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Edge coloring BIBDS and constructing MOELRs

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Edge Coloring BIBDs and Constructing MOELRs

By
JOHN S. ASPLUND

A THESIS

Submitted in partial fulfillment of the requirements
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This thesis, "Edge Coloring BIBDs and Constructing MOELRs", is hereby approved in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE IN MATHEMATICAL SCIENCES.

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Abstract

Chapter 1 is used to introduce the basic tools and mechanics used within this thesis. Some historical uses and background are touched upon as well. The majority of the definitions are contained within this chapter as well.

In Chapter 2 we consider the question whether one can decompose λ copies of monochromatic K_v into copies of K_k such that each copy of the K_k contains at most one edge from each K_v. This is called a proper edge coloring (Hurd, Sarvate, [29]). The majority of the content in this section is a wide variety of examples to explain the constructions used in Chapters 3 and 4.

In Chapters 3 and 4 we investigate how to properly color $\mathrm{BIBD}(v, k, \lambda)$ for $k = 4$, and 5. Not only will there be direct constructions of relatively small BIBDs, we also prove some generalized constructions used within.

In Chapter 5 we talk about an alternate solution to Chapters 3 and 4. A purely graph theoretical solution using matchings, augmenting paths, and theorems about the edge-chromatic number is used to develop a theorem that than covers all possible cases. We also discuss how this method performed compared to the methods in Chapters 3 and 4.

In Chapter 6, we switch topics to Latin rectangles that have the same number of symbols and an equivalent sized matrix to Latin squares. Suppose $ab = n^2$. We define an equitable Latin rectangle as an $a \times b$ matrix on a set of n symbols where each symbol appears either $\lceil \frac{b}{n} \rceil$ or $\lfloor \frac{b}{n} \rfloor$ times in each row of the matrix and either $\lceil \frac{a}{n} \rceil$ or $\lfloor \frac{a}{n} \rfloor$ times in each column of the matrix. Two equitable Latin rectangles are orthogonal in the usual way. Denote a set of k $a \times b$ mutually orthogonal equitable Latin rectangles as a k–$\mathrm{MOELR}(a, b; n)$. We show that there exists a k–$\mathrm{MOELR}(a, b; n)$ for all a, b, n where k is at least 3 with some exceptions.

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Chapter 1

Introduction

This thesis is the result of the author's interest in furthering recently developed fields in combinatorial designs. The goal of this thesis is to construct proper colorings of complete graphs using balanced incomplete block designs and expand the known results of mutually orthogonal equitable Latin rectangles. With that said, now we begin with an introduction to graph theory.

1.1 Graphs

To begin, a *graph*, G, is an ordered pair (V, E) comprised of a set of vertices V and a collection of unordered pairs of vertices called edges E. The order of the graph is $|V|$, the number of vertices. To be more precise, we will only deal with graphs that are simple and undirected. A *simple graph* is a graph with no loops or multiple edges between two distinct vertices. If an edge connects a vertex to itself we say the edge is a *loop*. A graph may have several edges between the same two vertices. These edges are called *multiple edges*. Figure 1.1 is an example of a simple graph. A simple graph is a *complete graph* if every pair of vertices is joined by an edge. The complete graph with v vertices is denoted K_v.

Graph colorings have a wide variety of real world applications including radio wave assigning. Suppose we have six different radio stations at different distances from each other. If two radio stations are within 100 miles of each other then they cannot use the same frequency. To apply graph colorings to this problem, draw six vertices to represent the six radio stations. Place an edge between two vertices if they are within 100 miles of each other. The frequencies will be represented by colors. So if radio station a and b are within 100 miles, we need to color the vertices different colors because they cannot use the same frequency. In other words, if we give a coloring that uses the least number of colors possible on the vertices of the graph we created, we find the least number of radio signals we can possibly use without landing on some other stations signal. This is useful considering how many radio signals are continually flooding the air waves around populated areas such as Chicago and New York City.

There are several ways to color graphs including two of the most popular methods: vertex-coloring and edge-coloring. We are most interested in coloring the edges of a graph.

Let v be the number of vertices. The maximum number of edges possible in a simple graph is $\binom{v}{2} = \frac{v(v-1)}{2}$. Much has been done in the field of graph coloring which is why we will touch on both of the two previously mentioned sub-divisions of graph coloring. For a more in-depth look into the properties, definitions, and theorems of graph theory, see [20].

1.1.1 Vertex-Colored Graphs

Two vertices are *adjacent* if they share a common edge. Two edges can be adjacent as well as long as they share a common vertex. Two edges are adjacent if they share a common vertex. A labeling of the vertices of a graph G by the colors $\{1, 2, \ldots, k\}$ in such a way that adjacent vertices receive different colors is called a *k–coloring* of G. The *chromatic number*, $\chi(G)$, of a graph G is the smallest integer k such that G has a k–coloring.

There are many applications of vertex coloring. When scheduling a set of interfering jobs, one can use a *conflict graph*. A conflict graph is a simple graph where an edge between two vertices represents when two corresponding jobs are unable to be executed at the same time. This can happen if two jobs share a resource like machinery or personnel or interfere in some other way. Let the colors represent time slots and every job require one time slot. This creates a one to one correspondence with the vertices and colors because a vertex or a job can only be completed once. If the colors represent one hour intervals then the chromatic number is the minimum number of hours required to finish all v jobs. For more examples of vertex-coloring see [35].

1.1.2 Edge-Colored Graphs

Intuitively, a proper edge-coloring is an assignment of colors to the edges of a graph such that no adjacent edges share a color. Formally, we may define a proper edge-coloring as follows.

Definition 1.1. Let $G = (V, E)$ be a graph and $C = \{1, 2, \ldots, k\}$ be a set of colors. A *proper coloring*[1] or *proper edge-coloring* is a labeling of the edges with colors from C in such a way where no two adjacent edges share a color in common.

If we properly color a complete graph K_k, each color is used at most once when coloring the edges of K_k making each K_k panchromatic. A graph is *panchromatic* if the graph has a unique color on each edge. Figure 1.1 depicts a properly colored graph of five vertices and six edges with minimum number of colors so that no two adjacent edges are given the same color.

A *matching* of a graph G is a set of edges where no two edges share a vertex. Another way to say this is the graph has a 1-factor. Suppose we select the two blue edges of Figure 1.1. Since these two edges are non-adjacent, we have a matching. The same is true if we select the red or green sets of edges as well.

[1]For simplicity we will use this word rather than proper edge-coloring.

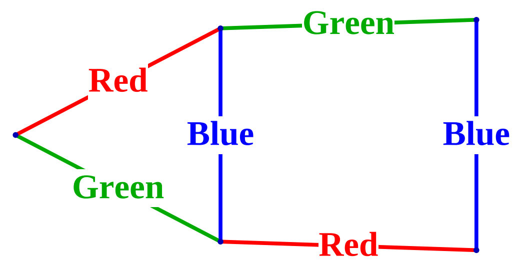


Figure 1.1: A properly colored graph.

Matching problems have many applications. We consider the date-match problem represented in Figure 1.2, an example from [39]. This figure is called a *bipartite graph* because

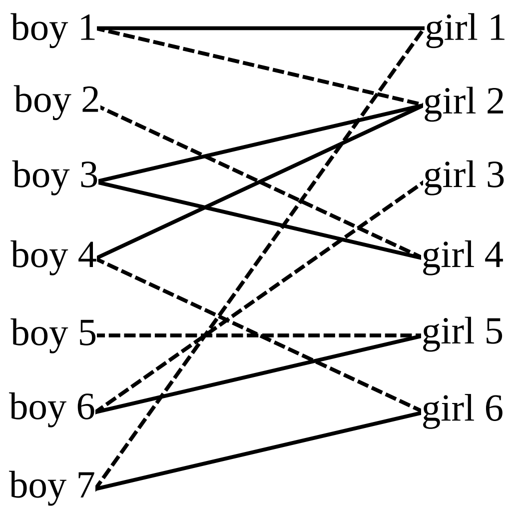


Figure 1.2: A date-match problem

it separates the vertices into two partition classes or sets where two vertices within the same partition class are non-adjacent. Suppose there are several boys and girls, and each girl favors some of the boys. What condition is necessary to make a date-matching in which every girl is assigned to one of her favorite boys? Consider a graph in which boys and girls correspond to vertices and a vertex corresponding to a girl is joined to vertices corresponding to boys whom the girl favors. Then there is a date-matching if and only if the graph has a matching containing all vertices corresponding to girls. The dashed edges give a possible matching given the requirements in the question.

Make note that if S is a set, then $S \backslash \{u\}$ or $S - \{u\}$ means take S and remove all elements of $\{u\}$ from S.

How do we generate a matching of G with as many edges as possible? One such way is through the use of alternating paths and augmenting paths. First consider an arbitrary matching called M which consists of edges from a graph G. A path in G which starts at an unmatched vertex in X and then contains an edge from $E \backslash M$, an edge from M, an edge from $E \backslash M$, an edge from M, and so on, is an *alternating path* with respect to M. An alternating path P that ends in an unmatched vertex of the set Y is called an *augmenting path* as seen in Figure 1.3. The bold edges on the left graph are the current matching. By removing the edges from the matching M that are from the augmenting path P in Figure 1.3 and add the edges that were not in M but were in P, then the matching M' has a larger matching than M as seen in the right graph in Figure 1.3.

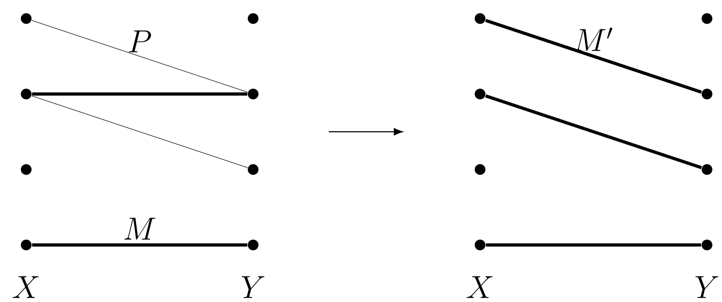


Figure 1.3: Using augmenting paths on alternating path P of matching M

The first paper on edge-coloring problems was published by Tait back in 1889. In this paper he proved that the four-color conjecture is true if and only if one can edge-color every planar 3-connected cubic graph using three colors. The four-color conjecture states that one can color a map using only four colors so that any adjacent countries have different colors. The four-color theorem was proven in 1976 with the use of computers. Later, in 1980, Hoyler proved edge-coloring problems are *NP–complete*. This means it is very unlikely that edge-coloring problems are able to be solved in polynomial-time. With this result, it shows us that we need to develop methods that will efficiently color graphs giving rise to one reason the problem in this thesis was pursued.

1.2 Designs

A *balanced incomplete block design* ($BIBD(v, k, \lambda)$) with parameters (v, b, r, k, λ) is a pair $(\mathcal{V}, \mathcal{B})$ where $\mathcal{V}$ is a v-set of points and $\mathcal{B}$ is a collection of b k-subsets of $\mathcal{V}$, called *blocks*, such that each element of $\mathcal{V}$ is contained in exactly r blocks and any pairs of points of $\mathcal{V}$ is contained in exactly λ blocks. A BIBD is *complete* if it has no repeating blocks and contain $\binom{v}{k}$ blocks. We denote this as BIBD(v, k, λ) because the values for b and r can be calculated with only $v, k,$ and λ.

To explain designs further we will use one of the most common examples in design theory; the Fano plane seen in Figure 1.4. Each line on this figure and the circle represent the seven blocks of the design. Our goal is to construct a BIBD$(7, 3, 1)$ so let $\mathcal{V} = \{0, 1, 2, 3, 4, 5, 6\}$. The parameters of a BIBD$(7, 3, 1)$ tell us we are going to make blocks of size 3, 3-subsets, and we will see any pair of points appear exactly once. Let

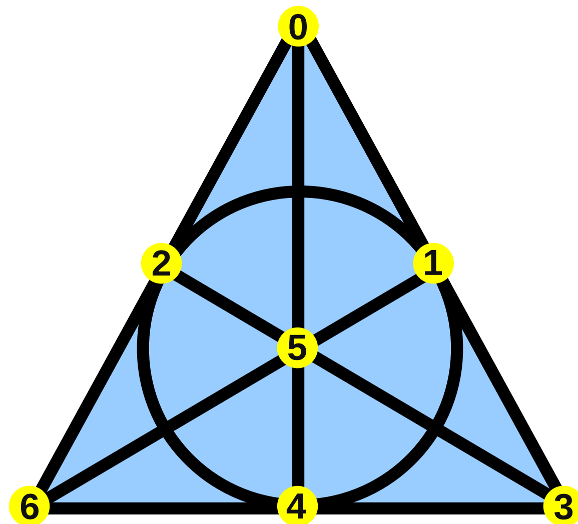


Figure 1.4: The Fano plane.

$B_1 = \{0,1,3\}$ be the first block where $B_1 \in \mathcal{B}$. From this we will develop a set of blocks cyclically from B_1 modulo 7. Here is the set of seven blocks we develop from B_1.

$$\{0,1,3\} \qquad \{1,2,4\} \qquad \{2,3,5\} \qquad \{3,4,6\} \qquad \{4,5,0\} \qquad \{5,6,1\} \qquad \{6,0,2\}$$

These seven blocks make up $\mathcal{B}$ and so we have constructed a $\text{BIBD}(7,3,1)$. We know we have constructed all the blocks because there is a formula for finding the number of blocks; $b = \dfrac{v(v-1)}{k(k-1)}\lambda$. In our case,

$$b = \frac{v(v-1)}{k(k-1)}\lambda = \frac{7(6)}{3(2)} \cdot 1 = 7.$$

Due to the complexity of designs, this method of cyclically constructing designs does not always work. It is only because this design is cyclic that we may construct it in this fashion. Keep this in mind when we talk about designs being sufficient.

The numbers v, b, r, k, and λ are *parameters* of the BIBD. A special type of BIBD called a *triple system* ($\text{TS}(v,\lambda)$) is a BIBD where $k = 3$. And a special type of triple system called *Steiner triple system* ($\text{STS}(v)$) is a BIBD where $\lambda = 1$ and $k = 3$ or a triple system where $\lambda = 1$. The necessary conditions under which a BIBD exists are given below.

Theorem 1.2. *(Hanani, [28]) The necessary conditions for the existence of a* $\text{BIBD}(v,b,r,k,\lambda)$ *are*

1. $vr = bk$

2. $r(k-1) = \lambda(v-1)$

Though the first questions in design theory were contrived many centuries ago, our focus will begin in 1847 when Kirkman [31] dealt with the existence of $\mathrm{STS}(v)$ which exist for all $v \equiv 1, 3 \pmod 6$. Kirkman had a famous schoolgirl problem [32] that dealt with STS designs:

> `Fifteen young girls in a school walk out three abreast in succession; it is required to arrange them daily, so that no two shall walk two abreast.`

Because there are 15 young girls, $v = 15$. When the girls walk three abreast, this means that we group the girls together in groups of size three. Thus we have an example of an $\mathrm{STS}(15)$ or a $\mathrm{BIBD}(15, 3, 1)$.

Significant contributions to this field include Euler in 1782, Kirkman in 1847, Moore in 1896, Bose in 1939, and Wilson in 1972. Hanani also contributed by determining the necessary and sufficient conditions of BIBDs with block size 4 and 5 with any λ value. Later, he published a survey over 100 pages long, detailing the previous existence results for blocks of size 3, 4, and 5. He then extended the results to block size 6 and any $\lambda \geq 2$.

Design theory has numerous applications throughout many fields, but the general consensus is that one of the most useful applications of design theory is in setting up experimental designs. One such application can be summed up as making a consumer experiment where consumers try products and give responses to questionnaires about the products they try. All the products are alterations of one main concept such as fast food experiments where a group of 30 to 40 consumers try out different types of foods. Since it is a principal feature that all samples be evaluated the same number of times, a balanced incomplete block design contains this and more features making it ideal for this type of experiment. We also want to have pairs of samples to be evaluated the same number of times by some consumer. To do this, we just let the consumer represent a block of different samples. Since each sample occurs r times and each pair of samples occurs λ times, the experimental design is said to be a BIBD.

A large list of parameters for BIBDs are listed in [16] by Cochran and Cox. Since their designs were meant for agricultural applications which had limited resources (i.e. animals and field plots) the number of repetitions is limited. The number of repetitions that consumer experiments tend to have at least 30 to 40. So the size of designs limits the usefulness with real world applications but with ever increasing technology, it does not seem infeasible that larger designs may be used in the near future.

Another type of design we will focus heavily on is the *group divisible design* or (GDD). A group divisible design of index λ, $(k, \lambda) - \mathrm{GDD}(m^u)$, is a triple $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ where $\mathcal{V}$ is a finite set of $v = mu$ points, $\mathcal{G}$ is a partition of $\mathcal{V}$ into parts (*groups*) whose sizes lie in a set of positive integers, and $\mathcal{B}$ is a family of subsets (*blocks*) of $\mathcal{V}$ that satisfy the following.

1. If $B \in \mathcal{B}$, then $|B| = k$.
2. If two elements are in the same group, then this pair cannot be in any block.
3. $|\mathcal{G}| > 1$.

We use exponential notation to denote the type of the GDD. For example, a GDD having u groups of size m would be referred to as a GDD(m^u) as in Figure 1.5. A pair of distinct points coming from two distinct groups of a GDD is called a *transverse pair*. A GDD is *uniform* if all groups have the same size and denoted as $k-\text{GDD}(m^u)$.

$$\begin{array}{ccccc}
\text{Group 1} & \text{Group 2} & \dots & \text{Group } u-1 & \text{Group } u \\
\bullet_1 & \bullet_1 & \cdots & \bullet_1 & \bullet_1 \\
\bullet_2 & \bullet_2 & \cdots & \bullet_2 & \bullet_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\bullet_m & \bullet_m & \cdots & \bullet_m & \bullet_m
\end{array}$$

Figure 1.5: k–GDD(m^u)

GDDs can be extended to BIBDs by making sure every pair of points occurs in λ blocks. This method is similar to Wilson's Fundamental Construction [51]. As an example, consider a 4–GDD(4^4) setup as in Figure 1.6. Our goal is to create a BIBD$(16, 4, 1)$. We

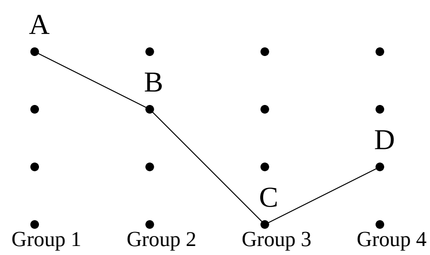


Figure 1.6: One block of a $4-\text{GDD}(4^4)$

know this GDD exists because the necessary and sufficient conditions for the existence of a $(4, \lambda)-\text{GDD}(m^u)$ were found by Zhu in [55]. They are given in the following theorem.

Theorem 1.3. *The necessary and sufficient conditions for the existence of a* $(4, \lambda)-$ GDD(m^u) *are*

1. $u \geq 4$,
2. $\lambda(u-1)m \equiv 0 \pmod 3$, *and*
3. $\lambda u(u-1)m^2 \equiv 0 \pmod{12}$,

with exception of $(m, u, \lambda) \in \{(2, 4, 1), (6, 4, 1)\}$*, in which case no such* GDD *exists.*

In a GDD, every transverse pair occurs in exactly λ blocks. This means that if $\lambda = 1$, every pair occurs in exactly one block or group. The only pairs that have not been accounted for are those that lie in the same group. We know by Hanani in [28] that the necessary conditions in Theorem 1.2 are sufficient for the following v values with $k = 4$:

k	λ	Conditions for v
4	1	1,4 (mod 12)
4	2	1 (mod 3)
4	3	0,1 (mod 4)
4	6	all

Consider the set of points in a group as the set of points from which we form a design. From these set of four points we can form a $\text{BIBD}(4, 4, 1)$ design by the table above. Because we do this to each group, we add the blocks from each $\text{BIBD}(4, 4, 1)$ on each group to the blocks we formed from the blocks of the GDD. This action of forming BIBDs on groups will be known as placing a BIBD on each group. This ensures that each pair among the groups has $\lambda = 1$. Therefore, there exists a $\text{BIBD}(16, 4, 1)$.

Non-uniform GDDs have also been studied.

Theorem 1.4. *(Ge, Ling, [24]) A* $4 - \text{GDD}(4^u m^1)$ *exists if and only if either* $u = 3$ *and* $m = 4$*, or* $u \geq 6$, $u \equiv 0 \pmod 3$ *and* $m \equiv 1 \pmod 3$ *with* $1 \leq m \leq 2(u - 1)$.

Theorem 1.5. *(Zhu, [55]) The necessary conditions for the existence of a uniform* $(5, \lambda) - \text{GDD}(m^u)$ *are as follows.*

1. $u \geq 5$,
2. $\lambda(u - 1)m \equiv 0 \pmod 4$*, and*
3. $\lambda u(u - 1)m^2 \equiv 0 \pmod{20}$.

Theorem 1.6. *(Ge, Rees, Zhu, [25]) A* $4 - \text{GDD}(1^u m^1)$ *exists if and only if* $u \geq 2m + 1$ *and either* m, $u + m \equiv 1$ *or* $4 \pmod{12}$ *or* $m, u + m \equiv 7$ *or* $10 \pmod{12}$.

Another class of GDDs that have been studied are $(5, \lambda)$–GDD.

Theorem 1.7. *(Ge, Ling, [24]) The necessary conditions for the existence of a* $5-\text{GDD}(m^u)$ *in Theorem 1.5 are also sufficient, except when* $m^u \in \{2^5, 2^{11}, 3^5, 6^5\}$*, and possibly where*

1. $m^u = 3^{45}, 3^{65}$;
2. $m \equiv 2, 6, 14, 18 \pmod{20}$ *and*
   - *(a)* $m = 2$ *and* $u \in \{15, 35, 71, 75, 95, 111, 115, 195, 215\}$;
   - *(b)* $m = 6$ *and* $u \in \{15, 35, 75, 95\}$;
   - *(c)* $m = 18$ *and* $u \in \{11, 15, 71, 111, 115\}$;
   - *(d)* $m \in \{14, 22, 26, 34, 38, 46, 58, 62\}$ *and* $u \in \{11, 15, 71, 75, 111, 115\}$;

(e) $m \in \{42, 54\}$ *or* $m = 2\alpha$ *with* $\alpha \equiv 1, 3, 7, 9 \pmod{10}$ *and* $33 \leq \alpha \leq 2443$, *and* $u = 15$*;*

3. $m \equiv 10 \pmod{20}$ *and*

(a) $m = 10$ *and* $u \in \{5, 7, 15, 23, 27, 33, 35, 39, 47\}$*;*

(b) $m = 30$ *and* $u \in \{9, 15\}$*;*

(c) $m = 50$ *and* $u \in \{15, 23, 27\}$*;*

(d) $m = 90$ *and* $u = 23$*;*

(e) $m = 10\alpha, \alpha \equiv 1 \pmod{6}, 7 \leq \alpha \leq 319$, *and* $u \in \{15, 23\}$*;*

(f) $m = 10\beta, \beta \equiv 5 \pmod{6}, 11 \leq \beta \leq 443$, *and* $u \in \{15, 23\}$*;*

(g) $m = 10\gamma, \gamma \equiv 1 \pmod{6}, 325 \leq \gamma \leq 487$, *and* $u = 15$*;*

(h) $m = 10\delta, \delta \equiv 5 \pmod{6}, 449 \leq \delta \leq 485$, *and* $u = 15$*;*

Theorem 1.8. *(Assaf, Bluskov, Greig, Shalaby, [11]) Let* $\lambda \geq 2$*. The necessary conditions for the existence of* $(5, \lambda) - \mathrm{GDD}$*s of type* m^u *in Theorem 1.5 are also sufficient, except possibly when* $\lambda = 2, u = 15$ *and either* $m = 9$ *or gcd*$(m, 15) = 1$*.*

Let $\mathcal{B}$ be a set of blocks in a GDD or BIBD. A *parallel class* or *resolution class* is a collection of blocks that partition the point-set of the design. A design is *resolvable* if the blocks of the design can be partitioned into parallel classes.

A *resolvable balanced incomplete block design* (RBIBD) is a balanced incomplete block design where the design is resolvable. The following are the necessary conditions for the existence of a $\mathrm{RBIBD}(v, k, \lambda)$:

1. $\lambda(v - 1) \equiv 0 \pmod{k - 1}$,

2. $v \equiv 0 \pmod{k}$.

From [22], these conditions are known to be sufficient for any k and λ, if v is large enough.

Theorem 1.9. *[7, 36, 23]. An* $\mathrm{RBIBD}(v, 5, \lambda)$ *exists for* $\lambda = 1, 2, 4$*, if any of the following conditions is satisfied:*

1. $\lambda = 1, v \equiv 5 \pmod{20}$ *and* $v \neq 45, 185, 225, 345, 465, 645$*;*

2. $\lambda = 2, v \equiv 5 \pmod{10}$ *and* $v \geq 50722395$*;*

3. $\lambda = 4, v \equiv 0 \pmod{5}$ *except for* $v = 10$ *and possibly for* $v = 15, 70, 75, 90, 95, 135, 160, 185, 190, 195$.

In the same manner, a resolvable GDD is a GDD where the blocks of the design can be partitioned into parallel classes, denoted as RGDD.

Theorem 1.10. *(Ge, Ling, [24]) The necessary conditions for the existence of a* $4-\mathrm{RGDD}(m^u)$*, namely,* $u \geq 4$*,* $mu \equiv 0 \pmod 4$ *and* $m(u-1) \equiv 0 \pmod 3$*, are also sufficient except for* $(m,u) \in \{(2,4),(2,10),(3,4),(6,4)\}$ *and possibly excepting:* $m = 2$ *and* $u \in \{34,46,52,70,82,94,100,118,130,142,178,184,202,214,238,250,334,346\}$; $m = 10$ *and* $u \in \{4,34,52,94\}$; $m \in [14,454] \cup \{478,502,514,526,614,626,686\}$ *and* $u \in \{10,70,82\}$*;* $m = 6$ *and* $u \in \{6,54,68\}$; $m = 18$ *and* $u \in \{18,38,62\}$; $m = 9$ *and* $u = 44$; $m = 12$ *and* $u = 27$; $m = 24$ *and* $u = 23$; *and* $m = 36$ *and* $u \in \{11,14,15,18,23\}$.

Chapter 2

Proper Edge Colorings of $\mathrm{BIBD}(v, k, \lambda)$

Even though the problem is stated in graph theoretic terms, the focus of this thesis remains to be design theory. In this chapter we will give a multitude of examples of the problem at hand and an example of every construction we use in Chapters 3 and 4. Pictorial representations will be used when appropriate.

2.1 The Focus

Before we start, make note that $\mathbb{Z}^+$ represents the set of positive integers not including zero. We consider the following question

Question. *Let $G = (V, E)$ be a graph and $\{1, 2, \ldots, k\}$ be a set of colors. Can one decompose λ copies of monochromatic K_v into copies of K_k such that each copy of K_k contains at most one edge from each K_v?*

A graph is *monochromatic* if all the edges of the graph are colored with the same color. Constructing panchromatic K_k graphs from λ monochromatic K_v graphs is the same as properly coloring K_k graphs using the edges of the monochromatic graphs. Note that λ denotes the number of colors used as well as the number of monochromatic graphs.

In [29] Hurd and Sarvate translated this graph theory problem into a problem based on designs. Let us consider the graph $G = (V, E)$ being the complete graph on v vertices such that $|V| = v$ and $|E| = \binom{v}{2} = \frac{v(v-1)}{2}$. We copy this graph λ times and give a unique color to each copy giving us λ colors. We label all vertices in each copy of K_v so as to distinguish which edge we remove based on the vertices that connect them. So each K_k must have k labeled vertices from V. Because we are trying to properly color a K_k graph, we will remove at most one edge from each monochromatic K_v. For a K_k to be properly colored, there need to be at least $\binom{k}{2}$ unique colors so each edge in the K_k can have a unique color. It follows that $\lambda \geq \binom{k}{2}$ for us to have a chance of properly coloring each K_k. Let b be the number of K_k graphs we create through this decomposition of the λ copies of K_v.

Suppose we represent the vertices of each $K_k = (V_0, E_0)$ as a set $\{v_1, v_2, \ldots, v_k\}$ of vertices where $v_1, v_2, \ldots, v_k \in V_0 \subset V$. Because the graph is complete on K_k it is easy to form the complete graph on k vertices from the set $\{v_1, v_2, \ldots, v_k\}$. Each one of these sets will be of the same size k, and we shall denote each of these sets as B_i such that $B_i \in \mathcal{B}$ for all $i = 1, 2, \dots, b$. Because a particular edge appears once in each of the λ copies of K_v, there are λ copies of this edge, each with a different color. We must ensure the number of times a particular edge is seen among the K_k graphs is λ. This is the same as confirming every pair of distinct vertices is in exactly λ of the B_i. At this point, notice that each set, or as we will call them blocks, has identical size k where each vertex is chosen from a set of v vertices and we must make sure that the number of pairs we see among all the blocks is λ. This is identical to the definition of balanced incomplete block design. As such, rather than saying we will color the pairs of points in a design so as to create properly colored K_k graphs from K_v graphs, we will say we have a properly colored design. So our objective for this paper equates to showing we can properly color a $\mathrm{BIBD}(v, k, \lambda)$.

We want to decompose λ copies of monochromatic K_v into copies K_k such that each copy of K_k contains at most one edge from each K_v. A *proper coloring*, or *proper edge-coloring*, of a $\mathrm{BIBD}(v, k, \lambda)$ is an assignment of colors to the edges of a K_k denoted by a block of the design (a block of the design denotes the K_k graph) with the properties:

1. each edge in a K_k graph generated from the points in a block receives a different color;
2. every edge is used exactly λ times;
3. every color is used exactly once among the λ copies of an edge.

In [29] Hurd and Sarvate discussed and solved the cases where $\lambda = mk(k-1)/2$ for any $\mathrm{BIBD}(v, k, \lambda)$. They also found the sufficient conditions for the existence of a properly colored triple system $\mathrm{TS}(v, \lambda)$. The following theorems are some results found by Hurd and Survate in [29].

Theorem 2.1. *The necessary conditions are sufficient for the existence of a* $\mathrm{TS}(v, \lambda)$ *which has a proper coloring.*

Theorem 2.2. *Suppose that there exists a* $\mathrm{BIBD}(v, k, \lambda)$*,* $(\mathcal{V}, \mathcal{B})$ *with* $\lambda = mk(k-1)/2$*,* $m \in \mathbb{Z}^+$*. Then the blocks of* $\mathcal{B}$ *can be properly colored with* λ *colors so that no two edges in any block have the same color.*

Theorem 2.3. *(Hurd, Sarvate, [29]) If a* $\mathrm{BIBD}(v, k, 1)$ *exists, then for the index* $\lambda \geq k(k-1)/2$ *there exists a* $\mathrm{BIBD}(v, k, \lambda)$ *whose edges can be taken from* λ *monochromatic copies of* K_v *so that no two edges in a block have the same color.*

As an example of properly coloring the block structure of a K_k using the structure of a BIBD we will give a graphical representation of a proper coloring of three monochromatic K_7 into panchromatic K_3. Suppose we have three K_7 graphs, a blue, a red, and a green one. Let $\{0, 1, 3\}$ be a block of vertices that will make up one vertex set of a panchromatic K_3. Note all pairs and triples here are listed with braces to indicate the sets are unordered. Color edge $\{0, 1\}$ red, edge $\{1, 3\}$ green, and edge $\{0, 3\}$ blue. To do this we remove the respective colored edges from the monochromatic K_7 graph to construct the panchromatic K_3. This is shown in Figure 2.1 with gray dashed lines indicating the edge has been removed from a K_7. Now we can repeat this block two more times to get three copies of this vertex set with different colors. The second block is based on the same vertex set, so color edge $\{0, 1\}$ blue, edge $\{1, 3\}$ red, and edge $\{0, 3\}$ green. Again, we remove edges from the same monochromatic K_7 graphs to construct the panchromatic K_3. This is shown in Figure 2.2. We perform the same task for the last copy of this block by coloring edge $\{0, 1\}$ green, edge $\{1, 3\}$ blue, and edge $\{0, 3\}$ red. This is shown in Figure 2.3.

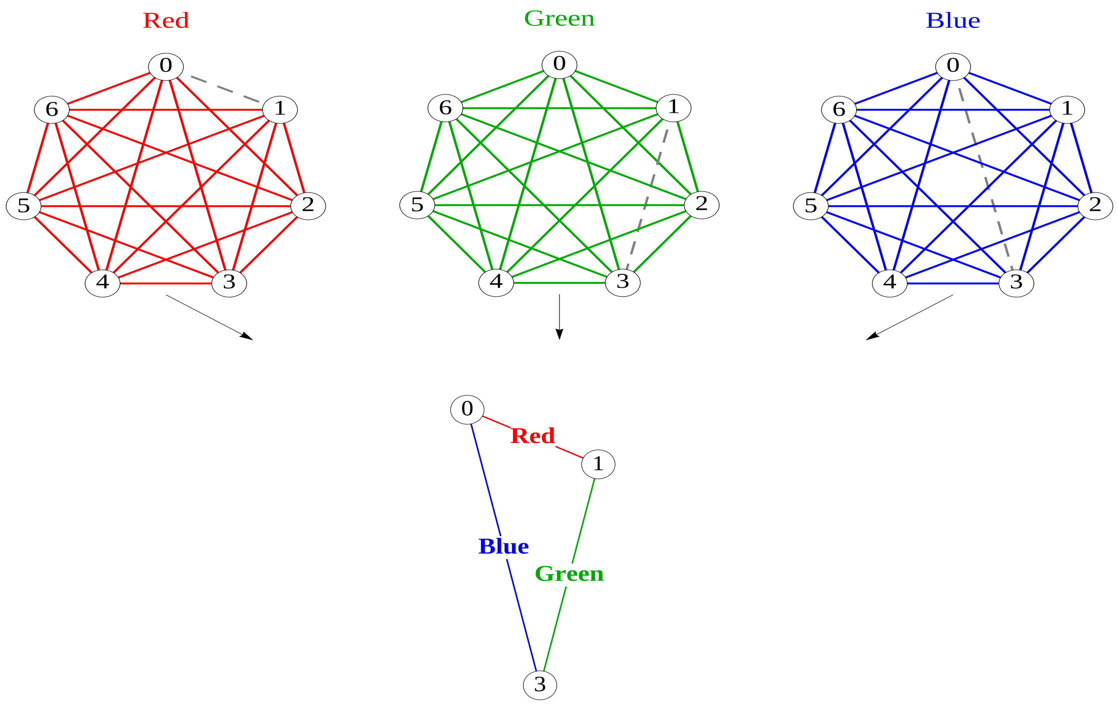


Figure 2.1: 1^{st} of three colored copies of a block from a BIBD$(7, 3, 3)$

As one can see, we have 3 copies of a single K_3 from $\lambda = 3$ copies of K_7. Because we can cyclically permute the block $\{0, 1, 3\}$ modulus 7 to get the remaining blocks the next block we would use is $\{1, 2, 4\}$. The same procedure just used on the block $\{0, 1, 3\}$ would be applied to this new block. This procedure will continue through each block until all the K_3 have been drawn with unique colorings on each edge, giving us a proper coloring.

Because the case where $k = 3$ is completely solved in [29], we focused on the cases where $k = 4$ and 5. We will need to know the necessary and sufficient conditions from the existence of a BIBD$(v, 4, \lambda)$ and a BIBD$(v, 5, \lambda)$ if we are to form any link between properly colored graphs and coloring the incidence structure of a BIBD.

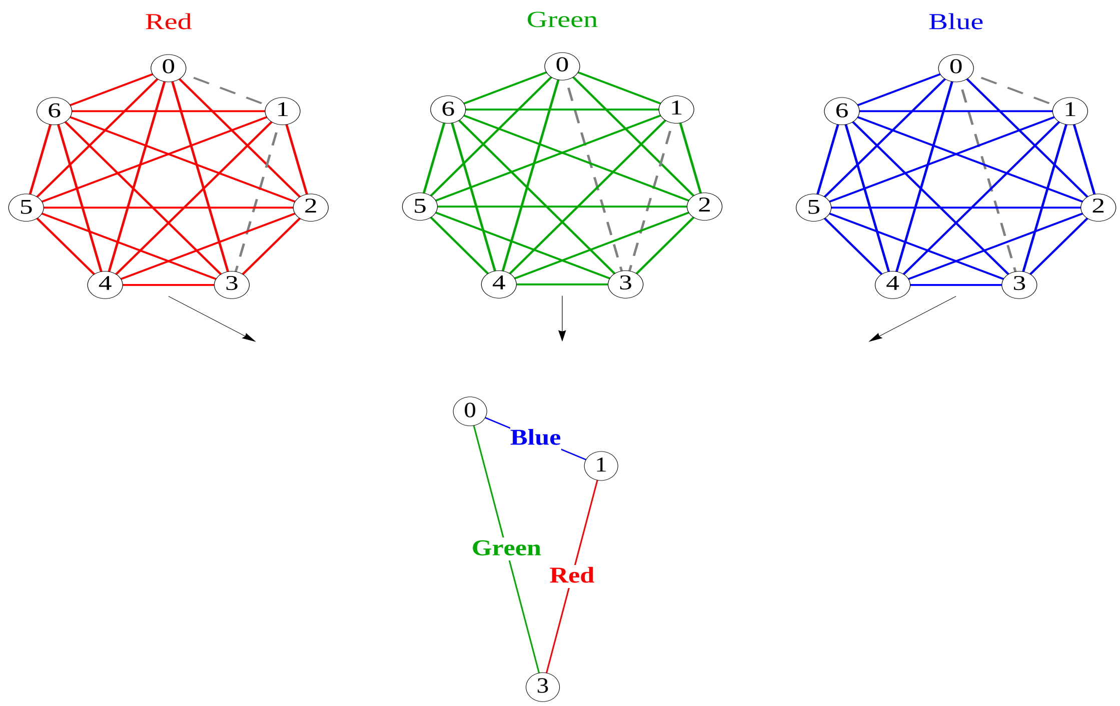


Figure 2.2: 2^{nd} of three colored copies of a block from a $\mathrm{BIBD}(7, 3, 3)$

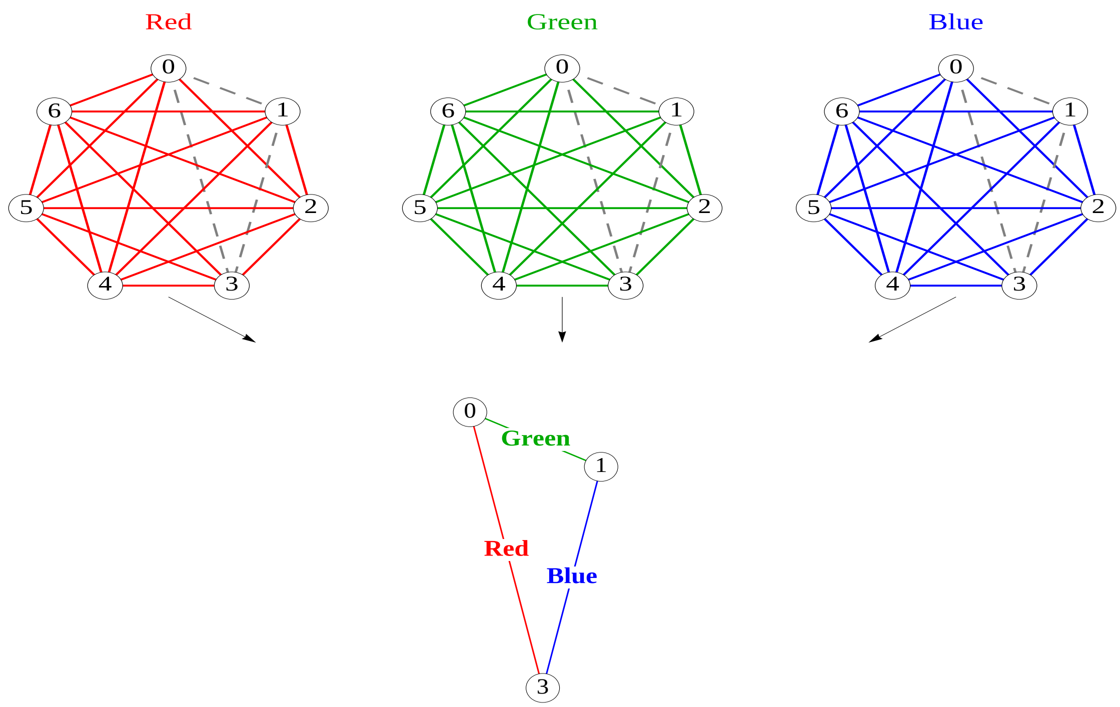


Figure 2.3: 3^{rd} of three colored copies of a block from a $\mathrm{BIBD}(7, 3, 3)$

The necessary and sufficient conditions for the existence of a $\mathrm{BIBD}(v, 4, \lambda)$ were obtained by Hanani in [28]. They are as follows:

If $\lambda \equiv 1, 5 \pmod 6$, then $v \equiv 1, 4 \pmod{12}$.

