the market and buy $m = \min\{\alpha, \nu\}$ units for S's posted price $P_2^{S\to I}$. A transaction fee T_M would be charged per traded unit. In such a case S's payoff will have increased if

$$E\left[\hat{U}^S\left(s_{2,M}\right)\right] > E\left[\hat{U}^S\left(s_{0,B}\right)\right],$$

i.e. if

$$m\left(P_2^{S\to I} - \hat{\Lambda}\right) + (\alpha - m)\left(P^{I\to S} - \hat{\Lambda}\right) - m\,T_M > \alpha\left(P^{I\to S} - \hat{\Lambda}\right).$$

It follows that

$$m\left(P_2^{S\to I} - P^{I\to S}\right) - m\,T_M > 0$$

and hence that

$$P_2^{S\to I} > P^{I\to S} + T_M.$$

Since $P_2^{S\to I} \leq P_0^{B\to I}$, S will not be able to find such a price as long as

$$P_0^{B\to I} \leq P^{I\to S} + T_M. \tag{7.21}$$

It follows that a necessary condition for $s_{0,B}$-type equilibrium solution is

$$\frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu} \leq P_0^{B\to I} \leq P^{I\to S} + T_M. \tag{7.22}$$

Suppose now that $P_0^{B\to I}$ satisfies inequality (7.22). Then S will not be able to profitably change his strategy. For $s_{0,B}$ to be a Nash equilibrium however, also B should not be able to single-handedly increase his payoff. It can however be shown that - no matter how $P_0^{B\to I}$ and $P_0^{S\to I}$ were chosen - B will always be able to induce a profitable strategy change.

Consider for this purpose that $P_0^{S\to I} > P_0^{B\to I}$. With

$$\frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu} \leq P_0^{B\to I}$$

this implies

$$\frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu} < P_0^{S\to I}$$

and hence that there is a constant $C_S > 0$ such that

$$P_0^{S\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B} + C_S}{\alpha+\nu}.$$

Suppose now that B is changing his strategy to a price offer $P_3^{B\to I}$ satisfying

$$P_3^{B\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B} - C_B}{\alpha+\nu}$$

for some constant $C_B > 0$. Since

$$P_3^{B\to I} < \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu} \leq P_0^{B\to I}$$

it follows that B would have increased his payoff by making such an offer if only I_R is willing to accept the strategy s_B^R.
The resulting outcome will hence depend on I_R's decision. For a strategy combination $s_{3,B}$, I_R's payoff can be determined as

$$\begin{aligned} E\left[\hat{U}^{I_R}(s_{3,B})\right] &= \nu\left(P_3^{B\to I} - \hat{\Lambda}\right) + \alpha\left(\hat{\Lambda} - P^{I\to S}\right) \\ &= \nu\left(\frac{\alpha P^{I\to S} + \nu P^{I\to B} - C_B}{\alpha+\nu} - \hat{\Lambda}\right) + \alpha\left(\hat{\Lambda} - P^{I\to S}\right) \\ &= \frac{\nu^2 P^{I\to B} + \left(\alpha^2 - \nu^2\right)\hat{\Lambda} - \alpha^2 P^{I\to S} - \nu C_B}{\alpha+\nu}. \end{aligned}$$

If however $s_{3,S}$ is played I_R will receive the following payoff:

$$\begin{aligned}
E\left[\hat{U}^{I_R}(s_{3,S})\right] &= \nu\left(P^{I\to B} - \hat{\Lambda}\right) + \alpha\left(\hat{\Lambda} - P_0^{S\to I}\right) \\
&= \nu\left(P^{I\to B} - \hat{\Lambda}\right) + \alpha\left(\hat{\Lambda} - \frac{\alpha P^{I\to S} + \nu P^{I\to B} + C_S}{\alpha+\nu}\right) \\
&= \frac{\nu^2 P^{I\to B} + \left(\alpha^2 - \nu^2\right)\hat{\Lambda} - \alpha^2 P^{I\to S} - \alpha C_S}{\alpha+\nu}.
\end{aligned}$$

Comparing these two results it follows that I_R will prefer s_B^R over s_S^R if

$$\nu C_B < \alpha C_S,$$

i.e. if

$$C_B < \frac{\alpha}{\nu} C_S. \tag{7.23}$$

Finally, I_R could also opt for a strategy s_-^R. In this case his payoff would be given by

$$\begin{aligned}
E\left[\hat{U}^{I_R}(s_{3,DT})\right] &= (\alpha - m)\left(\hat{\Lambda} - P^{I\to S}\right) + (\nu - m)\left(P^{I\to B} - \hat{\Lambda}\right) \\
&= \begin{cases} (\alpha-\nu)\left(\hat{\Lambda} - P^{I\to S}\right) & \text{if } m = \nu \\ (\nu-\alpha)\left(P^{I\to B} - \hat{\Lambda}\right) & \text{if } m = \alpha \end{cases} \\
&= \begin{cases} \frac{\nu^2 P^{I\to S} + \left(\alpha^2-\nu^2\right)\hat{\Lambda} - \alpha^2 P^{I\to S}}{\alpha+\nu} & \text{if } m = \nu \\ \frac{\nu^2 P^{I\to B} + \left(\alpha^2-\nu^2\right)\hat{\Lambda} - \alpha^2 P^{I\to B}}{\alpha+\nu} & \text{if } m = \alpha \end{cases}
\end{aligned}$$

Consider now that $P^{I\to S} = (1 - R_S)\hat{\Lambda}$ and $P^{I\to B} = (1 + R_B)\hat{\Lambda}$ imply that

$$\begin{aligned}
P^{I\to B} &= \frac{1+R_B}{1-R_S} P^{I\to S} \\
&= P^{I\to S} + \frac{R_S + R_B}{1-R_S} P^{I\to S} \\
&= P^{I\to S} + (R_S + R_B)\hat{\Lambda}
\end{aligned}$$

and that

$$\begin{aligned}
P^{I\to S} &= \frac{1-R_S}{1+R_B} P^{I\to B} \\
&= P^{I\to B} - \frac{R_S + R_B}{1+R_B} P^{I\to B} \\
&= P^{I\to B} - (R_S + R_B)\hat{\Lambda}
\end{aligned}$$

With these results it follows that

$$E\left[\hat{U}^{I_R}\left(s_{3,DT}\right)\right] = \begin{cases} \frac{\nu^2 P^{I\to B} - \nu^2 (R_S + R_B)\hat{\Lambda} + \left(\alpha^2 - \nu^2\right)\hat{\Lambda} - \alpha^2 P^{I\to S}}{\alpha+\nu} & \text{if } m = \nu \\ \frac{\nu^2 P^{I\to B} + \left(\alpha^2 - \nu^2\right)\hat{\Lambda} - \alpha^2 P^{I\to S} - \alpha^2 (R_S + R_B)\hat{\Lambda}}{\alpha+\nu} & \text{if } m = \alpha \end{cases}$$

$$= \frac{\nu^2 P^{I\to B} + \left(\alpha^2 - \nu^2\right)\hat{\Lambda} - \alpha^2 P^{I\to S} - m^2 (R_S + R_B)\hat{\Lambda}}{\alpha + \nu}$$

This implies that I_R will prefer $s_{3,B}$ over $s_{3,DT}$ if

$$\nu C_B < m^2 (R_S + R_B)\hat{\Lambda}$$

and hence if

$$C_B < \frac{m^2}{\nu}(R_S + R_B)\hat{\Lambda}. \tag{7.24}$$

Combining inequalities (7.23) and (7.24) yields that in order for I_R to opt for the strategy s_B^R, C_B has to satisfy

$$C_B < \min\left\{\frac{\alpha}{\nu}C_S, \frac{m^2}{\nu}(R_S + R_B)\hat{\Lambda}\right\}.$$

Such a $C_B > 0$ will exist if

$$0 < \min\left\{\frac{\alpha}{\nu}C_S, \frac{m^2}{\nu}(R_S + R_B)\hat{\Lambda}\right\}.$$

But since both quantities on the right hand of this inequality are distinctly positive, it follows that such a C_B will exist. Hence B will be able to offer a price $P_3^{B\to I} < P_0^{B\to I}$ that increases his payoff. Therefore a strategy combination of the type $s_{0,B}$ cannot be a Nash equilibrium. □

Finally, also a strategy combination of the s_S-type cannot be a Nash equilibrium:

Lemma 7.2.3 (Model III: non-existence of a s_S-type equilibrium)
Consider a market with transaction fee T_M as described above and suppose that $\alpha, \nu, R_S, R_B > 0$. Then no strategy combination of the form

$$s_{0,S} = \left(P_0^{S\to I}, P_0^{B\to I}, s_-^M, s_S^R\right),$$

i.e. no strategy combination in which I_R buys α units from S for $P_0^{S\to I}$ per unit, will be a Nash equilibrium solution.