If $\lambda \equiv 2, 4 \pmod 6$, then $v \equiv 1 \pmod 3$.

If $\lambda \equiv 3 \pmod 6$, then $v \equiv 0, 1 \pmod 4$.

If $\lambda \equiv 0 \pmod 6$, then $v \geq 4$.

If $\lambda \equiv 1, 5 \pmod 6$, then we can properly color a $\mathrm{BIBD}(v, 4, \lambda)$ by applying Theorem 2.3, and if $\lambda \equiv 0 \pmod 6$, then we can properly color a $\mathrm{BIBD}(v, 4, \lambda)$ by applying Theorem 2.2. Therefore, in Chapter 3 we need only consider $\mathrm{BIBD}(v, 4, \lambda)$ with $\lambda \equiv 2, 3, 4 \pmod 6$.

The necessary and sufficient conditions for the existence of a $\mathrm{BIBD}(v, 5, \lambda)$ were obtained by Hanani in [28]. They are as follows:

If $\lambda \equiv 1, 3, 5, 7, 9, 11, 13, 15, 17, 19 \pmod{20}$, then $v \equiv 1, 5 \pmod{20}$.

If $\lambda \equiv 2, 6, 14, 18 \pmod{20}$, then $v \equiv 1, 5 \pmod{10}$.

If $\lambda \equiv 4, 8, 12, 16 \pmod{20}$, then $v \equiv 0, 1 \pmod 5$.

If $\lambda \equiv 10 \pmod{20}$, then $v \pmod 2$.

If $\lambda \equiv 0 \pmod{20}$, then $v \geq 5$.

If $\lambda \equiv 1, 3, 5, 7, 9, 11, 13, 15, 17, 19 \pmod{20}$, then we can properly color a $\mathrm{BIBD}(v, 5, \lambda)$ by applying Theorem 2.3, and if $\lambda \equiv 0, 10 \pmod{20}$, then we can properly color a $\mathrm{BIBD}(v, 5, \lambda)$ by applying Theorem 2.2. Therefore, in Chapter 4 we need only consider $\mathrm{BIBD}(v, 5, \lambda)$ with $\lambda \equiv 2, 4, 6, 8, 12, 14, 16, 18 \pmod{20}$.

Our main result, based on the above cases, is as follows. We prove the following theorems later.

Theorem. *There is a proper edge coloring for every* $\mathrm{BIBD}(v, 4, \lambda)$ *if* $\lambda \geq 6$.

Theorem. *There is a proper edge coloring for every* $\mathrm{BIBD}(v, 5, \lambda)$ *if* $\lambda \geq 10$ *except possibly when* $\lambda = 2k, k \geq 5, v \equiv 15, 35, 75, 95 \pmod{100}$ *and* $\lambda = 14, 18$.

2.2 Examples of Direct Constructions

The methods we will use in this section will be described in full detail later in this thesis. The *incidence matrix* of a design is a $(0, 1)$-matrix where the rows are indexed by the v points and the columns are indexed by the b blocks. In this matrix, each entry receives a 1 if the point lies in a block and 0 otherwise. This incidence structure has several useful properties. The sum of entries down each column is k and the sum of the entries across

	B_1	B_2	B_3	B_4	B_5	B_6	B_7
0	1	0	0	0	1	0	1
1	1	1	0	0	0	1	0
2	0	1	1	0	0	0	1
3	1	0	1	1	0	0	0
4	0	1	0	1	1	0	0
5	0	0	1	0	1	1	0
6	0	0	0	1	0	1	1

Figure 2.4: Incidence matrix of a $\mathrm{BIBD}(7, 3, 1)$

each row is r which is the *repetition number* or the number of blocks a point will appear in. As an example we will use the $\mathrm{BIBD}(7, 3, 1)$ we constructed in Section 1.2. Figure 2.4 depicts the incidence matrix for a $\mathrm{BIBD}(7, 3, 1)$.

Using the incidence matrix in Figure 2.4 we can form a more useful structure known as the *edge-incidence matrix*. An edge-incidence matrix is a $(0, 1)$-matrix where the rows are indexed by all possible pairs from $\mathcal{V}$ and the columns are indexed by the b blocks of the design. The matrix has a 1 in entry (i, j) if pair i is contained in block j, and an entry of 0 otherwise. A couple special properties of this matrix are that the sum of entries across each row is λ and the sum of entries down each column is $\binom{k}{2}$. Figure 2.5 depicts the edge-incidence matrix of the previous incidence matrix of a $\mathrm{BIBD}(7, 3, 1)$.

	B_1	B_2	B_3	B_4	B_5	B_6	B_7
$\{0,1\}$	1	0	0	0	0	0	0
$\{1,2\}$	0	1	0	0	0	0	0
$\{2,3\}$	0	0	1	0	0	0	0
$\{3,4\}$	0	0	0	1	0	0	0
$\{4,5\}$	0	0	0	0	1	0	0
$\{5,6\}$	0	0	0	0	0	1	0
$\{6,0\}$	0	0	0	0	0	0	1
$\{0,2\}$	0	0	0	0	0	0	1
$\{1,3\}$	1	0	0	0	0	0	0
$\{2,4\}$	0	1	0	0	0	0	0
$\{3,5\}$	0	0	1	0	0	0	0
$\{4,6\}$	0	0	0	1	0	0	0
$\{5,0\}$	0	0	0	0	1	0	0
$\{6,1\}$	0	0	0	0	0	1	0
$\{0,3\}$	1	0	0	0	0	0	0
$\{1,4\}$	0	1	0	0	0	0	0
$\{2,5\}$	0	0	1	0	0	0	0
$\{3,6\}$	0	0	0	1	0	0	0
$\{4,0\}$	0	0	0	0	1	0	0
$\{5,1\}$	0	0	0	0	0	1	0
$\{6,2\}$	0	0	0	0	0	0	1

Figure 2.5: Edge-incidence matrix of a $\mathrm{BIBD}(7, 3, 1)$

To simplify our results we will use circulant matrices. A *circulant matrix* is an $n \times n$ square matrix, in which each row (except the first) is obtained from the preceding row by shifting the elements cyclically one column to the right. Figure 2.6 is a 7×7 circulant matrix whose first row is $[1000000]$.

1	0	0	0	0	0	0
0	1	0	0	0	0	0
0	0	1	0	0	0	0
0	0	0	1	0	0	0
0	0	0	0	1	0	0
0	0	0	0	0	1	0
0	0	0	0	0	0	1

Figure 2.6: A 7×7 circulant matrix.

An edge-coloring for a BIBD(v, k, λ) $(\mathcal{V}, \mathcal{B})$ can be described by providing an edge-coloring incidence matrix. An *edge-coloring incidence matrix* is an edge-incidence matrix that has been given a coloring to each pair of points in the design. In the example from the beginning of this section we color each pair of points in a convenient pattern. It should be noted that there does not need to be one of each color in each column. We will use the colors from the set $\{c_1, c_2, c_3\}$ to color Figure 2.7.

	B_1	B_2	B_3	B_4	B_5	B_6	B_7
$\{0,1\}$	c_1	0	0	0	0	0	0
$\{1,2\}$	0	c_1	0	0	0	0	0
$\{2,3\}$	0	0	c_1	0	0	0	0
$\{3,4\}$	0	0	0	c_1	0	0	0
$\{4,5\}$	0	0	0	0	c_1	0	0
$\{5,6\}$	0	0	0	0	0	c_1	0
$\{6,0\}$	0	0	0	0	0	0	c_1
$\{0,2\}$	0	0	0	0	0	0	c_2
$\{1,3\}$	c_2	0	0	0	0	0	0
$\{2,4\}$	0	c_2	0	0	0	0	0
$\{3,5\}$	0	0	c_2	0	0	0	0
$\{4,6\}$	0	0	0	c_2	0	0	0
$\{5,0\}$	0	0	0	0	c_2	0	0
$\{6,1\}$	0	0	0	0	0	c_2	0
$\{0,3\}$	c_3	0	0	0	0	0	0
$\{1,4\}$	0	c_3	0	0	0	0	0
$\{2,5\}$	0	0	c_3	0	0	0	0
$\{3,6\}$	0	0	0	c_3	0	0	0
$\{4,0\}$	0	0	0	0	c_3	0	0
$\{5,1\}$	0	0	0	0	0	c_3	0
$\{6,2\}$	0	0	0	0	0	0	c_3

Figure 2.7: Edge-colored incidence matrix of a BIBD$(7, 3, 1)$

In general, this is the edge-colored matrix M defined by

$$M = M[\{x,y\},B] = \begin{cases} c_i & \text{if } \{x,y\} \in B \text{ has color } c_i, \\ 0 & \text{if } \{x,y\} \notin B \end{cases}$$

where $c_1, c_2, \ldots, c_\lambda$ are the colors for all pairs $\{x,y\} \in \binom{\mathcal{V}}{2}$ and $B \in \mathcal{B}$. Note that $\binom{\mathcal{V}}{2}$ is the set of all pairs of $\mathcal{V}$. So the Figure 2.7 could be represented as

$(c_1)I$
$(c_2)A^6$
$(c_3)I$

where I is the 7×7 identity matrix and A is the 7×7 circulant matrix whose first row is $[0100000]$.

As an example, our goal is to properly color a $\text{BIBD}(5,4,\lambda)$. We form the edge-incidence matrix for a $\text{BIBD}(5,4,3)$ as in Figure 2.8. Each block represents the six edges

	B_1	B_2	B_3	B_4	B_5
{1,2}	0	0	1	1	1
{2,3}	1	0	0	1	1
{3,4}	1	1	0	0	1
{4,5}	1	1	1	0	0
{5,1}	0	1	1	1	0
{1,3}	0	1	0	1	1
{2,4}	1	0	1	0	1
{3,5}	1	1	0	1	0
{4,1}	0	1	1	0	1
{5,2}	1	0	1	1	0

Figure 2.8: Edge-incidence matrix for a $\text{BIBD}(5,4,3)$

of a unique K_4. The notation $\{1,2\}$ represents the edge between vertex 1 and vertex 2 in the graph. To color this design, we replace each "1" with an entry c_i which represents the color i for $i = 1,2,\ldots,\lambda$. Our matrix must have the property that each entry c_i appears exactly once in every row and column. We can color the first 5 edges of the blocks in the $\text{BIBD}(5,4,3)$ using 3 colors, and we can color the last 5 edges of the $\text{BIBD}(5,4,3)$ using 3 colors to get the following edge-coloring incidence matrix using 6 colors in Figure 2.9.

Let A be the 5×5 circulant matrix whose first row is $[01000]$. Then we can represent this edge-coloring incidence matrix in terms of A as follows.

$(c_1)A^2 + (c_2)A^3 + (c_3)A^4$
$(c_4)A + (c_5)A^3 + (c_6)A^4$

	B_1	B_2	B_3	B_4	B_5
{1,2}	0	0	c_1	c_2	c_3
{2,3}	c_3	0	0	c_1	c_2
{3,4}	c_2	c_3	0	0	c_1
{4,5}	c_1	c_2	c_3	0	0
{5,1}	0	c_1	c_2	c_3	0
{1,3}	0	c_4	0	c_5	c_6
{2,4}	c_6	0	c_4	0	c_5
{3,5}	c_5	c_6	0	c_4	0
{4,1}	0	c_5	c_6	0	c_4
{5,2}	c_4	0	c_5	c_6	0

Figure 2.9: Edge-Coloring Incidence Matrix for a BIBD$(5, 4, 3)$

Next we form a properly colored BIBD$(5, 4, \lambda)$. To form a properly colored BIBD$(5, 4, \lambda)$ with $\lambda = 3s$, $s \geq 2$, we simply repeat the blocks of the BIBD$(5, 4, 3)$ s times and follow the same coloring scheme with different colors. Define M_i as the following sub-matrix. Note that all subscripts are computed $\pmod{\lambda}$ where we identify c_0 with c_λ.

$$M_i = \begin{array}{|c|}\hline (c_{1+3i})A^2 + (c_{2+3i})A^3 + (c_{3+3i})A^4 \\ \hline (c_{4+3i})A + (c_{5+3i})A^3 + (c_{6+3i})A^4 \\ \hline \end{array}$$

Then the edge-coloring incidence matrix of a properly colored BIBD$(5, 4, \lambda)$ with $\lambda = 3s$, $s \geq 2$ is given by

$$M = \begin{array}{|c|c|c|c|}\hline M_0 & M_1 & \cdots & M_{k-1} \\ \hline \end{array}.$$

The matrix M has the property that every color $c_1, c_2, \ldots, c_\lambda = c_0$ is seen exactly once in each row, and every color is seen at most once in every column. Therefore, M is an edge-coloring incidence matrix of a properly colored BIBD$(5, 4, \lambda)$. We state this result as a lemma.

Lemma 2.4. *There exists a properly colored* BIBD$(5, 4, \lambda)$ *where* $\lambda = 3s, s \in \mathbb{Z}^+, s \geq 2$.

As an example of this lemma, we provide the edge-coloring incidence matrix of a properly colored BIBD$(5, 4, 9)$ in Figure 2.10.

At this time, let us bring focus to why we want to repeat each block of a BIBD(v, k, λ) enough times to get $\lambda \geq \binom{k}{2}$. In each K_k, there are $\binom{k}{2}$ colors. Because each row of the matrix represents an edge and we need at least each edge to appear in $\binom{k}{2}$ colors, we must

	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}
{1,2}	0	0	c_1	c_2	c_3	0	0	c_4	c_5	c_6	0	0	c_7	c_8	c_9
{2,3}	c_3	0	0	c_1	c_2	c_6	0	0	c_4	c_5	c_9	0	0	c_7	c_8
{3,4}	c_2	c_3	0	0	c_1	c_5	c_6	0	0	c_4	c_8	c_9	0	0	c_7
{4,5}	c_1	c_2	c_3	0	0	c_4	c_5	c_6	0	0	c_7	c_8	c_9	0	0
{5,1}	0	c_1	c_2	c_3	0	0	c_4	c_5	c_6	0	0	c_7	c_8	c_9	0
{1,3}	0	c_4	0	c_5	c_6	0	c_7	0	c_8	c_9	0	c_1	0	c_2	c_3
{2,4}	c_6	0	c_4	0	c_5	c_9	0	c_6	0	c_8	c_3	0	c_1	0	c_2
{3,5}	c_5	c_6	0	c_4	0	c_8	c_9	0	c_7	0	c_2	c_3	0	c_1	0
{4,1}	0	c_5	c_6	0	c_4	0	c_8	c_9	0	c_7	0	c_2	c_3	0	c_1
{5,2}	c_4	0	c_5	c_6	0	c_7	0	c_8	c_9	0	c_1	0	c_2	c_3	0

Figure 2.10: Edge-Coloring Incidence Matrix of a properly colored $\mathrm{BIBD}(5,4,9)$.

have $\binom{k}{2}$ unique colors in each row. But the number of entries in each row is λ. Thus $\lambda \geq \binom{k}{2}$.

The following example illustrates one of the main ideas used in the construction of a properly colored $\mathrm{BIBD}(v,4,\lambda)$ design.

Example 2.5. *A properly colored* $\mathrm{BIBD}(17,4,9)$.

Because a $\mathrm{BIBD}(17,4,1)$ does not exist we cannot apply Theorem 2.3. It is known by Theorem 1.3 that a $(4,1)-\mathrm{GDD}(4^4)$ $(\mathcal{V},\mathcal{B},\mathcal{G})$ exists where $\mathcal{V}$ is the point set, $\mathcal{B}$ is the set of blocks, and $\mathcal{G}$ is the set of groups G_1,G_2,G_3,G_4. We will construct a $\mathrm{BIBD}(17,4,9)$ design on $\mathcal{V}\cup\{\infty\}$. Let $A,B,C,D\in\mathcal{V}$. For each block $\{A,B,C,D\}$ of the GDD, make 9 copies of the block. Let K_4 represent this block as in Figure 2.11. We can see that given a block $\{A,B,C,D\}$ implies there is an edge between each point in the block.

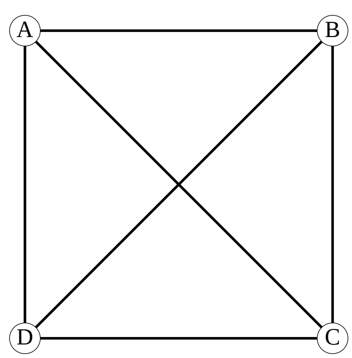


Figure 2.11: K_4

Our goal is to color each edge of this block a different color using the colors $c_i \in \{c_1, \ldots, c_9\}$. Each corresponding edge in the 9 copies of the block must also be a different color. So we form a 6×9 matrix. The rows of the matrix will be indexed by the 6 edges of K_4, and the columns will be indexed by the 9 copies of our block. The entries will represent the color of each edge. We must see every color $c_1, \ldots, c_9$ exactly once in each row and at most once in any column. The matrix for our 9 colored copies of $\{A, B, C, D\}$ is shown in Figure 2.12.

	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9
$\{A, B\}$	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
$\{A, C\}$	c_9	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
$\{A, D\}$	c_8	c_9	c_1	c_2	c_3	c_4	c_5	c_6	c_7
$\{B, C\}$	c_7	c_8	c_9	c_1	c_2	c_3	c_4	c_5	c_6
$\{B, D\}$	c_6	c_7	c_8	c_9	c_1	c_2	c_3	c_4	c_5
$\{C, D\}$	c_5	c_6	c_7	c_8	c_9	c_1	c_2	c_3	c_4

Figure 2.12: First 6 rows of a $\mathrm{LS}(9)$

Notice this matrix is simply the first 6 rows of an $\mathrm{LS}(9)$, a *Latin square* of side 9. Because an $\mathrm{LS}(9)$ is a 9×9 matrix and we only need 6 rows, we can always properly color the blocks of a GDD. We do this for each block of the GDD.

The only pairs that have not been covered are pairs that lie within the groups and pairs that involve $\{\infty\}$ as in Figure 2.13. These pairs need to be covered 9 times. So con-

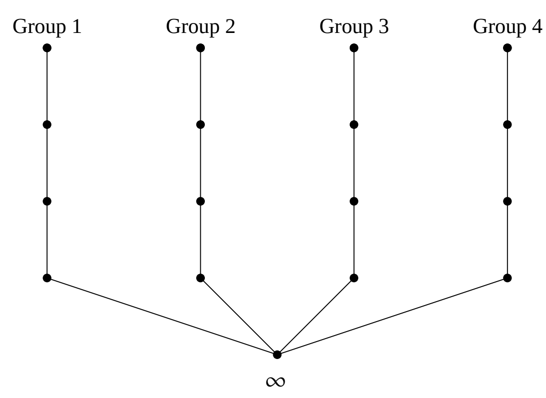


Figure 2.13: Coloring within groups of a GDD

struct a properly colored $\mathrm{BIBD}(5, 4, 9)$ on each group $G_i \cup \{\infty\}$ for $i = 1, 2, 3, 4$. This exists by Lemma 2.4. Now we have colored every block of the GDD and every group. We know through the use of Wilson's Construction from [51] that our GDD will yield a

$\mathrm{BIBD}(17, 4, 3)$, which we can copy 3 times to generate a properly colored $\mathrm{BIBD}(17, 4, 9)$.

Example 2.6. *A properly colored* $\mathrm{BIBD}(24, 4, 9)$.

Our goal here is to describe the process of using 4–RGDDs when $k = 4$. As we defined before, a group divisible design is resolvable when we can separate the blocks into parallel classes. This allows us to add points to parallel classes and still hold the properties of a design and the RGDD. Let P_i for $i = 1, 2, 3, 4, 5$ denote the parallel classes in the 4–RGDD(5^4) which exists by Theorem 1.10. Also let $\{\infty_1, \infty_2, \infty_3, \infty_4\}$ be four new points as in Figure 2.14.

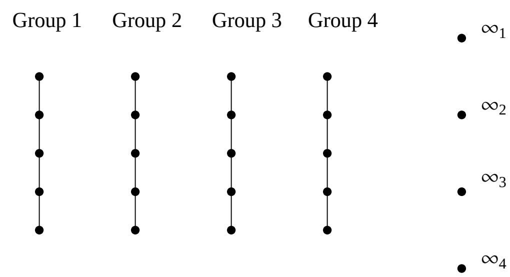


Figure 2.14: A 4–RGDD(5^4)

For each of the first four parallel classes, P_i, we place a properly colored $\mathrm{BIBD}(5, 4, 9)$ on each block plus ∞_i just as in Figure 2.15. This ensures that each pair including exactly one of the ∞_i points is covered exactly once.

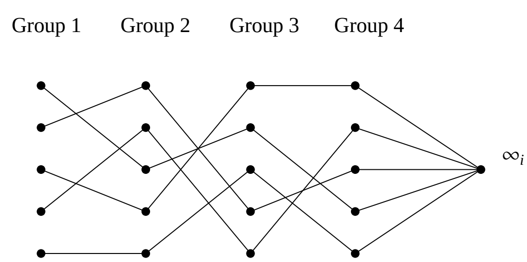


Figure 2.15: $P_i \cup \infty_i$ on a 4–RGDD(5^4)

Since there are only five parallel classes, color each block $\{A, B, C, D\}$ from parallel class P_5 as follows. We form a 6×9 matrix. The rows of the matrix will be indexed by the 6 edges of K_4, and the columns will be indexed by the 9 copies of our block. The entries will represent the color of each edge. We must see every color $c_1, \ldots, c_9$ exactly once in each row, and we must see each color at most once in any column. Coloring P_5 is the same as coloring the blocks of a 5–GDD as in Example 2.5

Notice this matrix is simply the first 6 rows of an $\mathrm{LS}(9)$. Because an $\mathrm{LS}(9)$ is a 9×9 matrix and we only need 6 rows, we can always properly color the blocks of the RGDD. We do this for each block of P_5 in the RGDD. The only pairs remaining are those that lie within the groups and those that lie within $\{\infty_1, \infty_2, \infty_3, \infty_4\}$. We know we can properly color a $\mathrm{BIBD}(5, 4, 9)$. Therefore, we can properly color a $\mathrm{BIBD}(24, 4, 9)$.

Now we shift our focus to constructions for properly colored $\mathrm{BIBD}(v, 5, \lambda)$.

Example 2.7. *A properly colored* $\mathrm{BIBD}(25, 5, 20)$.

The process shown in Example 2.5 can be extended to 5–GDDs using a similar method. Thus, it does not take much to realize that we could also use $(5, 2)$–GDDs. The construction is very similar, as well, to how we construct a properly colored $\mathrm{BIBD}(v, k, \lambda)$ using a 5–GDD. The difference is that each pair of points occurs 2 times within the blocks because we will use a $(5, 2)$–GDD and this will force λ to be at least 20. It is known by Theorem 1.7 that a $(5, 2)$–$\mathrm{GDD}(5^5)$ $(\mathcal{V}, \mathcal{B}, \mathcal{G})$ exists where $\mathcal{V}$ is the point set, $\mathcal{B}$ is the set of blocks, and $\mathcal{G}$ is the set of groups.

Consider two distinct color sets $C = \{c_1, \ldots, c_{10}\}$ and $D = \{d_1, \ldots, d_{10}\}$ that we will use to properly color each block. Our goal is to construct a properly colored edge-coloring incidence matrix made up of the blocks of the $(5, 2)$–$\mathrm{GDD}(5^5)$ as shown in Figure 2.16. To construct this matrix we replace the first "1" in each row with a C and the second "1" by D. Since each block is being repeated 10 times we will use every color in the sets C and D in the edge-coloring incidence matrix.

$$
\begin{array}{c|ccccccc|}
 & B_1 & B_2 & \ldots & B' & \ldots & B_{b-1} & B_b \\
\hline
 & 0 & C & \ldots & D & \ldots & 0 & 0 \\
 & C & 0 & \ldots & 0 & \ldots & D & 0 \\
 & 0 & 0 & \ldots & C & \ldots & 0 & D \\
 & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
\{x, y\} & 0 & 0 & \ldots & C & \ldots & D & 0 \\
 & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
 & C & 0 & \ldots & D & \ldots & 0 & 0 \\
\hline
\end{array}
$$

Figure 2.16: Edge-coloring incidence matrix for $(5, 2)$–GDD

This gives us 20 unique colors in each row of this matrix. Now we must make sure there are 10 unique colors in each column. We will permute each of the color sets to make sure that if we were to construct an $\mathrm{LS}(10)$ with the non-zero entries of the 10 copies of the blocks we will have unique entries down each column. This can be seen in Figure 2.18

Consider only the block B'. Let t_C represent the number of times we use the color set C and t_D represent the number of times we use the color set D. From Figure 2.17, $t_C = 8$ and $t_D = 2$. Figure 2.18 depicts the nonzero entries within the ten copies of B'.

We do repeat this for every block which gives use a unique color down each column. Thus we can properly color the blocks of the $(5, 2)$–$\mathrm{GDD}(5^5)$. The only pairs left are those that lie within the groups. These pairs need to be covered 20 times. So construct a properly colored $\mathrm{BIBD}(5, 5, 20)$ on each group. This exists by Theorem 2.3. Now we have colored every block of the GDD and every group. We know through the use of Wilson's Constructions in [51] that our GDD will yield a $\mathrm{BIBD}(25, 5, 20)$ which we properly colored.

$$\begin{array}{c|ccccccc|}
 & B_1 & B_2 & \dots & B' & \dots & B_{b-1} & B_b \\ \hline
 & 0 & C_1 & \dots & D_1 & \dots & 0 & 0 \\
 & C_1 & 0 & \dots & 0 & \dots & D_1 & 0 \\
 & 0 & 0 & \dots & C_1 & \dots & 0 & D_1 \\
 & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
\{x,y\} & 0 & 0 & \dots & C_2 & \dots & D_2 & 0 \\
 & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
 & C_{10} & 0 & \dots & D_2 & \dots & 0 & 0 \\ \hline
\end{array}$$

Figure 2.17: Edge-coloring incidence matrix for $(5, 2)$–GDD with cyclic shifting

B'_1	B'_2	B'_3	B'_4	B'_5	B'_6	B'_7	B'_8	B'_9	B'_{10}
d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}
c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}
c_{10}	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
c_9	c_{10}	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
c_8	c_9	c_{10}	c_1	c_2	c_3	c_4	c_5	c_6	c_7
c_7	c_8	c_9	c_{10}	c_1	c_2	c_3	c_4	c_5	c_6
c_6	c_7	c_8	c_9	c_{10}	c_1	c_2	c_3	c_4	c_5
c_5	c_6	c_7	c_8	c_9	c_{10}	c_1	c_2	c_3	c_4
c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_1	c_2	c_3
c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_1	c_2
d_{10}	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9

Figure 2.18: $\mathrm{LS}(10)$ of 10 copies of block B'

Example 2.8. *A properly colored* $\mathrm{BIBD}(31, 5, 12)$

This very useful construction is used in Chapter 4 where we use RBIBDs. We know there exists an $\mathrm{RBIBD}(25, 5, 1)$ by Theorem 1.9. This also tells us we have 6 parallel classes which we denote by P_i for $i = 1, \dots, 6$. Let $\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6\}$ be six new points. Now place a properly colored $\mathrm{BIBD}(6, 5, 12)$ on each block of $P_i \cup \{\infty_i\}$. To color this we use the same method found in Example 2.6. Normally, we have not covered the pairs that lie in the parallel classes P_i for $i > 6$ but we used all the parallel classes already. Now the only remaining pairs are of the form $\{\{\infty_i, \infty_j\} : i, j \in \{1, \dots, 6\}\}$, so we place a properly colored $\mathrm{BIBD}(6, 5, 12)$ on the set of points $\{\infty_1, \dots, \infty_6\}$. Because we can properly color every pair and we added six points, we can properly color a $\mathrm{BIBD}(31, 5, 12)$.

Example 2.9. *A properly colored* $\mathrm{BIBD}(471, 5, \lambda)$ *where* $\lambda = 2k, k \in \mathbb{Z}^+, k \geq 5$.

Let $\lambda = 2k, k \in \mathbb{Z}^+, k \geq 5$. This construction is very similar to Example 2.6 since we are going to be using 4–RGDDs as well. Let P_i for $i = 1, \dots, t$ denote the parallel classes in the 4–$\text{RGDD}(30^{12})$ which exists by Theorem 1.10. Because this design is resolvable it has parallel classes. In fact, 110 parallel classes. We need to have 110 new points which we will denote as $\{\infty_1, \infty_2, \ldots, \infty_{110}\}$. We set up all the point just like as in Figure 2.14. For each of the 110 we place a properly colored $\text{BIBD}(5, 5, \lambda)$ on each block plus $\{\infty\}$ just as in Figure 2.15. This ensures that each pair including exactly one of the ∞_i points is covered exactly once.

At this time, we note that we only have 470 points. So we add one last point called $\{\infty\}$ as in Figure 2.13. The only pairs remaining are those that lie within the groups with $\{\infty\}$ and those that lie in $\{\infty_1, \ldots, \infty_{110}\}$ with $\{\infty\}$. We can properly color a $\text{BIBD}(31, 5, \lambda)$ by Lemma 4.2. If $v = 111 = 10(11) + 1$, there exists a 5–$\text{GDD}(10^{11})$ by Lemma 1.7. There also exists a properly colored $\text{BIBD}(11, 5, \lambda)$. Thus, we can properly color a $\text{BIBD}(111, 5, \lambda)$ by the same reasoning applied to Example 2.5. This reasoning will be stated as Lemma 4.18 later in the paper. Therefore, we can properly color a $\text{BIBD}(471, 5, \lambda)$. We state this result as a lemma.

Lemma 2.10. *There exists a properly colored* $\text{BIBD}(471, 5, \lambda)$ *where* $\lambda = 2k, k \in \mathbb{Z}^+, k \geq 5$.

Suppose there exists a $\text{BIBD}(v, k, \lambda)$ where in each row and column of the edge-coloring incidence matrix there is a unique color from the color set $\{1, 2, \ldots, \lambda\}$. We will examine what conclusions we can make if $\lambda < \binom{k}{2}$.

Definition 2.11. We define a *semi-properly colored* BIBD is a $\text{BIBD}(v, k, \lambda)$ with the following properties:

1. $\lambda < \binom{k}{2}$
2. The set of color indices in each row of the edge-coloring incidence matrix forms the set $\{0, 1, \ldots, \lambda - 1\} \pmod{\lambda}$.
3. The set of colors in each column contain $\binom{k}{2}$ unique colors

Theorem 2.12. *If there exists a* $\text{BIBD}(v, k, \lambda)$ *and it is semi-properly colored, then a* $\text{BIBD}(v, k, m\lambda)$ *is properly colored for* $m\lambda \geq \binom{k}{2}, m \in \mathbb{Z}^+$.

Proof: Suppose we have the edge-colored incidence matrix of a semi-properly colored $\text{BIBD}(v, k, \lambda)$. To build a properly colored $\text{BIBD}(v, k, m\lambda)$, we first copy each block of the $\text{BIBD}(v, k, \lambda)$ m times. This is equivalent to concatenating $m - 1$ copies of the edge-colored incidence matrix of the semi-properly colored $\text{BIBD}(v, k, \lambda)$. For $t = 1, 2, \ldots, m-1$, add $t\lambda \pmod{m\lambda}$ to each color index, in the t^{th} copy of the matrix. Then the resulting matrix has the property that the $m\lambda$ entries are unique in each row. It is clear that we still have $\binom{k}{2}$ unique colors in each column. Therefore, the new matrix is the edge-coloring incidence matrix of a properly colored $\text{BIBD}(v, k, m\lambda)$. ✠

From now on, when we prove a direct construction for a properly colored BIBD we will almost exclusively use Theorem 2.12 to simplify the proof. Examples of the use of this theorem can be seen in Section 3.1 and 4.1.

Chapter 3

Proper Edge Colorings With Block Size 4

Understanding how to use the constructions in this chapter is imperative to following the reasoning in the main theorem. Before we get to the generalized constructions we must first show there exists properly colored $\mathrm{BIBD}(v, 4, \lambda)$ for small v.

As a standard among people in design theory, the use of ∞ is that of an extra point rather than a number. In some fields in discrete mathematics, ∞ is used as both a number and a point as in projective geometry and elliptic curves.

3.1 Direct Constructions With Block Size 4

Lemma 3.1. *There exists a properly edge-colored* $\mathrm{BIBD}(9, 4, \lambda)$ *design where* $\lambda = 3k, k \in \mathbb{Z}^+, k \geq 2$.

Proof: As in Lemma 2.4 we form an edge-coloring incidence matrix for a $\mathrm{BIBD}(9, 4, 3)$ as follows. Figure 3.1 represents the edge-coloring incidence matrix for the $\mathrm{BIBD}(9, 4, 3)$.

Let A be the 9×9 circulant matrix whose first row is $[010000000]$. Then the edge coloring incidence matrix of a $\mathrm{BIBD}(9, 4, 3)$ using 6 colors is given below.

$$M_i = \begin{array}{|c|c|} \hline (c_{2+3i})I + (c_{1+3i})A^8 & (c_{3+3i})I \\ \hline (c_{3+3i})A^8 & (c_{2+3i})A^3 + (c_{1+3i})A^8 \\ \hline (c_{4+3i})I + (c_{5+3i})A^7 & (c_{6+3i})A^5 \\ \hline (c_{6+3i})I & (c_{4+3i})I + (c_{5+3i})A^3 \\ \hline \end{array}$$

Since M_i is semi-properly colored, by applying Theorem 2.12 we can properly color a $\mathrm{BIBD}(9, 4, \lambda)$.

✠

	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}	B_{16}	B_{17}	B_{18}
$\{0,1\}$	c_2	0	0	0	0	0	0	0	c_1	c_3	0	0	0	0	0	0	0	0
$\{1,2\}$	c_1	c_2	0	0	0	0	0	0	0	0	c_3	0	0	0	0	0	0	0
$\{2,3\}$	0	c_1	c_2	0	0	0	0	0	0	0	0	c_3	0	0	0	0	0	0
$\{3,4\}$	0	0	c_1	c_2	0	0	0	0	0	0	0	0	c_3	0	0	0	0	0
$\{4,5\}$	0	0	0	c_1	c_2	0	0	0	0	0	0	0	0	c_3	0	0	0	0
$\{5,6\}$	0	0	0	0	c_1	c_2	0	0	0	0	0	0	0	0	c_3	0	0	0
$\{6,7\}$	0	0	0	0	0	c_1	c_2	0	0	0	0	0	0	0	0	c_3	0	0
$\{7,8\}$	0	0	0	0	0	0	c_1	c_2	0	0	0	0	0	0	0	0	c_3	0
$\{8,0\}$	0	0	0	0	0	0	0	c_1	c_2	0	0	0	0	0	0	0	0	c_3
$\{0,3\}$	0	0	0	0	0	0	0	0	c_3	0	0	0	c_2	0	0	0	0	c_1
$\{1,4\}$	c_3	0	0	0	0	0	0	0	0	c_1	0	0	0	c_2	0	0	0	0
$\{2,5\}$	0	c_3	0	0	0	0	0	0	0	0	c_1	0	0	0	c_2	0	0	0
$\{3,6\}$	0	0	c_3	0	0	0	0	0	0	0	0	c_1	0	0	0	c_2	0	0
$\{4,7\}$	0	0	0	c_3	0	0	0	0	0	0	0	0	c_1	0	0	0	c_2	0
$\{5,8\}$	0	0	0	0	c_3	0	0	0	0	0	0	0	0	c_1	0	0	0	c_2
$\{6,0\}$	0	0	0	0	0	c_3	0	0	0	c_2	0	0	0	0	c_1	0	0	0
$\{7,1\}$	0	0	0	0	0	0	c_3	0	0	0	c_2	0	0	0	0	c_1	0	0
$\{8,2\}$	0	0	0	0	0	0	0	c_3	0	0	0	c_2	0	0	0	0	c_1	0
$\{0,2\}$	c_4	0	0	0	0	0	0	c_5	0	0	0	0	0	0	c_6	0	0	0
$\{1,3\}$	0	c_4	0	0	0	0	0	0	c_5	0	0	0	0	0	0	c_6	0	0
$\{2,4\}$	c_5	0	c_4	0	0	0	0	0	0	0	0	0	0	0	0	0	c_6	0
$\{3,5\}$	0	c_5	0	c_4	0	0	0	0	0	0	0	0	0	0	0	0	0	c_6
$\{4,6\}$	0	0	c_5	0	c_4	0	0	0	0	c_6	0	0	0	0	0	0	0	0
$\{5,7\}$	0	0	0	c_5	0	c_4	0	0	0	0	c_6	0	0	0	0	0	0	0
$\{6,8\}$	0	0	0	0	c_5	0	c_4	0	0	0	0	c_6	0	0	0	0	0	0
$\{7,0\}$	0	0	0	0	0	c_5	0	c_4	0	0	0	0	c_6	0	0	0	0	0
$\{8,1\}$	0	0	0	0	0	0	c_5	0	c_4	0	0	0	0	c_6	0	0	0	0
$\{0,4\}$	c_6	0	0	0	0	0	0	0	0	c_4	0	0	c_5	0	0	0	0	0
$\{1,5\}$	0	c_6	0	0	0	0	0	0	0	0	c_4	0	0	c_5	0	0	0	0
$\{2,6\}$	0	0	c_6	0	0	0	0	0	0	0	0	c_4	0	0	c_5	0	0	0
$\{3,7\}$	0	0	0	c_6	0	0	0	0	0	0	0	0	c_4	0	0	c_5	0	0
$\{4,8\}$	0	0	0	0	c_6	0	0	0	0	0	0	0	0	c_4	0	0	c_5	0
$\{5,0\}$	0	0	0	0	0	c_6	0	0	0	0	0	0	0	0	c_4	0	0	c_5
$\{6,1\}$	0	0	0	0	0	0	c_6	0	0	c_5	0	0	0	0	0	c_4	0	0
$\{7,2\}$	0	0	0	0	0	0	0	c_6	0	0	c_5	0	0	0	0	0	c_4	0
$\{8,3\}$	0	0	0	0	0	0	0	0	c_6	0	0	c_5	0	0	0	0	0	c_4

Figure 3.1: Edge-Coloring Incidence Matrix of a $\mathrm{BIBD}(9, 4, 3)$.

Lemma 3.2. *There exists a properly colored* $\mathrm{BIBD}(8, 4, \lambda)$ *for* $\lambda = 3k, k \in \mathbb{Z}^+, k \geq 2$.

Proof: As in Lemma 2.4 we form an edge-coloring incidence matrix for a $\mathrm{BIBD}(8, 4, 3)$ as follows. Figure 3.2 represents the edge-coloring incidence matrix for the $\mathrm{BIBD}(8, 4, 3)$.

	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}
{0,1}	c_1	0	0	0	0	0	0	c_2	0	0	0	0	0	c_3
{1,2}	0	c_1	0	0	0	0	0	c_3	c_2	0	0	0	0	0
{2,3}	0	0	c_1	0	0	0	0	0	c_3	c_2	0	0	0	0
{3,4}	0	0	0	c_1	0	0	0	0	0	c_3	c_2	0	0	0
{4,5}	0	0	0	0	c_1	0	0	0	0	0	c_3	c_2	0	0
{5,6}	0	0	0	0	0	c_1	0	0	0	0	0	c_3	c_2	0
{6,0}	0	0	0	0	0	0	c_1	0	0	0	0	0	c_3	c_2
{0,2}	0	0	0	0	0	0	c_2	c_1	0	0	0	0	c_4	0
{1,3}	c_2	0	0	0	0	0	0	0	c_1	0	0	0	0	c_4
{2,4}	0	c_2	0	0	0	0	0	c_4	0	c_1	0	0	0	0
{3,5}	0	0	c_2	0	0	0	0	0	c_4	0	c_1	0	0	0
{4,6}	0	0	0	c_2	0	0	0	0	0	c_4	0	c_1	0	0
{5,0}	0	0	0	0	c_2	0	0	0	0	0	c_4	0	c_1	0
{6,1}	0	0	0	0	0	c_2	0	0	0	0	0	c_4	0	c_1
{0,3}	c_3	0	0	0	0	0	0	0	0	0	c_6	0	0	c_5
{1,4}	0	c_3	0	0	0	0	0	c_5	0	0	0	c_6	0	0
{2,5}	0	0	c_3	0	0	0	0	0	c_5	0	0	0	c_6	0
{3,6}	0	0	0	c_3	0	0	0	0	0	c_5	0	0	0	c_6
{4,0}	0	0	0	0	c_3	0	0	c_6	0	0	c_5	0	0	0
{5,1}	0	0	0	0	0	c_3	0	0	c_6	0	0	c_5	0	0
{6,2}	0	0	0	0	0	0	c_3	0	0	c_6	0	0	c_5	0
$\{0, \infty\}$	c_4	0	0	0	c_6	0	c_5	0	0	0	0	0	0	0
$\{1, \infty\}$	c_5	c_4	0	0	0	c_6	0	0	0	0	0	0	0	0
$\{2, \infty\}$	0	c_5	c_4	0	0	0	c_6	0	0	0	0	0	0	0
$\{3, \infty\}$	c_6	0	c_5	c_4	0	0	0	0	0	0	0	0	0	0
$\{4, \infty\}$	0	c_6	0	c_5	c_4	0	0	0	0	0	0	0	0	0
$\{5, \infty\}$	0	0	c_6	0	c_5	c_4	0	0	0	0	0	0	0	0
$\{6, \infty\}$	0	0	0	c_6	0	c_5	c_4	0	0	0	0	0	0	0

Figure 3.2: Edge-Coloring Incidence Matrix of a $\mathrm{BIBD}(8, 4, 3)$.

Let A be the 8×8 circulant matrix whose first row is $[01000000]$. Then we can represent an edge-coloring incidence matrix for a $\mathrm{BIBD}(8, 4, 3)$ in terms of A on the colors $\{c_1, c_2, c_3, c_4, c_5, c_6\}$. This representation is given in Figure 3.3.

For any i, let $M_i^{(6)}$ be the edge-coloring incidence matrix of a $\mathrm{BIBD}(8, 4, 6)$ on the colors $c_{1+6i}, c_{2+6i}, c_{3+6i}, c_{4+6i}, c_{5+6i}, c_{6+6i}$ found in Figure 3.4. Denote $M_0^{(9)}$ as the edge-coloring incidence matrix of a $\mathrm{BIBD}(8, 4, 9)$ on the colors $c_{1'}, c_{2'}, \ldots, c_{9'}$ found in Fig-

$c_1 I$	$c_2 I + c_3 A^6$
$c_2 A^6$	$c_1 I + c_4 A^5$
$c_3 I$	$c_5 A^3 + c_6 A^6$
$c_4 I + c_6 A^4 + c_5 A^6$	0

Figure 3.3: Edge-Coloring Incidence Matrix for a $\mathrm{BIBD}(8, 4, 3)$

$$M_i^{(6)} = \begin{array}{|c|c|c|c|} \hline (c_{1+6i})I & (c_{2+6i})I + (c_{3+6i})A^6 & (c_{5+6i})I & (c_{4+6i})I + (c_{6+6i})A^6 \\ \hline (c_{2+6i})A^6 & (c_{1+6i})I + (c_{4+6i})A^5 & (c_{6+6i})A^6 & (c_{3+6i})I + (c_{5+6i})A^5 \\ \hline (c_{3+6i})I & (c_{5+6i})A^3 + (c_{6+6i})A^6 & (c_{4+6i})I & (c_{1+6i})A^3 + (c_{2+6i})A^6 \\ \hline \begin{array}{l} (c_{4+6i})I \\ +(c_{6+6i})A^4 \\ +(c_{5+6i})A^6 \end{array} & 0 & \begin{array}{l} (c_{1+6i})I \\ +(c_{2+6i})A^4 \\ +(c_{3+6i})A^6 \end{array} & 0 \\ \hline \end{array}$$

Figure 3.4: Edge-Coloring Incidence Matrix for a $\mathrm{BIBD}(8, 4, 6)$

ure 3.5. The subscripts are computed $\pmod{\lambda}$ where we identify c_0 with c_λ. In the case

$$M_0^{(9)} = \begin{array}{|c|c|c|c|c|c|} \hline (c_{1'})I & (c_{2'})I + (c_{3'})A^6 & (c_{5'})I & (c_{4'})I + (c_{6'})A^6 & (c_{7'})I & (c_{8'})I + (c_{9'})A^6 \\ \hline (c_{2'})A^6 & (c_{1'})I + (c_{4'})A^5 & (c_{6'})A^6 & (c_{5'})I + (c_{8'})A^5 & (c_{9'})A^6 & (c_{3'})I + (c_{7'})A^5 \\ \hline (c_{3'})I & (c_{5'})A^3 + (c_{6'})A^6 & (c_{4'})I & (c_{7'})A^3 + (c_{9'})A^6 & (c_{8'})I & (c_{1'})A^3 + (c_{2'})A^6 \\ \hline \begin{array}{l} (c_{4'})I \\ +(c_{6'})A^4 \\ +(c_{5'})A^6 \end{array} & 0 & \begin{array}{l} (c_{7'})I \\ +(c_{8'})A^4 \\ +(c_{9'})A^6 \end{array} & 0 & \begin{array}{l} (c_{1'})I \\ +(c_{2'})A^4 \\ +(c_{3'})A^6 \end{array} & 0 \\ \hline \end{array}$$

Figure 3.5: Edge-Coloring Incidence Matrix for a $\mathrm{BIBD}(8, 4, 9)$

where $\lambda = 6k$, our matrix is

$$M = \begin{array}{|c|c|c|c|} \hline M_0^{(6)} & M_1^{(6)} & \cdots & M_{k-1}^{(6)} \\ \hline \end{array}.$$

The set of colors used is $C_0, C_1, \ldots, C_{k-1}$ where

$$C_i = \{c_{1+6i}, c_{2+6i}, c_{3+6i}, c_{4+6i}, c_{5+6i}, c_{6+6i}\}$$

for $i = 0, 1, \ldots, k-1$. Because each $M_i^{(6)}$ is an edge coloring incidence matrix of a properly colored $\mathrm{BIBD}(8, 4, 6)$ on a different set of 6 colors, it follows that M is an edge-coloring incidence matrix of a properly colored $\mathrm{BIBD}(8, 4, 6k)$.

In the case where $\lambda = 6k + 3$, our matrix is

$$M = \begin{array}{|c|c|c|c|} \hline M_0^{(9)} & M_0^{(6)} & \cdots & M_{k-2}^{(6)} \\ \hline \end{array}.$$

The set of colors used is $C, C_0, C_1, \ldots, C_{k-2}$ where $C = \{c_{1'}, c_{2'}, \ldots, c_{9'}\}$ and

$$C_i = \{c_{1+6i}, c_{2+6i}, c_{3+6i}, c_{4+6i}, c_{5+6i}, c_{6+6i}\}$$

for $i = 0, 1, \ldots, k-2$. Because $M_0^{(9)}$ is an edge coloring incidence matrix of a properly colored $\mathrm{BIBD}(8,4,9)$ on 9 colors, and each $M_i^{(6)}$ is an edge coloring incidence matrix of a properly colored $\mathrm{BIBD}(8,4,6)$ on a different set of 6 colors which are all disjoint from the colors in C; it follows that M is an edge-coloring incidence matrix of a properly colored $\mathrm{BIBD}(8,4,6k+3)$.

✠

Lemma 3.3. *There exists a properly colored* $\mathrm{BIBD}(12,4,\lambda)$ *for* $\lambda = 3k, k \geq 2, k \in \mathbb{Z}^+$.

Proof: As in Lemma 2.4 we form a colored edge-coloring incidence matrix for a $\mathrm{BIBD}(12,4,3)$ as follows. Let A be the 12×12 circulant matrix whose first row is $[010000000000]$. Then the edge coloring incidence matrix of a $\mathrm{BIBD}(12,4,3)$ using 6 colors is given below.

$M_i =$

$(c_{1+3i})I$	$(c_{2+3i})A^2$	$(c_{3+3i})A^5$
$(c_{2+3i})A^{10}$	$(c_{3+3i})A^9$	$(c_{4+3i})A^4$
$(c_{3+3i})I$	$(c_{4+3i})A$	$(c_{5+3i})A^5$
$(c_{4+3i})A^4 + (c_{5+3i})A^8$	$(c_{6+3i})A^2$	0
$(c_{6+3i})A^4$	$(c_{1+3i})A + (c_{5+3i})A^7$	0
0	0	$(c_{1+3i})A^2 + (c_{2+3i})A^4 + (c_{6+3i})A^5$

Since M_i is semi-properly colored, by Theorem 2.12 we can properly color a $\mathrm{BIBD}(12,4,\lambda)$.

✠

Lemma 3.4. *There exists a properly colored* $\mathrm{BIBD}(7,4,\lambda)$ *for* $\lambda = 2k, k \in \mathbb{Z}^+, k \geq 3$.

Proof: As in Lemma 2.4 we form a colored edge-coloring incidence matrix for a $\mathrm{BIBD}(7,4,2)$ as follows. Figure 3.6 represents the edge-coloring incidence matrix for the $\mathrm{BIBD}(7,4,2)$.

Let A be the 7×7 circulant matrix whose first row is $[0100000]$. Then the edge coloring incidence matrix of a $\mathrm{BIBD}(7,4,2)$ using 6 colors is given below.

$M_i =$

$(c_{1+2i})I + (c_{2+2i})A$
$(c_{3+2i})I + (c_{4+2i})A^2$
$(c_{5+2i})I + (c_{6+2i})A^3$

Since M_i is semi-properly colored, by Theorem 2.12 we can properly color a $\mathrm{BIBD}(7,4,\lambda)$.

✠

Lemma 3.5. *There exists a properly colored* $\mathrm{BIBD}(19,4,\lambda)$ *for* $\lambda = 2k, k \in \mathbb{Z}^+, k \geq 3$.

Proof: Let A be the 19×19 circulant matrix whose first row is $[0100000000000000000]$. Then the edge coloring incidence matrix of a $\mathrm{BIBD}(19,4,2)$ using 6 colors is given in Figure 3.7. Since M_i is semi-properly colored, by Theorem 2.12 we can properly color a $\mathrm{BIBD}(19,4,\lambda)$.

✠

	B_1	B_2	B_3	B_4	B_5	B_6	B_7
$\{0,1\}$	c_1	c_2	0	0	0	0	0
$\{5,0\}$	0	c_1	c_2	0	0	0	0
$\{3,4\}$	0	0	c_1	c_2	0	0	0
$\{3,6\}$	0	0	0	c_1	c_2	0	0
$\{2,5\}$	0	0	0	0	c_1	c_2	0
$\{2,4\}$	0	0	0	0	0	c_1	c_2
$\{0,2\}$	c_2	0	0	0	0	0	c_1
$\{0,3\}$	c_3	0	c_4	0	0	0	0
$\{6,1\}$	0	c_3	0	c_4	0	0	0
$\{3,5\}$	0	0	c_3	0	c_4	0	0
$\{1,4\}$	0	0	0	c_3	0	c_4	0
$\{6,2\}$	0	0	0	0	c_3	0	c_4
$\{1,2\}$	c_4	0	0	0	0	c_3	0
$\{6,0\}$	0	c_4	0	0	0	0	c_3
$\{1,3\}$	c_5	0	0	c_6	0	0	0
$\{5,6\}$	0	c_5	0	0	c_6	0	0
$\{4,5\}$	0	0	c_5	0	0	c_6	0
$\{4,6\}$	0	0	0	c_5	0	0	c_6
$\{2,3\}$	c_6	0	0	0	c_5	0	0
$\{5,1\}$	0	c_6	0	0	0	c_5	0
$\{4,0\}$	0	0	c_6	0	0	0	c_5

Figure 3.6: Edge-Coloring Incidence Matrix of a $\mathrm{BIBD}(7,4,2)$.

$$M_i = \begin{array}{|c|c|c|}
\hline
(c_{1+2i})A^{16} & 0 & (c_{2+2i})I \\ \hline
(c_{2+2i})A^{18} & 0 & (c_{3+2i})A^{15} \\ \hline
(c_{3+2i})I & 0 & (c_{4+2i})A^{13} \\ \hline
0 & (c_{4+2i})A^{18} & (c_{5+2i})I \\ \hline
0 & (c_{5+2i})I & (c_{6+2i})A^{15} \\ \hline
0 & (c_{6+2i})A^{6} & (c_{1+2i})I \\ \hline
(c_{4+2i})A^{7} & (c_{1+2i})A^{5} & 0 \\ \hline
(c_{5+2i})A^{7} & (c_{2+2i})A^{14} & 0 \\ \hline
(c_{6+2i})I & (c_{3+2i})I & 0 \\ \hline
\end{array}$$

Figure 3.7: Edge-Coloring Incidence Matrix of a $\mathrm{BIBD}(19,4,2)$.

Lemma 3.6. *There exists a properly colored* $\mathrm{BIBD}(10, 4, \lambda)$ *for* $\lambda = 2k, k \in \mathbb{Z}^+, k \geq 4$.

Proof: We form an edge-coloring incidence matrix using 8 colors for a $\mathrm{BIBD}(10, 4, 2)$ as follows. Let $A_i^{(j)}$ be the following 5×5 matrices for $j = 1, 2, 3$.

$$A_i^{(1)} = \begin{array}{|ccccc|}\hline c_{1+2i} & 0 & c_{2+2i} & 0 & 0 \\ c_{2+2i} & 0 & 0 & c_{1+2i} & 0 \\ 0 & c_{2+2i} & c_{1+2i} & 0 & 0 \\ 0 & c_{1+2i} & 0 & 0 & c_{2+2i} \\ 0 & 0 & 0 & c_{2+2i} & c_{1+2i} \\ \hline\end{array} \quad A_i^{(2)} = \begin{array}{|ccccc|}\hline 0 & 0 & 0 & c_{2+2i} & c_{1+2i} \\ c_{1+2i} & 0 & c_{2+2i} & 0 & 0 \\ 0 & c_{2+2i} & 0 & c_{1+2i} & 0 \\ c_{2+2i} & c_{1+2i} & 0 & 0 & 0 \\ 0 & 0 & c_{1+2i} & 0 & c_{2+2i} \\ \hline\end{array}$$

$$A_i^{(3)} = \begin{array}{|ccccc|}\hline c_{2+2i} & 0 & 0 & 0 & c_{1+2i} \\ 0 & 0 & c_{1+2i} & c_{2+2i} & 0 \\ c_{1+2i} & 0 & c_{2+2i} & 0 & 0 \\ 0 & c_{1+2i} & 0 & 0 & c_{2+2i} \\ 0 & c_{2+2i} & 0 & c_{1+2i} & 0 \\ \hline\end{array}$$

Let $B_i^{(j)}$ be the following 8×5 matrices for $j = 1, 2, 3, 4, 5, 6$.

$$B_i^{(1)} = \begin{array}{|ccccc|}\hline c_{3+2i} & 0 & 0 & 0 & 0 \\ c_{4+2i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{3+2i} & 0 \\ 0 & c_{3+2i} & 0 & 0 & 0 \\ 0 & c_{4+2i} & 0 & 0 & 0 \\ 0 & 0 & c_{4+2i} & 0 & 0 \\ 0 & 0 & 0 & c_{4+2i} & 0 \\ 0 & 0 & 0 & 0 & c_{4+2i} \\ \hline\end{array} \quad B_i^{(2)} = \begin{array}{|ccccc|}\hline 0 & c_{4+2i} & 0 & 0 & 0 \\ 0 & 0 & c_{3+2i} & 0 & 0 \\ c_{4+2i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{4+2i} & 0 \\ 0 & 0 & 0 & 0 & c_{3+2i} \\ c_{3+2i} & 0 & 0 & 0 & 0 \\ 0 & c_{3+2i} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{3+2i} & 0 \\ \hline\end{array}$$

$$B_i^{(3)} = \begin{array}{|ccccc|}\hline c_{5+2i} & 0 & 0 & 0 & 0 \\ 0 & c_{5+2i} & 0 & 0 & 0 \\ 0 & 0 & c_{5+2i} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{5+2i} \\ 0 & 0 & c_{6+2i} & 0 & 0 \\ 0 & 0 & 0 & c_{5+2i} & 0 \\ 0 & 0 & 0 & 0 & c_{6+2i} \\ 0 & 0 & 0 & c_{6+2i} & 0 \\ \hline\end{array} \quad B_i^{(4)} = \begin{array}{|ccccc|}\hline c_{6+2i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{6+2i} & 0 \\ 0 & c_{6+2i} & 0 & 0 & 0 \\ 0 & 0 & c_{6+2i} & 0 & 0 \\ 0 & 0 & c_{5+2i} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{6+2i} \\ c_{5+2i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{5+2i} & 0 \\ \hline\end{array}$$

$$B_i^{(5)} = \begin{array}{|ccccc|}\hline 0 & 0 & 0 & 0 & c_{7+2i} \\ c_{7+2i} & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{8+2i} & 0 & 0 \\ 0 & 0 & 0 & c_{7+2i} & 0 \\ 0 & 0 & 0 & 0 & c_{8+2i} \\ 0 & 0 & c_{7+2i} & 0 & 0 \\ 0 & c_{8+2i} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{8+2i} & 0 \\ \hline\end{array} \quad B_i^{(6)} = \begin{array}{|ccccc|}\hline 0 & 0 & c_{8+2i} & 0 & 0 \\ 0 & c_{8+2i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{7+2i} \\ 0 & 0 & 0 & 0 & c_{8+2i} \\ 0 & 0 & 0 & c_{7+2i} & 0 \\ c_{8+2i} & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{7+2i} & 0 & 0 \\ 0 & c_{7+2i} & 0 & 0 & 0 \\ \hline\end{array}$$

Let $C_i^{(j)}$ be the following 5×5 matrices for $j = 1, 2, 3$.

$$C_i^{(1)} = \boxed{\begin{matrix} c_{7+2i} & c_{8+2i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{7+2i} \\ 0 & 0 & c_{8+2i} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}} \quad C_i^{(2)} = \boxed{\begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_{5+2i} & 0 & c_{6+2i} \\ c_{6+2i} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & c_{6+2i} & 0 & 0 & 0 \end{matrix}}$$

$$C_i^{(3)} = \boxed{\begin{matrix} 0 & 0 & 0 & 0 & 0 \\ c_{3+2i} & 0 & 0 & c_{4+2i} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{3+2i} \\ 0 & c_{3+2i} & 0 & 0 & 0 \end{matrix}}$$

Now the edge-coloring incidence matrix can be represented by the above sub-matrices along with the all 0 sub-matrix.

$$M_i = \begin{array}{|c|c|c|} \hline A_i^{(1)} & 0 & 0 \\ \hline 0 & 0 & A_i^{(2)} \\ \hline 0 & A_i^{(3)} & 0 \\ \hline B_i^{(1)} & B_i^{(2)} & 0 \\ \hline B_i^{(3)} & 0 & B_i^{(4)} \\ \hline 0 & B_i^{(5)} & B_i^{(6)} \\ \hline C_i^{(1)} & C_i^{(2)} & C_i^{(3)} \\ \hline \end{array}$$

Since M_i is semi-properly colored, by Theorem 2.12 we can properly color a $\text{BIBD}(10, 4, \lambda)$.

✠

Because the process of checking each individual lemma for a correct direct construction is difficult, a program was built for the specific purpose of checking the small cases. Along with this is a csv (comma separated variables) and a tsv (tab separated variables) file of the edge-colored incidence matrix which are semi-properly colored. The program called *bibdchecker.cpp* is used to verify the design is a semi-properly colored BIBD. *Mathematica* was used to generate the properly colored versions of the BIBDs in csv and tsv form. Go to

`http://www.mathlab.mtu.edu/~msjukuri/Data.html`

for the program and corresponding csv and tsv files.

3.2 Main Results

Before we can state the main theorem of this chapter, we must prove the general constructions we use. Note that the exponential notation in 4–GDD$(a_1^{b_1} a_2^{b_2} \cdots a_x^{b_x})$ means we have b_1 groups of size a_1, b_2 groups of size a_2,..., and b_x groups of size a_x in the GDD. We can use 4–GDDs and 4–RGDDs to build our $\mathrm{BIBD}(v, 4, \lambda)$s in a way that will allow us to properly color the edges. We now give some recursive constructions which are based off this idea.

Lemma 3.7. *If there exists a* $4 - \mathrm{GDD}(a_1^{b_1} a_2^{b_2} \cdots a_x^{b_x})$*, and a properly colored* $\mathrm{BIBD}(a_i, 4, \lambda)$ *for all* $i = 1, 2, \ldots, x$*, then there exists a properly colored* $\mathrm{BIBD}\left(\sum_{i=1}^{x} a_i b_i, 4, \lambda\right)$.

Proof: Repeat each of the blocks in a $4 - \mathrm{GDD}(a_1^{b_1} a_2^{b_2} \cdots a_x^{b_x})$ λ times. For each block, we must color each edge a different color using the colors $c_i \in \{c_1, \ldots, c_\lambda\}$. Each corresponding edge in the λ copies of the blocks must also be a different color. So we color the edges in the λ copies of each block as follows. Form a $6 \times \lambda$ matrix. The rows of the matrix will be indexed by the 6 edges of K_4, and the columns will be indexed by the λ copies of the block. The entries of the matrix will be the first 6 rows of an $\mathrm{LS}(\lambda)$. Now the only pairs that have not been covered are the pairs which lie within the groups. So we place a properly colored $\mathrm{BIBD}(a_i, 4, \lambda)$ on each group for all $i = 1, 2, \ldots, x$. This forms a properly colored $\mathrm{BIBD}\left(\sum_{i=1}^{x} a_i b_i, 4, \lambda\right)$. ✠

Lemma 3.8. *If there exists a* $4 - \mathrm{GDD}(m^u)$ *and a properly colored* $\mathrm{BIBD}(m+1, 4, \lambda)$*, then there exists a properly colored* $\mathrm{BIBD}(mu+1, 4, \lambda)$.

Proof: Let G_i for $i = 1, \ldots, u$ be the u groups of size m. Repeat each of the blocks in a $4 - \mathrm{GDD}(m^u)$ λ times. For each block, we must color each edge a different color, using the colors $c_i \in \{c_1, \ldots, c_\lambda\}$. Each corresponding edge in the λ copies of the block must also be a different color. So we color the edges in the λ copies of each block as follows. Form a $6 \times \lambda$ matrix. The rows of the matrix will be indexed by the 6 edges of K_4, and the columns will be indexed by the λ copies of the block. The entries of the matrix will be the first 6 rows of an $\mathrm{LS}(\lambda)$. Now the only pairs that have not been covered are the pairs which lie within the groups and pairs which contain the point $\{\infty\}$. So we place a properly colored $\mathrm{BIBD}(m+1, 4, \lambda)$ on each $G_i \cup \{\infty\}$ for all $i = 1, 2, \ldots, u$. This forms a properly colored $\mathrm{BIBD}\,(mu+1, 4, \lambda)$. ✠

Lemma 3.9. *If there exists a* $4{-}\mathrm{RGDD}(m^u)$*, a properly colored* $\mathrm{BIBD}(5, 4, \lambda)$*, a properly colored* $\mathrm{BIBD}(m, 4, \lambda)$*, and a properly colored* $\mathrm{BIBD}(t, 4, \lambda)$ *for some* $t \leq \dfrac{m(u-1)}{3}$*, then there exists a properly colored* $\mathrm{BIBD}(mu+t, 4, \lambda)$.

Proof: Let P_i for $i = 1, \ldots, \dfrac{m(u-1)}{3}$ denote the parallel classes in the $4-\mathrm{RGDD}(m^u)$. Also let $\{\infty_1, \infty_2, \ldots,$
$\infty_t\}$ be t new points where $0 \leq t \leq \dfrac{m(u-1)}{3}$. Consider the parallel class P_i for $i = 1, \ldots, t$. We take each block of P_i and join it with $\{\infty_i\}$. Now place a properly colored $\mathrm{BIBD}(5, 4, \lambda)$ on each block of $P_i \cup \{\infty_i\}$. For each block in P_i for $i = t+1, \ldots, \dfrac{m(u-1)}{3}$ we repeat it λ times. We must color each edge of each block a different color, using the colors $c_i \in \{c_1, \ldots, c_\lambda\}$. Each corresponding edge in the λ copies of the blocks must also be a different color. So we color the edges in the λ copies of each block as follows. Form a $6 \times \lambda$ matrix. The rows of the matrix will be indexed by the 6 edges of K_4, and the columns will be indexed by the λ copies of the block. The entries of the matrix will be the first 6 rows of a $\mathrm{LS}(\lambda)$. Now the only pairs that have not been covered are the pairs which lie within the groups and the pairs of the form $\{\{\infty_i, \infty_j\} : i, j \in \{1, \ldots, t\}\}$. So we place a properly colored $\mathrm{BIBD}(m, 4, \lambda)$ on each group, G_i, for all $i = 1, 2, \ldots, u$ and we place a properly colored $\mathrm{BIBD}(t, 4, \lambda)$ on the set of points $\{\infty_1, \ldots, \infty_t\}$. This forms a properly colored $\mathrm{BIBD}(mu + t, 4, \lambda)$. ✠

The following theorem illustrates the use of the above lemmas.

Theorem 3.10. *There exists a properly colored* $\mathrm{BIBD}(v, 4, \lambda)$ *for* $v \equiv 0 \pmod{12}$ *where* $\lambda = 3k$*,* $k \geq 2$*.*

Proof: Let $v = 24$. By Theorem 1.10 a $4 - \mathrm{RGDD}(5^4)$ exists with 5 parallel classes, and Lemma 2.4 allows us to properly color a $\mathrm{BIBD}(5, 4, \lambda)$ for $\lambda = 3k, k \in \mathbb{Z}^+, k \geq 2$. Therefore, we can apply Lemma 3.9 with $m = 5$, $u = 4$, and $t = 4$ to obtain a properly colored $\mathrm{BIBD}(24, 4, \lambda)$ for $\lambda = 3k$ $k \geq 2$.

Let $v = 36$. By Theorem 1.3, there exists a $4 - \mathrm{GDD}(9^4)$. From Lemma 3.1 we have a properly colored $\mathrm{BIBD}(9, 4, \lambda)$ for $\lambda = 3k$, $k \geq 2$. Hence, we apply Lemma 3.7 with $x = 1$, $a_1 = 9$, and $b_1 = 4$ to properly color a $\mathrm{BIBD}(36, 4, \lambda)$ for $\lambda = 3k$, $k \geq 2$.

Now suppose $v = 12u$ where $u \geq 4$. There exists a $4 - \mathrm{GDD}(12^u)$ for $u \geq 4$ by Theorem 1.3. By Lemma 3.3, we can properly color a $\mathrm{BIBD}(12, 4, \lambda)$ design for $\lambda = 3k$, $k \geq 2$. Thus, we can let $x = 1$, $a_1 = 12$, $b_1 = u$, so it follows by Lemma 3.7 that we can properly color a $\mathrm{BIBD}(v, 4, \lambda)$ for $\lambda = 3k$, $k \geq 2$. ✠

3.2.1 $\lambda \equiv 0 \pmod 3$

In this section, we properly color all $\mathrm{BIBD}(v, 4, \lambda)$s where $\lambda \equiv 0 \pmod 3$. In this case, the necessary and sufficient conditions for the existence of a $\mathrm{BIBD}(v, 4, \lambda)$ are $v \equiv 0, 1 \pmod 4$. Note when $v \equiv 0, 1 \pmod 4$ and $\lambda \equiv 0 \pmod 6$ these are already covered by Theorem 2.2, but the results in this section satisfy this case as well.

Theorem 3.11. *There exists a proper coloring for every* $\mathrm{BIBD}(v, 4, \lambda)$ *for* $\lambda = 3k, k \in \mathbb{Z}^+, k \geq 2$*, where* $v \equiv 0, 1 \pmod 4$*.*

Proof: Note that in each case we assume $\lambda = 3k, k \in \mathbb{Z}^+, k \geq 2$ unless otherwise stated. We will break this problem up into two main cases, $v \equiv 1 \pmod 4$, and $v \equiv 0 \pmod 4$.

Case 1: $v \equiv 1 \pmod 4$

Let $v \equiv 1 \pmod 4$. Figure 3.8 represents the possible v values.

	5	9
13	17	21
25	29	33
37	41	45
49	53	57
61	65	69
$\vdots$	$\vdots$	$\vdots$

Figure 3.8: Possible v values for Case 1

Each column of Figure 3.8 represents $v \equiv 1 \pmod 4$ in three different ways; $v \equiv 1 \pmod{12}$, $v \equiv 5 \pmod{12}$, and $v \equiv 9 \pmod{12}$.

Case 1.1: $v \equiv 1 \pmod{12}$

By Theorem 2.3, we can properly color a $\mathrm{BIBD}(v, 4, \lambda)$ where $v \equiv 1, 4 \pmod{12}$.

Case 1.2: $v \equiv 5 \pmod{12}$

For $v = 5$, we can properly color a $\mathrm{BIBD}(5, 4, \lambda)$ by Lemma 2.4

Let $v \equiv 5 \pmod{12}$. So $v = 5 + 12x = 1 + 4(1 + 3x)$ where $x \geq 1$. We construct a $4 - \mathrm{GDD}(4^u)$ where $u = 1 + 3x$ and $x \geq 1$. This exists by Theorem 1.3. We also know that a properly colored $\mathrm{BIBD}(5, 4, \lambda)$ exists by Lemma 2.4. So we apply Lemma 3.8 with $m = 4, u = 1 + 3x, x \geq 1$.

Case 1.3: $v \equiv 9 \pmod{12}$

If $v \equiv 9 \pmod{12}$, then we have that either $v \equiv 9 \pmod{24}$ or $v \equiv 21 \pmod{24}$.

Case 1.3.1: $v \equiv 9 \pmod{24}$

For $v = 9$, we can properly color a $\mathrm{BIBD}(9, 4, \lambda)$ by Lemma 3.1.

For $v > 9$, let $v = 24x + 9 = 8(3x + 1) + 1$, $x \geq 1$. Theorem 1.3 says that there exists a $4 - \mathrm{GDD}(8^u)$ for $u = 3x + 1$. Also by Lemma 3.1 there exists a properly colored $\mathrm{BIBD}(9, 4, \lambda)$. So apply Lemma 3.8 with $m = 8, u = 3x + 1, x \geq 1$.

Case 1.3.2: $v \equiv 21 \pmod{24}$

If $v \equiv 21 \pmod{24}$, then we can write v as $v \equiv 21 \pmod{48}$ or $v \equiv 45 \pmod{48}$.

Case 1.3.2.1: $v \equiv 45 \pmod{48}$

Suppose $v = 48x + 45 = 4(12x + 11) + 1$. We can construct a $4 - \mathrm{GDD}(m^4)$ for $m = 12x + 11$ by Theorem 1.3. By Theorem 3.10, we can properly color a $\mathrm{BIBD}(m + 1, 4, \lambda)$. So apply Lemma 3.8 to obtain a properly colored $\mathrm{BIBD}(4m + 1, 4, \lambda)$.

Case 1.3.2.2: $v \equiv 21 \pmod{48}$

If $v = 21$, then we can write v as $v = 4(5) + 1$. We can construct a $4 - \mathrm{RGDD}(5^4)$ by Theorem 1.10. This has 5 parallel classes. Let $\{\infty\}$ be a new point, and let P_i denote the i^{th} parallel classes. We take each block of P_1 and join it with $\{\infty\}$. Place a properly colored $\mathrm{BIBD}(5, 4, \lambda)$ design on each block of $P_1 \cup \{\infty\}$. Now we repeat each block in P_i λ times for $i = 2, 3, 4, 5$. We must color each edge of each block a different color, using the colors $c_i \in \{c_1, \ldots, c_\lambda\}$. Each corresponding edge in the λ copies of the block must also be a different color. So we color the edges in the λ copies of each block as follows. Form a $6 \times \lambda$ matrix. The rows of the matrix will be indexed by the 6 edges of K_4, and the columns will be indexed by the λ copies of the block. The entries of the matrix will be the first 6 rows of a $\mathrm{LS}(\lambda)$. Now the only pairs that have not been covered are the pairs which lie within the groups. So we place a properly colored $\mathrm{BIBD}(5, 4, \lambda)$ on each group G_i for $i = 1, 2, 3, 4$. This forms a properly colored $\mathrm{BIBD}(21, 4, \lambda)$.

Now suppose $v = 21 + 48x$ for $x \geq 1$. Since $v = 21 + 48x = 4(12x + 4) + 5$, we can construct a $4 - \mathrm{RGDD}(m^u)$ with $m = 12x + 5$ and $u = 4$, by Theorem 1.10. We also have that a properly colored $\mathrm{BIBD}(5, 4, \lambda)$ design exists by Lemma 2.4, and a properly colored $\mathrm{BIBD}(12x + 4, 4, \lambda)$ exists by Theorem 2.3. Since $5 \leq 12x + 4$ for all $x > 0$, we can apply Lemma 3.9.

Thus, we can properly color a $\mathrm{BIBD}(v, 4, \lambda)$ for $v \equiv 1 \pmod{4}$.

Case 2: $v \equiv 0 \pmod{4}$

Let $v \equiv 0 \pmod{4}$. Figure 3.9 represents the possible v values.

	8	12
16	20	24
28	32	36
40	44	48
52	56	60
64	68	72
$\vdots$	$\vdots$	$\vdots$

Figure 3.9: Possible v values for Case 2

Each column of the above table represents $v \equiv 0 \pmod{4}$ in three different ways;

$v \equiv 0 \pmod{12}$, $v \equiv 4 \pmod{12}$, and $v \equiv 8 \pmod{12}$. Recall that if $v \equiv 0 \pmod{12}$, these designs were colored in Theorem 3.10.

Case 2.1: $v \equiv 4 \pmod{12}$

By Theorem 2.3, we can properly color a $\mathrm{BIBD}(v, 4, \lambda)$ for $v \equiv 4 \pmod{12}$.

Case 2.2: $v \equiv 8 \pmod{12}$

If $v \equiv 8 \pmod{12}$, then we can rewrite v as $v \equiv 8 \pmod{24}$ or $v \equiv 20 \pmod{24}$.

Case 2.2.1: $v \equiv 8 \pmod{24}$

We can properly color a $\mathrm{BIBD}(8, 4, \lambda)$ by Lemma 3.2. There exists a $4 - \mathrm{GDD}(8^u)$ for $u = 3x + 1$ and $x \geq 1$ by Theorem 1.3. Therefore, we use Lemma 3.7 with $x = 1$, $a_1 = 8$, and $b_1 = u$.

Case 2.2.2: $v \equiv 20 \pmod{24}$

We break this case into two subcases, $v \equiv 20 \pmod{48}$ and $v \equiv 44 \pmod{48}$.

Case 2.2.2.1: $v \equiv 20 \pmod{48}$

Let $v = 48x + 20 = 4(12x + 5)$ for $x \geq 0$. There exists a $4 - \mathrm{GDD}(m^4)$ where $m = 12x + 5$ by Theorem 1.3. We properly color all $\mathrm{BIBD}(m, 4, \lambda)$s in Case 1.2 and Lemma 2.4. Thus we can apply Lemma 3.7 with $x = 1$, $a_1 = m$, and $b_1 = 4$.

Case 2.2.2.2: $v \equiv 44 \pmod{48}$

Let $v = 44 + 48x = 4(12x + 9) + 8$ for $x \geq 0$. There exists a $4 - \mathrm{RGDD}(m^4)$ for $m = 12x + 9$ by Theorem 1.10. This has $12x + 9$ parallel classes. We can properly color each $\mathrm{BIBD}(v, 4, \lambda)$ for $v \equiv 9 \pmod{12}$ by Case 1.3, and we can properly color a $\mathrm{BIBD}(5, 4, \lambda)$ using Lemma 2.4. Also, we can properly color a $\mathrm{BIBD}(8, 4, \lambda)$ by Lemma 3.2. So we apply Lemma 3.9 with $m = 12x + 9$, $u = 4$, and $t = 8$.

Therefore, we can properly color a $\mathrm{BIBD}(v, 4, \lambda)$ for $v \equiv 0 \pmod{4}$ and $\lambda = 3k$, $k \geq 2$. Furthermore, there exists a proper coloring for every $\mathrm{BIBD}(v, 4, \lambda)$ where $v \equiv 0, 1 \pmod{4}$. ✠

3.2.2 $\lambda \equiv 2, 4 \pmod{6}$

In this section, we properly color all $\mathrm{BIBD}(v, 4, \lambda)$s where $\lambda \equiv 2$ or $4 \pmod{6}$. In this case, the necessary and sufficient conditions for the existence of a $\mathrm{BIBD}(v, 4, \lambda)$ are that $v \equiv 1 \pmod{3}$. Note that when $v \equiv 1 \pmod{3}$ and $\lambda \equiv 0 \pmod{6}$, we could also use Theorem 2.2.

Theorem 3.12. *There exists a proper coloring for every* $\mathrm{BIBD}(v, 4, \lambda)$ *design for* $\lambda = 2k, k \in \mathbb{Z}^+, k \geq 3$, *where* $v \equiv 1 \pmod{3}$.

Proof: Note that in each case we assume $\lambda = 2k, k \in \mathbb{Z}^+, k \geq 3$ unless otherwise stated. Figure 3.10 represents the possible v values when $v \equiv 1 \pmod 3$.

	4	7	10
13	16	19	22
25	28	31	34
37	40	43	46
49	52	55	58
61	64	67	70
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Figure 3.10: Possible v values for Theorem 3.12

Each column of the above table represents $v \equiv 1 \pmod 3$ in four different ways; $v \equiv 1 \pmod{12}$, $v \equiv 4 \pmod{12}$, $v \equiv 7 \pmod{12}$, and $v \equiv 10 \pmod{12}$.

Case 1: $v \equiv 1, 4 \pmod{12}$

By Theorem 2.3, we can properly color a $\mathrm{BIBD}(v, 4, \lambda)$ for $v \equiv 1, 4 \pmod{12}$.

Case 2: $v \equiv 7 \pmod{12}$

We can properly color a $\mathrm{BIBD}(7, 4, \lambda)$ for $\lambda = 2k, k \geq 3$ by Lemma 3.4. We can also properly color a $\mathrm{BIBD}(19, 4, \lambda)$ for all such λ by Lemma 3.5. Let $v = 12x + 7 = 6(2x + 1) + 1$ for $x \geq 2$. By Theorem 1.3, there exists a $4 - \mathrm{GDD}(6^{2x+1})$ for all $x \geq 2$. So we can apply Lemma 3.8 with $m = 6$ and $u = 2x + 1$.

Case 3: $v \equiv 10 \pmod{12}$

If $v = 10$, we can properly color a $\mathrm{BIBD}(10, 4, \lambda)$ with $\lambda = 6$ by Theorem 2.2. Then for all $\lambda = 2k$, $k \geq 4$, we apply Lemma 3.6.

If $v = 22$, we can apply Lemma 3.7 with $x = 2$, $a_1 = 1$, $b_1 = 15$, $a_2 = 7$, and $b_2 = 1$. Note that the required $4 - \mathrm{GDD}(1^{15}7^1)$ exists by Theorem 1.6, and a properly colored $\mathrm{BIBD}(7, 4, \lambda)$ exists by Lemma 3.4.

Now let $v = 12x + 10$, with $x \geq 2$. There exists a $4 - \mathrm{GDD}(4^u m^1)$ where $m = 10$ and $u = 3x$ for all $x \geq 2$ by Theorem 1.4. So we let $x = 2$, $a_1 = 4$, $b_1 = u$, $a_2 = m$, and $b_2 = 1$; and we apply Lemma 3.7. The required properly colored $\mathrm{BIBD}(10, 4, \lambda)$ as stated above. ✠

3.3 Conclusion

We are now in a position to prove the main theorem.

Theorem 3.13. *There is a proper edge coloring for every* $\mathrm{BIBD}(v, 4, \lambda)$ *where* $\lambda \geq 6$.

Proof: Recall the necessary and sufficient conditions for the existence of a $\mathrm{BIBD}(v, 4, \lambda)$.

If $\lambda \equiv 1, 5 \pmod 6$, then $v \equiv 1, 4 \pmod{12}$;
If $\lambda \equiv 2, 4 \pmod 6$, then $v \equiv 1 \pmod 3$;
If $\lambda \equiv 3 \pmod 6$, then $v \equiv 0, 1 \pmod 4$; and
If $\lambda \equiv 0 \pmod 6$, then $v \geq 4$.

If $\lambda \equiv 1, 5 \pmod 6$, then $v \equiv 1, 4 \pmod{12}$ and we apply Theorem 2.3 to properly color a $\mathrm{BIBD}(v, 4, \lambda)$. If $\lambda \equiv 0 \pmod 6$, then $v \geq 4$ and we can properly color a $\mathrm{BIBD}(v, 4, \lambda)$ by applying Theorem 2.2. If $\lambda \equiv 3 \pmod 6$, then $v \equiv 0, 1 \pmod 4$ and we can apply Theorem 3.11. Finally, if $\lambda \equiv 2, 4 \pmod 6$, then $v \equiv 1 \pmod 3$ and we apply Theorem 3.12. ✠

Chapter 4

Proper Edge Coloring With Block Size 5

Again, we first need to understand the construction in Chapter 2 before adventuring into this chapter. Our goal in this chapter is to prove we can properly color all $\text{BIBD}(v, 5, \lambda)$ except possibly for a few cases. To prove this we must first show there exists properly colored $\text{BIBD}(v, 5, \lambda)$ for small v. Afterwords, we prove the generalized constructions hold which are then used on the main theorem.