Proof. In analogy to the proof of the previous lemma it can be shown that B will be able to profitably switch to a strategy combination of the s_B-type unless $P_0^{S\to I}$ satisfies

$$P_0^{S\to I} \leq \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}. \tag{7.25}$$

Furthermore, B could profitably switch to an outcome scenario involving I_M if

$$P_0^{S\to I} < P^{I\to B} - T_M.$$

Hence for $s_{0,S}$ to be a Nash equilibrium it follows that

$$P_0^{S\to I} \geq P^{I\to B} - T_M. \tag{7.26}$$

Since it will be impossible for B to initiate a change to a direct trade scenario s_{DT}, inequalities (7.22) and (7.22) imply that for $s_{0,S}$ to be a Nash equilibrium S's price offer needs to satisfy

$$P^{I\to B} - T_M \leq P_0^{S\to I} \leq \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}. \tag{7.27}$$

Consider now that in order for a s_S-scenario to be possible in the first place, the initial price offers had to be set according to

$$P_0^{S\to I} > P_0^{B\to I}.$$

From inequality (7.27) it follows that

$$P_0^{B\to I} < \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}.$$

Hence,

$$P_0^{B\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B} - C_B}{\alpha + \nu}$$

for some $C_B > 0$. It can then be shown that S could improve his payoff by offering a price

$$P_1^{S\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B} + C_S}{\alpha + \nu}$$

where C_S satisfies

$$C_S < \min\left\{\frac{\nu}{\alpha}C_B, \frac{m^2}{\alpha}(R_S + R_B)\hat{\Lambda}\right\}.$$

The existence of such a C_S follows from the fact that clearly

$$0 < \min\left\{\frac{\nu}{\alpha}C_B, \frac{m^2}{\alpha}(R_S + R_B)\hat{\Lambda}\right\}.$$

It was therefore shown that either S or B will always be able to profitably switch away from a strategy combination $s_{0,S}$. It follows that no such strategy combination can be a Nash equilibrium. □

7.3 Interpretation of the predicted market behaviour

Summarizing the results of the previous two sections, the following can be said about Nash equilibrium solutions in a market with transaction fee T_M:

Theorem 7.3.1 (A Nash equilibrium solution in a market with fees)
Consider a market with transaction fee T_M as described above.
A Nash equilibrium solution

$$s_{0,M} = \left(P_0^{S\to I}, P_0^{B\to I}, s_+^M, s_-^R\right)$$

with

$$P_0^{S\to I} = P_0^{B\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu} \tag{7.28}$$

will exist if and only if

$$T_M \leq \frac{m}{\alpha + \nu}\left(P^{I\to B} - P^{I\to S}\right), \tag{7.29}$$

where $m = \min\{\alpha, \nu\}$.
No other equilibrium solutions can exist.

Proof. The above claim follows directly from Theorem (7.1.1), Corollary (7.1.4) and Lemmata (7.2.1)-(7.2.3). □

This result implies that the market behaviour will be completely predictable as long as inequality (7.29) is satisfied (assuming that all players act perfectly rational, see section 1.2.4):

1. S will sell $m = \min\{\alpha, \nu\}$ units to I_M for

$$P_0^{S\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}.$$

   A transaction fee T_M will apply for each traded unit.

2. B will buy $m = \min\{\alpha, \nu\}$ units from I_M for

$$P_0^{B\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}.$$

   In order to trade with I_M, B will have to pay a transaction fee of T_M per unit.

3. I_R will buy/sell the remaining $|\alpha - \nu|$ units for $P^{I\to S}$ or $P^{I\to B}$ respectively.

The price for which $m = \min\{\alpha, \nu\}$ units of G are traded is the same price as the one discussed in the previous model. Hence, also this price will depend on the model parameters $\hat{\Lambda}$, R_S, R_B, α and ν in a rather natural fashion[3].
The behaviour of the different market participants can however not be predicted if inequality (7.29) does not hold.

7.3.1 The roles of the intermediaries

The roles played by the two intermediaries in this model depend on how the transaction costs T_M relate to the other model parameters $\hat{\Lambda}$, R_S, R_B, α and ν.