4.1 Direct Constructions With Block Size 5

Lemma 4.1. *There exists a properly colored* $\text{BIBD}(11, 5, \lambda)$ *for* $\lambda = 2k, k \in \mathbb{Z}^+, k \geq 5$.

Proof: We form an edge-coloring incidence matrix using 10 colors for a $\text{BIBD}(11, 5, 2)$. Figure 4.1 represents the edge-coloring incidence matrix for the $\text{BIBD}(11, 5, 2)$.

Let A be the 11×11 circulant matrix whose first row is $[01000000000]$. Then the edge-coloring incidence matrix of a $\text{BIBD}(11, 5, 2)$ using 10 colors is given below.

$$M_i = \begin{array}{|c|}\hline (c_{1+2i})I + (c_{2+2i})A \\ \hline (c_{3+2i})I + (c_{4+2i})A^2 \\ \hline (c_{5+2i})I + (c_{6+2i})A^3 \\ \hline (c_{7+2i})I + (c_{8+2i})A^4 \\ \hline (c_{9+2i})I + (c_{10+2i})A^5 \\ \hline \end{array}$$

Since M_i is semi-properly colored, by applying Theorem 2.12 we can properly color a $\text{BIBD}(11, 5, \lambda)$.

✠

Lemma 4.2. *There exists a properly colored* $\text{BIBD}(31, 5, \lambda)$ *for* $\lambda = 2k, k \in \mathbb{Z}^+, k \geq 5$.

Proof: Let A be the 31×31 circulant matrix whose first row is

$$[0100000000000000000000000000000].$$

Then the edge coloring incidence matrix of a $\text{BIBD}(31, 5, 2)$ using 10 colors in Figure 4.2.

	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}
$\{0,1\}$	c_1	c_2	0	0	0	0	0	0	0	0	0
$\{0,5\}$	0	c_1	c_2	0	0	0	0	0	0	0	0
$\{8,0\}$	0	0	c_1	c_2	0	0	0	0	0	0	0
$\{10,0\}$	0	0	0	c_1	c_2	0	0	0	0	0	0
$\{10,4\}$	0	0	0	0	c_1	c_2	0	0	0	0	0
$\{10,1\}$	0	0	0	0	0	c_1	c_2	0	0	0	0
$\{9,1\}$	0	0	0	0	0	0	c_1	c_2	0	0	0
$\{6,9\}$	0	0	0	0	0	0	0	c_1	c_2	0	0
$\{2,6\}$	0	0	0	0	0	0	0	0	c_1	c_2	0
$\{5,7\}$	0	0	0	0	0	0	0	0	0	c_1	c_2
$\{3,7\}$	c_2	0	0	0	0	0	0	0	0	0	c_1
$\{0,2\}$	c_3	0	c_4	0	0	0	0	0	0	0	0
$\{6,0\}$	0	c_3	0	c_4	0	0	0	0	0	0	0
$\{9,0\}$	0	0	c_3	0	c_4	0	0	0	0	0	0
$\{8,10\}$	0	0	0	c_3	0	c_4	0	0	0	0	0
$\{9,10\}$	0	0	0	0	c_3	0	c_4	0	0	0	0
$\{8,1\}$	0	0	0	0	0	c_3	0	c_4	0	0	0
$\{9,3\}$	0	0	0	0	0	0	c_3	0	c_4	0	0
$\{6,7\}$	0	0	0	0	0	0	0	c_3	0	c_4	0
$\{3,4\}$	0	0	0	0	0	0	0	0	c_3	0	c_4
$\{2,7\}$	c_4	0	0	0	0	0	0	0	0	c_3	0
$\{4,5\}$	0	c_4	0	0	0	0	0	0	0	0	c_3
$\{0,3\}$	c_5	0	0	c_6	0	0	0	0	0	0	0
$\{0,4\}$	0	c_5	0	0	c_6	0	0	0	0	0	0
$\{8,2\}$	0	0	c_5	0	0	c_6	0	0	0	0	0
$\{10,3\}$	0	0	0	c_5	0	0	c_6	0	0	0	0
$\{7,9\}$	0	0	0	0	c_5	0	0	c_6	0	0	0
$\{2,4\}$	0	0	0	0	0	c_5	0	0	c_6	0	0
$\{5,10\}$	0	0	0	0	0	0	c_5	0	0	c_6	0
$\{7,8\}$	0	0	0	0	0	0	0	c_5	0	0	c_6
$\{2,3\}$	c_6	0	0	0	0	0	0	0	c_5	0	0
$\{5,6\}$	0	c_6	0	0	0	0	0	0	0	c_5	0
$\{5,8\}$	0	0	c_6	0	0	0	0	0	0	0	c_5
$\{7,0\}$	c_7	0	0	0	c_8	0	0	0	0	0	0
$\{1,4\}$	0	c_7	0	0	0	c_8	0	0	0	0	0
$\{5,9\}$	0	0	c_7	0	0	0	c_8	0	0	0	0
$\{6,8\}$	0	0	0	c_7	0	0	0	c_8	0	0	0
$\{4,9\}$	0	0	0	0	c_7	0	0	0	c_8	0	0
$\{10,2\}$	0	0	0	0	0	c_7	0	0	0	c_8	0
$\{3,5\}$	0	0	0	0	0	0	c_7	0	0	0	c_8
$\{7,1\}$	c_8	0	0	0	0	0	0	c_7	0	0	0
$\{4,6\}$	0	c_8	0	0	0	0	0	0	c_7	0	0
$\{2,5\}$	0	0	c_8	0	0	0	0	0	0	c_7	0
$\{3,8\}$	0	0	0	c_8	0	0	0	0	0	0	c_7
$\{1,2\}$	c_9	0	0	0	0	c_{10}	0	0	0	0	0
$\{1,5\}$	0	c_9	0	0	0	0	c_{10}	0	0	0	0
$\{8,9\}$	0	0	c_9	0	0	0	0	c_{10}	0	0	0
$\{3,6\}$	0	0	0	c_9	0	0	0	0	c_{10}	0	0
$\{7,10\}$	0	0	0	0	c_9	0	0	0	0	c_{10}	0
$\{4,8\}$	0	0	0	0	0	c_9	0	0	0	0	c_{10}
$\{1,3\}$	c_{10}	0	0	0	0	0	c_9	0	0	0	0
$\{1,6\}$	0	c_{10}	0	0	0	0	0	c_9	0	0	0
$\{9,2\}$	0	0	c_{10}	0	0	0	0	0	c_9	0	0
$\{6,10\}$	0	0	0	c_{10}	0	0	0	0	0	c_9	0
$\{4,7\}$	0	0	0	0	c_{10}	0	0	0	0	0	c_9

Figure 4.1: Edge-Coloring Incidence Matrix of a $\mathrm{BIBD}(11,5,2)$.

$$
M_i = \begin{array}{|c|c|c|}
\hline
(c_{1+2i})I & 0 & (c_{2+2i})A^{27} \\ \hline
(c_{2+2i})A^{30} & 0 & (c_{3+2i})A^{18} \\ \hline
(c_{3+2i})I & (c_{2+2i})I & 0 \\ \hline
(c_{4+2i})A^{28} & 0 & (c_{1+2i})I \\ \hline
0 & (c_{1+2i})A^{22} & (c_{4+2i})I \\ \hline
(c_{5+2i})A^{30} & (c_{4+2i})A^{28} & 0 \\ \hline
(c_{6+2i})I & (c_{3+2i})A^{17} & 0 \\ \hline
(c_{7+2i})A^{24} & 0 & (c_{6+2i})A^{26} \\ \hline
0 & (c_{6+2i})I & (c_{5+2i})A^{27} \\ \hline
0 & (c_{5+2i})A^{10} & (c_{8+2i})A^{26} \\ \hline
0 & (c_{8+2i})A^{28} & (c_{7+2i})A^{27} \\ \hline
(c_{8+2i})A^{28} & (c_{9+2i})A^{22} & 0 \\ \hline
0 & (c_{10+2i})A^{10} & (c_{9+2i})I \\ \hline
(c_{10+2i})A^{30} & (c_{7+2i})I & 0 \\ \hline
(c_{9+2i})I & 0 & (c_{10+2i})I \\ \hline
\end{array}
$$

Figure 4.2: Edge-Coloring Incidence Matrix of a $\mathrm{BIBD}(31, 5, 2)$.

Since M_i is semi-properly colored, by applying Theorem 2.12 we can properly color a $\mathrm{BIBD}(31, 5, \lambda)$.

✠

Lemma 4.3. *There exists a properly colored* $\mathrm{BIBD}(51, 5, \lambda)$ *for* $\lambda = 2k, k \in \mathbb{Z}^+, k \geq 5$.

Proof: Let A be the 51×51 circulant matrix whose first row is

$$[010000000000000000000000000000000000000000000000000].$$

Then the edge coloring incidence matrix of a $\mathrm{BIBD}(51, 5, 2)$ using 10 colors in Figure 4.3.

Since M_i is semi-properly colored, by applying Theorem 2.12 we can properly color a $\mathrm{BIBD}(51, 5, \lambda)$.

✠

Lemma 4.4. *There exists a properly colored* $\mathrm{BIBD}(6, 5, \lambda)$ *for* $\lambda = 4k, k \in \mathbb{Z}^+, k \geq 3$.

Proof: The incidence matrix of a $\mathrm{BIBD}(6, 5, 4)$ is

	B_1	B_2	B_3	B_4	B_5	B_6
0	0	1	1	1	1	1
1	1	0	1	1	1	1
2	1	1	0	1	1	1
3	1	1	1	0	1	1
4	1	1	1	1	0	1
5	1	1	1	1	1	0

Figure 4.4 represents the edge-coloring incidence matrix for the $\mathrm{BIBD}(6, 5, 4)$.

$$
M_i = \begin{array}{|c|c|c|c|c|}
\hline
(c_{1+4i})I & (c_{2+4i})I & 0 & 0 & 0 \\ \hline
0 & 0 & (c_{1+4i})A^{35} & 0 & (c_{2+4i})A^{41} \\ \hline
0 & 0 & 0 & (c_{1+4i})A^{15} & (c_{4+4i})A^{39} \\ \hline
(c_{2+4i})A^{20} & 0 & 0 & 0 & (c_{3+4i})I \\ \hline
0 & 0 & (c_{2+4i})A^{40} & 0 & (c_{1+4i})A^{41} \\ \hline
0 & 0 & 0 & (c_{2+4i})A^{44} & (c_{5+4i})A^{47} \\ \hline
0 & 0 & (c_{3+4i})A^{40} & (c_{4+4i})I & 0 \\ \hline
0 & (c_{1+4i})A^{50} & 0 & 0 & (c_{6+4i})A^{47} \\ \hline
0 & (c_{3+4i})I & (c_{4+4i})A^{9} & 0 & 0 \\ \hline
0 & (c_{4+4i})A^{28} & 0 & 0 & (c_{7+4i})I \\ \hline
0 & 0 & (c_{5+4i})I & 0 & (c_{8+4i})A^{47} \\ \hline
0 & 0 & 0 & (c_{6+4i})A^{12} & (c_{9+4i})I \\ \hline
(c_{4+4i})A^{50} & 0 & 0 & (c_{3+4i})I & 0 \\ \hline
(c_{3+4i})I & (c_{6+4i})A^{42} & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & (c_{7+4i})A^{15} & (c_{10+4i})I \\ \hline
(c_{5+4i})A^{16} & 0 & (c_{6+4i})I & 0 & 0 \\ \hline
(c_{6+4i})A^{16} + (c_{7+4i})A^{37} & 0 & 0 & 0 & 0 \\ \hline
0 & (c_{5+4i})A^{18} & (c_{8+4i})I & 0 & 0 \\ \hline
0 & (c_{7+4i})A^{18} & 0 & (c_{8+4i})A^{12} & 0 \\ \hline
(c_{8+4i})A^{20} & 0 & (c_{7+4i})A^{9} & 0 & 0 \\ \hline
(c_{9+4i})A^{20} + (c_{10+4i})A^{37} & 0 & 0 & 0 & 0 \\ \hline
0 & (c_{8+4i})A^{50} & 0 & (c_{5+4i})A^{15} & 0 \\ \hline
0 & (c_{9+4i})I & 0 & (c_{10+4i})A^{38} & 0 \\ \hline
0 & (c_{10+4i})A^{42} & (c_{9+4i})A^{33} & 0 & 0 \\ \hline
0 & 0 & (c_{10+4i})A^{9} & (c_{9+4i})A^{12} & 0 \\ \hline
\end{array}
$$

Figure 4.3: Edge-Coloring Incidence Matrix of a $\mathrm{BIBD}(51, 5, 2)$.

	B_1	B_2	B_3	B_4	B_5	B_6
$\{0,1\}$	0	0	c_1	c_2	c_3	c_4
$\{1,2\}$	c_4	0	0	c_1	c_2	c_3
$\{2,3\}$	c_3	c_4	0	0	c_1	c_2
$\{3,4\}$	c_2	c_3	c_4	0	0	c_1
$\{4,5\}$	c_1	c_2	c_3	c_4	0	0
$\{5,0\}$	0	c_1	c_2	c_3	c_4	0
$\{0,2\}$	0	c_5	0	c_6	c_7	c_8
$\{1,3\}$	c_8	0	c_5	0	c_6	c_7
$\{2,4\}$	c_7	c_8	0	c_5	0	c_6
$\{3,5\}$	c_6	c_7	c_8	0	c_5	0
$\{4,0\}$	0	c_6	c_7	c_8	0	c_5
$\{5,1\}$	c_5	0	c_6	c_7	c_8	0
$\{0,3\}$	0	c_9	c_{10}	0	c_{11}	c_{12}
$\{1,4\}$	c_{12}	0	c_9	c_{10}	0	c_{11}
$\{2,5\}$	c_{11}	c_{12}	0	c_9	c_{10}	0

Figure 4.4: Edge-Coloring Incidence Matrix of a $\mathrm{BIBD}(6,5,4)$.

Let A be the 6×6 circulant matrix whose first row is $[010000]$. Also let B be the first 3 rows of the 6×6 circulant matrix whose first row is $[010000]$. Then the edge coloring incidence matrix of a $\mathrm{BIBD}(6,5,4)$ using 12 colors is given below.

$$M_i = \begin{array}{|c|}\hline (c_{1+4i})A^2 + (c_{2+4i})A^3 + (c_{3+4i})A^4 + (c_{4+4i})A^5 \\ \hline (c_{5+4i})A + (c_{6+4i})A^3 + (c_{7+4i})A^4 + (c_{8+4i})A^5 \\ \hline (c_{9+4i})B + (c_{10+4i})B^2 + (c_{11+4i})B^4 + (c_{12+4i})B^5 \\ \hline \end{array}$$

Since M_i is semi-properly colored, by applying Theorem 2.12 we can properly color a $\mathrm{BIBD}(6,5,\lambda)$.

✠

Lemma 4.5. *There exists a properly colored* $\mathrm{BIBD}(10,5,\lambda)$ *for* $\lambda = 4k, k \in \mathbb{Z}^+, k \geq 3$.

Proof: Figure 4.5 represents the edge-coloring incidence matrix for the $\mathrm{BIBD}(10,5,4)$. Let A be the 9×9 circulant matrix whose first row is $[010000000]$. Then the edge coloring incidence matrix of a $\mathrm{BIBD}(10,5,4)$ using 12 colors is given below.

$$M_i = \begin{array}{|c|c|}\hline (c_{4+4i})I & (c_{2+4i})I + (c_{3+4i})A + (c_{1+4i})A^8 \\ \hline (c_{5+4i})A^5 & (c_{7+4i})I + (c_{8+4i})A + (c_{6+4i})A^7 \\ \hline (c_{9+4i})A^3 + (c_{10+4i})A^8 & (c_{11+4i})A + (c_{12+4i})A^8 \\ \hline (c_{2+4i})I + (c_{3+4i})A^3 & (c_{4+4i})I + (c_{5+4i})A^5 \\ \hline (c_{1+4i})I + (c_{8+4i})A^3 + (c_{6+4i})A^5 + (c_{7+4i})A^8 & 0 \\ \hline \end{array}$$

	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}	B_{16}	B_{17}	B_{18}
$\{0,1\}$	c_4	0	0	0	0	0	0	0	0	c_2	c_3	0	0	0	0	0	0	c_1
$\{1,2\}$	0	c_4	0	0	0	0	0	0	0	c_1	c_2	c_3	0	0	0	0	0	0
$\{2,3\}$	0	0	c_4	0	0	0	0	0	0	0	c_1	c_2	c_3	0	0	0	0	0
$\{3,4\}$	0	0	0	c_4	0	0	0	0	0	0	0	c_1	c_2	c_3	0	0	0	0
$\{4,5\}$	0	0	0	0	c_4	0	0	0	0	0	0	0	c_1	c_2	c_3	0	0	0
$\{5,6\}$	0	0	0	0	0	c_4	0	0	0	0	0	0	0	c_1	c_2	c_3	0	0
$\{6,7\}$	0	0	0	0	0	0	c_4	0	0	0	0	0	0	0	c_1	c_2	c_3	0
$\{7,8\}$	0	0	0	0	0	0	0	c_4	0	0	0	0	0	0	0	c_1	c_2	c_3
$\{8,0\}$	0	0	0	0	0	0	0	0	c_4	c_3	0	0	0	0	0	0	c_1	c_2
$\{0,2\}$	0	0	0	0	0	c_5	0	0	0	c_7	c_8	0	0	0	0	0	c_6	0
$\{1,3\}$	0	0	0	0	0	0	c_5	0	0	0	c_7	c_8	0	0	0	0	0	c_6
$\{2,4\}$	0	0	0	0	0	0	0	c_5	0	c_6	0	c_7	c_8	0	0	0	0	0
$\{3,5\}$	0	0	0	0	0	0	0	0	c_5	0	c_6	0	c_7	c_8	0	0	0	0
$\{4,6\}$	c_5	0	0	0	0	0	0	0	0	0	0	c_6	0	c_7	c_8	0	0	0
$\{5,7\}$	0	c_5	0	0	0	0	0	0	0	0	0	0	c_6	0	c_7	c_8	0	0
$\{6,8\}$	0	0	c_5	0	0	0	0	0	0	0	0	0	0	c_6	0	c_7	c_8	0
$\{7,0\}$	0	0	0	c_5	0	0	0	0	0	0	0	0	0	0	c_6	0	c_7	c_8
$\{8,1\}$	0	0	0	0	c_5	0	0	0	0	c_8	0	0	0	0	0	c_6	0	c_7
$\{0,3\}$	0	0	0	c_{10}	0	0	0	0	c_9	0	c_{11}	0	0	0	0	0	0	c_{12}
$\{1,4\}$	c_9	0	0	0	c_{10}	0	0	0	0	c_{12}	0	c_{11}	0	0	0	0	0	0
$\{2,5\}$	0	c_9	0	0	0	c_{10}	0	0	0	0	c_{12}	0	c_{11}	0	0	0	0	0
$\{3,6\}$	0	0	c_9	0	0	0	c_{10}	0	0	0	0	c_{12}	0	c_{11}	0	0	0	0
$\{4,7\}$	0	0	0	c_9	0	0	0	c_{10}	0	0	0	0	c_{12}	0	c_{11}	0	0	0
$\{5,8\}$	0	0	0	0	c_9	0	0	0	c_{10}	0	0	0	0	c_{12}	0	c_{11}	0	0
$\{6,0\}$	c_{10}	0	0	0	0	c_9	0	0	0	0	0	0	0	0	c_{12}	0	c_{11}	0
$\{7,1\}$	0	c_{10}	0	0	0	0	c_9	0	0	0	0	0	0	0	0	c_{12}	0	c_{11}
$\{8,2\}$	0	0	c_{10}	0	0	0	0	c_9	0	c_{11}	0	0	0	0	0	0	c_{12}	0
$\{0,4\}$	c_2	0	0	c_3	0	0	0	0	0	c_4	0	0	0	0	c_5	0	0	0
$\{1,5\}$	0	c_2	0	0	c_3	0	0	0	0	0	c_4	0	0	0	0	c_5	0	0
$\{2,6\}$	0	0	c_2	0	0	c_3	0	0	0	0	0	c_4	0	0	0	0	c_5	0
$\{3,7\}$	0	0	0	c_2	0	0	c_3	0	0	0	0	0	c_4	0	0	0	0	c_5
$\{4,8\}$	0	0	0	0	c_2	0	0	c_3	0	c_5	0	0	0	c_4	0	0	0	0
$\{5,0\}$	0	0	0	0	0	c_2	0	0	c_3	0	c_5	0	0	0	c_4	0	0	0
$\{6,1\}$	c_3	0	0	0	0	0	c_2	0	0	0	0	c_5	0	0	0	c_4	0	0
$\{7,2\}$	0	c_3	0	0	0	0	0	c_2	0	0	0	0	c_5	0	0	0	c_4	0
$\{8,3\}$	0	0	c_3	0	0	0	0	0	c_2	0	0	0	0	c_5	0	0	0	c_4
$\{\infty,0\}$	c_1	0	0	c_8	0	c_6	0	0	c_7	0	0	0	0	0	0	0	0	0
$\{\infty,1\}$	c_7	c_1	0	0	c_8	0	c_6	0	0	0	0	0	0	0	0	0	0	0
$\{\infty,2\}$	0	c_7	c_1	0	0	c_8	0	c_6	0	0	0	0	0	0	0	0	0	0
$\{\infty,3\}$	0	0	c_7	c_1	0	0	c_8	0	c_6	0	0	0	0	0	0	0	0	0
$\{\infty,4\}$	c_6	0	0	c_7	c_1	0	0	c_8	0	0	0	0	0	0	0	0	0	0
$\{\infty,5\}$	0	c_6	0	0	c_7	c_1	0	0	c_8	0	0	0	0	0	0	0	0	0
$\{\infty,6\}$	c_8	0	c_6	0	0	c_7	c_1	0	0	0	0	0	0	0	0	0	0	0
$\{\infty,7\}$	0	c_8	0	c_6	0	0	c_7	c_1	0	0	0	0	0	0	0	0	0	0
$\{\infty,8\}$	0	0	c_8	0	c_6	0	0	c_7	c_1	0	0	0	0	0	0	0	0	0

Figure 4.5: Edge-Coloring Incidence Matrix of a $\mathrm{BIBD}(10,5,4)$.

Since M_i is semi-properly colored, by applying Theorem 2.12 we can properly color a $\mathrm{BIBD}(10, 5, \lambda)$.

✠

Lemma 4.6. *There exists a properly colored* $\mathrm{BIBD}(20, 5, \lambda)$ *for* $\lambda = 4k, k \in \mathbb{Z}^+, k \geq 3$.

Proof: Let A be the 19×19 circulant matrix whose first row is $[0100000000000000000]$. Then the edge-coloring incidence matrix of a $\mathrm{BIBD}(20, 5, 4)$ using 12 colors in Figure 4.6.

$$M_i = \begin{array}{|c|c|c|c|}
\hline
(c_{4+4i})A^{17} & (c_{3+4i})A^{14} & (c_{2+4i})A^{12} & (c_{1+4i})A^{2} \\
\hline
(c_{1+4i})I & (c_{2+4i})A^{9} & 0 & (c_{3+4i})A^{4} + (c_{4+4i})A^{10} \\
\hline
(c_{3+4i})I & 0 & (c_{1+4i})A^{6} + (c_{4+4i})A^{15} & (c_{2+4i})A^{4} \\
\hline
0 & (c_{1+4i})A^{13} + (c_{4+4i})A^{18} & (c_{3+4i})A^{15} & (c_{6+4i})A^{8} \\
\hline
(c_{2+4i})A^{5} & (c_{5+4i})A^{14} + (c_{8+4i})A^{18} & (c_{7+4i})A^{11} & 0 \\
\hline
0 & (c_{6+4i})A^{13} & (c_{8+4i})A^{11} & (c_{5+2i})A^{8} + (c_{7+4i})A^{10} \\
\hline
(c_{6+4i})A^{5} & (c_{7+4i})A^{14} & (c_{5+4i})A^{3} & (c_{8+4i})A^{8} \\
\hline
(c_{5+4i})A^{5} & (c_{10+4i})A^{7} & (c_{11+4i})A^{11} & (c_{12+4i})A^{10} \\
\hline
0 & (c_{9+4i})A^{18} & (c_{10+4i})A^{12} + (c_{12+4i})A^{15} & (c_{11+4i})A^{10} \\
\hline
\begin{array}{c}(c_{7+4i})I + (c_{8+4i})A^{5} + \\ (c_{9+4i})A^{16} + (c_{10+4i})A^{17}\end{array} & 0 & 0 & 0 \\
\hline
\end{array}$$

Figure 4.6: Edge-Coloring Incidence Matrix of a $\mathrm{BIBD}(20, 5, 4)$.

Since M_i is semi-properly colored, by applying Theorem 2.12 we can properly color a $\mathrm{BIBD}(20, 5, \lambda)$.

✠

Lemma 4.7. *There exists a properly colored* $\mathrm{BIBD}(36, 5, \lambda)$ *for* $\lambda = 4k, k \in \mathbb{Z}^+, k \geq 3$.

Proof: Let A be the 36×36 circulant matrix whose first row is

$$[010000000000000000000000000000000000].$$

Let B be the first 18 rows of a 36×36 circulant matrix whose first row is

$$[010000000000000000000000000000000000].$$

Then the edge-coloring incidence matrix of a $\mathrm{BIBD}(36, 5, 4)$ using 12 colors in Figure 4.7.

Since M_i is semi-properly colored, by applying Theorem 2.12 we can properly color a $\mathrm{BIBD}(36, 5, \lambda)$.

✠

Lemma 4.8. *There exists a properly colored* $\mathrm{BIBD}(40, 5, \lambda)$ *for* $\lambda = 4k, k \in \mathbb{Z}^+, k \geq 3$.

Proof: Let A be the 39×39 circulant matrix whose first row is

$$[010000000000000000000000000000000000000].$$

Then the edge coloring incidence matrix of a $\mathrm{BIBD}(40, 5, 4)$ using 12 colors in Figure 4.8.

Since M_i is semi-properly colored, by applying Theorem 2.12 we can properly color a $\mathrm{BIBD}(40, 5, \lambda)$.

✠

$M_i =$

$(c_{1+4i})A^{34}$	$(c_{2+4i})A^{26}$	0	0	0	$(c_{3+4i})A^{32}$	$(c_{4+4i})I$
$(c_{2+4i})I$	0	$(c_{3+4i})A^{2}$	0	$(c_{4+4i})I$	$(c_{1+4i})A^{2}$	0
$(c_{3+4i})I$	$(c_{4+4i})A^{25}$	$(c_{1+4i})I$	0	0	0	$(c_{2+4i})A^{30}$
0	$(c_{3+4i})A^{26}$	0	$(c_{1+4i})A^{14} + (c_{4+4i})A^{18}$	0	$(c_{2+4i})I$	0
0	0	$(c_{2+4i})A^{2}$	$(c_{3+4i})A^{23}$	0	$(c_{4+4i})I$	$(c_{1+4i})A^{35}$
0	$(c_{1+4i})A^{6}$	0	0	$(c_{3+4i})A^{6}$	$(c_{6+4i})A^{2}$	$(c_{8+4i})I$
0	0	$+ (c_{4+4i})A^{26} + (c_{5+4i})A^{33}$	0	$(c_{2+4i})A^{34}$	$(c_{7+4i})A^{2}$	0
0	0	0	$(c_{2+4i})A^{18}$	$(c_{1+4i})A^{6} + (c_{5+4i})A^{27}$	0	$(c_{3+4i})A^{35}$
$(c_{4+4i})A^{9}$	0	0	$(c_{5+4i})A^{23}$	$(c_{6+4i})I$	0	$(c_{7+4i})I$
$(c_{5+4i})A^{33}$	$(c_{6+4i})I$	$(c_{7+4i})I$	$(c_{8+4i})A^{10}$	0	0	0
$(c_{6+4i})A^{9} + (c_{7+4i})A^{34}$	$(c_{5+4i})I$	0	0	0	$(c_{8+4i})A^{31}$	0
$(c_{8+4i})A^{9}$	0	$(c_{6+4i})A^{2}$	0	0	$(c_{5+4i})A^{32}$	$(c_{11+4i})A^{27}$
$(c_{9+4i})I$	0	0	$(c_{6+4i})I + (c_{7+4i})A^{23}$	$(c_{8+4i})A^{19}$	0	0
$(c_{10+4i})A^{23}$	$(c_{7+4i})I$	$(c_{8+4i})A^{33}$	$(c_{9+4i})A^{14}$	0	0	0
0	0	0	0	$(c_{7+4i})A^{6} + (c_{9+4i})A^{34}$	0	$(c_{10+4i})A^{15} + (c_{12+4i})A^{30}$
0	$(c_{8+4i})A^{6} + (c_{9+4i})A^{22}$	0	0	0	$(c_{11+4i})I$	$(c_{6+4i})A^{15}$
0	$(c_{10+4i})A^{6}$	$(c_{9+4i})I + (c_{11+4i})A^{19}$	$(c_{12+4i})I$	0	0	0
0	0	0	$(c_{10+4i})B^{36} + (c_{11+4i})B^{18}$	0	$(c_{9+4i})B^{2} + (c_{12+4i})B^{20}$	0

Figure 4.7: Edge-Coloring Incidence Matrix of a $\mathrm{BIBD}(36, 5, 4)$

$M_i =$

0	0	0	0	$(c_{1+4i})A^{28} + (c_{2+4i})A^{29}$	0	0	$(c_{3+4i})A^{16} + (c_{4+4i})A^{21}$
0	0	0	0	$(c_{3+4i})A^{27} + (c_{4+4i})A^{29}$	$(c_{1+4i})A^{8} + (c_{2+4i})A^{24}$	0	0
$(c_{2+4i})I$	$(c_{3+4i})A^{37}$	$(c_{4+4i})A^{38}$	0	$(c_{5+4i})A^{28}$	0	0	0
0	0	$(c_{1+4i})A^{3}$	0	$(c_{6+4i})A^{29}$	0	$(c_{3+4i})A^{5}$	$(c_{8+4i})A^{20}$
0	0	0	$(c_{3+4i})A^{9} + (c_{4+4i})A^{14}$	0	0	0	$(c_{1+4i})A^{20} + (c_{2+4i})A^{21}$
$(c_{1+4i})A^{36}$	$(c_{2+4i})A^{13}$	0	0	0	0	$(c_{4+4i})A^{11}$	$(c_{7+4i})A^{21}$
0	0	$(c_{2+4i})A^{3} + (c_{3+4i})A^{10}$	0	0	0	$(c_{1+4i})A + (c_{8+4i})A^{18}$	0
0	$(c_{1+4i})A^{34}$	0	$(c_{2+4i})A^{31}$	$(c_{7+4i})A^{25}$	$(c_{4+4i})A^{32}$	0	0
$(c_{3+4i})I$	$(c_{4+4i})A^{7}$	$(c_{5+4i})A^{19}$	$(c_{6+4i})A^{23}$	0	0	0	0
0	0	0	$(c_{1+4i})A^{14}$	$(c_{8+4i})A^{27}$	$(c_{3+4i})A^{32}$	$(c_{2+4i})A^{11}$	0
0	$(c_{5+4i})A^{37}$	$(c_{8+4i})A^{10}$	0	$(c_{10+4i})A^{28}$	0	$(c_{7+4i})A^{5}$	0
$(c_{4+4i})A^{12}$	$(c_{6+4i})A^{7}$	0	$(c_{7+4i})A^{4}$	$(c_{9+4i})A^{29}$	0	0	0
0	$(c_{7+4i})A^{26}$	0	0	0	$(c_{8+4i})A^{6}$	$(c_{5+4i})A^{18}$	$(c_{6+4i})A^{15}$
0	0	$(c_{6+4i})A^{10}$	$(c_{8+4i})A^{23}$	0	$(c_{7+4i})A^{22}$	0	$(c_{5+4i})A^{16}$
$(c_{7+4i})A^{12}$	$(c_{8+4i})A^{13}$	0	0	0	$(c_{9+4i})A^{8}$	$(c_{6+4i})A^{33}$	0
0	0	$(c_{7+4i})A^{19} + (c_{9+4i})A^{35}$	0	0	$(c_{5+4i})A^{22} + (c_{12+4i})A^{24}$	0	0
0	0	0	$(c_{11+4i})A^{9} + (c_{12+4i})A^{31}$	0	0	$(c_{9+4i})A^{11} + (c_{10+4i})A^{18}$	0
$(c_{8+4i})A^{30}$	$(c_{9+4i})A^{13}$	0	0	0	$(c_{6+4i})A^{24}$	0	$(c_{11+4i})A^{20}$
0	$(c_{11+4i})A^{26}$	$(c_{10+4i})A^{38}$	$(c_{9+4i})A^{23}$	0	0	0	$(c_{12+4i})A^{21}$
$(c_{5+4i})I + (c_{6+4i})A^{12} + (c_{11+4i})A^{30} + (c_{12+4i})A^{36}$	0	0	0	0	0	0	0

Figure 4.8: Edge-Coloring Incidence Matrix of a $\mathrm{BIBD}(40, 5, 4)$.

Lemma 4.9. *There exists a properly colored* $\mathrm{BIBD}(56, 5, \lambda)$ *for* $\lambda = 4k, k \in \mathbb{Z}^+, k \geq 3$.

Proof: Let A be the 56×56 circulant matrix whose first row is

$$[01000000000000000000000000000000000000000000000000000000].$$

Let B be the first 28 rows of a 56×56 circulant matrix whose first row is

$$[01000000000000000000000000000000000000000000000000000000].$$

Then the edge coloring incidence matrix of a $\mathrm{BIBD}(56, 5, 4)$ using 12 colors in Figures 4.9 and 4.10.

Since the concatenation of M_i and N_i is semi-properly colored, by Theorem 2.12 we can properly color a $\mathrm{BIBD}(56, 5, \lambda)$.

✠

Lemma 4.10. *There exists a properly colored* $\mathrm{BIBD}(15, 5, \lambda)$ *for* $\lambda = 2k, k \in \mathbb{Z}^+, k \geq 5$.

Proof: Note that the colors c_i and d_i are distinct for all i. We form an edge-coloring incidence matrix for a $\mathrm{BIBD}(15, 5, \lambda)$ using a $\mathrm{BIBD}(15, 5, 4)$ and a $\mathrm{BIBD}(15, 5, 6)$ as follows. Let A be the 14×14 circulant matrix whose first row is $[01000000000000]$. Let B be the first 7 rows of a 14×14 circulant matrix whose first row is $[01000000000000]$

Then the edge coloring incidence matrix of a $\mathrm{BIBD}(15, 5, 4)$ using 12 colors is given below.

$$M_i = \begin{array}{|c|c|c|}
\hline
(c_{1+4i})I & (c_{2+4i})I + (c_{3+4i})A^{9} + (c_{4+4i})A^{13} & 0 \\
\hline
(c_{2+4i})A^{13} & (c_{1+4i})I & (c_{3+4i})I + (c_{4+4i})A^{12} \\
\hline
(c_{3+4i})I & (c_{8+4i})A^{12} & (c_{1+4i})A^{7} + (c_{2+4i})A^{10} \\
\hline
0 & (c_{5+4i})A^{12} + (c_{6+4i})A^{13} & (c_{7+4i})I + (c_{8+4i})A^{4} \\
\hline
(c_{4+4i})A^{11} & (c_{7+4i})I + (c_{9+4i})A^{13} & (c_{6+4i})A^{12} \\
\hline
(c_{5+4i})A^{6} & (c_{10+4i})I & (c_{11+4i})A^{4} + (c_{12+4i})A^{10} \\
\hline
(c_{11+4i})B^{6} + (c_{12+4i})B^{13} & 0 & (c_{9+4i})I + (c_{10+4i})B^{7} \\
\hline
(c_{7+4i})I + (c_{8+4i})A^{6} + (c_{9+4i})A^{11} + (c_{10+4i})A^{13} & 0 & 0 \\
\hline
\end{array}$$

Let C be the 15×15 circulant matrix whose first row is $[010000000000000]$. Let $E_i^{(j)}$ be the following 15×3 matrix.

$$E_i^{(1)} = \begin{array}{|ccc|}
\hline
d_{7+6i} & 0 & 0 \\
0 & d_{7+6i} & 0 \\
0 & 0 & d_{7+6i} \\
d_{8+6i} & 0 & 0 \\
0 & d_{8+6i} & 0 \\
0 & 0 & d_{8+6i} \\
d_{9+6i} & 0 & 0 \\
0 & d_{9+6i} & 0 \\
0 & 0 & d_{9+6i} \\
d_{10+6i} & 0 & 0 \\
0 & d_{10+6i} & 0 \\
0 & 0 & d_{10+6i} \\
d_{11+6i} & 0 & 0 \\
0 & d_{11+6i} & 0 \\
0 & 0 & d_{11+6i} \\
\hline
\end{array}
\quad
E_i^{(2)} = \begin{array}{|ccc|}
\hline
d_{4+6i} & 0 & 0 \\
0 & d_{4+6i} & 0 \\
0 & 0 & d_{4+6i} \\
d_{12+6i} & 0 & 0 \\
0 & d_{12+6i} & 0 \\
0 & 0 & d_{12+6i} \\
d_{5+6i} & 0 & 0 \\
0 & d_{5+6i} & 0 \\
0 & 0 & d_{5+6i} \\
d_{6+6i} & 0 & 0 \\
0 & d_{6+6i} & 0 \\
0 & 0 & d_{6+6i} \\
d_{3+6i} & 0 & 0 \\
0 & d_{3+6i} & 0 \\
0 & 0 & d_{3+6i} \\
\hline
\end{array}$$

$M_i =$

0	$(c_{1+4i})A^{16} + (c_{2+4i})A^{17}$	0	$(c_{3+4i})A^{50}$	0
$(c_{1+4i})A^{47}$	$(c_{3+4i})A^{17}$	0	0	0
0	0	$(c_{1+4i})A^{36}$	0	$(c_{2+4i})A^{3} + (c_{3+4i})A^{14}$
0	0	0	0	0
0	0	$(c_{2+4i})A^{33} + (c_{3+4i})A^{41}$	0	0
0	0	0	$(c_{1+4i})I$	0
0	0	0	$(c_{2+4i})I$	0
0	0	$(c_{5+4i})A^{36} + (c_{6+4i})A^{41}$	0	$(c_{4+4i})A^{11}$
$(c_{2+4i})I + (c_{4+4i})A^{31}$	0	0	0	0
0	0	0	$(c_{4+4i})A^{10}$	$(c_{1+4i})I$
$(c_{3+4i})I$	0	0	0	$(c_{5+4i})A^{11} + (c_{6+4i})A^{14}$
0	0	0	0	0
0	0	$(c_{4+4i})A^{41}$	0	$(c_{7+4i})A^{3}$
$(c_{5+4i})A^{45}$	0	0	0	$(c_{8+4i})A^{14}$
0	$(c_{4+4i})A^{15}$	$(c_{7+4i})I$	0	0
$(c_{7+4i})A^{47}$	$(c_{5+4i})A^{16}$	0	$(c_{8+4i})A^{10}$	0
0	$(c_{6+4i})A^{17}$	0	$(c_{5+4i})A^{10} + (c_{7+4i})A^{27}$	0
0	$(c_{7+4i})A^{35}$	0	0	0
0	$(c_{8+4i})A^{35}$	0	0	0
0	$(c_{9+4i})A^{35}$	$(c_{8+4i})I$	0	0
0	$(c_{10+4i})I$	0	0	$(c_{9+4i})A^{11}$
$(c_{8+4i})A^{22}$	0	0	$(c_{6+4i})A^{49}$	0
$(c_{6+4i})A^{45}$	0	$(c_{11+4i})I$	$(c_{12+4i})A^{50}$	0
0	0	0	0	$(c_{11+4i})A^{14}$
$(c_{9+4i})I + (c_{10+4i})A^{47}$	0	0	0	0
0	0	0	0	0
0	0	0	$(c_{9+4i})A^{27}$	0
0	0	$(c_{9+4i})B^{56} + (c_{12+4i})B^{28}$	0	0