If

$$T_M \leq \frac{m}{\alpha + \nu}\left(P^{I\to B} - P^{I\to S}\right),$$

I_M and I_R will trade the complete market volume according to a strategy combination $s_{0,M}$. While I_M will trade $m = \min\{\alpha, \nu\}$ from S to B, I_R will trade the remaining $|\alpha - \nu|$ for his own posted price. The intermediaries' payoffs can hence be found as

$$E\left[\hat{U}^{I_M}\left(s_{0,M}\right)\right] = 2m\, T_M$$

[3] for a detailed discussion of the equilibrium price see section 5.3.1

and

$$E\left[\hat{U}^{I_R}(s_{0,M})\right] = (\alpha - m)\left(\hat{\Lambda} - P^{I\to S}\right) + (\nu - m)\left(P^{I\to B} - \hat{\Lambda}\right).$$

Since both payoffs can be expected to be positive, it follows that in this case the participation of both intermediaries is sufficiently explained.

In a market with

$$T_M > \frac{m}{\alpha + \nu}\left(P^{I\to B} - P^{I\to S}\right)$$

nothing can be said about how the exact nature of the trades is assembled. But also in this case I_R's and I_M's payoffs can always be expected to be positive. Only if one of the intermediaries does not at all participate in the market, his expected payoff will be zero - but in such a case this would simply be justified since no work would have been done by the respective intermediary. Hence also in this case a financial motiviation for the intermediaries to enter the trade will be apparent.

7.3.2 An upper bound for transaction costs

Theorem 7.3.1 shows that the market maker I_M is guaranteed to participate in the trade as long as his transaction fee T_M satisfies

$$T_M \leq \frac{m}{\alpha + \nu}\left(P^{I\to B} - P^{I\to S}\right).$$

Using

$$\begin{aligned} P^{I\to B} - P^{I\to S} &= (1 + R_B)\,\hat{\Lambda} - (1 - R_S)\,\hat{\Lambda} \\ &= (R_B + R_S)\,\hat{\Lambda} \end{aligned}$$

this bound for T_M can be rewritten as

$$T_M \leq \frac{m}{\alpha + \nu}(R_B + R_S)\,\hat{\Lambda}. \tag{7.30}$$

Using $\frac{m}{\alpha+\nu} \leq \frac{1}{2}$ and $R_{avg} = \frac{R_B+R_S}{2}$ a less accurate, but more accessible formulation of an upper bound for T_M can be found as

$$T_M \leq R_{avg}\,\hat{\Lambda}. \tag{7.31}$$

It was one of the key assumptions of the model that I_M should not be able to adjust his transaction fee T_M on a round-by-round basis. If he would be allowed to act in such a way he could simply use his perfect information on the respective model parameters to force S and B into a trade with him.

It is however crucial for a market place to offer its participants more or less stable trading conditions, in this case more or less stable transaction fees. Only then can a market place be an attractive trading environment. Furthermore, it was assumed that T_M represents a fair reflection of the work I_M has to invest in order to execute a trade. It follows that the bounds (7.30) and (7.31) can be used to estimate how much this work may cost at most in order to allow trades with I_M. Bound (7.31) for example implies that I_M's participation in a trade will not be guaranteed if I_M's work is more expensive than the average risk premium S and B are willing to pay.

Consider now that it is in I_M's interest to participate in trades in as many rounds as possible during the time in which T_M remains unchanged. Since however $\hat{\Lambda}$, R_S, R_B, α and ν vary from round to round, I_M will try to offer his work for a price significantly less than the expected average risk premium over the period during which T_M is kept constant. For futures trading, Locke and Venkatesh[4] observe effective transaction costs between 0.0004% and 0.033% of the nominal value of a contract. Since even 0.033% can be expected to be significanly less than R_{avg}, the results of this model agree with such low transaction costs.

7.3.3 A closer view on markets in disequilibrium

If the transaction fee T_M satisfies

$$T_M > \frac{m}{\alpha + \nu} \left(P^{I \to B} - P^{I \to S} \right) \tag{7.32}$$

Theorem 7.3.1 implies that no equilibrium solution can be found.