Figure 4.9: Edge-Coloring Incidence Matrix of a $\mathrm{BIBD}(56, 5, 4)$

$N_i =$

0	0	$(c_{4+4i})A^{10}$	0	0	0
$(c_{2+4i})I + (c_{4+4i})A^{54}$	0	0	0	0	0
0	0	0	0	$(c_{4+4i})A^{29}$	0
$(c_{1+4i})I + (c_{3+4i})A^{24}$	0	0	0	$(c_{2+4i})A^{47}$	$(c_{8+4i})I$
0	$(c_{4+4i})A^{5}$	$(c_{1+4i})A^{15}$	0	0	0
0	0	$(c_{2+4i})A^{15}$	$(c_{3+4i})I$	0	$(c_{4+4i})A^{52}$
0	$(c_{3+4i})A^{12}$	0	$(c_{1+4i})A^{43} + (c_{4+4i})A^{50}$	0	0
0	0	0	0	0	$(c_{3+4i})A^{8}$
0	0	$(c_{3+4i})A^{9}$	0	$(c_{1+4i})I$	0
0	0	$(c_{5+4i})A^{10}$	0	0	$(c_{2+4i})I$
0	$(c_{7+4i})I$	0	0	0	0
0	$(c_{1+4i})A^{12}$	$(c_{8+4i})I$	$(c_{2+4i})A^{36}$	0	$(c_{7+4i})A^{8}$
0	0	0	$(c_{6+4i})I$	$(c_{5+4i})I$	0
0	0	0	$(c_{7+4i})A^{50}$	$(c_{6+4i})A^{43}$	0
0	$(c_{5+4i})A^{45}$	$(c_{6+4i})A^{15}$	0	0	0
0	$(c_{2+4i})A^{5}$	0	0	0	0
0	0	0	0	$(c_{8+4i})A^{43}$	0
0	$(c_{8+4i})A^{30}$	0	0	$(c_{10+4i})A^{47}$	$(c_{1+4i})A^{8}$
0	0	0	$(c_{5+4i})A^{43}$	0	$(c_{6+4i})A^{27} + (c_{7+4i})A^{46}$
$(c_{5+4i})A^{20}$	0	0	$(c_{10+4i})I$	0	0
0	0	$(c_{12+4i})A^{9}$	0	$(c_{3+4i})A^{47}$	0
$(c_{7+4i})A^{20}$	0	$(c_{9+4i})A^{10}$	0	0	0
0	$(c_{10+4i})A^{12}$	0	0	0	0
$(c_{6+4i})A^{20} + (c_{8+4i})A^{24}$	0	0	$(c_{9+4i})A^{24}$	0	0
0	$(c_{11+4i})A^{30}$	0	0	0	$(c_{12+4i})A^{52}$
$(c_{9+4i})A^{24}$	$(c_{6+4i})I$	0	$(c_{8+4i})A^{50}$	$(c_{7+4i})A^{26}$	0
0	0	$(c_{10+4i})A^{15}$	0	$(c_{12+4i})I$	$(c_{11+4i})A^{27}$
$(c_{10+4i})B^{24} + (c_{11+4i})B^{52}$	0	0	0	0	0

Figure 4.10: Edge-Coloring Incidence Matrix of a BIBD$(56, 5, 4)$

Let $F_i^{(j)}$ for $j = 1, 2, 3, 4, 5, 6$ be the following 15×15 matrix.

$$F_i^{(1)} = \begin{bmatrix}
d_{7+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{12+6i} & 0 & 0 \\
0 & d_{7+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{12+6i} & 0 \\
0 & 0 & d_{7+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{12+6i} \\
d_{8+6i} & 0 & 0 & d_{12+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & d_{8+6i} & 0 & 0 & d_{12+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d_{8+6i} & 0 & 0 & d_{12+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & d_{8+6i} & 0 & 0 & d_{7+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_{8+6i} & 0 & 0 & d_{7+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_{8+6i} & 0 & 0 & d_{7+6i} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_{8+6i} & 0 & 0 & d_{7+6i} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{8+6i} & 0 & 0 & d_{7+6i} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{8+6i} & 0 & 0 & d_{7+6i} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{8+6i} & 0 & 0 & d_{7+6i} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{8+6i} & 0 & 0 & d_{7+6i} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{8+6i} & 0 & 0 & d_{7+6i}
\end{bmatrix}$$

$$F_i^{(2)} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{9+6i} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{9+6i} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{9+6i} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{9+6i} \\
d_{9+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & d_{9+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d_{9+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & d_{9+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_{9+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_{9+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_{9+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{9+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{12+6i} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{12+6i} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{12+6i} & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$F_i^{(3)} = \begin{bmatrix}
d_{10+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{11+6i} & 0 & 0 \\
0 & d_{10+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{11+6i} & 0 \\
0 & 0 & d_{10+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{11+6i} \\
d_{11+6i} & 0 & 0 & d_{10+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & d_{11+6i} & 0 & 0 & d_{10+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d_{11+6i} & 0 & 0 & d_{10+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & d_{12+6i} & 0 & 0 & d_{10+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_{12+6i} & 0 & 0 & d_{10+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_{12+6i} & 0 & 0 & d_{10+6i} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_{11+6i} & 0 & 0 & d_{12+6i} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{11+6i} & 0 & 0 & d_{12+6i} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{11+6i} & 0 & 0 & d_{12+6i} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{11+6i} & 0 & 0 & d_{10+6i} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{11+6i} & 0 & 0 & d_{10+6i} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{11+6i} & 0 & 0 & d_{10+6i}
\end{bmatrix}$$

$$F_i^{(4)} = \begin{bmatrix}
d_{12+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & d_{12+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d_{12+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & d_{4+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_{4+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d_{4+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d_{4+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{4+6i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{4+6i} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{4+6i} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{4+6i} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{4+6i} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{4+6i} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{4+6i} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{4+6i}
\end{bmatrix}$$

$$F_i^{(5)} = \begin{array}{|ccccccccccccccc|}\hline
0&0&0&0&0&0&0&0&0&0&0&0&0&d_{5+6i}&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&d_{5+6i}\\
d_{5+6i}&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&d_{5+6i}&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&d_{5+6i}&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&d_{5+6i}&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&d_{12+6i}&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&d_{12+6i}&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&d_{12+6i}&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&d_{5+6i}&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&d_{5+6i}&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&d_{5+6i}&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&d_{5+6i}&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&d_{5+6i}&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&d_{5+6i}&0&0\\\hline
\end{array}$$

$$F_i^{(6)} = \begin{array}{|ccccccccccccccc|}\hline
d_{3+6i}&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&d_{3+6i}&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&d_{3+6i}&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&d_{3+6i}&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&d_{3+6i}&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&d_{3+6i}&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&d_{3+6i}&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&d_{3+6i}&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&d_{3+6i}&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&d_{3+6i}&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&d_{3+6i}&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&d_{3+6i}&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&d_{12+6i}&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&d_{12+6i}&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&d_{12+6i}\\\hline
\end{array}$$

Thus, the edge-coloring incidence matrix of a $\mathrm{BIBD}(15,5,6)$ using 12 colors is given Figure 4.11.

Suppose $k \equiv 0 \pmod 2$. To form a properly colored $\mathrm{BIBD}(15,5,\lambda)$ with $\lambda = 2k, k \in \mathbb{Z}^+, k \geq 5$, we simply repeat the blocks of the $\mathrm{BIBD}(15,5,4)$ $\frac{k}{2}$ times and follow the same coloring scheme with different colors. The subscripts of the colors are all computed $\pmod \lambda$ where we identify c_0 with c_λ. Thus the edge-coloring incidence matrix of a properly colored $\mathrm{BIBD}(15,5,\lambda)$ for $k \equiv 0 \pmod 2$ can be given in terms of the M_i as

$$M = \begin{array}{|c|c|c|c|}\hline M_0 & M_1 & \cdots & M_{\frac{k}{2}-1} \\\hline\end{array}\,.$$

Suppose $k \equiv 1 \pmod 2$. To form a properly colored $\mathrm{BIBD}(15,5,\lambda)$ with $\lambda = 2k, k \in \mathbb{Z}^+, k \geq 13$, we simply repeat the blocks of the $\mathrm{BIBD}(15,5,4)$ $\frac{\lambda-6}{4}$ times joined with the blocks of a $\mathrm{BIBD}(15,5,6)$. We know we can properly color a $\mathrm{BIBD}(15,5,10)$ by Theorem 2.2. The colors used on the $\mathrm{BIBD}(15,5,4)$ and $\mathrm{BIBD}(15,5,6)$ will be distinct from each other requiring $\lambda \geq 26$. The subscripts of the colors are all computed $\pmod \lambda$ where we identify c_0 with c_λ. Thus the edge coloring incidence matrix of a properly colored $\mathrm{BIBD}(15,5,\lambda)$ for $13 \leq k \equiv 1 \pmod 2$ can be given in terms of the M_i as

$$M = \begin{array}{|c|c|c|c|c|}\hline M_{\alpha_0} & M_0 & M_1 & \cdots & M_{((\lambda-6)/4)-1} \\\hline\end{array}\,.$$

This leaves three cases where $\lambda = 14, 18, 22$. Suppose $\lambda = 14$. We form an edge-coloring incidence matrix using 14 colors for a $\mathrm{BIBD}(15,5,14)$. Let H be a 15×15 circulant matrix whose first row is [010000000000000]. Let $G^{(j)}$ for $j = 1, 2$ be the following 15×3 matrices.

$M_{\alpha_i} =$

$(d_{1+6i})I + (d_{2+6i})C^{13} + (d_{3+6i})C^{14}$	$(d_{4+6i})C^{8}$	$(d_{5+6i})C^{6}$	$(d_{6+6i})C^{5}$	0
$(d_{5+6i})I + (d_{6+6i})C^{14}$	$(d_{1+6i})I + (d_{2+6i})C^{13}$	$(d_{3+6i})I + (d_{4+6i})C^{13}$	0	0
$F_i^{(1)}$	$F_i^{(2)}$	0	$F_i^{(3)}$	$E_i^{(1)}$
$(d_{10+6i})C^{13}$	$(d_{3+6i})I + (d_{11+6i})C^{11}$	$(d_{6+6i})I$	$(d_{1+6i})C^{4} + (d_{2+6i})C^{9}$	0
$(d_{9+6i})C^{14}$	$(d_{6+6i})C^{13}$	$(d_{7+6i})C^{5} + (d_{8+6i})C^{11}$	$(d_{4+6i})C^{5} + (d_{5+6i})C^{9}$	0
$F_i^{(4)}$	$F_i^{(5)}$	$(d_{1+6i})C^{6} + (d_{2+6i})C^{11}$	$F_i^{(6)}$	$E_i^{(2)}$
0	$(d_{7+6i})I + (d_{8+6i})C^{7}$	$(d_{11+6i})C^{5} + (d_{12+6i})C^{13}$	$(d_{9+6i})C^{4} + (d_{10+6i})C^{12}$	0

Figure 4.11: Edge-coloring Incidence Matrix of a $\mathrm{BIBD}(15, 5, 6)$

$$G^{(1)} = \begin{array}{|ccc|}\hline c_1 & 0 & 0 \\ 0 & c_1 & 0 \\ 0 & 0 & c_1 \\ c_3 & 0 & 0 \\ 0 & c_3 & 0 \\ 0 & 0 & c_3 \\ c_{12} & 0 & 0 \\ 0 & c_{12} & 0 \\ 0 & 0 & c_{12} \\ c_{13} & 0 & 0 \\ 0 & c_{13} & 0 \\ 0 & 0 & c_{13} \\ c_{14} & 0 & 0 \\ 0 & c_{14} & 0 \\ 0 & 0 & c_{14} \\ \hline \end{array} \quad G^{(2)} = \begin{array}{|ccc|}\hline c_4 & 0 & 0 \\ 0 & c_4 & 0 \\ 0 & 0 & c_4 \\ c_5 & 0 & 0 \\ 0 & c_5 & 0 \\ 0 & 0 & c_5 \\ c_6 & 0 & 0 \\ 0 & c_6 & 0 \\ 0 & 0 & c_6 \\ c_8 & 0 & 0 \\ 0 & c_8 & 0 \\ 0 & 0 & c_8 \\ c_9 & 0 & 0 \\ 0 & c_9 & 0 \\ 0 & 0 & c_9 \\ \hline \end{array}$$

Let $G^{(j)}$ for $j = 3, 4$ be the following 15×15 matrices.

$$G^{(3)} = \begin{array}{|ccccccccccccccc|}\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_3 \\
c_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{14} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{14} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 \\ \hline
\end{array}$$

$$G^{(4)} = \boxed{\begin{array}{ccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_9 \\
c_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_8 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_8 & 0
\end{array}}$$

Let $G^{(j)}$ for $j = 5, 6$ be the following 15×9 matrices.

$$G^{(5)} = \boxed{\begin{array}{ccccccccc}
c_{14} & 0 & 0 & c_{13} & 0 & 0 & c_{12} & 0 & 0 \\
0 & c_{14} & 0 & 0 & c_{13} & 0 & 0 & c_{12} & 0 \\
0 & 0 & c_{14} & 0 & 0 & c_{13} & 0 & 0 & c_{12} \\
c_1 & 0 & 0 & c_{14} & 0 & 0 & c_{13} & 0 & 0 \\
0 & c_1 & 0 & 0 & c_{14} & 0 & 0 & c_{13} & 0 \\
0 & 0 & c_1 & 0 & 0 & c_{14} & 0 & 0 & c_{13} \\
c_3 & 0 & 0 & c_1 & 0 & 0 & c_{14} & 0 & 0 \\
0 & c_3 & 0 & 0 & c_1 & 0 & 0 & c_{14} & 0 \\
0 & 0 & c_3 & 0 & 0 & c_1 & 0 & 0 & c_{14} \\
c_{12} & 0 & 0 & c_3 & 0 & 0 & c_1 & 0 & 0 \\
0 & c_{12} & 0 & 0 & c_3 & 0 & 0 & c_1 & 0 \\
0 & 0 & c_{12} & 0 & 0 & c_3 & 0 & 0 & c_1 \\
c_{13} & 0 & 0 & c_{12} & 0 & 0 & c_3 & 0 & 0 \\
0 & c_{13} & 0 & 0 & c_{12} & 0 & 0 & c_3 & 0 \\
0 & 0 & c_{13} & 0 & 0 & c_{12} & 0 & 0 & c_3
\end{array}}$$

$$G^{(6)} = \begin{array}{|ccccccccc|}
\hline
c_8 & 0 & 0 & c_6 & 0 & 0 & c_5 & 0 & 0 \\
0 & c_8 & 0 & 0 & c_6 & 0 & 0 & c_5 & 0 \\
0 & 0 & c_8 & 0 & 0 & c_6 & 0 & 0 & c_5 \\
c_9 & 0 & 0 & c_8 & 0 & 0 & c_6 & 0 & 0 \\
0 & c_9 & 0 & 0 & c_8 & 0 & 0 & c_6 & 0 \\
0 & 0 & c_9 & 0 & 0 & c_8 & 0 & 0 & c_6 \\
c_4 & 0 & 0 & c_9 & 0 & 0 & c_8 & 0 & 0 \\
0 & c_4 & 0 & 0 & c_9 & 0 & 0 & c_8 & 0 \\
0 & 0 & c_4 & 0 & 0 & c_9 & 0 & 0 & c_8 \\
c_5 & 0 & 0 & c_4 & 0 & 0 & c_9 & 0 & 0 \\
0 & c_5 & 0 & 0 & c_4 & 0 & 0 & c_9 & 0 \\
0 & 0 & c_5 & 0 & 0 & c_4 & 0 & 0 & c_9 \\
c_6 & 0 & 0 & c_5 & 0 & 0 & c_4 & 0 & 0 \\
0 & c_6 & 0 & 0 & c_5 & 0 & 0 & c_4 & 0 \\
0 & 0 & c_6 & 0 & 0 & c_5 & 0 & 0 & c_4 \\
\hline
\end{array}$$

Thus the edge-coloring incidence matrix of a properly colored $\mathrm{BIBD}(15, 5, 14)$ is given by concatenating $N^{(0)}$ and $N^{(1)}$ from Figure 4.12.

Now suppose $\lambda = 18$. All subscripts are computed (mod 18). We form the properly colored $\mathrm{BIBD}(15, 5, 18)$ given in terms of the M_{α_i} as

$$M = \begin{array}{|c|c|c|}\hline M_{\alpha_0} & M_{\alpha_1} & M_{\alpha_2} \\ \hline \end{array}\,.$$

Now suppose $\lambda = 22$. Let $M_{0'}$ be the properly colored $\mathrm{BIBD}(15, 5, 10)$ on 10 colors. Then

$$M = \begin{array}{|c|c|c|c|}\hline M_{0'} & M_0 & M_1 & M_2 \\ \hline \end{array}\,.$$

✠

Lemma 4.11. *There exists a properly colored* $\mathrm{BIBD}(35, 5, \lambda)$ *for* $\lambda = 2k, k \in \mathbb{Z}^+, k \geq 5$.

Proof: Let A be the 35×35 circulant matrix whose first row is

[01000000000000000000000000000000000].

Let $A_i^{(j)}$ be the following 35×28 matrices for $j = 1, 2$ in Figures 4.13 and 4.14.

$N^{(0)} =$

$(c_1)I + (c_2)H^{13} + (c_3)H^{14}$	$(c_4)H^6$	$(c_5)I$	$(c_6)H^5$	0
$(c_7)I + (c_8)H^{12} + (c_9)H^{14}$	$(c_{10})I + (c_{11})H^{13}$	$(c_{12})H^7$	0	0
$(c_4)I + (c_5)H^{13}$	0	$(c_6)H^{14}$	$(c_7)I + (c_8)H^{12}$	$G^{(1)}$
$(c_6)H^{14}$	$(c_7)I$	$(c_1)I + (c_{11})H^{11}$	$(c_3)H^4 + (c_4)H^9$	0
$(c_{10})I$	$(c_1)H^5 + (c_2)H^{11}$	$(c_3)H^5$	$(c_9)H^5 + (c_{11})H^9$	0
0	$(c_3)H^6 + (c_{12})H^{11}$	$(c_{13})H^5 + (c_{14})H^{11}$	$(c_1)I$	$G^{(2)}$
0	$(c_5)H^5 + (c_6)H^{13}$	$(c_7)H^7 + (c_8)H^{14}$	$(c_{12})H^4 + (c_{13})H^{12}$	0

$N^{(1)} =$

$(c_7)I + (c_8)H^{10} + (c_9)H^{14}$	$(c_{10})I + (c_{11})H^{14}$	$(c_{12})I + (c_{13})H^{14}$	$(c_{14})H^{13}$	0	0
$(c_6)I$	$(c_1)I + (c_{14})H^{10}$	$(c_2)I + (c_3)H^9$	$(c_4)I$	$(c_5)I + (c_{13})H^{13}$	0
$G^{(3)}$	$(c_2)H^{13}$	0	$(c_9)I + (c_{10})H^8$	$(c_{11})H^{11}$	$G^{(5)}$
$(c_2)H^{13} + (c_{10})H^{14}$	$(c_{13})H^{14}$	$(c_{14})H^{13}$	$(c_5)H^{12}$	$(c_8)I + (c_9)H^4 + (c_{12})H^8$	0
$(c_4)I + (c_5)H^{14}$	$(c_7)I + (c_{12})H^{13}$	$(c_8)H^{14}$	$(c_6)H^5 + (c_{13})H^{13}$	$(c_{14})H^{13}$	0
$(c_{11})I$	$G^{(4)}$	$(c_7)I + (c_{10})H^{13}$	0	$(c_2)H^4$	$G^{(6)}$
0	$(c_3)I$	$(c_1)H^7 + (c_9)H^{14}$	$(c_2)I + (c_4)H^5 + (c_{11})H^{12}$	$(c_{10})I + (c_{14})H^{11}$	0

Figure 4.12: Edge-Coloring Incidence Matrix of a BIBD$(15, 5, 14)$.

$$
A_i^{(1)} = \left[\begin{array}{cccccccccccccc}
c_{1+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{2+2i} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_{1+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{2+2i} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_{1+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{2+2i} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_{1+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{2+2i} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_{1+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{2+2i} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_{1+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{2+2i} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{1+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{2+2i} \\
c_{2+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{3+2i} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_{2+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{3+2i} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_{2+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{3+2i} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_{2+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{3+2i} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_{2+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{3+2i} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_{2+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{3+2i} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{2+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{3+2i} \\
c_{3+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{4+2i} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_{3+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{4+2i} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_{3+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{4+2i} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_{3+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{4+2i} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_{3+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{4+2i} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_{3+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{4+2i} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{3+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{4+2i} \\
c_{4+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{5+2i} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_{4+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{5+2i} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_{4+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{5+2i} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_{4+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{5+2i} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_{4+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{5+2i} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_{4+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{5+2i} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{4+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{5+2i} \\
c_{5+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{1+2i} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_{5+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{1+2i} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_{5+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{1+2i} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_{5+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{1+2i} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_{5+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{1+2i} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_{5+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{1+2i} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{5+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{1+2i}
\end{array}\right]
$$

Figure 4.13: 35×28 colored matrix

$$
A_i^{(2)} = \begin{array}{|cccccccccccccc|}
\hline
c_{6+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{7+2i} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_{6+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{7+2i} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_{6+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{7+2i} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_{6+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{7+2i} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_{6+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{7+2i} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_{6+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{7+2i} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{6+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{7+2i} \\
c_{7+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{8+2i} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_{7+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{8+2i} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_{7+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{8+2i} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_{7+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{8+2i} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_{7+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{8+2i} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_{7+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{8+2i} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{7+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{8+2i} \\
c_{8+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{9+2i} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_{8+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{9+2i} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_{8+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{9+2i} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_{8+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{9+2i} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_{8+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{9+2i} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_{8+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{9+2i} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{8+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{9+2i} \\
c_{9+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{10+2i} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_{9+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{10+2i} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_{9+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{10+2i} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_{9+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{10+2i} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_{9+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{10+2i} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_{9+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{10+2i} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{9+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{10+2i} \\
c_{10+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{6+2i} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_{10+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{6+2i} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_{10+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{6+2i} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_{10+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{6+2i} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_{10+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{6+2i} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_{10+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{6+2i} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_{10+2i} & 0 & 0 & 0 & 0 & 0 & 0 & c_{6+2i} \\
\hline
\end{array}
$$

Figure 4.14: 35×28 colored matrix

Then the edge coloring incidence matrix of a $\mathrm{BIBD}(35, 5, 2)$ using 10 colors is given in Figure 4.15.

Since M_i is semi-properly colored, by Theorem 2.12 we can properly color a $\mathrm{BIBD}(35, 5, \lambda)$.

✠

Lemma 4.12. *There exists a properly colored* $\mathrm{BIBD}(16, 5, \lambda)$ *for* $\lambda = 4k, k \in \mathbb{Z}^+, k \geq 3$.

Proof: Let A be a 16×16 circulant matrix whose first row is $[0100000000000000]$. Then the edge coloring incidence matrix of a $\mathrm{BIBD}(16, 5, 4)$ using 12 colors is given below. Let B be the first 8 rows of a 16×16 circulant matrix whose first row is $[0100000000000000]$. Then the edge-coloring incidence matrix of a $\mathrm{BIBD}(16, 5, 4)$ using 12 colors is given in Figure 4.16

Since M_i is semi-properly colored, by Theorem 2.12 we can properly color a $\mathrm{BIBD}(16, 5, \lambda)$.

✠

$M_i =$

$(c_{1+2i})A^{23}$	0	$(c_{2+2i})A^{3}$	0
$(c_{2+2i})I$	0	$(c_{1+2i})A^{2}$	0
0	$(c_{3+2i})I$	$(c_{4+2i})A^{3}$	0
$(c_{3+2i})A^{27}$	$(c_{4+2i})A^{17}$	0	0
$(c_{4+2i})A^{27}$	$(c_{5+2i})A^{13}$	0	0
$(c_{5+2i})A^{33}$	0	$(c_{8+2i})A^{18}$	0
0	0	0	$A^{(1)}$
$(c_{7+2i})I$	$(c_{8+2i})A^{8}$	0	0
0	$(c_{2+2i})A^{17}$	$(c_{3+2i})A^{12}$	0
$(c_{10+2i})A^{33}$	0	$(c_{7+2i})A^{12}$	0
$(c_{9+2i})A^{33}$	$(c_{6+2i})A^{8}$	0	0
$(c_{6+2i})I$	0	$(c_{5+2i})A^{12}$	0
$(c_{8+2i})I$	$(c_{1+2i})A^{13}$	0	0
0	0	0	$A^{(2)}$
0	$(c_{9+2i})A^{32}$	$(c_{10+2i})A^{18}$	0
0	$(c_{7+2i})A^{13}$	$(c_{6+2i})A^{18}$	0
0	$(c_{10+2i})A^{17}$	$(c_{9+2i})I$	0

Figure 4.15: Edge-Coloring Incidence Matrix of a $\mathrm{BIBD}(35, 5, 2)$.

$M_i =$

0	$(c_{1+4i})I$	$(c_{2+4i})I + (c_{3+4i})A^{14} + (c_{4+4i})A^{15}$
$(c_{1+4i})I$	$(c_{2+4i})A^{8}$	$(c_{7+4i})I + (c_{8+4i})A^{15}$
$(c_{2+4i})A^{11} + (c_{3+4i})A^{14}$	$(c_{4+4i})A^{11}$	$(c_{1+4i})I$
$(c_{4+4i})A^{4} + (c_{5+4i})A^{8}$	$(c_{7+4i})A^{15}$	$(c_{6+4i})H^{13}$
$(c_{7+4i})I$	$(c_{6+4i})I + (c_{8+4i})A^{11}$	$(c_{5+4i})A^{14}$
$(c_{6+4i})A^{4} + (c_{8+4i})A^{1}4$	$(c_{5+4i})A^{6}$	$(c_{11+4i})A^{15}$
$(c_{9+4i})A^{11}$	$(c_{10+4i})A^{6} + (c_{11+4i})A^{15}$	$(c_{12+4i})I$
$(c_{10+4i})I + (c_{11+4i})B^{8}$	$(c_{9+4i})I + (c_{12+4i})B^{8}$	0

Figure 4.16: Edge-Coloring Incidence Matrix of a $\mathrm{BIBD}(16, 5, 4)$

Definition 4.13. A $g-$*circulant* matrix is an $n \times n$ square matrix of complex numbers, in which each row (except the first) is obtained from the preceding row by shifting the elements cyclically g columns to the right.

Example 4.14. *A* $3-$*circulant matrix of order 7 whose first row is* $[1000000]$.

1	0	0	0	0	0	0
0	0	0	1	0	0	0
0	0	0	0	0	0	1
0	0	1	0	0	0	0
0	0	0	0	0	1	0
0	1	0	0	0	0	0
0	0	0	0	1	0	0

Lemma 4.15. *There exists a properly colored* $\mathrm{BIBD}(50, 5, \lambda)$ *for* $\lambda = 4k, k \in \mathbb{Z}^+, k \geq 3$.

Proof: Let A be the 49×49 circulant matrix whose first row is

$$[0100000000000000000000000000000000000000000000000].$$

Let B_1 be the 49×49 25−circulant matrix whose first row is

$$[1000000000000000000000000000000000000000000000000].$$

Let B_2 be the 49×49 37−circulant matrix whose first row is

$$[1000000000000000000000000000000000000000000000000].$$

Then the edge-coloring incidence matrix of a $\mathrm{BIBD}(50, 5, 4)$ using 12 colors is given in Figures 4.17 and 4.18.

Since M_i concatenated with N_i is semi-properly colored, by Theorem 2.12 we can properly color a $\mathrm{BIBD}(50, 5, \lambda)$.

✠

Lemma 4.16. *There exists a properly colored* $\mathrm{BIBD}(60, 5, \lambda)$ *for* $\lambda = 4k, k \in \mathbb{Z}^+, k \geq 3$.

Proof: Let A be the 59×59 circulant matrix whose first row is

$$[01000000000000000000000000000000000000000000000000000000000].$$

Let B_1 be the 59×59 6−circulant matrix whose first row is

$$[10000000000000000000000000000000000000000000000000000000000].$$

Let B_2 be the 59×59 12−circulant matrix whose first row is

$$[10000000000000000000000000000000000000000000000000000000000].$$

Let B_3 be the 59×59 15−circulant matrix whose first row is

$$[10000000000000000000000000000000000000000000000000000000000].$$

Let B_4 be the 59×59 24−circulant matrix whose first row is

$$[10000000000000000000000000000000000000000000000000000000000].$$

Let B_5 be the 59×59 30−circulant matrix whose first row is

$$[10000000000000000000000000000000000000000000000000000000000].$$

Let B_6 be the 59×59 37−circulant matrix whose first row is

$$[10000000000000000000000000000000000000000000000000000000000].$$

$M_i =$

0	$(c_{1+4i})A + (c_{2+4i})A^2$	0	0	0
0	$(c_{4+4i})A^2$	$(c_{1+4i})B_1A + (c_{2+4i})B_1A^2$	0	0
0	0	0	0	$(c_{2+4i})A^7$
0	0	$(c_{3+4i})B_1A^2$	$(c_{1+4i})B_2A + (c_{2+4i})B_1A^2$	$(c_{4+4i})A^4$
$(c_{1+4i})I$	0	0	0	$(c_{3+4i})I$
0	0	0	0	0
0	$(c_{3+4i})A^{41}$	0	0	$(c_{1+4i})A^7$
0	$(c_{5+4i})I$	0	$(c_{4+4i})B_2A^2$	0
$(c_{4+4i})A^{44}$	$(c_{6+4i})A$	0	$(c_{3+4i})B_2A^{41}$	$(c_{5+4i})A^4$
$(c_{3+4i})A^{10}$	$(c_{8+4i})A^2$	0	0	0
0	0	0	$(c_{7+4i})B_2$	0
0	0	0	0	$(c_{6+4i})A^7$
0	0	0	$(c_{5+4i})B_2A^{41}$	0
$(c_{2+4i})I$	0	$(c_{4+4i})B_1A^{41}$	0	$(c_{7+4i})A^{44}$
$(c_{5+4i})A^{10}$	$(c_{7+4i})I$	$(c_{6+4i})B_1A^{34}$	$(c_{8+4i})B_2A$	0
0	$(c_{10+4i})A$	$(c_{7+4i})B_2$	0	0
0	$(c_{9+4i})A^2$	$(c_{8+4i})B_1A^{34}$	$(c_{6+4i})B_2A^{41}$	0
0	0	$(c_{5+4i})B_1A$	0	0
0	0	$(c_{9+4i})B_1A^{34}$	$(c_{10+4i})B_2A^2$	$(c_{8+4i})I$
0	0	$(c_{10+4i})B_1A^2$	0	0
0	0	0	$(c_{9+4i})B_2A^{34}$	0
0	0	0	0	0
0	0	0	0	$(c_{9+4i})A^4 + (c_{11+4i})A^{30}$
$(c_{11+4i})A^{10}$	0	0	0	0
$(c_{6+4i})I + (c_{7+4i})A^{10}+$ $(c_{8+4i})A^{35} + (c_{9+4i})A^{44}$	0	0	0	0

Figure 4.17: First of two edge-coloring incidence matrices of a $\mathrm{BIBD}(50, 5, 4)$

$$N_i = \begin{array}{|c|c|c|c|c|}
\hline
0 & (c_{3+4i})B_2A^{44} & (c_{4+4i})I & 0 & 0 \\ \hline
0 & 0 & (c_{3+4i})A^{48} & 0 & 0 \\ \hline
(c_{3+4i})B_1A^7 + (c_{4+4i})B_1A^{30} & 0 & (c_{1+4i})I & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & (c_{2+4i})I & (c_{4+4i})I \\ \hline
(c_{1+4i})B_1A^7 & (c_{2+4i})B_2A^7 + (c_{4+4i})B_2A^{30} & 0 & 0 & (c_{3+4i})A^6 \\ \hline
0 & (c_{8+4i})B_2A^{44} & 0 & (c_{6+4i})A^{44} & 0 \\ \hline
(c_{7+4i})B_1A^4 & 0 & (c_{2+4i})A^{26} & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 \\ \hline
(c_{2+4i})B_1 & 0 & 0 & 0 & (c_{1+4i})A^{32} \\ \hline
(c_{5+4i})B_1A^{30} & 0 & 0 & (c_{3+4i})A^{24} & (c_{2+4i})A^6 \\ \hline
0 & (c_{1+4i})B_2A^7 & 0 & (c_{4+4i})I & (c_{7+4i})A^{44} \\ \hline
0 & (c_{6+4i})B_2A^{44} & 0 & (c_{7+4i})A^{13} + (c_{8+4i})A^{37} & 0 \\ \hline
(c_{9+4i})B_1A^7 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 \\ \hline
0 & (c_{5+4i})B_2A^4 & 0 & 0 & (c_{8+4i})A^{22} \\ \hline
0 & 0 & 0 & 0 & (c_{11+4i})I \\ \hline
(c_{6+4i})B_1A^4 & 0 & (c_{8+4i})A^{18} & (c_{11+4i})A^{13} & 0 \\ \hline
0 & 0 & (c_{7+4i})A^{18} & 0 & 0 \\ \hline
0 & (c_{7+4i})B_2 & (c_{5+4i})A^{46} & (c_{12+4i})A^{44} & 0 \\ \hline
(c_{8+4i})B_1A^{30} & (c_{10+4i})B_2 & (c_{11+4i})A^{18} & 0 & 0 \\ \hline
0 & (c_{11+4i})B_2A^{30} & (c_{10+4i})A^{48} & 0 & (c_{9+4i})A^{22} + (c_{12+4i})A^{44} \\ \hline
0 & 0 & (c_{12+4i})I & 0 & (c_{10+4i})A^6 \\ \hline
(c_{12+4i})B_1A^7 & 0 & 0 & (c_{5+4i})A^{24} + (c_{10+4i})A^{37} & 0 \\ \hline
0 & 0 & 0 & 0 & 0 \\ \hline
\end{array}$$

Figure 4.18: Second of two edge-coloring incidence matrices of a $\mathrm{BIBD}(50, 5, 4)$

Let B_7 be the 59×59 $48-$circulant matrix whose first row is

$$[10000000000000000000000000000000000000000000000000000000000].$$

Then the edge coloring incidence matrix of a $\mathrm{BIBD}(60, 5, 4)$ using 12 colors is given in Figures 4.19 and 4.20.

Since M_i concatenated with N_i is semi-properly colored, by Theorem 2.12 we can properly color a $\mathrm{BIBD}(60, 5, \lambda)$.

✠

Because the process of checking each individual lemma for a correct direct construction is difficult, a program was built for the specific purpose of checking the small cases. Along with this is a csv (comma separated variables) and a tsv (tab separated variables) file of the edge-colored incidence matrix which are semi-properly colored. The program called *bibdchecker.cpp* is used to verify the design is a semi-properly colored BIBD. *Mathematica* was used to generate the properly colored versions of the BIBDs in csv and tsv form. Go to

`http://www.mathlab.mtu.edu/~msjukuri/Data.html`

for the program and corresponding csv and tsv files.

4.2 Main Results

Lemma 4.17. *If there exists a* $5-\mathrm{GDD}(a_1^{b_1} a_2^{b_2} \cdots a_x^{b_x})$*, and a properly colored* $\mathrm{BIBD}(a_i, 5, \lambda)$ *for all* $i = 1, 2, \ldots, x$*, then there exists a properly colored*

$$\mathrm{BIBD}\left(\sum_{i=1}^{x} a_i b_i, 5, \lambda\right).$$

Proof: Repeat each of the blocks in a $5 - \mathrm{GDD}(a_1^{b_1} a_2^{b_2} \cdots a_x^{b_x})$ λ times. For each block, we must color each edge a different color using the colors $c_i \in \{c_1, \ldots, c_\lambda\}$. Each corresponding edge in the λ copies of the blocks must also be a different color. So we color the edges in the λ copies of each block as follows. Form a $10 \times \lambda$ matrix. The rows of the matrix will be indexed by the 10 edges of K_5, and the columns will be indexed by the λ copies of the block. The entries of the matrix will be the first 10 rows of an $\mathrm{LS}(\lambda)$. Now the only pairs that have not been covered are the pairs which lie within the groups. So we place a properly colored $\mathrm{BIBD}(a_i, 5, \lambda)$ on each group for all $i = 1, 2, \ldots, x$. This forms a properly colored $\mathrm{BIBD}\left(\sum_{i=1}^{x} a_i b_i, 5, \lambda\right)$. ✠

$M_i =$

0	$(c_{1+4i})I$	0	0	0	$(c_{2+4i})B_7A^{55}$
0	0	$(c_{1+4i})B_5$	0	0	0
0	$(c_{3+4i})A^{58}$	0	0	$(c_{4+4i})B_6$	0
0	$(c_{2+4i})I$	0	$(c_{1+4i})B_3$	0	0
0	0	0	0	$(c_{3+4i})B_6A^7$	$(c_{4+4i})B_7$
0	$(c_{4+4i})A^{55}$	$(c_{2+4i})B_5A^{58}$	0	0	$(c_{3+4i})B_7$
0	$(c_{5+4i})A^7$	0	0	0	0
0	$(c_{6+4i})A^7$	$(c_{3+4i})B_5$	0	$(c_{1+4i})B_6$	0
$(c_{1+4i})A^{45}$	$(c_{7+4i})A^{58}$	0	$(c_{2+4i})B_3A^7$	0	0
0	$(c_{8+4i})I$	0	0	0	$(c_{6+4i})B_7A^7$
0	$(c_{9+4i})A^7$	0	0	$(c_{2+4i})B_6A^{49}$	$(c_{7+4i})B_7A^{55}$
0	0	$(c_{4+4i})B_5A^{55}$	$(c_{3+4i})B_3A^{58}$	0	0
0	0	0	0	$(c_{5+4i})B_6A^{58}$	0
$(c_{4+4i})I$	0	$(c_{5+4i})B_5A^7$	0	0	0
$(c_{2+4i})A^{15}$	0	0	$(c_{4+4i})B_3A^{55}$	0	0
0	0	$(c_{7+4i})B_5A^7$	$(c_{8+4i})B_3$	0	$(c_{1+4i})B_7$
0	$(c_{10+4i})A^7$	0	0	0	$(c_{5+4i})B_7A^{49}$
0	0	$(c_{6+4i})B_5A^{58}$	0	$(c_{7+4i})B_6A^7$	0
0	0	0	$(c_{7+4i})B_3A^{49}$	0	0
0	0	$(c_{8+4i})B_5$	0	0	0
$(c_{3+4i})A^{36}$	0	0	0	$(c_{8+4i})B_6$	0
0	0	$(c_{9+4i})B_5A^7$	0	0	$(c_{8+4i})B_1A^{49}$
$(c_{8+4i})I$	0	0	$(c_{5+4i})B_3A^{49}$	0	$(c_{10+4i})B_7A^{49}$
0	0	0	$(c_{6+4i})B_3A^{55}$	$(c_{9+4i})B_6A^{58}$	0
0	0	$(c_{10+4i})B_5A^{49}$	0	0	0
0	0	0	0	0	$(c_{12+4i})B_7A^{58}$
0	0	0	$(c_{9+4i})B_3A^{58}$	$(c_{12+4i})B_6A^{55}$	0
0	0	0	$(c_{10+4i})B_3A^7$	0	0
$(c_{6+4i})A^{15}$	0	0	0	$(c_{11+4i})B_6A^7$	0
$(c_{9+4i})I + (c_{10+4i})A^{15} + (c_{11+4i})A^{36} + (c_{12+4i})A^{45}$	0	0	0	0	0

Figure 4.19: First of two edge-coloring incidence matrices of a $\mathrm{BIBD}(60, 5, 4)$

$N_i =$

0	0	$(c_{3+4i})B_1A^{55}$	$(c_{4+4i})I$	0	0
$(c_{2+4i})B_4A^{55}$	0	0	$(c_{3+4i})A^{58}$	$(c_{4+4i})I$	0
0	0	0	$(c_{1+4i})I$	$(c_{2+4i})A^{57}$	0
0	$(c_{4+4i})B_2A^{55}$	0	0	$(c_{3+4i})A^{22}$	0
0	$(c_{2+4i})B_2$	0	0	$(c_{1+4i})I$	0
0	0	0	0	0	$(c_{1+4i})I$
$(c_{3+4i})B_4A^{49}$	0	$(c_{2+4i})B_1A^{49}$	0	0	$(c_{4+4i})A^7$
0	0	$(c_{4+4i})B_1A^{55}$	0	0	0
0	$(c_{3+4i})B_2A^{49}$	0	0	0	0
$(c_{1+4i})B_4$	0	$(c_{7+4i})B_1$	0	0	0
0	0	$(c_{8+4i})B_1A^7$	0	0	0
$(c_{5+4i})B_4$	0	0	0	0	$(c_{2+4i})A^{40}$
$(c_{4+4i})B_4A^7$	0	0	0	0	$(c_{3+4i})A^7 + (c_{6+4i})A^{53}$
0	$(c_{6+4i})B_2A^{49}$	0	$(c_{7+4i})A^{56}$	0	0
$(c_{7+4i})B_4A^{55}$	$(c_{1+4i})B_2A^{58}$	0	0	0	0
0	0	0	$(c_{2+4i})A^{58}$	0	0
0	0	0	$(c_{8+4i})I + (c_{11+4i})A^{42}$	0	0
0	0	$(c_{1+4i})B_1A^{49}$	0	$(c_{8+4i})A^{18}$	0
0	$(c_{8+4i})B_2A^{58}$	$(c_{6+4i})B_1A^{55}$	0	0	$(c_{5+4i})I$
$(c_{9+4i})B_4A^7$	$(c_{7+4i})B_2$	0	0	$(c_{6+4i})A^{18}$	0
0	0	$(c_{5+4i})B_1A^7$	0	0	$(c_{10+4i})A^{28}$
$(c_{6+4i})B_4A^{55}$	0	0	0	$(c_{7+4i})A^{22}$	0
0	0	0	0	$(c_{11+4i})A^{18}$	0
0	$(c_{11+4i})B_2$	0	0	$(c_{12+4i})A^{22}$	0
$(c_{8+4i})B_4$	0	0	$(c_{5+4i})A^{25}$	0	$(c_{7+4i})A^{53}$
0	$(c_{5+4i})B_2A^7$	0	$(c_{6+4i})A^{25}$	0	$(c_{11+4i})A^7$
$(c_{11+4i})B_4A^{58}$	0	0	0	$(c_{10+4i})A^{22}$	0
0	0	$(c_{11+4i})B_1A^{49}$	$(c_{12+4i})A^{25}$	0	$(c_{9+4i})A^{28}$
0	$(c_{9+4i})B_2A^{49}$	$(c_{12+4i})B_1A^{55}$	0	0	0
0	0	0	0	0	0

Figure 4.20: Second of two edge-coloring incidence matrices of a $\mathrm{BIBD}(60, 5, 4)$

Lemma 4.18. *If there exists a* $5-\mathrm{GDD}(m^u)$ *and a properly colored* $\mathrm{BIBD}(m+1,5,\lambda)$*, then there exists a properly colored* $\mathrm{BIBD}(mu+1,5,\lambda)$.