There are various possible reasons for T_M and the other model parameters to be related in such a way. It is for example possible that I_M's work represented by T_M is simply too expensive in comparison with $R_{avg}\,\hat{\Lambda}$. In such a case, the combined transaction fee that S and B would have to pay for trading one unit of G via the intermediary I_M (i.e. $2\,T_M$) should be expected to be at least not much smaller than the maximal price spread $P^{I \to B} - P^{I \to S}$. A participation of I_M in the trade is hence rather unlikely.

[4]see [LV91] p. 239f.

Suppose now that I_M is able to offer his work sufficiently cheap, i.e. suppose that - under reasonable expectations on the model parameters - T_M satisifies $T_M << R_{avg}\,\hat{\Lambda}$ during most rounds in which T_M is kept constant. Inequality (7.32) might then be due to one or more of the following reasons:

- R_S and R_B are unusually small. Since this would correspond to a drastic reevaluation of S's and B's risk aversity, such a case should occur extremely rarely.
- $\hat{\Lambda}$ is very small compared to its usual level. This means that the price per unit of G has crashed significantly. Since under normal expectations on $\hat{\Lambda}$, T_M should satisfy $T_M << R_{avg}\,\hat{\Lambda}$, the price might have to decrease by 50% or more to violate this condition. Also such a price crash should hence occur quite rarely.
- Supply and demand are in a significant disequilibrium, i.e. either $\alpha >> \nu$ or $\nu >> \alpha$. If inequality (7.32) can mainly be explained by this reason, it can be shown that the market should be in some kind of quasi-equilibrium.

Suppose now that

$$T_M > \frac{m}{\alpha + \nu}\left(P^{I\to B} - P^{I\to S}\right)$$

mainly holds because of a significant supply-demand disequilibrium, i.e. suppose that

$$T_M << R_{avg}\,\hat{\Lambda}$$

is still satisfied. If for example $\nu << \alpha$, inequality (7.32) implies

$$\begin{aligned} P^{I\to S} + T_M &> P^{I\to S} + \frac{\nu}{\alpha + \nu}\left(P^{I\to B} - P^{I\to S}\right) \\ &= \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}, \end{aligned}$$

i.e.

$$P^{I\to S} < \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu} < P^{I\to S} + T_M.$$

Since T_M is assumed to be rather small compared to $R_{avg}\,\hat{\Lambda}$ and hence also small compared to

$$\begin{aligned} P^{I\to B} - P^{I\to S} &= (R_B + R_S)\,\hat{\Lambda} \\ &= 2R_{avg}\,\hat{\Lambda}, \end{aligned}$$

it follows that in such a case $\frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}$ should be expected to be very close to $P^{I\to S}$ and rather distant from $P^{I\to B}$, i.e.

$$P^{I\to S} \approx \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}.$$

Furthermore, unless $R_S \approx 0$, $\frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}$ should also be expected to be clearly less than $\hat{\Lambda}$ since

$$\frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu} \approx (1 - R_S)\hat{\Lambda}.$$

Suppose now that B is offering a price

$$P_0^{B\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}.$$

Then an offer $P_0^{S\to I} \leq P_0^{B\to I}$ would not be feasible in the sense of Proposition 6.3.1. Hence S could only offer a price

$$P_0^{S\to I} > \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}.$$

Consider that the outcome would then depend on I_R's decision. It was shown previously that in such a case I_R would opt for his strategy s_B^R. Furthermore, it was shown in the proof of Lemma 7.2.2 that S would be unable to change away from this outcome to a from his point of view more profitable strategy combination. Hence, by setting such an offer, B could guarantuee himself the realization of a price

$$P_0^{B\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}$$

per unit.

As shown in the proof of Lemma 7.2.2 this solution will not be a Nash equilibrium solution only because of B's chance of further improving his payoff. However, while a price offer

$$P_1^{B\to I} < \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu}.$$

might improve B's payoff, such a price offer might also backfire. B would no longer be guaranteed the rather favorable $s_{0,B}$-outcome and might instead face the worst possible case: a strategy combination $s_{0,S}$. The question that now arises is whether the possible increase in B's payoff warrants facing the downside of such a changed offer.