Proof: Let G_i for $i=1,\ldots,u$ be the u groups of size m. Repeat each of the blocks in a $5-\mathrm{GDD}(m^u)$ λ times. For each block, we must color each edge a different color, using the colors $c_i \in \{c_1,\ldots,c_\lambda\}$. Each corresponding edge in the λ copies of the block must also be a different color. So we color the edges in the λ copies of each blocks as follows. Form a $10\times\lambda$ matrix. The rows of the matrix will be indexed by the 10 edges of K_5, and the columns will be indexed by the λ copies of the block. The entries of the matrix will be the first 10 rows of an $\mathrm{LS}(\lambda)$. Now the only pairs that have not been covered are the pairs which lie within the groups and pairs which contain the point $\{\infty\}$. So we place a properly colored $\mathrm{BIBD}(m+1,5,\lambda)$ on each $G_i\cup\{\infty\}$ for all $i=1,2,\ldots,u$. This forms a properly colored BIBD $(mu+1,5,\lambda)$. ✠

Lemma 4.19. *If there exists a* $(5,2)-\mathrm{GDD}(m^u)$ *and a properly colored* $\mathrm{BIBD}(m,5,\lambda)$*, then there exists a properly colored* $\mathrm{BIBD}(mu,5,\lambda)$ *where* $\lambda=2k, k\in\mathbb{Z}^+, k\geq 10$.

Proof: Repeat each of the blocks in a $(5,2)-\mathrm{GDD}(m^u)$ k times. For each block, we will color each edge a different color and each corresponding edge in the k copies of the blocks must also be a different color. Form a $10\times k$ matrix whose rows are indexed by the 10 edges of the K_5 and whose columns are indexed by the k copies of the block. Because each edge is seen twice among all $10\times k$ matrices which represent the blocks of the GDD, we assign color sets to each edge as follows. Let $C=\{c_1,c_2,\ldots,c_k\}$ and $D=\{d_1,d_2,\ldots,d_k\}$ be two distinct sets of k colors each. For each edge in the block, if it is the first time it has occurred in a block of the GDD, we assign the color set C to it, and if it is the second time it has occurred, we assign the color set D to it. Let t_C denote the number of edges which have color set C assigned to it, and t_D denote the number of edges which have color set D assigned to it. The entries of our $10\times k$ matrix will be the first t_C rows of an $\mathrm{LS}(k)$ on the rows corresponding to the edges assigned with color set C and the first t_D rows of an $\mathrm{LS}(k)$ on the rows corresponding to the edges assigned with color set D. Consider some edge e. This edge, e, is colored with every color from C exactly once in the k copies of the first block of the GDD containing e, and it is colored with every color from D exactly once in the k copies of the second block in the GDD containing e. Furthermore, we can be sure that every edge in any block is colored with different colors because of the properties of Latin squares and because C and D are distinct.

The only pairs that have not been covered by the blocks of the GDD are pairs which lie within the groups. So we place a properly colored $\mathrm{BIBD}(m,5,\lambda)$ on each group of the GDD. This forms a properly colored $\mathrm{BIBD}(mu,5,\lambda)$.

✠

Lemma 4.20. *If there exists a* $(5,2)-\mathrm{GDD}(m^u)$ *and a properly colored* $\mathrm{BIBD}(m+1,5,\lambda)$*, then there exists a properly colored* $\mathrm{BIBD}(mu+1,5,\lambda)$ *where* $\lambda=2k, k\in\mathbb{Z}^+, k\geq 10$.

Proof: For $i=1,2,\ldots,u$ let G_i denote the i^{th} group of size m in the GDD and let $\{\infty\}$ be a new point. Follow the same argument as in the first paragraph of the proof of Lemma 4.19.

Then the only pairs that have not been covered by the blocks of the GDD are pairs which lie within the groups, and pairs which contain the point $\{\infty\}$. So place a properly colored $\mathrm{BIBD}(m+1, 5, \lambda)$ on each $G_i \cup \{\infty\}$. This forms a properly colored $\mathrm{BIBD}(mu, 5, \lambda)$. ✠

Lemma 4.21. *Suppose there exists an* $\mathrm{RBIBD}(v-m, 5, 1)$ *with* t *parallel classes. If there exists a properly colored* $\mathrm{BIBD}(6, 5, \lambda)$*, and a properly colored* $\mathrm{BIBD}(m, 5, \lambda)$ *for* $m \leq t$*, then there exits a properly colored* $\mathrm{BIBD}(v, 5, \lambda)$*.*

Proof: Let $\{\infty_1, \infty_2, \ldots, \infty_m\}$ be m new points. Consider the parallel class P_i for $i = 1, \ldots, m$. We take each block of the P_i and join it with $\{\infty_i\}$. Now place a properly colored $\mathrm{BIBD}(6, 5, \lambda)$ on each block of $P_i \cup \{\infty_i\}$. Now the only pairs that have not been covered are in the parallel classes P_i for $i = m+1, \ldots, t$ and the pairs of the form $\{\{\infty_i, \infty_j\} : i, j \in \{1, \ldots, m\}\}$. So we place a properly colored $\mathrm{BIBD}(5, 5, \lambda)$ on each block of the remaining parallel classes and we place a properly colored $\mathrm{BIBD}(m, 5, \lambda)$ on the set of points $\{\infty_1, \ldots, \infty_m\}$. This forms a properly colored $\mathrm{BIBD}(v, 5, \lambda)$ where $\lambda = 4k, \lambda \geq 3$. ✠

Corollary 4.22. *Let* $m \leq 5n+1$*. If there exists properly colored* $\mathrm{BIBD}(m, 5, \lambda)$*, then there exists a properly colored* $\mathrm{BIBD}(20n+5+m, 5, \lambda)$ *for each* $\lambda = 4k, k \geq 3$*, and* $n \neq 2, 11, 17, 23, 32$*.*

Proof: There exists an $\mathrm{RBIBD}(20n+5, 5, 1)$ for each $n \neq 2, 11, 17, 23, 32$ by Theorem 1.9. This design has $5n+1$ parallel classes. There exists a properly colored $\mathrm{BIBD}(6, 5, \lambda)$ for $\lambda = 4k$ and $k \geq 3$. So apply Lemma 4.21 with $v = 20n+5$. ✠

If the block size is five, then the possible v values for all possible λ values are as follows from (Hanani, [28]).

k	λ	Conditions for v	Exceptions
5	1	1,5 (mod 20)	none
5	2	1,5 (mod 10)	15
5	4	0,1 (mod 5)	none
5	10	1 (mod 2)	none
5	20	all	none

Because we can properly color any $\lambda = 10k, k \in \mathbb{Z}^+, k \geq 1$ with Theorem 2.2 and any λ when $v \equiv 1, 5 \pmod{20}$ by Theorem 2.3, we will focus on $\lambda \equiv 0 \pmod 2$ and $\lambda \equiv 0 \pmod 4$ in the following sections.

4.2.1 $\lambda \equiv 0 \pmod 2$

In this section, we properly color all $\mathrm{BIBD}(v, 5, \lambda)$s where $\lambda \equiv 0 \pmod 2$ except possibly when $v \equiv 15, 35, 75, 95 \pmod{100}$ and $\lambda = 14, 18$.

In this case, the necessary and sufficient conditions for the existence of a $\mathrm{BIBD}(v, 5, \lambda)$ are that $v \equiv 1, 5 \pmod{10}$ except $v = 15$ and $\lambda = 2$. Note that when $v \equiv 1, 5 \pmod{10}$

and $\lambda \equiv 0 \pmod{10}$ these are already covered by Theorem 2.2, but the results in this section will also cover this case.

Theorem 4.23. *There exists a proper coloring for every* $\mathrm{BIBD}(v, 5, \lambda)$ *for* $\lambda = 2k, k \in \mathbb{Z}^+, k \geq 5$, *where* $v \equiv 1, 5 \pmod{10}$ *except possibly when* $\lambda = 2k, v \equiv 15, 35, 75, 95 \pmod{100}$ *and* $\lambda = 14, 18$.

Proof: Note that in each case we assume $\lambda = 2k, k \in \mathbb{Z}^+, k \geq 5$ unless otherwise stated. We first consider $v \equiv 5 \pmod{10}$. In other words $v \equiv 5$ or $15 \pmod{20}$.

Now suppose $v \equiv 1 \pmod{10}$. If $v \equiv 1 \pmod{20}$ then we can properly color a $\mathrm{BIBD}(v, 5, \lambda)$ by Theorem 2.3. We deal with each subcase, $v \equiv 1 \pmod{20}$ and $v \equiv 11 \pmod{20}$, separately.

Case 1: $v \equiv 1 \pmod{20}$

By Theorem 2.3, we can properly color a $\mathrm{BIBD}(v, 5, \lambda)$ where $v \equiv 1, 5 \pmod{20}$.

Case 2: $v \equiv 11 \pmod{20}$

If $v \equiv 11 \pmod{20}$, then we have that either $v \equiv 11 \pmod{40}$ or $v \equiv 31 \pmod{40}$.

Case 2.1: $v \equiv 11 \pmod{40}$

Let $v = 11 + 40x = 1 + 10(1 + 4x)$ for $x \geq 1$. There exists a 5–$\mathrm{GDD}(10^u)$ where $u = 1 + 4x$ except possibly when $x = 1, 8$ by Theorem 1.7. We can properly color a $\mathrm{BIBD}(11, 5, \lambda)$ by Lemma 4.1. So we can apply Lemma 4.18 with $m = 10, u = 1 + 4x$ and $v \neq 51, 331$.

If $v = 51$, then by Lemma 4.3 we can properly color a $\mathrm{BIBD}(51, 5, \lambda)$. If $v = 331 = 1+30(11)$, then there exists a $5-\mathrm{GDD}(30^{11})$ by Theorem 1.7. There also exists a properly colored $\mathrm{BIBD}(31, 5, \lambda)$ by Lemma 4.2. Thus we can apply Lemma 4.18 to properly color a $\mathrm{BIBD}(331, 5, \lambda)$.

Case 2.2: $v \equiv 31 \pmod{40}$

If $v = 31$, then we can properly colored a $\mathrm{BIBD}(31, 5, \lambda)$ by Lemma 4.2.

Let $v = 31 + 40x = 1 + 10(3 + 4x)$ for $x \geq 1$. There exists a $5 - \mathrm{GDD}(10^u)$ where $u = 3 + 4x$ except possible when $x = 1, 3, 5, 6, 8, 9, 11$ by Theorem 1.7. We can properly color a $\mathrm{BIBD}(11, 5, \lambda)$ by Lemma 4.1. We can also properly color a $\mathrm{BIBD}(31, 5, \lambda)$ by Lemma 4.2. So we can apply Lemma 4.18 with $m = 10, u = 3 + 4x$ and $v \neq 71, 151, 231, 271, 351, 391, 471$.

If $v = 71 = 5(14) + 1$, then there exists a 5–$\mathrm{GDD}(14^5)$ by Theorem 1.7. There also exists a properly colored $\mathrm{BIBD}(15, 5, \lambda)$ by Lemma 4.10. Then by Lemma 4.18, we can properly color a $\mathrm{BIBD}(71, 5, \lambda)$. If $v = 151 = 1 + 30(5)$, then there exists a $5 - \mathrm{GDD}(30^5)$ by Theorem 1.7. There also exists a properly colored $\mathrm{BIBD}(31, 5, \lambda)$ by Lemma 4.2. So we can apply Lemma 4.18 to properly color a $\mathrm{BIBD}(151, 5, \lambda)$. If $v = 231 = 11(21)$, then there exists a $5 - \mathrm{GDD}(11^{21})$ by Theorem 1.7. There also exists a properly colored $\mathrm{BIBD}(11,5,\lambda)$ by Lemma 4.1. So we can apply Lemma 4.17 to properly color a $\mathrm{BIBD}(231,5,\lambda)$. If $v = 271 = 1 + 54(5)$, then there exists a $5-\mathrm{GDD}(54^5)$ by Theorem 1.7. If $v = 55$, then there exists a $5-\mathrm{GDD}(11^5)$ by Theorem 1.7. Since we can properly color a $\mathrm{BIBD}(11,5,\lambda)$ by Lemma 4.1, we can properly color a $\mathrm{BIBD}(55,5,\lambda)$ with Lemma 1.5. So we can apply Lemma 4.18 to properly color a $\mathrm{BIBD}(271,5,\lambda)$. If $v = 351 = 70(5)+1$, then there exists a $5-\mathrm{GDD}(70^5)$ by Theorem 1.7. There also exists a properly colored $\mathrm{BIBD}(71,5,\lambda)$ as stated above. So we can apply Lemma 4.18 to properly color a $\mathrm{BIBD}(351,5,\lambda)$. If $v = 391 = 30(13)+1$ then there exists a $(5)-\mathrm{GDD}(30^{13})$ by Theorem 1.7. There also exists a properly colored $\mathrm{BIBD}(31,5,\lambda)$ by Lemma 4.2. So we apply Lemma 4.18 to properly color a $\mathrm{BIBD}(391,5,\lambda)$. If $v = 471$, then we can properly colored a $\mathrm{BIBD}(471,5,\lambda)$ by Lemma 2.10.

Case 3: $v \equiv 15 \pmod{20}$

If $v \equiv 5 \pmod{20}$, then we properly color each $\mathrm{BIBD}(v,5,\lambda)$ by Theorem 2.3. If $\lambda = 10$, then by Theorem 2.2 we can properly color all $\mathrm{BIBD}(v,5,10)$ where $v \equiv 15 \pmod{20}$.

If $v = 15$, then by Lemma 4.10 we can properly color all $\mathrm{BIBD}(15,5,\lambda)$. Let $v = 15 + 20x = 5(3+4x)$ for $x \geq 1$. There exists a $(5,2)-\mathrm{GDD}(5^u)$ for $u = 3 + 4x$ by Theorem 1.8. It is trivial to properly color a $\mathrm{BIBD}(5,5,\lambda)$. So we can apply Lemma 4.19 with $m = 5, u = 3 + 4x$.

Now let $v = 20n + 15$. If $n \geq 3$, then apply Corollary 4.22 with $m = 10$ to properly color a $\mathrm{BIBD}(v,5,\lambda)$ where $v \equiv 15 \pmod{20}$ and $\lambda = 12, 16$. This leaves the open cases of $v = 15, 35, 55, 235, 355, 475, 655$ (corresponding to $n = 0, 1, 2, 11, 17, 23, 32$). We construct the properly colored $\mathrm{BIBD}(15,5,\lambda)$ directly in Lemma 4.10. Also, we construct the properly colored $\mathrm{BIBD}(35,5,\lambda)$ directly in Lemma 4.11. For the other v, note that there exists an $\mathrm{RBIBD}(20(n-1)+5,5,1)$ with $5(n-1)+1 = 5n+4$ parallel classes. In each case, $n \geq 10$, meaning we have at least 51 parallel classes. However, in each case we must only use $20n + 15 - (20(n-1)+5) = 30$ parallel classes. Thus we again apply Corollary 4.22 with $m = 25$ and $n = n - 1$.

To further break down the cases of $v \equiv 15 \pmod{20}$ we consider $v \equiv 55 \pmod{100}$. Let $v = 55 + 100x = (11+20x)5$ for $x \geq 0$. There exists a 5–$\mathrm{GDD}((11+20x)^5)$ by Theorem 1.7. There also exists a properly colored $\mathrm{BIBD}(11+20x,5,\lambda)$ by Case 2. Again, we can properly color a $\mathrm{BIBD}(5,5,\lambda)$. Thus, we can properly color a $\mathrm{BIBD}(v,5,\lambda)$ for $v \equiv 55 \pmod{100}$.

Note that the remaining cases for a $\mathrm{BIBD}(v,5,\lambda)$ where $v \equiv 15, 35, 75, 95 \pmod{100}$ are $\lambda = 14, 18$.

Henceforth, there exists a proper coloring for every $\mathrm{BIBD}(v,5,\lambda)$, where $v \equiv 1, 5 \pmod{10}$. ✠

4.2.2 $\lambda \equiv 0 \pmod 4$

In this section, we properly color all $\mathrm{BIBD}(v,5,\lambda)$s where $\lambda \equiv 0 \pmod 4$. Note the necessary and sufficient conditions for the existence of a $\mathrm{BIBD}(v,5,\lambda)$ are that $v \equiv 0, 1$

(mod 5). Note that when $v \equiv 0, 1 \pmod 5$ and $\lambda \equiv 0 \pmod{20}$ these are already covered by Theorem 2.2, but the results in this section satisfy this case as well.

Theorem 4.24. *There exists a proper coloring for every* $\mathrm{BIBD}(v, 5, \lambda)$ *for* $\lambda = 4k, k \geq 3$*, where* $v \equiv 0, 1 \pmod 5$.

Proof: Note that in each case we assume $\lambda = 4k, k \geq 3$ unless otherwise stated. We will break this problem up into two main cases, $v \equiv 0 \pmod 5$ and $v \equiv 1 \pmod 5$.

Case 1: $v \equiv 0 \pmod 5$

We consider four subcases: $v \equiv 0, 5, 10,$ or $15 \pmod{20}$. If $v \equiv 0 \pmod{20}$, then we have that either $v \equiv 0 \pmod{40}$ or $v \equiv 20 \pmod{40}$.

Case 1.1: $v \equiv 0 \pmod{40}$

If $v = 40$, then by Lemma 4.8, we can properly color a $\mathrm{BIBD}(40, 5, \lambda)$. If $v = 80 = 16(5)$, then there exists a $5 - \mathrm{GDD}(16^5)$ by Theorem 1.7. We can properly color a $\mathrm{BIBD}(16, 5, \lambda)$ by Lemma 4.12. So we can properly color a $\mathrm{BIBD}(80, 5, \lambda)$ by Lemma 4.17.

Let $v = 40x = 20(2x)$ for $x \geq 3$. There exists a $5 - \mathrm{GDD}(20^{2x})$ by Theorem 1.7. We can properly color a $\mathrm{BIBD}(20, 5, \lambda)$ by Lemma 4.6. So we can apply Lemma 4.17.

Case 1.2: $v \equiv 20 \pmod{40}$

If $v = 20$, then by Lemma 4.6 we can properly color a $\mathrm{BIBD}(20, 5, \lambda)$. If $v = 60$, then by Lemma 4.16, we can properly color a $\mathrm{BIBD}(60, 5, \lambda)$.

Let $v = 20+40x = 20(1+2x)$ for $x \geq 2$. There exists a 5–$\mathrm{GDD}(20^u)$ for $u = 1+2x$ by Theorem 1.7. Because we can properly color a $\mathrm{BIBD}(20, 5, \lambda)$, we can apply Lemma 4.17.

Case 1.3: $v \equiv 5 \pmod{20}$

By Theorem 2.3, we can properly color a $\mathrm{BIBD}(v, 5, \lambda)$ where $v \equiv 1, 5 \pmod{20}$.

Case 1.4: $v \equiv 10 \pmod{20}$

Let $v = 20n + 10$. If $n \geq 1$, then apply Corollary 4.22 with $m = 5$. This leaves the open cases of $v = 50, 230, 350, 470, 650$ (corresponding to $n = 0, 2, 11, 17, 23, 32$). We construct the properly colored $\mathrm{BIBD}(10, 5, \lambda)$ directly in Lemma 4.5. We also construct the properly colored $\mathrm{BIBD}(50, 5, \lambda)$ directly in Lemma 4.15. For the other v, note that these exists an $\mathrm{RBIBD}(20(n-1)+5, 5, 1)$ with $5(n-1)+1 = 5n-4$ parallel classes. In each case, $n \geq 10$, meaning we have at least 51 parallel classes. However, in each case we must only use $20n + 10 - (20(n-1)+5) = 25$ parallel classes. Thus we again apply Corollary 4.22 with $m = 25$ and $n = n - 1$.

Case 1.5: $v \equiv 15 \pmod{20}$

Let $v = 20n + 15$. If $n \geq 3$, then apply Corollary 4.22 with $m = 10$. The open cases left are $v = 15, 35, 55, 235, 355, 475, 655$ (corresponding to $n = 0, 1, 2, 11, 17, 23, 32$). We construct the properly colored $\text{BIBD}(15, 5, \lambda)$ directly in Lemma 4.10. We also construct the properly colored $\text{BIBD}(35, 5, \lambda)$ directly in Lemma 4.11. If $v = 55$, there exists a 5–GDD(11^5) by Theorem 1.7. There also exists a properly colored $\text{BIBD}(11, 5, \lambda)$ by Theorem 4.23. So we apply Lemma 4.17 to properly color a $\text{BIBD}(55, 5, \lambda)$. For the other v, note that there exists an $\text{RBIBD}(20(n-1)+5, 5, 1)$ with $5(n-1)+1 = 5n-4$ parallel classes. In each case, $n \geq 10$, meaning we have at least 51 parallel classes. However, in each case we must only use $20n + 15 - (20(n-1) + 5) = 30$ parallel classes. Thus we again apply Corollary 4.22 with $m = 30$ and $n = n - 1$.

Case 2: $v \equiv 1 \pmod 5$

If $v \equiv 1 \pmod 5$, then $v \equiv 1, 6, 11, \text{ or } 16 \pmod{20}$. If $v \equiv 1 \pmod{20}$, we can properly color a $\text{BIBD}(v, 5, \lambda)$ by Theorem 2.3. If $v \equiv 11 \pmod{20}$, then apply Theorem 4.23 to obtain a properly colored $\text{BIBD}(v, 5, \lambda)$. This leaves two cases; $v \equiv 6, 16 \pmod{20}$.

Case 2.1: $v \equiv 6 \pmod{20}$

Let $v = 6+20x = 1+5(1+4x)$ for $x \geq 1$. There exists a $5-\text{GDD}(5^u)$ for $u = 1+4x$ by Theorem 1.7. We can properly color a $\text{BIBD}(6, 5, \lambda)$ by Lemma 4.4. So we can apply Lemma 4.18 with $m = 5$ and $u = 1 + 4x$.

Case 2.2: $v \equiv 16 \pmod{20}$

Let $v = 20n+16$. If $v = 16$, then by Lemma 4.12 we can properly color a $\text{BIBD}(v, 5, \lambda)$. If $n \geq 3$, then apply Corollary 4.22 with $m = 11$. This leaves the open cases of $v = 16, 36, 56, 231, 351, 471, 651$ (corresponding to $n = 0, 1, 2, 11, 17, 23, 32$). We construct the properly colored $\text{BIBD}(16, 5, \lambda)$ directly in Lemma 4.12. If $v = 36$, there exists a properly colored $\text{BIBD}(36, 5, \lambda)$ by Lemma 4.7. There also exists a properly colored $\text{BIBD}(56, 5, \lambda)$ by Lemma 4.9. For the other v, note that there exists an $\text{RBIBD}(20(n-1) + 5, 5, 1)$ either $(5, n-1) + 1 = 5n - 4$ parallel classes. In each case, $n \geq 10$, meaning we have at least 51 parallel classes. However, in each case we must only use $20n+16-(20(n-1)+5) = 31$ parallel classes. Thus we again apply Corollary 4.22 with $m = 31$ and $n = n - 1$.

✠

4.3 Conclusion

We are now in a position to prove the main theorem.

Theorem 4.25. *There is a proper edge coloring for every* $\text{BIBD}(v, 5, \lambda)$ *where* $\lambda \geq 10$*, except possibly when* $\lambda = 2k, v \equiv 15, 35, 75, 95 \pmod{100}$ *and* $\lambda = 14, 18$.

Proof: Recall the necessary and sufficient conditions for the existence of a $\mathrm{BIBD}(v, 5, \lambda)$.

If $\lambda \equiv 1, 3, 5, 7, 9, 11, 13, 15, 17, 19 \pmod{20}$, then $v \equiv 1, 5 \pmod{20}$;
If $\lambda \equiv 2, 6, 14, 18 \pmod{20}$, then $v \equiv 1, 5 \pmod{10}$;
If $\lambda \equiv 4, 8, 12, 16 \pmod{20}$, then $v \equiv 0, 1 \pmod{5}$;
If $\lambda \equiv 10 \pmod{20}$, then $v \equiv 1 \pmod{2}$; and
If $\lambda \equiv 0 \pmod{20}$, then $v \geq 5$.

If $\lambda \equiv 1, 3, 5, 7, 9, 11, 13, 15, 17, 19 \pmod{20}$, then $v \equiv 1, 5 \pmod{20}$ and we can properly color a $\mathrm{BIBD}(v, 5, \lambda)$ by applying Theorem 2.3. If $\lambda \equiv 10 \pmod{20}$ or $\lambda \equiv 0 \pmod{20}$, then $v \equiv 1 \pmod{2}$ or v may be anything, respectfully, and we can properly color a $\mathrm{BIBD}(v, 5, \lambda)$ by applying Theorem 2.2. If $\lambda \equiv 2, 6, 14, 18 \pmod{20}$, then $v \equiv 1, 5 \pmod{10}$ and we apply Theorem 4.23. Finally, if $\lambda \equiv 4, 8, 12, 16 \pmod{20}$, then $v \equiv 0, 1, \pmod{5}$ and we apply Theorem 4.24. ✠

Chapter 5

Alternate Method to Properly Color

In this chapter, we take a graph theoretic approach to the proper coloring question in Chapters 3 and 4. We also explain the strengths and weaknesses in application of the method in this chapter against the methods from Chapters 3 and 4. The books by West [50] and Diestel [20] were used often as references in this chapter.

5.1 Background in Graph Theory

An X, Y*-bigraph* is a bipartite graph where the two partitions are called X and Y. For the remainder of this chapter we adopt the notation X and Y as the partitioned sets of vertices from some arbitrary bipartite graph.

A *saturation* of X is defined as a matching containing all points in X. We define a *maximal matching* as the largest possible matching set for a given graph G. If a matching M saturates X, then for every $S \subseteq X$ there must be at least $|S|$ vertices in Y that have neighbors (adjacent vertices) in S. Let $N_G(S)$ or $N(S)$ denote the set of neighboring or adjacent vertices to vertices in S. Clearly, $|N(S)| \geq |S|$ is a necessary condition. Phillip Hall proved that for all $S \subseteq X$, $|N(S)| \geq |S|$ is also a sufficient condition, hence why this condition is called Hall's Condition (Marriage Condition).

Theorem 5.1. *(Marriage Theorem, [27]) An X, Y-bigraph G has a matching that saturates X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.*

Proof: We begin by showing if an X, Y-bigraph G has a matching that saturates X, then $|N(S)| \geq |S|$ for all $S \subseteq X$. Consequently, if a matching saturates X, then for every $S \subseteq X$ there must be at least $|S|$ vertices in Y, and out of necessity $|N(S)| \geq |S|$.

Now we prove that Hall's Condition is sufficient using the contrapositive of the statement. We want to show if M is a maximal matching in G and M does not saturate X, then we achieve a set $S \subseteq X$ such that $|N(S)| < |S|$. Let $u \in X$ be a vertex not in the matching M. Also let S_0 be the set of all vertices in X reachable from u by M-alternating paths in G and T_0 be the set of all vertices in Y reachable from u by M-alternating paths in G. Note that $u \in S_0$ in Figure 5.1.

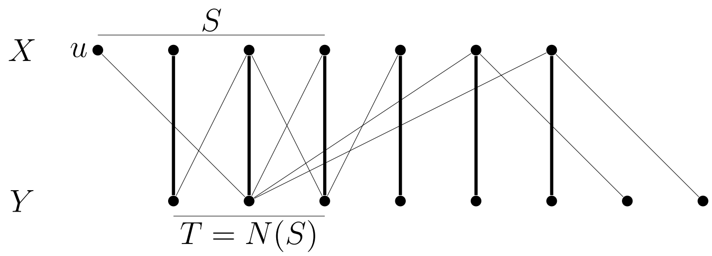


Figure 5.1: Proving the Marriage Theorem with alternating paths.

Our claim is that we must prove M matches T_0 with $S_0 - \{u\}$. Since the M-alternating paths from u reach Y through edges not in M and return to X along edges in M, to get to every vertex of $S_0 - \{u\}$, one must pass through an edge in M from a vertex in T_0. This shows there are no augmenting paths through M. Because there is no M-augmenting path, every vertex in T_0 is saturated. It follows that an M-alternating path reaching $y \in T_0$ may be extended via M to a vertex of S_0. This means the edges in the matching M yield a bijection from T_0 to $S_0 - \{u\}$. Therefore we have $|T_0| = |S_0 - \{u\}|$.

Because a matching between T_0 and $S_0 - \{u\}$ exists, it tells us $T_0 \subseteq N(S_0)$. Furthermore, because of the way we constructed S_0 and T_0, $T_0 = N(S_0)$. Now suppose that $y \in Y \backslash T_0$ has an adjacent vertex $v \in S_0$. Since $y \notin T_0$, the edge vy cannot be in M since u is unsaturated and the rest of S_0 is matched to T_0 by M. If we were to add vy to an M-alternating path reaching v, then the M-alternating path would reach Y. This contradicts $y \notin T_0$. Therefore vy cannot exist.

This shows $T_0 = N(S_0)$, and thus $|N(S_0)| = |T| = |S_0| - 1 < |S_0|$ for our S_0. But since our S_0 is arbitrary, this is true for all S and thus completes our proof. ✠

Corollary 5.2. *[50] For $k > 0$, every k-regular bipartite graph has a perfect matching.*

Proof: A *perfect matching* is a matching consisting of every vertex in the graph. Let G be a k-regular X, Y-bigraph. Since each edge has two endpoints, we count the number of endpoints in X and the number of endpoints in Y. Because for each edge one endpoint that lies in X and there is an endpoint that in Y, the total number of endpoints are equivalent on both sides. It follows from the graph being k-regular that the number of endpoints in X is $k|X|$ and the number of endpoints in Y is $k|Y|$, so $k|X| = k|Y|$. Thus, $|X| = |Y|$. Thus a matching that saturates X will also saturate Y and this is matching is perfect. So we only need to prove there is a matching that saturates X. This is equivalent to verifying Hall's Condition.

Now we consider $S \subseteq X$. Let m represent the number of edges from S to $N(S)$. Because G is a k-regular X, Y-bigraph, $m = k|S|$. The number of edges in $N(S)$ is $k|N(S)|$ and because there may be more edges coming into $N(S)$, $k|S| = m \leq k|N(S)|$, and so $|N(S)| \geq |S|$. Finally, we note that we choose S arbitrarily and thus we have established Hall's Condition. ✠

To establish the understanding of the following theorem we need a little more terminology. A k**-edge-coloring** of G is a labeling of the edges defined by the mapping $f : E \to S$, where $|S| = k$ and E is the set of edges. These labels used will be colors. We call a k-edge-coloring *proper* if incident edges have different labels. This is equivalent to saying that each unique color generates a matching. An edge is *incident* to another edge if they share a common vertex. It is straightforward to call a graph k**-edge-colorable** if it has a proper k-edge-coloring. The *edge-chromatic number*, $\chi'(G)$, of a simple graph[1] G is given as a number k such that k is the minimum number where G is k-edge-colorable. It follows that since edges sharing a vertex need different colors, $\chi'(G) \geq \Delta(G)$ where $\Delta(G)$ is the maximum degree of all the vertices. It was proved independently by Vizing [48] and Gupta [26] that if G is a simple graph, then $\Delta(G) + 1$ colors suffice to color G.

Theorem 5.3. *[50] If H is bipartite, then $\chi'(H) = \Delta(H)$.*

Proof: The idea behind this proof is to establish the following. Given a k-regular bipartite graph G and a subgraph H of G, since G has a perfect matching and yields a proper $\Delta(G)$-edge-coloring, we can achieve a proper $\Delta(G)$-edge-coloring on H by removing edges. To prove this, though, we will start with a graph H and show it generates G. First, Corollary 5.2 states every k-regular bipartite graph G has a perfect matching. Hence, we can use induction on G to show $\Delta(G) = k$ is the edge-chromatic number of G and thus yields a proper $\Delta(G)$-edge-coloring. As a result, it suffices to show for every bipartite graph H with maximum degree k, H is a subgraph of a k-regular graph G, i.e. there is a k-regular bipartite graph G containing H.

To construct G from H, first consider the two partite sets of H as X and Y. We add vertices to the smaller set of X and Y, if necessary, until the two sets are equal. If this new graph H' is not regular, then both X and Y have a vertex with degree less than $\Delta(G)$. Add an edge between these two vertices. Continue adding edges until H' becomes k-regular. Since H' is now k-regular, it has the same properties as G and thus H can yield a proper k-edge-coloring. ✠

5.2 Restating our Problem

Consider the structure of an edge-incidence matrix of a $\mathrm{BIBD}(v, k, \lambda)$. We construct an X, Y-bigraph where X is the set of all possible pairs of vertices (set of edges) and Y is the set of blocks. There is an edge in this bipartite graph between $x \in X$ and $y \in Y$ if the pair represented by x is in the block y. This will give us a bipartite graph with degree λ on each x and degree $\binom{k}{2}$ on each y. Since $\lambda \geq \binom{k}{2}$, $\Delta(G) = \lambda$. Thus, by Theorem 5.3 $\lambda = \Delta(G) = \chi'(G)$. Suppose that where there is an edge between a pair and a block, it means that edge in the block represents the K_k of that block. Then we only need to properly color this bipartite graph in order to properly color the decompositions of λ copies of K_v into copies of K_k. Since we have constructed this bipartite graph from a $\mathrm{BIBD}(v, k, \lambda)$

[1]We say simple graph instead of loopless because the only types of graphs that we deal with are simple graphs.

and we can properly color this graph, we can properly color any $\mathrm{BIBD}(v, k, \lambda)$. The result of this is given in the theorem below.

Theorem 5.4. *There exists a properly colored decomposition of λ copies of monochromatic K_v into panchromatic K_k if there exists a* $\mathrm{BIBD}(v, k, \lambda)$.

5.3 BIBD vs. PD

Though this method solves our problem completely, the method also loses any structure that was given when using proper colorings of BIBDs. One such example is the correlation between BIBDs and another type of coloring problem associated with graph decompositions. A *k-path decomposition* (k–PD) of a graph $\mathcal{H}$ is an edge-partition of $\mathcal{H}$ into subgraphs isomorphic to P_{k+1}, paths of length k. In [29], a question first posed by M. L. Yu is addressed. If $\mathcal{H}$ consists of λ copies of K_v, and the edges of each copy of K_v are monochromatically colored using λ distinct colors, can one find a properly colored k–PD? Again in [29], it was shown that if $v \geq 3$ is odd, then the necessary conditions are sufficient for the existence of a properly colored 2–PD with $\lambda = 2$.

This idea does not extend directly to BIBDs with block size 4 because it is impossible to obtain panchromatic blocks when $\lambda < \binom{k}{2}$. So we must construct equitably colored blocks. Let $|E| = e$ where E is the set of edges in K_k and $d = \lfloor \frac{e}{\lambda} \rfloor$. An *equitably colored* block is such that each color is used either d or $d + 1$ times to color the edges of the block. An equitable edge-coloring of a $\mathrm{BIBD}(v, k, \lambda)$ is then a decomposition of $\mathcal{H}$ into equitable blocks.

Before we revisit the idea of path decomposition, we define and remark upon equitable rectangles. Suppose $r, c,$ and v are positive integers. An *equitable* $(r, c; v)$*-rectangle* is an $r \times c$ array L where every entry is chosen from a v-set X, such that the following two properties are satisfied:

1. every symbol $x \in X$ occurs $\lfloor \frac{c}{v} \rfloor$ or $\lceil \frac{c}{v} \rceil$ times in each row of L.
2. every symbol $x \in X$ occurs $\lfloor \frac{r}{v} \rfloor$ or $\lceil \frac{r}{v} \rceil$ times in each column of L.

We will make use of the equitable rectangles studied in [14] and [45] as follows. If there exists a $\mathrm{BIBD}(v, k, 1)$, then we can equitably color the blocks of a $\mathrm{BIBD}(v, k, \lambda)$ in much the same way that we would color the design. Take λ copies of each block in the $\mathrm{BIBD}(v, k, 1)$. For each block in the $\mathrm{BIBD}(v, k, 1)$ form a $\binom{k}{2} \times \lambda$ matrix. The rows of the matrix will be indexed by the $\binom{k}{2}$ pairs of points, and the columns will be indexed by the λ copies of the block. The entries will be that of an equitable $(\binom{k}{2}, \lambda; \lambda)$-rectangle. We suspect that one may be able to equitably color $\mathrm{BIBD}(v, k, \lambda)$s with $2 \leq \lambda \leq \binom{k}{2}$ with $k = 3, 4,$ or 5 by coloring some small designs directly, and then applying recursive techniques involving GDDs and equitable $(r, c; v)$-rectangles but this is still open.

This leads us to believe it may be possible to extend this idea to properly color k–PDs by using equitably colored $\mathrm{BIBD}(v, k+1, \lambda)$s. For example, consider the following equitably colored $\mathrm{BIBD}(5, 4, 3)$.

	B_1	B_2	B_3	B_4	B_5
$\{1,2\}$	0	0	c_1	c_2	c_3
$\{2,3\}$	c_3	0	0	c_1	c_2
$\{3,4\}$	c_2	c_3	0	0	c_1
$\{4,5\}$	c_1	c_2	c_3	0	0
$\{5,1\}$	0	c_1	c_2	c_3	0
$\{1,3\}$	0	c_1	0	c_3	c_2
$\{2,4\}$	c_2	0	c_1	0	c_3
$\{3,5\}$	c_3	c_2	0	c_1	0
$\{4,1\}$	0	c_3	c_2	0	c_1
$\{5,2\}$	c_1	0	c_3	c_2	0

Each block of the design consists of 2 edges that are colored by c_i, for $i = 1, 2, 3$. Furthermore, the edges in each block can easily be decomposed into properly colored paths of length 3.

If we consider the case where $k = 5$ we cannot create k-path decompositions because the length of the longest path without a cycle is four. Thus, we have to construct other path lengths to get any results. If we look at the edge-colored incidence matrix of a $\text{BIBD}(11, 5, 2)$ in Figure 4.1, we see that we can construct a 2–PD if the index on the colors is evaluated $\pmod 2$. It is not necessary that our path decompositions be uniform but for a more rigid and useful structure we assume it is so. If we let path decompositions be non-uniform, it may be possible to construct two 4–PDs and one 2–PD. But again, this is not possible given the structure of the alternate method.

Finally, because we don't have any structure to the bipartite graph consisting of the pairs and blocks, we do not know the general construction of the coloring and therefore, do not generate a properly colored path decomposition.

Chapter 6

Mutually Orthogonal Equitable Latin Rectangles

At this time we will abandon all conventions of notation developed in Chapters 2, 3, 4, and 5 to switch over to a completely new topic about Latin squares. In this chapter we investigate the maximum number of possible mutually orthogonal equitable Latin rectangles for all possible parameters. In essence, we will show that there are at least 3–MOELR$(a, b; n)$ for all $a, b,$ and n except possibly when

$a = 9$ and $n = 3s$, where $s \equiv 2, 4, 5, 7 \pmod 9, s > 9$;
$a = 18$ and $n = 3s$, where $s \equiv 8, 10, \pmod{18}, s > 18$;
$a = 36$ and $n = 6s$, where $s \equiv 5, 7, 11, 13, 17, 19, 23, 25, 29, 31 \pmod{36}, s > 36$.

and a finite number of cases. To do this we use a plethora of examples to establish a solid understanding of each construction used in the main theorem.

6.1 Mutually Orthogonal Latin Squares

A *Latin square* of *side* n (or *order* n) is an $n \times n$ array in which each cell contains a single symbol from an n*–set* S, such that each symbol occurs exactly once in each row and exactly once in each column. We denote a Latin square of order n as LS(n). As an example, here is a Latin square of side 8 on the symbols $0, 1, \ldots, 7$.

0	1	2	3	4	5	6	7
1	0	3	4	5	6	7	2
2	3	5	0	6	7	4	1
3	4	0	7	1	2	5	6
4	5	6	1	7	0	2	3
5	6	7	2	0	3	1	4
6	7	4	5	2	1	3	0
7	2	1	6	3	4	0	5

Figure 6.1: Latin square of side 8

In 1779 Euler introduced the famous 36 officers problem. Say there are 36 officers, each having a rank and regiment.

```
There are 6 different ranks and 6 different
regiments.  The officers are to be arranged in
a square in such a way that each horizontal and
vertical line has an officer of each rank and each
regiment.  If we look only at the ranks the square
created by the officers is a Latin square.  The
same is true for if we consider only the regiments.
Considering both the ranks and regiments if we
superimpose the two Latin squares does every ordered
pair occur exactly once?
```

If two Latin squares have this property, then the squares are *orthogonal*. A set of k Latin squares of order n, say $L_1, \ldots, L_k$, are said to be *mutually orthogonal Latin squares* if L_i and L_j are orthogonal for all $1 \le i < j \le k$. The maximum number of MOLS of order n is denoted as $N_{\text{MOLS}}(n)$. Euler made a conjecture based on the knowledge that he knew $N_{\text{MOLS}}(2) = 1$ and he strongly suspected $N_{\text{MOLS}}(6) = 1$.

Conjecture 6.1. *If* $n \equiv 2 \pmod 4$*, then* $N_{\text{MOLS}}(n) = 1$.

Later, using BIBDs, Parker [41, 42] established a construction showing $N_{\text{MOLS}}(21) \ge 4$. It was not until 1960 that Bose and Shrikhande [12] saw the work of Parker and made a shocking generalization using pairwise balanced designs rather than balanced incomplete block designs. A *pairwise balanced design* or a $\text{PBD}(v, K, \lambda)$ is a BIBD with the size of the blocks being from the set K. As an example, Figure 6.2 is a $\text{PBD}(10, \{3, 4\})$ where the blocks are listed column-wise.

1	1	1	2	2	2	3	3	3	4	4	4
2	5	8	5	6	7	5	6	7	5	6	7
3	6	9	8	9	10	10	8	9	9	10	8
4	7	10									

Figure 6.2: A $\text{PBD}(10, \{3, 4\})$

Bose and Shrikhande proved $N_{\text{MOLS}}(22) \ge 2$ and $N_{\text{MOLS}}(66) \ge 5$. Later in [13], Parker joined with Bose and Shrikhande to show $N_{\text{MOLS}}(n) \ge 2$ for all $n \ge 10$.

In [52] Wilson developed a new class of constructions. And since then, the number of extensions and refinements of the techniques made by Bose-Shrikhande-Parker and the Wilson constructions are so plentiful that we can only mention where to refer for more literature on the subject. Colbourn and Dinitz in [18] give a more detailed description of constructions for small n. For a massive collection of results on the topics of combinatorial designs including this section and section 1.2, see [19].

Let us consider an example of three mutually orthogonal Latin squares of order 4, also denoted as 3–$\mathrm{MOLS}(4)$. The matrices M_1, M_2, M_3 below represent the three Latin squares of a 3–$\mathrm{MOLS}(4)$ that are orthogonal.

$$M_1 = \begin{array}{|cccc|} \hline 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ \hline \end{array} \quad M_2 = \begin{array}{|cccc|} \hline 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ \hline \end{array} \quad M_3 = \begin{array}{|cccc|} \hline 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ \hline \end{array}$$

The idea of Latin squares can be rewritten in such a way that we create a $(k+2) \times n^2$ array from k–$\mathrm{MOLS}(n)$. An **orthogonal array** OA$(k+2, n)$ is a $(k+2) \times n^2$ array with entries from an n-set S having the property that in any two rows, each pair of symbols from S occurs exactly once. Figure 6.3 is an OA$(5, 4)$ created from 3–$\mathrm{MOLS}(4)$. The first row indicates the row of the Latin square. The second row indicates the column of the Latin square. The third, forth, and fifth rows are the elements in the first, second, and third Latin square respectively. So column five of Figure 6.3 would tell us in row two column one of the first, second, and third Latin squares the entries 4, 3, and 2 appear respectively.

1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4
1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
1	2	3	4	4	3	2	1	2	1	4	3	3	4	1	2
1	2	3	4	3	4	1	2	4	3	2	1	2	1	4	3
1	2	3	4	2	1	4	3	3	4	1	2	4	3	2	1

Figure 6.3: An OA$(5, 4)$

In a similar way as we went from 3–$\mathrm{MOLS}(4)$ to an OA$(5, 4)$ we may create a transversal design from an orthogonal array. A *transversal design* of order or group size n, block size k, and index λ, denoted TD$_\lambda(k, n)$, is a triple $(V, \mathcal{G}, \mathcal{B})$, where

1. $\mathcal{V}$ is a set of kn elements;
2. $\mathcal{G}$ is a partition of $\mathcal{V}$ into k classes (the groups), each of size n;
3. $\mathcal{B}$ is a collection of k-subsets of $\mathcal{V}$ (the blocks);
4. every unordered pair of elements from $\mathcal{V}$ is contained either in exactly one group or in exactly λ blocks, but not both.

When $\lambda = 1$, one simply writes TD(k, n).

On the element set $\{1, 2, 3, 4\} \times \{1, 2, 3, 4, 5\}$, the blocks of a TD$(5, 4)$ derived from an OA$(5, 4)$ are shown below. To go from the OA$(5, 4)$ to a TD$(5, 4)$ we let each column of the OA$(5, 4)$ be a block and list elements in the block as the element from the OA joined with the number denoting the row.

$$
\begin{array}{llll}
\{11,12,13,14,15\} & \{11,22,23,24,25\} & \{11,32,33,34,35\} & \{11,42,43,44,45\} \\
\{21,12,43,34,25\} & \{21,22,33,44,15\} & \{21,32,23,14,45\} & \{21,42,13,24,35\} \\
\{31,12,23,44,35\} & \{31,22,13,34,45\} & \{31,32,43,24,15\} & \{31,42,33,14,25\} \\
\{41,12,33,24,45\} & \{41,22,43,14,35\} & \{41,32,13,44,25\} & \{41,42,23,34,15\}
\end{array}
$$

6.2 Background and Terms

A $k \times n$ *Latin rectangle* is a $k \times n$ array (where $k \leq n$) in which each cell contains a single symbol from an n-set S, such that each symbol occurs exactly once in each row and at most once in each column. Two Latin rectangles are *orthogonal* if when superimposed no ordered pair of symbols appears more than once. A set of $m \times n$ Latin rectangles is *mutually orthogonal*, or a $\mathrm{MOLR}(m, n)$, if every two Latin rectangles in the set are orthogonal. The maximum number of $\mathrm{MOLR}(m, n)$ is denoted $N_{\mathrm{MOLR}}(m, n)$.

Given positive integers a, b and n, an *equitable* $(a, b; n)$*–rectangle* is an $a \times b$ array, L, with entries from a n–set S, such that the following two properties are satisfied:

1. every symbol $s \in S$ occurs either $\lceil \frac{b}{n} \rceil$ or $\lfloor \frac{b}{n} \rfloor$; and times in each row of L.
2. every symbol $s \in S$ occurs either $\lceil \frac{a}{n} \rceil$ or $\lfloor \frac{a}{n} \rfloor$ times in each column of L.

An equitable $(a, b; n)$–rectangle is *row–regular* if $n|b$, and it is *column–regular* if $n|a$. It is *regular* if it is both row– and column–regular. In a row–regular $(a, b; n)$–rectangle, every symbol occurs exactly $\frac{b}{n}$ times in each row; in a column–regular $(a, b; n)$–rectangle, every symbol occurs exactly $\frac{a}{n}$ times in each column.

Notice that an equitable $(a, b; b)$–rectangle with $a \leq b$ is a Latin rectangle, and a Latin rectangle with $a = b$ is the same thing as a Latin square of side a.

Suppose that L is an equitable $(a, b; n)$–rectangle on symbol set S and R is an equitable $(a, b; n')$–rectangle on symbol set S', where $ab = nn'$. We say that L and R are *orthogonal* provided that, for every ordered pair $(s, s') \in S \times S'$, there is a unique cell C such that $L(C) = s$ and $R(C) = s'$. (Equivalently, the superposition of L and R yields every ordered pair of symbols in $S \times S'$.) It is easy to see that orthogonal equitable $(a, a; a)$–rectangles are identical to orthogonal Latin squares of order a. Pairs of orthogonal equitable Latin rectangles were introduced in [45]. A complete solution for the existence of these rectangles was given in [14].

Now suppose that L_1 is an equitable $(a, b; n)$–rectangle on symbol set S, L_2 is an equitable $(a, b; n')$–rectangle on symbol set S', and L_3 is an equitable $(a, b; n'')$–rectangle on symbol set S'' where $ab = nn' = n'n'' = nn''$. Then it follows that $n = n' = n''$. Therefore, a set of k *mutually orthogonal equitable Latin rectangles*, or a k–$\mathrm{MOELR}(a, b; n)$ is a set of k pairwise equitable $(a, b; n)$–rectangles on a symbol set S where $ab = n^2$. We will denote the maximum number of $\mathrm{MOELR}(a, b; n)$ by $N_{\mathrm{MOELR}}(a, b; n)$. For the remainder of the chapter we will say Latin rectangles instead of equitable Latin rectangles for simplicity.

To enhance the importance of mutually orthogonal equitable Latin rectangles, we will talk about mix functions and their applications. To start, we defined mix functions as

Ristenpart and Rogaway did in [43] as follows. Let $|S| = r$. Suppose the mapping $f : S \times S \to S \times S$, and denote $f(s_1, s_2) = (f_L(s_1, s_2), f_R(s_1, s_2))$ for all $s_1, s_2 \in S$. A *permutation* is a rearrangement of the elements in an ordered set. Suppose that the following properties are satisfied:

1. $f(\cdot, \cdot)$ is a permutation of $S \times S$
2. if $s_1 \in S$ is fixed, then $f_L(s_1, \cdot)$ is a permutation of S
3. if $s_1 \in S$ is fixed, then $f_R(s_1, \cdot)$ is a permutation of S
4. if $s_2 \in S$ is fixed, then $f_L(\cdot, s_2)$ is a permutation of S
5. if $s_2 \in S$ is fixed, then $f_R(\cdot, s_2)$ is a permutation of S.

Then we say that f is a $\text{MIX}(r)$ function.

In [43] it was observed that by using orthogonal Latin squares of order r we can construct $\text{MIX}(r)$ functions. Not only this, but the converse is also true.

Theorem 6.2. *[45] Suppose that $|S| = r$, $f : S \times S \to S \times S$, and let L and R be the $S \times S$ arrays defined by $L[s_1, s_2] = f_L(s_1, s_2)$ and $R[s_1, s_2] = f_R(s_1, s_2)$. Define two $r \times r$ arrays $L = (\lambda_{s_1,s_2})$ and $R = (\rho_{s_1,s_2})$ by the rules $\lambda_{s_1,s_2} = f_L(s_1, s_2)$ and $\rho_{s_1,s_2} = f_R(s_1, s_2)$ for all s_1, s_2. Then f is a $\text{MIX}(r)$ function if and only if L and R are orthogonal Latin squares of order r.*

We now extend this definition of mix functions to generalized mix functions. Because mix functions have cryptographic applications from [43], generalized mix functions have potential applications in this field too. Which in turn may provide very useful information to allow mutually orthogonal equitable Latin rectangles to have cryptographic applications in the same manner.

In [14] a type of orthogonal equitable rectangle was studied called *orthogonal generalized equitable rectangles* (OGER). An OGER is defined as follows. Suppose r, t, s_1, s_2 are positive integers such that $rt = s_1 s_2$. An OGER is a pair (A, B) of two $r \times t$ rectangles satisfying the following properties:

1. $A = (a_{i,j})$ is defined on a set S_1 of s_1 symbols and $B = (b_{i,j})$ is defined on a set of S_2 of s_2 symbols, where $s_1 s_2 = rt$.
2. A and B are equitable on rows and equitable on columns.
3. A and B are orthogonal.

We denote A and B as $(r, t; s_1, s_2)$–OGER as in [14].

Example 6.3. *A* $(2, 6; 3, 4)$–OGER.

1	1	2	2	3	3
2	2	3	3	1	1

1	2	1	2	3	4
3	4	2	1	4	3

A striking difference between OGERs and MOELRs is that OGERs have symbol sets of different sizes defined on different arrays while MOELRs require a fixed symbol set among all of the arrays.

6.3 Small Cases

Example 6.4. *A* 3–$\mathrm{MOELR}(9, 16; 12)$.

$$M_1 = \begin{array}{|cccccccccccccccc|}
\hline
1 & 3 & 4 & 2 & 1 & 3 & 4 & 2 & 7 & 9 & 10 & 8 & 7 & 9 & 10 & 8 \\
2 & 5 & 6 & 1 & 2 & 5 & 6 & 1 & 8 & 11 & 12 & 7 & 8 & 11 & 12 & 7 \\
3 & 6 & 1 & 4 & 3 & 6 & 1 & 4 & 9 & 12 & 7 & 10 & 9 & 12 & 7 & 10 \\
4 & 2 & 5 & 1 & 4 & 2 & 5 & 1 & 10 & 8 & 11 & 7 & 10 & 8 & 11 & 7 \\
5 & 1 & 3 & 6 & 5 & 1 & 3 & 6 & 11 & 7 & 9 & 12 & 11 & 7 & 9 & 12 \\
6 & 2 & 1 & 5 & 6 & 2 & 1 & 5 & 12 & 8 & 7 & 11 & 12 & 8 & 7 & 11 \\
3 & 4 & 2 & 6 & 3 & 4 & 2 & 6 & 9 & 10 & 8 & 12 & 9 & 10 & 8 & 12 \\
5 & 6 & 4 & 3 & 5 & 6 & 4 & 3 & 11 & 12 & 10 & 9 & 11 & 12 & 10 & 9 \\
2 & 4 & 3 & 5 & 2 & 4 & 3 & 5 & 8 & 10 & 9 & 11 & 8 & 10 & 9 & 11 \\
\hline
\end{array}$$

$$M_2 = \begin{array}{|cccccccccccccccc|}
\hline
1 & 2 & 5 & 3 & 7 & 8 & 11 & 9 & 1 & 2 & 5 & 3 & 7 & 8 & 11 & 9 \\
2 & 6 & 3 & 5 & 8 & 12 & 9 & 11 & 2 & 6 & 3 & 5 & 8 & 12 & 9 & 11 \\
3 & 1 & 4 & 6 & 9 & 7 & 10 & 12 & 3 & 1 & 4 & 6 & 9 & 7 & 10 & 12 \\
4 & 5 & 1 & 6 & 10 & 11 & 7 & 12 & 4 & 5 & 1 & 6 & 10 & 11 & 7 & 12 \\
5 & 3 & 6 & 4 & 11 & 9 & 12 & 10 & 5 & 3 & 6 & 4 & 11 & 9 & 12 & 10 \\
6 & 4 & 2 & 3 & 12 & 10 & 8 & 9 & 6 & 4 & 2 & 3 & 12 & 10 & 8 & 9 \\
4 & 1 & 6 & 2 & 10 & 7 & 12 & 8 & 4 & 1 & 6 & 2 & 10 & 7 & 12 & 8 \\
2 & 5 & 3 & 1 & 8 & 11 & 9 & 7 & 2 & 5 & 3 & 1 & 8 & 11 & 9 & 7 \\
1 & 2 & 5 & 4 & 7 & 8 & 11 & 10 & 1 & 2 & 5 & 4 & 7 & 8 & 11 & 10 \\
\hline
\end{array}$$

$$M_3 = \begin{array}{|cccccccccccccccc|}
\hline
1 & 4 & 3 & 6 & 7 & 10 & 9 & 12 & 7 & 10 & 9 & 12 & 1 & 4 & 3 & 6 \\
2 & 3 & 5 & 4 & 8 & 9 & 11 & 10 & 8 & 9 & 11 & 10 & 2 & 3 & 5 & 4 \\
3 & 4 & 6 & 2 & 9 & 10 & 12 & 8 & 9 & 10 & 12 & 8 & 3 & 4 & 6 & 2 \\
4 & 1 & 2 & 5 & 10 & 7 & 8 & 11 & 10 & 7 & 8 & 11 & 4 & 1 & 2 & 5 \\
5 & 2 & 1 & 3 & 11 & 8 & 7 & 9 & 11 & 8 & 7 & 9 & 5 & 2 & 1 & 3 \\
6 & 5 & 3 & 4 & 12 & 11 & 9 & 10 & 12 & 11 & 9 & 10 & 6 & 5 & 3 & 4 \\
2 & 6 & 4 & 1 & 8 & 12 & 10 & 7 & 8 & 12 & 10 & 7 & 2 & 6 & 4 & 1 \\
6 & 2 & 1 & 5 & 12 & 8 & 7 & 11 & 12 & 8 & 7 & 11 & 6 & 2 & 1 & 5 \\
3 & 5 & 6 & 1 & 9 & 11 & 12 & 7 & 9 & 11 & 12 & 7 & 3 & 5 & 6 & 1 \\
\hline
\end{array}$$

It was known to Euler that a pair of orthogonal Latin squares of order 6 do not exist. However, Examples 6.5 and 6.6 show that there exists a 3–MOELR$(a, b; 6)$ for $a = 4, b = 9$ and $a = 3, b = 12$. When $a = 2$ and $b = 18$, the existence of a 3–MOELR$(2, 18; 6)$ is unresolved. See Lemma 6.27 for more on this.

Example 6.5. *A* 3–MOELR$(4, 9; 6)$.

$$M_1 = \begin{array}{|ccccccccc|}\hline 1 & 2 & 3 & 4 & 5 & 6 & 3 & 5 & 2 \\ 3 & 5 & 6 & 2 & 1 & 2 & 4 & 6 & 4 \\ 4 & 6 & 1 & 5 & 3 & 1 & 2 & 4 & 3 \\ 2 & 1 & 4 & 1 & 6 & 5 & 6 & 3 & 5 \\ \hline\end{array}$$

$$M_2 = \begin{array}{|ccccccccc|}\hline 1 & 2 & 3 & 4 & 5 & 6 & 4 & 2 & 1 \\ 2 & 6 & 1 & 5 & 3 & 4 & 1 & 5 & 2 \\ 5 & 3 & 4 & 1 & 6 & 2 & 6 & 3 & 5 \\ 3 & 5 & 6 & 6 & 4 & 3 & 2 & 1 & 4 \\ \hline\end{array}$$

$$M_3 = \begin{array}{|ccccccccc|}\hline 1 & 2 & 3 & 4 & 5 & 6 & 2 & 6 & 3 \\ 4 & 3 & 4 & 1 & 2 & 5 & 6 & 2 & 5 \\ 3 & 5 & 6 & 2 & 1 & 3 & 4 & 1 & 6 \\ 6 & 4 & 2 & 5 & 3 & 4 & 1 & 5 & 1 \\ \hline\end{array}$$

Example 6.6. *A* 3–MOELR$(3, 12; 6)$

$$M_1 = \begin{array}{|cccccccccccc|}\hline 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & 5 & 6 & 6 \\ 2 & 2 & 1 & 1 & 4 & 4 & 3 & 3 & 6 & 6 & 5 & 5 \\ 3 & 4 & 3 & 6 & 5 & 6 & 5 & 2 & 4 & 1 & 2 & 1 \\ \hline\end{array}$$

$$M_2 = \begin{array}{|cccccccccccc|}\hline 1 & 2 & 1 & 2 & 4 & 6 & 2 & 4 & 3 & 5 & 5 & 6 \\ 2 & 6 & 3 & 6 & 3 & 5 & 5 & 1 & 4 & 1 & 2 & 4 \\ 3 & 1 & 2 & 2 & 6 & 3 & 1 & 5 & 6 & 4 & 4 & 5 \\ \hline\end{array}$$

$$M_3 = \begin{array}{|cccccccccccc|}\hline 1 & 3 & 3 & 6 & 2 & 1 & 5 & 4 & 4 & 5 & 2 & 6 \\ 2 & 4 & 2 & 5 & 1 & 3 & 6 & 5 & 3 & 4 & 6 & 1 \\ 3 & 6 & 4 & 1 & 3 & 5 & 2 & 1 & 2 & 6 & 5 & 4 \\ \hline\end{array}$$

Let $\mathbb{F}_n$ be a finite field of order n. For each $f \in \mathbb{F}_n, f \neq 0$, define the $\mathbb{F}_n \times \mathbb{F}_n$ matrix A_f by $A_f[x, y] = fx + y$ for all $x, y \in \mathbb{F}_n$. Then $\cup_f A_f$ forms a set of $|\mathbb{F}_n|$–MOLS(n). This construction is referred to as the *finite field construction* for MOLS (see [38]).

If M is a Latin rectangle, we denote $M\backslash$Row i as the Latin rectangle obtained by removing Row i from M.

Example 6.7. *A* 3–MOELR$(4, 25; 10)$.

Let A_1, A_2, A_3 be 3–MOLS(5) constructed using the finite field construction of sets of MOLS on the set of symbols $A = \{a_1, \cdots, a_5\}$. Let B_1, B_2, B_3 be a copy of this set of 3–MOLS(5) on the set of symbols $B = \{b_1, \cdots, b_5\}$. Let

$$M_1 = [A_1\backslash\text{Row 5}, A_2\backslash\text{Row 5}, B_1\backslash\text{Row 5}, B_2\backslash\text{Row 5}, C_1]$$
$$M_2 = [A_2\backslash\text{Row 5}, B_1\backslash\text{Row 5}, A_3\backslash\text{Row 5}, B_3\backslash\text{Row 5}, C_2]$$
$$M_3 = [B_3\backslash\text{Row 5}, A_3\backslash\text{Row 5}, A_2\backslash\text{Row 5}, B_1\backslash\text{Row 5}, C_3]$$

where

$$C_1 = \begin{bmatrix} \text{Row 5} & \text{of} & A_1 \\ \text{Row 5} & \text{of} & A_2 \\ \text{Row 5} & \text{of} & B_1 \\ \text{Row 5} & \text{of} & B_2 \end{bmatrix}, C_2 = \begin{bmatrix} \text{Row 5} & \text{of} & A_2 \\ \text{Row 5} & \text{of} & B_1 \\ \text{Row 5} & \text{of} & A_3 \\ \text{Row 5} & \text{of} & B_3 \end{bmatrix},$$
$$C_3 = \begin{bmatrix} \text{Row 5} & \text{of} & B_3 \\ \text{Row 5} & \text{of} & A_3 \\ \text{Row 5} & \text{of} & A_2 \\ \text{Row 5} & \text{of} & B_1 \end{bmatrix}.$$

Then M_1, M_2, M_3 form a 3–MOELR$(4, 25; 10)$. For each pair of rectangles M_i, M_j for $i, j, \in \{1, 2, 3\}$, we have the property that every ordered pair of each of the following types occurs exactly once: (a_i, a_j), (a_i, b_j), (b_i, b_j), (b_i, a_j) for all $i, j \in \{1, \cdots, 5\}$. Therefore, orthogonality holds. Consider any rectangle M_i for $i \in \{1, 2, 3\}$. In any row of this rectangle, either the symbols from set A each occur 2 times and the symbols from the set B occur 3 times, or vice versa. Furthermore, because we used the finite field construction to construct our set of MOLS(5), we have that the pair (k, k) for $k = 0, 1, 2, 3, 4$ has appeared in row 1 for each pair of squares. Therefore, no symbol will be repeated in any column of any of the C_i for $i \in \{1, 2, 3\}$.

We give a 3–MOELR$(4, 25; 10)$ in Figure 6.4.

Example 6.8. *A* 3–MOELR$(4, 49; 14)$.

Let A_1, A_2, A_3 be 3–MOLS(7) constructed using the finite field construction of sets of MOLS on the set of symbols $A = \{a_1, \cdots, a_7\}$. Let B_1, B_2, B_3 be a copy of this set of 3 MOLS(7) on the set of symbols $B = \{b_1, \cdots, b_7\}$. Let

$$M_1 = [A_1\backslash\text{Rows 5,6,7}, A_2\backslash\text{Rows 5,6,7}, B_1\backslash\text{Row 5,6,7}, B_2\backslash\text{Rows 5,6,7}, C_1]$$
$$M_2 = [A_2\backslash\text{Rows 5,6,7}, B_1\backslash\text{Rows 5,6,7}, A_3\backslash\text{Rows 5,6,7}, B_3\backslash\text{Rows 5,6,7}, C_2]$$
$$M_3 = [B_3\backslash\text{Rows 5,6,7}, A_3\backslash\text{Rows 5,6,7}, A_2\backslash\text{Rows 5,6,7}, B_1\backslash\text{Rows 5,6,7}, C_3]$$

where

$$C_1 = \begin{array}{|lll lll lll|}\hline \text{Row 5} & \text{of} & A_1 & \text{Row 6} & \text{of} & A_2 & \text{Row 5} & \text{of} & B_1 \\ \text{Row 6} & \text{of} & A_1 & \text{Row 7} & \text{of} & A_2 & \text{Row 5} & \text{of} & B_2 \\ \text{Row 7} & \text{of} & A_1 & \text{Row 6} & \text{of} & B_1 & \text{Row 6} & \text{of} & B_2 \\ \text{Row 5} & \text{of} & A_2 & \text{Row 7} & \text{of} & B_1 & \text{Row 7} & \text{of} & B_2 \\ \hline \end{array}$$

$$C_2 = \begin{array}{|lll lll lll|}\hline \text{Row 5} & \text{of} & A_2 & \text{Row 6} & \text{of} & B_1 & \text{Row 5} & \text{of} & A_3 \\ \text{Row 6} & \text{of} & A_2 & \text{Row 7} & \text{of} & B_1 & \text{Row 5} & \text{of} & B_3 \\ \text{Row 7} & \text{of} & A_2 & \text{Row 6} & \text{of} & A_3 & \text{Row 6} & \text{of} & B_3 \\ \text{Row 5} & \text{of} & B_1 & \text{Row 7} & \text{of} & A_3 & \text{Row 7} & \text{of} & B_3 \\ \hline \end{array}$$

$$C_3 = \begin{array}{|lll lll lll|}\hline \text{Row 5} & \text{of} & B_3 & \text{Row 6} & \text{of} & A_3 & \text{Row 5} & \text{of} & A_2 \\ \text{Row 6} & \text{of} & B_3 & \text{Row 7} & \text{of} & A_3 & \text{Row 5} & \text{of} & B_1 \\ \text{Row 7} & \text{of} & B_3 & \text{Row 6} & \text{of} & A_2 & \text{Row 6} & \text{of} & B_1 \\ \text{Row 5} & \text{of} & A_3 & \text{Row 7} & \text{of} & A_2 & \text{Row 7} & \text{of} & B_1 \\ \hline \end{array}.$$

Then M_1, M_2, M_3 form a 3–MOELR$(4, 49; 14)$. For each pair of rectangles M_i, M_j for $i, j, \in \{1, 2, 3\}$, we have the property that every ordered pair of each of the following types occurs exactly once: (a_i, a_j), (a_i, b_j), (b_i, b_j), (b_i, a_j) for all $i, j \in \{1, \cdots, 7\}$. Therefore, orthogonality holds. Consider any rectangle M_i for $i \in \{1, 2, 3\}$. In any row of this rectangle, either the symbols from set A each occur 4 times and the symbols from the set B occur 3 times, or vice versa. Furthermore, because we used the finite field construction to construct our set of 3–MOLS(7), we can be sure that no symbol is repeated in any column of any of the C_i for $i \in \{1, 2, 3\}$. This is because $2 \cdot 4 \neq 4, 5$ or $6 \pmod 7$, so columns 1-7 of C_1 contain no repeated symbols. For columns 8-14 of C_2 there are no repeated symbols because $2 \cdot 5 \not\equiv 3 \cdot 5$ or $3 \cdot 6 \pmod 7$ and $2 \cdot \not\equiv 3 \cdot 5$ or $3 \cdot 6 \pmod 7$. Similarly, $4 \not\equiv 2 \cdot 4$ or $2 \cdot 5$ or $2 \cdot 6 \pmod 7$, so columns 15-21 of C_1 contain no repeated symbols. It is easy to see that no symbol is repeated in any of the other columns of M_i.

We give a 3–MOELR$(4, 49; 14)$ in Figure 6.5.

$$M_1 = \begin{array}{|ccccccccccccccccccccccccc|}\hline 0&1&2&3&4&0&1&2&3&4&5&6&7&8&9&5&6&7&8&9&4&0&1&2&3 \\ 1&2&3&4&0&2&3&4&0&1&6&7&8&9&5&7&8&9&5&6&3&4&0&1&2 \\ 2&3&4&0&1&4&0&1&2&3&7&8&9&5&6&9&5&6&7&8&9&5&6&7&8 \\ 3&4&0&1&2&1&2&3&4&0&8&9&5&6&7&6&7&8&9&5&8&9&5&6&7 \\ \hline \end{array}$$

$$M_2 = \begin{array}{|ccccccccccccccccccccccccc|}\hline 0&1&2&3&4&5&6&7&8&9&0&1&2&3&4&5&6&7&8&9&3&4&5&6&7 \\ 2&3&4&0&1&6&7&8&9&5&3&4&0&1&2&8&9&5&6&7&9&5&6&7&8 \\ 4&0&1&2&3&7&8&9&5&6&1&2&3&4&0&6&7&8&9&5&2&3&4&0&1 \\ 1&2&3&4&0&8&9&5&6&7&4&0&1&2&3&9&5&6&7&8&7&8&9&5&6 \\ \hline \end{array}$$

$$M_3 = \begin{array}{|ccccccccccccccccccccccccc|}\hline 5&6&7&8&9&0&1&2&3&4&0&1&2&3&4&5&6&7&8&9&7&8&9&6&7 \\ 8&9&5&6&7&3&4&0&1&2&2&3&4&0&1&6&7&8&9&5&2&3&4&0&1 \\ 6&7&8&9&5&1&2&3&4&0&4&0&1&2&3&7&8&9&5&6&3&4&0&1&2 \\ 9&5&6&7&8&4&0&1&2&3&1&2&3&4&0&8&9&5&6&7&9&5&6&7&8 \\ \hline \end{array}$$

Figure 6.4: A 3–MOELR$(4, 25; 10)$

$$M_1 = \begin{array}{|ccccccccccccccccccccccccccccccccccccccccccccccccc|}\hline
0 & 1 & 2 & 3 & 4 & 5 & 6 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 4 & 5 & 6 & 0 & 1 & 2 & 3 & 3 & 4 & 5 & 6 & 0 & 1 & 2 & 11 & 12 & 13 & 7 & 8 & 9 & 10 \\
1 & 2 & 3 & 4 & 5 & 6 & 0 & 2 & 3 & 4 & 5 & 6 & 0 & 1 & 8 & 9 & 10 & 11 & 12 & 13 & 7 & 9 & 10 & 11 & 12 & 13 & 7 & 8 & 5 & 6 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 0 & 1 & 2 & 3 & 4 & 8 & 9 & 10 & 11 & 12 & 13 & 7 \\
2 & 3 & 4 & 5 & 6 & 0 & 1 & 4 & 5 & 6 & 0 & 1 & 2 & 3 & 9 & 10 & 11 & 12 & 13 & 7 & 8 & 11 & 12 & 13 & 7 & 8 & 9 & 10 & 6 & 0 & 1 & 2 & 3 & 4 & 5 & 12 & 13 & 7 & 8 & 9 & 10 & 11 & 10 & 11 & 12 & 13 & 7 & 8 & 9 \\
3 & 4 & 5 & 6 & 0 & 1 & 2 & 6 & 0 & 1 & 2 & 3 & 4 & 5 & 10 & 11 & 12 & 13 & 7 & 8 & 9 & 13 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 0 & 13 & 7 & 8 & 9 & 10 & 11 & 12 & 12 & 13 & 7 & 8 & 9 & 10 & 11 \\ \hline
\end{array}$$

$$M_2 = \begin{array}{|ccccccccccccccccccccccccccccccccccccccccccccccccc|}\hline
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 1 & 2 & 3 & 4 & 5 & 6 & 0 & 12 & 13 & 7 & 8 & 9 & 10 & 11 & 5 & 6 & 0 & 1 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 & 6 & 0 & 1 & 8 & 9 & 10 & 11 & 12 & 13 & 7 & 3 & 4 & 5 & 6 & 0 & 1 & 2 & 10 & 11 & 12 & 13 & 7 & 8 & 9 & 3 & 4 & 5 & 6 & 0 & 1 & 2 & 13 & 7 & 8 & 9 & 10 & 11 & 12 & 12 & 13 & 7 & 8 & 9 & 10 & 11 \\
4 & 5 & 6 & 0 & 1 & 2 & 3 & 9 & 10 & 11 & 12 & 13 & 7 & 8 & 6 & 0 & 1 & 2 & 3 & 4 & 5 & 13 & 7 & 8 & 9 & 10 & 11 & 12 & 5 & 6 & 0 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 5 & 6 & 0 & 8 & 9 & 10 & 11 & 12 & 13 & 7 \\
6 & 0 & 1 & 2 & 3 & 4 & 5 & 10 & 11 & 12 & 13 & 7 & 8 & 9 & 2 & 3 & 4 & 5 & 6 & 0 & 1 & 9 & 10 & 11 & 12 & 13 & 7 & 8 & 11 & 12 & 13 & 7 & 8 & 9 & 10 & 4 & 5 & 6 & 0 & 1 & 2 & 3 & 11 & 12 & 13 & 7 & 8 & 9 & 10 \\ \hline
\end{array}$$

$$M_3 = \begin{array}{|ccccccccccccccccccccccccccccccccccccccccccccccccc|}\hline
7 & 8 & 9 & 10 & 11 & 12 & 13 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 12 & 13 & 7 & 8 & 9 & 10 & 11 & 1 & 2 & 3 & 4 & 5 & 6 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 0 \\
10 & 11 & 12 & 13 & 7 & 8 & 9 & 3 & 4 & 5 & 6 & 0 & 1 & 2 & 2 & 3 & 4 & 5 & 6 & 0 & 1 & 8 & 9 & 10 & 11 & 12 & 13 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 7 & 4 & 5 & 6 & 0 & 1 & 2 & 3 & 11 & 12 & 13 & 7 & 8 & 9 & 10 \\
13 & 7 & 8 & 9 & 10 & 11 & 12 & 6 & 0 & 1 & 2 & 3 & 4 & 5 & 4 & 5 & 6 & 0 & 1 & 2 & 3 & 9 & 10 & 11 & 12 & 13 & 7 & 8 & 11 & 12 & 13 & 7 & 8 & 9 & 10 & 3 & 4 & 5 & 6 & 0 & 1 & 2 & 12 & 13 & 7 & 8 & 9 & 10 & 11 \\
9 & 10 & 11 & 12 & 13 & 7 & 8 & 2 & 3 & 4 & 5 & 6 & 0 & 1 & 6 & 0 & 1 & 2 & 3 & 4 & 5 & 10 & 11 & 12 & 13 & 7 & 8 & 9 & 5 & 6 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 0 & 1 & 2 & 3 & 4 & 13 & 7 & 8 & 9 & 10 & 11 & 12 \\ \hline
\end{array}$$

Figure 6.5: A 3–MOELR$(4, 49; 14)$

6.4 Computer Construction of MOELR

The construction of MOELRs given in Examples 6.5 and 6.6 was accomplished by a program coded in C++. The following is the description of the different parts of the programs created.

Note that the programs were designed only to find 3–MOELR$(a, b; n)$. The program was separated into 2 stages: a first stage program that generated the triples used in the 3 corresponding matrices and a second stage program to place the triples in matrices, check which are unique, and record unique up to isomorphism 3–MOELR in a list.

The second program used recursion to exhaustively generate all possible combinations of entries in the matrix. Once a single solution was found, it was printed and placed in a set that could be called later to check whether any new solution was isomorphic to a previous solution. To check for isomorphisms, the new solution and all old solutions were permuted in a way that set the first column and first row in lexicographical order. Once in lexicographical order, the two solutions were compared to see if a one to one mapping exists. Note that only a single set of triples from the first program was used to generate all the solutions. That means, even if the program found all possible solutions for this triple set, there are more triple sets that could be used.

These programs were used to find a 3–MOELR$(3, 12; 6)$ and a 3–MOELR$(4, 9; 6)$.

6.5 Supporting Lemmas

Notice that if we take the transpose of each rectangle in a k-MOELR$(a, b; n)$, then the equitability and orthogonality properties still hold. Thus we have the following result.

Lemma 6.9. *If there exists a k–MOELR$(a, b; n)$, then there exists a k–MOELR$(b, a; n)$.*

The next result is very similar to the result for MOLS that says that if there exists k–MOLS(n), then $k \leq n - 1$ (see [19]).

Lemma 6.10. *If there exists a k–MOELR$(a, b; n)$, then $k \leq n - 1$.*

Proof: Without loss, we can assume that the first n entries in the first row of each rectangle is $1, 2, \cdots, n$ (i.e. $L_j(1, i) = i$ is the first row and i^{th} column in the j^{th} rectangle for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, k$). Then $L_j(2, 1) \neq 1$ for all $j = 1, 2, \cdots, k$, because 1 has already appeared in the first column of every rectangle. Let $L_1(2, 1) = x_1$. Then $L_j(2, 1) \neq x_1$ for all $j = 2, 3, \cdots, k$ because the ordered pair (x_1, x_1) has already appeared for each pair of rectangles in row 1 (i.e. $L_j(1, i) = L_h(1, i) = i$ for all $j, h \in \{1, 2, \cdots, k\}$ and $i = 1, 2, \cdots, n$). Furthermore, $L_j(2, 1) \neq L_h(2, 1)$ for all $j, h \in \{2, 3, \cdots, k\}$ for the same reason. Thus we have that each entry $L_j(2, 1)$ is unique and also $L_j(2, 1) \neq 1$ for all $j = 1, 2, \cdots, k$. Because there are only $n - 1$ ways to fill this entry in each rectangle, it follows that we can have at most $n - 1$ rectangles. Thus $k \leq n - 1$. ✠

Lemma 6.11. *If there exists k–MOLS(n) and $a|n$, then there exists a k–MOELR$(a, b; n)$.*

Proof: Suppose $a \leq b$ and $a|n$. Write $n = ax$ so $ab = n^2 = a^2x^2$; and therefore, $b = ax^2 = xn$. Let $A_1, A_2, \cdots, A_k$ be k–MOLS(n). We form the j^{th} rectangle M_j as follows.

$$M_j(i, tn + l) = A_j(ta + i, l)$$

$$\text{for } j = 1, 2, \cdots, k; \quad i = 1, 2, \cdots, a; \quad t = 0, 1, \cdots, x-1; \quad l = 1, 2, \cdots, n$$

This rectangle has a rows and $b = xn$ columns. Every symbol appears in each row x times and each column 0 or 1 times. Furthermore, every ordered pair occurs exactly once. Therefore, the M_j for $j = 1, 2, \cdots, k$ form a k–MOELR$(a, b; n)$. ✠

Example 6.12. *A* 2–$MOELR(6, 24; 12)$ *using Lemma 6.11*

We will construct a 2–MOELR$(6, 24; 12)$ instead of the 5–MOELR$(6, 24; 12)$ to make the example easier to understand. Following are the 2–MOLS(12).

$$A_1 = \begin{array}{|c|} \hline 0123456789ab \\ 123450789ab6 \\ 23450189ab67 \\ 3450129ab678 \\ 450123ab6789 \\ 501234b6789a \\ \hline 6789ab012345 \\ 789ab6123450 \\ 89ab67234501 \\ 9ab678345012 \\ ab6789450123 \\ b6789a501234 \\ \hline \end{array} \quad A_2 = \begin{array}{|c|} \hline 03619b28547a \\ 1472a639058b \\ 2583b74a1096 \\ 3094685b21a7 \\ 41a5790632b8 \\ 52b08a174369 \\ \hline 69073582ba14 \\ 7a1840936b25 \\ 8b2951a47630 \\ 963a02b58741 \\ a74b13609852 \\ b8562471a903 \\ \hline \end{array}$$

By performing Lemma 6.11 we adjoin the last six rows to the right of the first six rows to get a total of 6 rows and 24 columns. In Figure 6.6, we have 2–MOELR$(6, 24; 12)$.

$$M_1 = \begin{array}{|c|c|} \hline 0123456789ab & 6789ab012345 \\ 123450789ab6 & 789ab6123450 \\ 23450189ab67 & 89ab67234501 \\ 3450129ab678 & 9ab678345012 \\ 450123ab6789 & ab6789450123 \\ 501234b6789a & b6789a501234 \\ \hline \end{array} \quad M_2 = \begin{array}{|c|c|} \hline 03619b28547a & 69073582ba14 \\ 1472a639058b & 7a1840936b25 \\ 2583b74a1096 & 8b2951a47630 \\ 3094685b21a7 & 963a02b58741 \\ 41a5790632b8 & a74b13609852 \\ 52b08a174369 & b8562471a903 \\ \hline \end{array}$$

Figure 6.6: A 2–MOELR$(6, 24; 12)$

MacNeish [33] and Mann [34] showed that the maximum number of MOLS$(p^{(i+j)/2})$ is $p^{(i+j)/2} - 1$. Therefore, by applying Lemma 6.11, we have the following result.