Note that already a switch to a direct trade scenario would most likely decrease B's payoff: as explained earlier $\frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha+\nu}$ should be expected to be clearly less than $\hat{\Lambda}$ (unless $R_S \approx 0$). A change to either $s_{1,S}$ or $s_{1,DT}$ would occur if

$$E\left[\hat{U}^{I_R}(s_{1,DT})\right] > E\left[\hat{U}^{I_R}(s_{1,B})\right].$$

With $m = \nu$ this translates to

$$(\alpha - \nu)\left(\hat{\Lambda} - P^{I\to S}\right) > \nu P_1^{B\to I} - \alpha P^{I\to S} + (\alpha - \nu)\hat{\Lambda}$$

and hence to

$$P^{I\to S} > P_1^{B\to I}.$$

It follows that no matter how S's offer $P_0^{S\to I}$ was set, B could at best realize a price

$$P_1^{B\to I} = P^{I\to S}.$$

Since however for small T_M already

$$P_0^{B\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu} \approx P^{I\to S}$$

B can barely improve by changing his offer. It is hence a reasonable expectation that B would stay at his offer $P_0^{B\to I}$ and force the market into a $s_{0,B}$-type outcome. This result corresponds to what one might expect in such a market: since $\alpha >> \nu$, S is under a lot more hedging pressure than B. B can use this advantage to buy his ν units for a price that is very close to $P^{I\to S}$, the cheapest price in the model framework.

Similarly, if $m = \alpha$, S should be expected to offer a price

$$P_0^{S\to I} = \frac{\alpha P^{I\to S} + \nu P^{I\to B}}{\alpha + \nu} \approx P^{I\to B}.$$

The resulting strategy combination would in this case be of the type $s_{0,S}$.

It follows that in markets where inequality (7.32) is mainly due to a drastic supply-demand disequilibrium, anything else than a $s_{0,B}$- or $s_{0,S}$-type trade (depending on whether $\alpha >> \nu$ or $\nu >> \alpha$) would be a rather surprising outcome. Hence, while a market with

$$T_M > \frac{m}{\alpha + \nu}\left(P^{I\to B} - P^{I\to S}\right)$$

will technically not be in an equilibrium state, it will in many cases be in some kind of quasi-equilibrium[5].

[5] Technically, the identified quasi-equilibrium is a so called ϵ-equilibrium, see Everett, [Eve57]

Conclusion

In the modern economy, exchange of goods is a process largely driven by the activity of intermediaries. Crucial advantages of intermediated exchange versus a more direct, cross-customer exchange are - among others - the centralization of exchange, facilitation of information transfer, reduction of search and bargaining costs, alleviation effects of adverse selection or management of inventory

In existing literature, the classic theoretical framework for understanding intermediated exchange is based on two major components: information asymmetry and inventory management. Examples for such classical models can be found in the papers by Dennert [Den93], Kyle [Kyl85], Laffont and Maskin [LM90], Copeland and Galai [CG83] or Glosten and Milgrom [GM85]. Spulber [Spu99] provides a detailed review of many classical models and aspects of intermediated trading.

While these models are applicable to the relatively slow, pit-oriented trading before the Internet revolution, they do not reflect the modern structure of the financial marketplace. The advent of electronic trading environments led to a fast and direct access to exchanges, pit-trading has on many exchanges disappeared completely. Information asymmetries have decreased, platforms like the Bloomberg terminal do now provide a quick access to almost all relevant information. Market inefficiencies have hence been eliminated.

At the same time, the storage and “buffering” functions of intermediaries are declining in the modern economy. Intermediaries do often carry no inventory at all: they serve as pure conduits in the transfer of goods or services. It follows that modern market structure makes it necessary to develop a new generation of models.

In the preceding work a series of three such models was proposed in a game-theoretical context. In all proposed models a commercial buyer, a commercial seller and one or two non-commercial intermediaries were assumed to trade a certain amount of one commodity. As a common feature, all three models were mainly based on differences in the goals of the different market participants, hedging pressure as one of the most critical aspects in the behaviour of commercial entities played a crucial role. The main components of most classical models, information asymmetry and inventory management, were largely ignored. Information was assumed to be distributed in a perfectly symmetric way, and the existence of intermediaries was explained by the different financial goals of the market participants.

In the first proposed model, only one intermediary acted in the market, hence fulfilling a dual function: he traded price spreads and took risk. As a main result of this first model, it has been shown that a naive approach in which buyer and seller were assumed to fulfil as much of their needs as possible in a direct trade could not be an equilibrium solution. This result showed a clear economic need for non-commercial intermediaries in such a market. However, it has also been shown, that the setup of this market model did result in a complete market disequilibrium. No clearing equilibrium price could be established. The behaviour of the different market participants was hence completely unpredictable.

In a second model, a second intermediary was introduced. Instead of having one intermediary trade spreads and take risk, specialized intermediaries I_M (a market maker) and I_R (a risk-neutral intermediary) were now either trading spreads or taking risk. As a main result, this economic specialization led to a rather stable market situation: it has been shown that in such a market model the market maker I_M would trade $\min\{\alpha, \nu\}$ from seller S to buyer B for a price F determined as

$$F = \frac{\alpha P^{I \to S} + \nu P^{I \to B}}{\alpha + \nu}.$$

The remaining $|\alpha - \nu|$ were traded with I_R for his binding price offers $P^{I \to S}$ and $P^{I \to B}$ respectively.

It should be noted that in this equilibrium solution intermediaries were involved in all occurring trades, no direct trades from buyer to seller were taking place. This result corresponds to observations that can be made in real markets: wherever intermediaries occur, they are usually trading almost the complete supply and demand.

The equilibrium price F turned out to be a rather natural solution: in accordance to standard economic theory, it has been shown that an increase in the supply α would lead to a decrease of F, while an increase in demand ν would be accompanied by an increasing clearing price F. Also the other model parameters R_S, R_B and $\hat{\Lambda}$ were shown to influence the equilibrium price F in a rather predictable and natural manner.

Overall, the introduction of a fourth player, the market maker I_M, proved to be a crucial step towards a more efficient and transparent market behaviour. This corresponds to findings made by Rust and Hall [RH03]: they extended a model proposed by Spulber [Spu96] consisting of buyers, sellers and middlemen by introducing a market maker and found that under certain conditions the introduction of such a market maker would lead to a more efficient market environment.

As a major shortcoming, this second market model could not explain why a market maker I_M would have entered such a market in the first place. In the market equilibrium, I_M's payoff was shown to equal zero, a financial motive for trading was hence missing. To overcome this model deficiency, a third model was then considered: the market maker was now assumed to charge a transaction fee T_M for his services.

In this market model the existence of an equilibrium solution depended on how the transaction fee T_M was set. If

$$T_M \leq \frac{m}{\alpha + \nu}\left(P^{I\to B} - P^{I\to S}\right),$$

where $m = \min\{\alpha, \nu\}$, the market equilibrium was found to be the same as in the previous model. The only difference was that this time I_M's motiviation was clear: he would now earn a payoff of $2m\, T_M$.

If however on the other hand

$$T_M > \frac{m}{\alpha + \nu}\left(P^{I\to B} - P^{I\to S}\right),$$

no Nash equilibrium could be observed. Under certain conditions it was nevertheless possible to predict that the market would approach some kind of quasi-equilibrium: if $\alpha >> \nu$, the shortage of demand would lead to B being able to buy G from I_R for a price close to $P^{I\to S}$, the cheapest possible price in the model framework. In the case of $\nu >> \alpha$, just the opposite would occur: a shortage in supply would then lead to S selling his supply to I_R for a price close to the most expensive possible price, i.e. for a price close to $P^{I\to B}$.

A more detailed examination of the above bound on T_M suggested that in order for I_M to be involved in the trades in as many rounds as possible, he would need to offer his services for a price

$$T_M << R_{avg}\,\hat{\Lambda},$$

i.e. for a price much less than the average risk premium S and B are willing to pay. This agrees with findings made by Locke and Venkatesh [LV91], who observed effective transaction costs between 0.0004% and 0.033% of the nominal value of a contract, i.e. prices much smaller than what market participants should be expected to pay for transferring their risk.

While the presented models provided some insightful results that should help to explain phenomena observed in real markets, they are of course just models of reality. Further research might be necessary to explain more aspect of intermediated trading. Possible directions of future research could for example be seen in the introduction of implicit instead of explicit transaction costs or in a discretization of the underlying price scale.

In many cases, market makers do not charge an explicit trading fee for their services as assumed in the third model of this work. Instead, they will only trade a spread if this spread appears to be big enough to guarantuee them a certain minimal profit. The minimal spread size could hence be seen as an implicit transaction fee that market participants have to pay. Such a model extension might therefore provide further insight into real-life market behaviour.

Furthermore, one of the key assumptions of the presented models was that trading took place on a continuous price scale. For every two prices $p_1 < p_2$, a tradeable price p_3 with $p_1 < p_3 < p_2$ was assumed to exist. This does however not correspond to reality: the minimal price step in a certain market is defined by the so called tick size τ. If therefore $p_2 = p_1 + \tau$, no tradeable price p_3 with $p_1 < p_3 < p_2$ can exist. The price scale in a model incorporating this tick size would therefore have to be discrete. For seller S and buyer B this would imply that their strategy sets would no longer be infinite but finite, since only tradeable prices could be used as strategies. It is highly likely that looking at a discrete instead of at a continuous price scale may provide some more interesting and important results.

These examples show that already simple extensions to the presented models might be of interest for future research. I am however confident that also the presented models can provide the reader with interesting insight and important information on how and why intermediated exchange works in the modern-day economy.

Bibliography

[AS98] Franklin Allen and Anthony M. Santomero, *The theory of financial intermediation*, Journal of Banking & Finance **21** (1998), 1461–1485.

[Bit81] Michael Bitz, *Entscheidungstheorie*, Verlag Vahlen, Munich, 1981.

[Bor85] Kim C. Border, *Fixed point theorems with applications to economics and game theory*, Cambridge University Press, Cambridge, 1985.

[Bro11] Luitzen E.J. Brouwer, *Beweis der Invarianz der Dimensionszahl*, Mathematische Annalen **70** (1911), 161–165.

[CG83] Thomas E. Copeland and Dan Galai, *Information effects on the bid-ask spread*, Journal of Finance **31** (1983), 1457–1469.

[Den93] Jürgen Dennert, *Price competition between market makers*, The Review of Economic Studies **60** (1993), no. 3, 735–751.

[Eki03] Erdal Ekici, *Nearly continuous multifunctions*, Acta Mathematica Universitatis Comenianae **LXXII** (2003), no. 2, 229–235.

[Eve57] Hugh Everett, *Recursive games*, Contributions to the Theory of Games, vol. 3, Princeton University Press, 1957.

[Fan60] Ky Fan, *A generalization of Tychonoff's fixed point theorem*, Mathematische Annalen **142** (1960), 305–310.

[FH99] William Fung and David A. Hsieh, *A primer on hedge funds*, Journal of Empirical Finance **6** (1999), 309–331.

[Gib92] Robert Gibbons, *A primer in game theory*, Harvester Wheatsheaf, Hertfordshire, 1992.

[GM85] Lawrence R. Glosten and Paul R. Milgrom, *Bid, ask and transaction prices in a specialist market with heterogeneously informed traders*, Journal of Financial Economics **14** (1985), 71–100.

[Kak41] Shizuo Kakutani, *A generalization of Brouwer's fixed point theorem*, Duke Mathematical Journal **8** (1941), no. 3, 457–459.

[Kyl85] Albert S. Kyle, *Continuous auctions and insider trading*, Econometrica **53** (1985), no. 6, 1315–1335.

[LH64] Carlton E. Lemke and Joseph T. Howson, *Equilibrium points of bimatrix games*, SIAM Journal of Applied Mathematics **12** (1964), 413–423.

[LM90] Jean-Jaques Laffont and Eric S. Maskin, *The efficient market hypothesis and insider trading on the stock market*, Journal of Political Economy **98** (1990), no. 6, 70–93.

[LV91] Peter R. Locke and P. C. Venkatesh, *Futures market transaction costs*, The Journal of Futures Markets **17** (1991), 229–245.

[MT05] Andy McLennan and Rabee Tourky, *From imitation games to Kakutani*, Manuscript, 2005.

[Nas51] John F. Nash, *Non-cooperative games*, Annals of Mathematics **54** (1951), 286–295.

[RH03] John Rust and George Hall, *Middlemen versus market makers: a theory of competitive exchange*, The Journal of Political Economy **11** (2003), 353–403.

[Sch04] Walter Schlee, *Einführung in die Spieltheorie*, Vieweg Verlag, Wiesbaden, 2004.

[Spu96] Daniel F. Spulber, *Market making by price setting firms*, Review of Economic Studies **63** (1996), 559–580.

[Spu99] ______, *Market microstructure: intermediaries and the theory of the firm*, Cambridge University Press, Cambridge, 1999.

[vN28] John v. Neumann, *Zur Theorie der Gesellschaftsspiele*, Mathematische Annalen **100** (1928), 295–300.

[Wie02] Harald Wiese, *Entscheidungs- und Spieltheorie*, Springer Verlag, Berlin/Heidelberg/New York, 2002.