Corollary 6.13. *There exists a complete set of mutually orthogonal equitable rectangles, or a* $(p^{(i+j)/2} - 1)$–MOELR$(p^i, p^j; p^{i+j})$ *for all* $i, j \geq 0$ *and* p *a prime.*

The next two results rely on the existence of orthogonal arrays. An orthogonal array, $\mathrm{OA}(k, n)$, is equivalent to $(k-2)$–$\mathrm{MOLS}(n)$ (see [19]).

Lemma 6.14. *If there exists an* $\mathrm{OA}(k, n_1)$ *and there exists a set of* k–$\mathrm{MOLS}(n_2)$*, then there exists a* k-$\mathrm{MOELR}(n_2, n_1^2 n_2; n_1 n_2)$.

Proof: Let $M_{i,j}$ represent the i^{th} square of the set of k-$\mathrm{MOLS}(n_2)$ on the j^{th} set of n_2 symbols, where $j = 1, 2, \cdots, n_1$ and $i = 1, 2, \ldots, k$. Form the array X as follows. On each entry of the $\mathrm{OA}(k, n_1)$, $O_{i,s}$, replace $O_{i,s}$ with $M_{i,O_{i,s}}$ for $i = 1, 2, \ldots, k$ and $s = 1, 2, \cdots, n_1^2$. Now each row of X corresponds to an $n_2 \times n_1^2 n_2$ rectangle on $n_1 n_2$ symbols.

For any two rows of X, $i_1, i_2 = 1, \ldots, k$, we have every ordered pair $(M_{i_1,j_1} M_{i_2,j_2})$ for $j_1, j_2 \in \{1, 2, \ldots, n_1\}$ exactly once because X is an $\mathrm{OA}(k, n_1)$. Furthermore, because M_{i_1,j_1} and M_{i_2,j_2} are orthogonal, we see every ordered pair among their n_2 symbols exactly once. Therefore, we see every ordered pair on $n_1 n_2$ symbols exactly once among the two rows of X. Consider some row i of X. This row corresponds to one of the $n_2 \times n_1^2 n_2$ rectangles. Each $M_{i,j}$ has n_2 rows and it was part of the set of k-$\mathrm{MOLS}(n_2)$, so each symbol occurs exactly once in each column. Furthermore, each $M_{i,j}$ is repeated n_1 times for $j = 1, 2, \ldots, n_1$. Therefore, each symbol occurs exactly n_1 times in each row. Thus, the k rows of X form a k–$\mathrm{MOELR}(n_2, n_1^2 n_2; n_1 n_2)$. ✠

Example 6.15. *A* 3–$\mathrm{MOELR}(5, 20; 10)$ *using Lemma 6.14*

Because there exists 2–$\mathrm{MOLS}(10)$, we could use Theorem 6.11 to obtain a 2–$\mathrm{MOELR}(5, 20; 10)$. However, there exists an $\mathrm{OA}(3, 2)$ and there exists 3–$\mathrm{MOLS}(5)$. Therefore, we can apply Theorem 6.14 to obtain a 3–$\mathrm{MOELR}(5, 20; 10)$. We construct the 3–$\mathrm{MOELR}(5, 20; 10)$ as follows. Suppose that the 3 squares of the 3–$\mathrm{MOLS}(5)$ are labeled $M_{1,1}, M_{2,1}, M_{3,1}$ and are defined on symbol set $S_1 = \{1, 2, 3, 4, 5\}$. Consider another copy of this set of MOLS on the symbol set $S_2 = \{1', 2', 3', 4', 5'\}$. So we have the copies $M_{1,2}, M_{2,2}, M_{3,2}$.

Let O be the given $\mathrm{OA}(3, 2)$.

$$O = \mathrm{OA}(3,2) = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 1 \end{bmatrix}$$

Following the proof of Theorem 6.14, we replace each entry of $O_{i,s}$ with $M_{i,O_{i,s}}$ for $i = 1, 2, 3$ and $s = 1, 2, 3, 4$. So,

$$O = \mathrm{OA}(3,2) = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 1 \end{bmatrix} \Rightarrow X = \begin{bmatrix} M_{1,1} & M_{1,1} & M_{1,2} & M_{1,2} \\ M_{2,1} & M_{2,2} & M_{2,1} & M_{2,2} \\ M_{3,1} & M_{3,2} & M_{3,2} & M_{3,1} \end{bmatrix}$$

Lemma 6.16. *Suppose there exists a set of* 3–$\mathrm{MOLS}(s)$. *Write* $s = ay + r$ *for some* $y \in \mathbb{Z}$ *and* $0 \le r < a$. *Then if* $n = 2s$ *and* $r = \frac{a}{4}$ *or* $r = \frac{3a}{4}$*, then there exists a* 3–$\mathrm{MOELR}(a, b; n)$ *for all* $a < s$ *and* $s | b$.

Proof: We have that $ab = n^2 = 4s^2$. Therefore $b = \frac{4s^2}{a} = \frac{4s \cdot s}{a}$. Because $s|b$, we have $\frac{b}{s} = \frac{4s}{a} \in \mathbb{Z}$. Now, because $\frac{4s}{a} \in \mathbb{Z}$, it follows that

$$\begin{aligned}\frac{4(ay+r)}{a} \in \mathbb{Z} \quad &\Rightarrow \quad \frac{4ay}{a} + \frac{4r}{a} \in \mathbb{Z} \\ &\Rightarrow \quad \frac{4r}{a} \in \mathbb{Z} \\ &\Rightarrow \quad r \cdot \frac{4}{a} = \frac{r}{a/4}\end{aligned}$$

Because $4|a$, we have that $\frac{a}{4}|r$. In our case, $r = \frac{a}{4}$ or $r = \frac{3a}{4}$.

Given a set of 3–MOLS(s), we do the following. Let S_1 and S_2 be a partition of a set of n symbols, where $|S_j| = s$ for $j = 1, 2$. Define $M_{i,j}$ to be the i^{th} Latin square of the set of 3–MOLS(s) on symbol set S_j. Let $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ be a 3–MOELR$(s, 4s; 2s)$ constructed by applying Lemma 6.14 with $k = 3, n_1 = 2$, and $n_2 = s$.

Let $X = [\mathcal{R}_1 \ \mathcal{R}_2 \ \mathcal{R}_3]^T$ be an array where the rows form this 3–MOELR$(s, 4s; 2s)$. Thus

$$X = \begin{array}{|c|}\hline \mathcal{R}_1 \\ \hline \mathcal{R}_2 \\ \hline \mathcal{R}_3 \\ \hline \end{array} = \begin{array}{|c|c|c|c|}\hline M_{1,1} & M_{1,1} & M_{1,2} & M_{1,2} \\ \hline M_{2,1} & M_{2,2} & M_{2,1} & M_{2,2} \\ \hline M_{3,1} & M_{3,2} & M_{3,2} & M_{3,1} \\ \hline \end{array}$$

Consider row $\mathcal{R}_i$ of X. For $t = 1, 2, \ldots, s$, let $r_{t,j}$ denote the t^{th} row of $M_{i,j}$. For example, we write $\mathcal{R}_1$ as in Figure 6.7.

We have that $\mathcal{R}_i$ is an $s \times 4$ array where each column represents a LS(s). Because $s = ay + r$, it follows that we can rearrange each column of $\mathcal{R}_i$ into a new structure which is an $a \times y$ array followed by an $r \times 1$ array.

Let $\mathcal{R}'_i$ be this arrangement of columns of $\mathcal{R}_i$. We give $\mathcal{R}'_1$ in Figure 6.8.

$$\mathcal{R}'_1 = \begin{array}{c} 1 \\ \begin{array}{|c|c|c|c|}\hline r_{1,1} & & r_{(y-1)a+1,1} & r_{ya+1,1} \\ \vdots & \cdots & \vdots & \vdots \\ \vdots & & \vdots & r_{s,1} \\ \cline{4-4} r_{a,1} & & r_{ya,1} & \\ \cline{1-3} \end{array} \end{array} \quad \begin{array}{c} 2 \\ \begin{array}{|c|c|c|c|}\hline r_{1,1} & & r_{(y-1)a+1,1} & r_{ya+1,1} \\ \vdots & \cdots & \vdots & \vdots \\ \vdots & & \vdots & r_{s,1} \\ \cline{4-4} r_{a,1} & & r_{ya,1} & \\ \cline{1-3} \end{array} \end{array}$$

$$\begin{array}{c} 3 \\ \begin{array}{|c|c|c|c|}\hline r_{1,2} & & r_{(y-1)a+1,2} & r_{ya+1,2} \\ \vdots & \cdots & \vdots & \vdots \\ \vdots & & \vdots & r_{s,2} \\ \cline{4-4} r_{a,2} & & r_{ya,2} & \\ \cline{1-3} \end{array} \end{array} \quad \begin{array}{c} 4 \\ \begin{array}{|c|c|c|c|}\hline r_{1,2} & & r_{(y-1)a+1,2} & r_{ya+1,2} \\ \vdots & \cdots & \vdots & \vdots \\ \vdots & & \vdots & r_{s,2} \\ \cline{4-4} r_{a,2} & & r_{ya,2} & \\ \cline{1-3} \end{array} \end{array}$$

Figure 6.8: $\mathcal{R}_1$ of matrix X.

$$
\mathcal{R}_1 = \begin{array}{|c|c|c|c|}\hline M_{1,1} & M_{1,1} & M_{1,2} & M_{1,2} \\ \hline \end{array} =
\begin{array}{r}
y\left\{ \begin{array}{r|c|c|c|c|}
\cline{2-5}
a\left\{ \begin{array}{c} \\ \\ \\ \\ \end{array} \right. & \begin{array}{c} r_{1,1} \\ r_{2,1} \\ \vdots \\ r_{a,1} \end{array} & \begin{array}{c} r_{1,1} \\ r_{2,1} \\ \vdots \\ r_{a,1} \end{array} & \begin{array}{c} r_{1,2} \\ r_{2,2} \\ \vdots \\ r_{a,2} \end{array} & \begin{array}{c} r_{1,2} \\ r_{2,2} \\ \vdots \\ r_{a,2} \end{array} \\
\cline{2-5}
a\left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. & \begin{array}{c} r_{a+1,1} \\ \vdots \\ r_{2a,1} \end{array} & \begin{array}{c} r_{a+1,1} \\ \vdots \\ r_{2a,1} \end{array} & \begin{array}{c} r_{a+1,2} \\ \vdots \\ r_{2a,2} \end{array} & \begin{array}{c} r_{a+1,2} \\ \vdots \\ r_{2a,2} \end{array} \\
\cline{2-5}
 & \vdots & \vdots & \vdots & \vdots \\
\cline{2-5}
a\left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. & \begin{array}{c} r_{(y-1)a+1,1} \\ \vdots \\ r_{ya,1} \end{array} & \begin{array}{c} r_{(y-1)a+1,1} \\ \vdots \\ r_{ya,1} \end{array} & \begin{array}{c} r_{(y-1)a+1,2} \\ \vdots \\ r_{ya,2} \end{array} & \begin{array}{c} r_{(y-1)a+1,2} \\ \vdots \\ r_{ya,2} \end{array} \\
\cline{2-5}
\end{array} \right. \\
\begin{array}{r|c|c|c|c|}
r\left\{ \begin{array}{c} \\ \\ \\ \\ \end{array} \right. & \begin{array}{c} r_{ya+1,1} \\ r_{ya+2,1} \\ \vdots \\ r_{s,1} \end{array} & \begin{array}{c} r_{ya+1,1} \\ r_{ya+2,1} \\ \vdots \\ r_{s,1} \end{array} & \begin{array}{c} r_{ya+1,2} \\ r_{ya+2,2} \\ \vdots \\ r_{s,2} \end{array} & \begin{array}{c} r_{ya+1,2} \\ r_{ya+2,2} \\ \vdots \\ r_{s,2} \end{array} \\
\cline{2-5}
\end{array}
\end{array}
$$

Figure 6.7: The array $\mathcal{R}_1$ from X.

Let $\mathcal{R}'_{i,j}$ be the $a \times y$ array from column j of $\mathcal{R}_i$. Let $z_{i,j}$ be the $r \times 1$ array from column j of $\mathcal{R}_i$.

Let v be the $4s$-tuple obtained by taking each entry from $\mathcal{R}'_i$ going down the columns. Thus, from $\mathcal{R}'_i$ we get

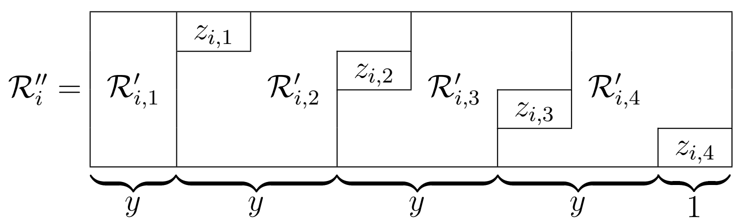
$$v = (r_{1,1}, r_{2,1}, \ldots, r_{s,1}, r_{1,1}, \ldots, r_{s,1}, r_{1,2}, \ldots, r_{s,2}, r_{1,2}, \ldots, r_{s,2}) .$$

Now write v as $v = \left(v_0, v_1, \ldots, v_{\frac{4s}{a}-1}\right)$ where each v_i has a entries.

Let $\mathcal{R}''_i = \left(v_0^T, v_1^T, \ldots, v^T_{\frac{4s}{a}-1}\right)$. We now show that $\mathcal{R}''_1, \mathcal{R}''_2, \mathcal{R}''_3$ form a 3–MOELR$(a, b; n)$. First note that $\mathcal{R}''_i$ is an $a \times \frac{4s}{a}$ array. However, each entry represents a row of an LS(s). Thus, $\mathcal{R}''_i$ contains $\left(\frac{4s}{a}\right) s = \frac{4s^2}{a} = b$ columns. Because we apply the same structuring to each $\mathcal{R}_i$, it follows that orthogonality holds.

Case 1: $r = \frac{a}{4}$

We have that $\mathcal{R}''_i$ has the following form.



Thus the number of columns in $\mathcal{R}''_i$ is

$$s(4y+1) = s\left(4\left(\frac{s-r}{a}\right)+1\right) = s\left(\frac{4s}{a} - \frac{4r}{a} + 1\right) = \frac{4s^2}{a} = b$$

It is easy to see that equitability holds in each column because each $z_{i,j}$ lies in a different column. Because $a < s$, it follows that each symbol appears exactly 0 or 1 time in each column.

Consider any row of $\mathcal{R}''_i$. In this row, each symbol appears $2y+1$ or $2y$ times. We have:

$$2y = 2\left(\frac{s-r}{a}\right) = \frac{2s}{a} - \frac{2r}{a} = \frac{2s}{a} - \frac{2}{a}\cdot\frac{a}{4} = \frac{2s}{a} - \frac{1}{2} = \left\lfloor \frac{2s}{a} \right\rfloor = \left\lfloor \frac{b}{n} \right\rfloor$$

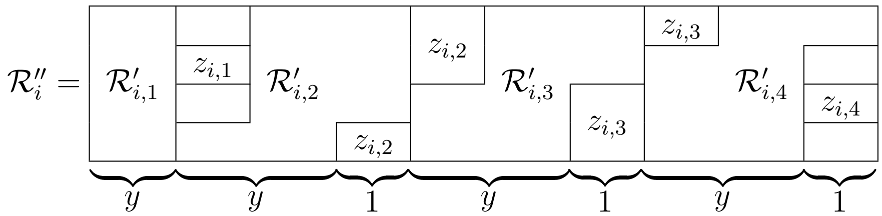
and

$$2y+1 = 2\left(\frac{s-r}{a}\right)+1 = \frac{2s}{a} - \frac{2r}{a} + 1 = \frac{2s}{a} - \frac{1}{2} + 1 = \frac{2s}{a} + \frac{1}{2} = \left\lceil \frac{2s}{a} \right\rceil = \left\lceil \frac{b}{n} \right\rceil,$$

thus equitability holds.

Case 2: $r = \frac{3a}{4}$

In this case, $\mathcal{R}''_i$ has the following form.



Thus the number of columns in $\mathcal{R}''_i$ is

$$s(4y+3) = s\left(4\left(\frac{s-r}{a}\right)+3\right) = s\left(\frac{4s}{a} - \frac{4r}{a} + 3\right) = \frac{4s^2}{a} = b.$$

Again, it is easy to see that equitability holds in each column. As in Case 1, each symbol appears $2y$ or $2y+1$ times in each row. ✠

Lemma 6.17. *Suppose there exists a set of* 3–MOLS(s). *Write $s = ay + r$ for some $y \in \mathbb{Z}$ and $0 \leq r < a$. Then if $n = 3s$ and $r = \frac{a}{9}$ or $r = \frac{8a}{9}$, then there exists a* 3–MOELR$(a, b; n)$ *for all $a < s$ and $s|b$.*

Proof: We have that $ab = n^2 = 9s^2$. Therefore $b = \frac{9s^2}{a} = \frac{9s \cdot s}{a}$ Because $s|b$, we have $\frac{b}{s} = \frac{9s}{a} \in \mathbb{Z}$. Now, because $\frac{9s}{a} \in \mathbb{Z}$, it follows that

$$\begin{aligned} \frac{9(ay+r)}{a} \in \mathbb{Z} \quad &\Rightarrow \quad \frac{9ay}{a} + \frac{9r}{a} \in \mathbb{Z} \\ &\Rightarrow \quad \frac{9r}{a} \in \mathbb{Z} \\ &\Rightarrow \quad r \cdot \frac{9}{a} = \frac{r}{a/9} \end{aligned}$$

Because $9|a$, we have that $\frac{a}{9}|r$. In our case, $r = \frac{a}{9}$ or $r = \frac{8a}{9}$. Given a set of 3–MOLS(s), we do the following. Let S_1, S_2 and S_3 be a partition of n symbols, where $|S_i| = s$ for $i = 1, 2, 3$.

Define $M_{i,j}$ to be the i^{th} Latin square of the set of 3–MOLS(s) on the symbol set S_j. Let $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ be a 3–MOELR$(s, 9s; 3s)$ constructed by applying Lemma 6.14 with $k = 3, n_1 = 3$, and $n_2 = s$.

Let $X = [\mathcal{R}_1 \ \mathcal{R}_2 \ \mathcal{R}_3]^T$ be an array where the rows form a 3–MOELR$(s, 9s; 3s)$. Thus

$$X = \begin{array}{|c|}\hline \mathcal{R}_1 \\ \hline \mathcal{R}_2 \\ \hline \mathcal{R}_3 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline M_{1,1} & M_{1,1} & M_{1,1} & M_{1,2} & M_{1,2} & M_{1,2} & M_{1,3} & M_{1,3} & M_{1,3} \\ \hline M_{2,1} & M_{2,2} & M_{2,3} & M_{2,1} & M_{2,2} & M_{2,3} & M_{2,1} & M_{2,2} & M_{2,3} \\ \hline M_{3,1} & M_{3,2} & M_{3,3} & M_{3,2} & M_{3,3} & M_{3,1} & M_{3,3} & M_{3,1} & M_{3,2} \\ \hline \end{array}$$

Consider row $\mathcal{R}_i$ of X. For $t = 1, 2, \ldots, s$, let $r_{t,j}$ denote the t^{th} row of $M_{i,j}$.

For example, we write $\mathcal{R}_1$ as in Figure 6.9.

We have that $\mathcal{R}_i$ is an $s \times 9$ array where each column represents an LS(s). Because $s = ay + r$, it follows that we can rearrange each column of $\mathcal{R}_i$ into a new structure which

$$
\mathcal{R}_1 = \begin{array}{cc|c|c|c|c|c|c|c|c|c|}
\hline
 & & r_{1,1} & r_{1,1} & r_{1,1} & r_{1,2} & r_{1,2} & r_{1,2} & r_{1,3} & r_{1,3} & r_{1,3} \\
 & a & r_{2,1} & r_{2,1} & r_{2,1} & r_{2,2} & r_{2,2} & r_{2,2} & r_{2,3} & r_{2,3} & r_{2,3} \\
 & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & r_{a,1} & r_{a,1} & r_{a,1} & r_{a,2} & r_{a,2} & r_{a,2} & r_{a,3} & r_{a,3} & r_{a,3} \\
\hline
 & & r_{a+1,1} & r_{a+1,1} & r_{a+1,1} & r_{a+1,2} & r_{a+1,2} & r_{a+1,2} & r_{a+1,3} & r_{a+1,3} & r_{a+1,3} \\
y & a & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & r_{2a,1} & r_{2a,1} & r_{2a,1} & r_{2a,2} & r_{2a,2} & r_{2a,2} & r_{2a,3} & r_{2a,3} & r_{2a,3} \\
\hline
 & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
 & & r_{(y-1)a+1,1} & r_{(y-1)a+1,1} & r_{(y-1)a+1,1} & r_{(y-1)a+1,2} & r_{(y-1)a+1,2} & r_{(y-1)a+1,2} & r_{(y-1)a+1,3} & r_{(y-1)a+1,3} & r_{(y-1)a+1,3} \\
 & a & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & r_{ya,1} & r_{ya,1} & r_{ya,1} & r_{ya,2} & r_{ya,2} & r_{ya,2} & r_{ya,3} & r_{ya,3} & r_{ya,3} \\
\hline
 & & r_{ya+1,1} & r_{ya+1,1} & r_{ya+1,1} & r_{ya+1,2} & r_{ya+1,2} & r_{ya+1,2} & r_{ya+1,3} & r_{ya+1,3} & r_{ya+1,3} \\
 & r & r_{ya+2,1} & r_{ya+2,1} & r_{ya+2,1} & r_{ya+2,2} & r_{ya+2,2} & r_{ya+2,2} & r_{ya+2,3} & r_{ya+2,3} & r_{ya+2,3} \\
 & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & r_{s,1} & r_{s,1} & r_{s,1} & r_{s,2} & r_{s,2} & r_{s,2} & r_{s,3} & r_{s,3} & r_{s,3} \\
\hline
\end{array}
$$

Figure 6.9: Example of $\mathcal{R}_1$

is an $a \times y$ array followed by an $r \times 1$ array. Let $\mathcal{R}_i'$ be this arrangement of columns of $\mathcal{R}_i$. We give $\mathcal{R}_1'$ in Figure 6.10

$$\mathcal{R}_1' = \begin{array}{ccc}
1 & 2 & 3 \\
\begin{array}{|c|c|c|c|}\hline r_{1,1} & & r_{(y-1)a+1,1} & r_{ya+1,1} \\ \vdots & \dots & \vdots & \vdots \\ \vdots & & \vdots & r_{s,1} \\ r_{a,1} & & r_{ya,1} & \\ \hline \end{array} &
\begin{array}{|c|c|c|c|}\hline r_{1,1} & & r_{(y-1)a+1,1} & r_{ya+1,1} \\ \vdots & \dots & \vdots & \vdots \\ \vdots & & \vdots & r_{s,1} \\ r_{a,1} & & r_{ya,1} & \\ \hline \end{array} &
\begin{array}{|c|c|c|c|}\hline r_{1,1} & & r_{(y-1)a+1,1} & r_{ya+1,1} \\ \vdots & \dots & \vdots & \vdots \\ \vdots & & \vdots & r_{s,1} \\ r_{a,1} & & r_{ya,1} & \\ \hline \end{array} \\
4 & 5 & 6 \\
\begin{array}{|c|c|c|c|}\hline r_{1,2} & & r_{(y-1)a+1,2} & r_{ya+1,2} \\ \vdots & \dots & \vdots & \vdots \\ \vdots & & \vdots & r_{s,2} \\ r_{a,2} & & r_{ya,2} & \\ \hline \end{array} &
\begin{array}{|c|c|c|c|}\hline r_{1,2} & & r_{(y-1)a+1,2} & r_{ya+1,2} \\ \vdots & \dots & \vdots & \vdots \\ \vdots & & \vdots & r_{s,2} \\ r_{a,2} & & r_{ya,2} & \\ \hline \end{array} &
\begin{array}{|c|c|c|c|}\hline r_{1,2} & & r_{(y-1)a+1,2} & r_{ya+1,2} \\ \vdots & \dots & \vdots & \vdots \\ \vdots & & \vdots & r_{s,2} \\ r_{a,2} & & r_{ya,2} & \\ \hline \end{array} \\
7 & 8 & 9 \\
\begin{array}{|c|c|c|c|}\hline r_{1,3} & & r_{(y-1)a+1,3} & r_{ya+1,3} \\ \vdots & \dots & \vdots & \vdots \\ \vdots & & \vdots & r_{s,3} \\ r_{a,3} & & r_{ya,3} & \\ \hline \end{array} &
\begin{array}{|c|c|c|c|}\hline r_{1,3} & & r_{(y-1)a+1,3} & r_{ya+1,3} \\ \vdots & \dots & \vdots & \vdots \\ \vdots & & \vdots & r_{s,3} \\ r_{a,3} & & r_{ya,3} & \\ \hline \end{array} &
\begin{array}{|c|c|c|c|}\hline r_{1,3} & & r_{(y-1)a+1,3} & r_{ya+1,3} \\ \vdots & \dots & \vdots & \vdots \\ \vdots & & \vdots & r_{s,3} \\ r_{a,3} & & r_{ya,3} & \\ \hline \end{array}
\end{array}$$

Figure 6.10: $\mathcal{R}_1$ of matrix X.

Let $\mathcal{R}_{i,j}'$ be the $a \times y$ array from column j of $\mathcal{R}_i$. Let $z_{i,j}$ be the $r \times 1$ array from column j of $\mathcal{R}_i$. Let v be the $9s$-tuple obtained by taking each entry from $\mathcal{R}_i'$ going down the columns. Thus, from $\mathcal{R}_1'$ we get

$$v = \left(r_{1,1}, r_{2,1} \dots, r_{s,1}, \dots, r_{1,3}, \dots, r_{s,3}\right).$$

Now write v as $v = \left(v_0, v_1, \dots, v_{\frac{9s}{a}-1}\right)$ where each v_i has a entries.

Let $\mathcal{R}_i'' = \left(v_0^T, \dots, v_{\frac{9s}{a}-1}^T\right)$. Now we show that $\mathcal{R}_1'', \mathcal{R}_2'', \mathcal{R}_3''$ form a 3–MOELR$(a, b; n)$. First note that $\mathcal{R}_1''$ is an $a \times \frac{9s}{a}$ array. However, each entry represents a row of an LS(s). Thus, $\mathcal{R}_i''$ contains $\left(\frac{9s}{a}\right) s = \frac{9s^2}{a} = b$ columns. Because we apply the same structuring to each $\mathcal{R}_i$, it follows that orthogonality holds.

Case 1: $r = \frac{a}{9}$

We have that $\mathcal{R}_i''$ has the form in Figure 6.11. Thus the number of columns in $\mathcal{R}_i''$ is

$$s(9y+1) = s\left(9\left(\frac{s-r}{a}\right)+1\right) = s\left(\frac{9s}{a} - \frac{9r}{a} + 1\right) = \frac{9s^2}{a} = b.$$

It is easy to see that equitability holds in each column because each $z_{i,j}$ lies in a different column. Because $a < s$, it follows that each symbol appears exactly 0 or 1 time in each

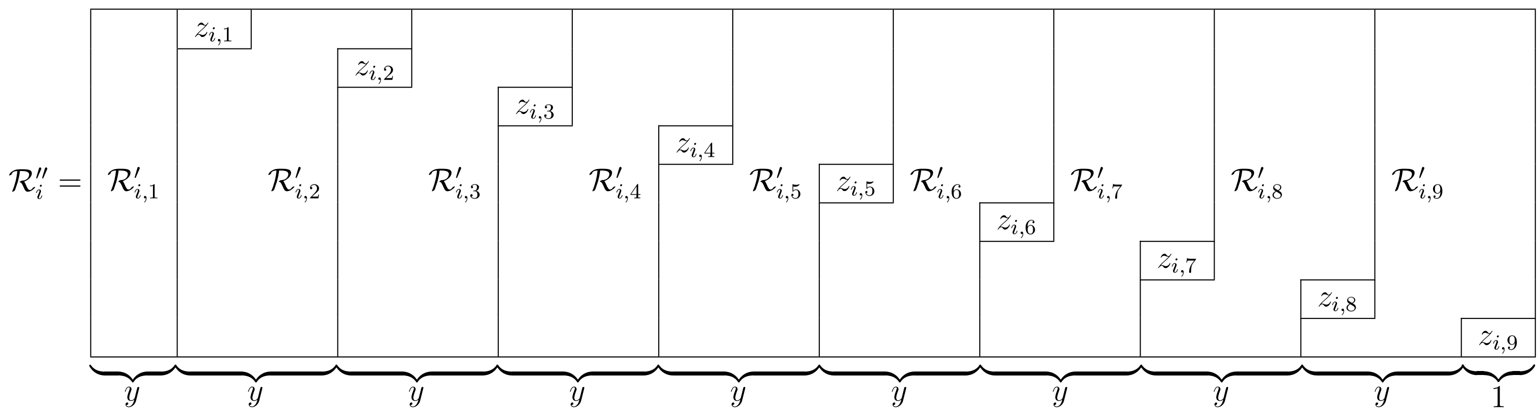


Figure 6.11: Form of $\mathcal{R}''_i$ to check equitability.

column. Consider any row of $\mathcal{R}_i''$. In this row, each symbol appears $3y+1$ or $3y$ times. We have:

$$\begin{aligned} 3y &= 3\left(\frac{s-r}{a}\right) \\ &= \frac{3s}{a} - \frac{3r}{a} \\ &= \frac{3s}{a} - \frac{3}{a}\cdot\frac{a}{9} \\ &= \frac{3s}{a} - \frac{1}{9} \\ &= \left\lfloor \frac{3s}{a} \right\rfloor = \left\lfloor \frac{b}{n} \right\rfloor \end{aligned}$$

and

$$\begin{aligned} 3y+1 &= 3\left(\frac{s-r}{a}\right)+1 \\ &= \frac{3s}{a} - \frac{3r}{a} + 1 \\ &= \frac{3s}{a} - \frac{1}{9} + 1 \\ &= \frac{3s}{a} + \frac{8}{9} \\ &= \left\lceil \frac{3s}{a} \right\rceil = \left\lceil \frac{b}{n} \right\rceil, \end{aligned}$$

thus equitability holds.

Case 2: $\quad r = \frac{8a}{9}$

In this case, $\mathcal{R}_i''$ has the form in Figure 6.12. This is because the number of columns in $\mathcal{R}_i''$ is

$$s(9y+8) = s\left(9\left(\frac{s-r}{a}\right)+8\right) = s\left(\frac{9s}{a} - \frac{9r}{a} + 8\right) = \frac{9s^2}{a} = b.$$

Again, it is easy to see that equitability holds in each column. As in Case 1, each symbol appears $3y$ or $3y+1$ times in each row.

✠

Lemma 6.18. *Suppose there exists a set of* 3–MOLS(s). *Write* $s = ay + r$ *for some* $y \in \mathbb{Z}$ *and* $0 \le r < a$. *Then if* $n = 6s$ *and* $r = \frac{a}{36}$ *or* $r = \frac{35a}{36}$, *then there exists a* 3–MOELR$(a, b; n)$ *for all* $a < s$ *and* $s|b$.

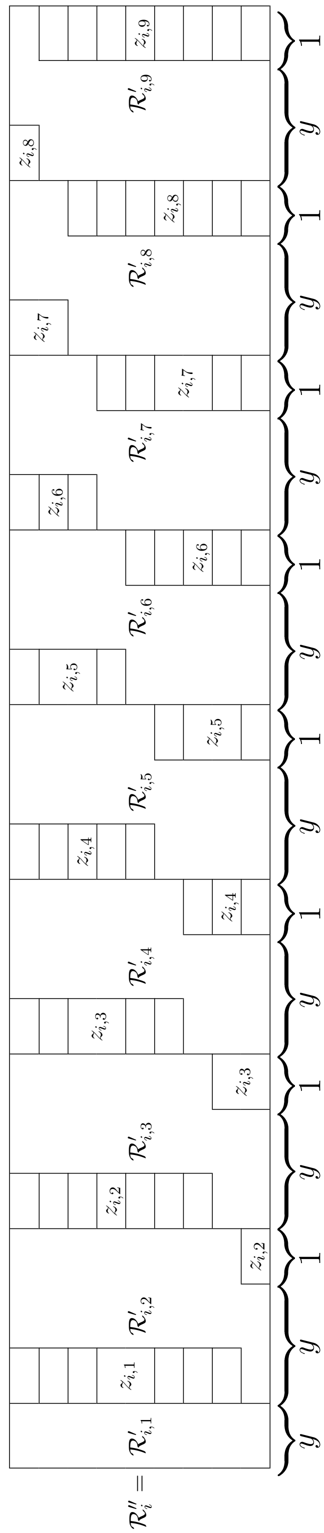


Figure 6.12: Form of $\mathcal{R}''_i$ to check equitability.

Proof: We have that $ab = n^2 = 36s^2$. Therefore $b = \frac{36s^2}{a} = \frac{36s \cdot s}{a}$ Because $s|b$, we have $\frac{b}{s} = \frac{36s}{a} \in \mathbb{Z}$. Now, because $\frac{36s}{a} \in \mathbb{Z}$, it follows that

$$\begin{aligned} \frac{36(ay+r)}{a} \in \mathbb{Z} \quad &\Rightarrow \quad \frac{36ay}{a} + \frac{36r}{a} \in \mathbb{Z} \\ &\Rightarrow \quad \frac{36r}{a} \in \mathbb{Z} \\ &\Rightarrow \quad r \cdot \frac{36}{a} = \frac{r}{a/36} \end{aligned}$$

Because $36|a$, we have that $\frac{a}{36}|r$. In our case, $r = \frac{a}{36}$ or $r = \frac{35a}{36}$. Given a set of 3–MOLS(s), we do the following. Let S_1, S_2 and S_3 be a partition of n symbols, where $|S_j| = s$ for $j = 1, 2, 3$.

Define $M_{i,j}$ to be the i^{th} Latin square of the set of 3–MOLS(s) on the symbol set S_j. Let $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ be a 3–MOELR$(s, 36s; 6s)$ constructed by applying Lemma 6.14 with $k = 3, n_1 = 6$, and $n_2 = s$.

Let $X = [\mathcal{R}_1 \ \mathcal{R}_2 \ \mathcal{R}_3]^T$ be an array where the rows form a 3–MOELR$(s, 36s; 6s)$. Thus

$$X = \begin{array}{|c|}\hline \mathcal{R}_1 \\ \hline \mathcal{R}_2 \\ \hline \mathcal{R}_3 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline M_{1,1} & M_{1,1} & M_{1,1} & \cdots & M_{1,6} & M_{1,6} & M_{1,6} & M_{1,6} & M_{1,6} & M_{1,6} \\ \hline M_{2,1} & M_{2,2} & M_{2,3} & \cdots & M_{2,1} & M_{2,2} & M_{2,3} & M_{2,4} & M_{2,5} & M_{2,6} \\ \hline M_{3,1} & M_{3,2} & M_{3,3} & \cdots & M_{3,2} & M_{3,3} & M_{3,4} & M_{3,5} & M_{3,6} & M_{3,1} \\ \hline \end{array}$$

Consider row $\mathcal{R}_i$ of X. For $t = 1, 2, \ldots, s$, let $r_{t,j}$ denote the t^{th} row of $M_{i,j}$.

For example, we write $\mathcal{R}_1$ as in Figure 6.13.

We have the $\mathcal{R}_i$ is an $s \times 4$ array where each column represents an LS(s). Because $s = ay + r$, it follows that we can rearrange each column of $\mathcal{R}_i$ into a new structure which is an $a \times y$ array followed by an $r \times 1$ array. Let $\mathcal{R}'_i$ be this arrangement of columns of $\mathcal{R}_i$. We give $\mathcal{R}'_1$ in Figure 6.14

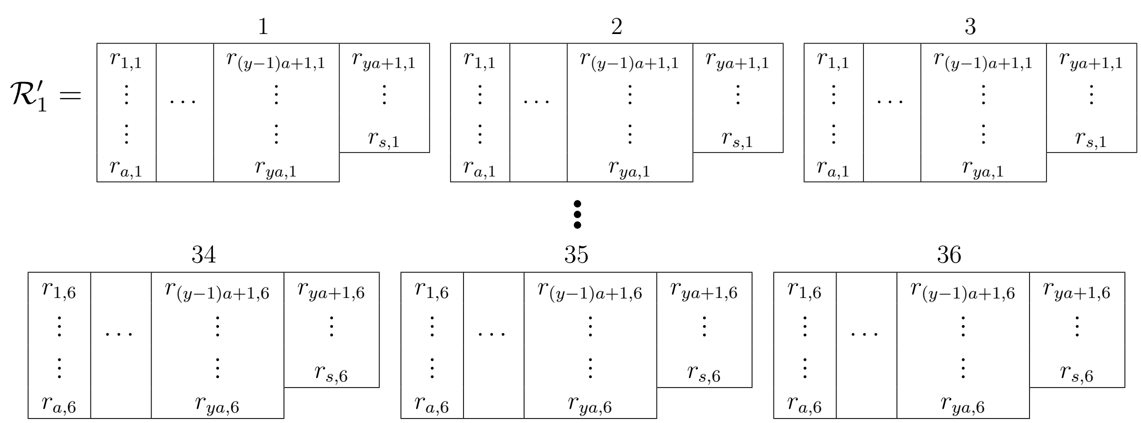


Figure 6.14: $\mathcal{R}_1$ of matrix X.

$$
\mathcal{R}_1 = \begin{array}{cc|c|c|c|c|c|c|c|}
\cline{3-9}
 & a\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. & \begin{array}{c} r_{1,1} \\ r_{2,1} \\ \vdots \\ r_{a,1} \end{array} & \begin{array}{c} r_{1,1} \\ r_{2,1} \\ \vdots \\ r_{a,1} \end{array} & \begin{array}{c} r_{1,1} \\ r_{2,1} \\ \vdots \\ r_{a,1} \end{array} & \begin{array}{c} \cdots \\ \cdots \\ \vdots \\ \cdots \end{array} & \begin{array}{c} r_{1,6} \\ r_{2,6} \\ \vdots \\ r_{a,6} \end{array} & \begin{array}{c} r_{1,6} \\ r_{2,6} \\ \vdots \\ r_{a,6} \end{array} & \begin{array}{c} r_{1,6} \\ r_{2,6} \\ \vdots \\ r_{a,6} \end{array} \\
\cline{3-9}
y\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. & a\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. & \begin{array}{c} r_{a+1,1} \\ \vdots \\ r_{2a,1} \end{array} & \begin{array}{c} r_{a+1,1} \\ \vdots \\ r_{2a,1} \end{array} & \begin{array}{c} r_{a+1,1} \\ \vdots \\ r_{2a,1} \end{array} & \begin{array}{c} \cdots \\ \vdots \\ \cdots \end{array} & \begin{array}{c} r_{a+1,6} \\ \vdots \\ r_{2a,6} \end{array} & \begin{array}{c} r_{a+1,6} \\ \vdots \\ r_{2a,6} \end{array} & \begin{array}{c} r_{a+1,6} \\ \vdots \\ r_{2a,6} \end{array} \\
\cline{3-9}
 & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cline{3-9}
 & a\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. & \begin{array}{c} r_{(y-1)a+1,1} \\ \vdots \\ r_{ya,1} \end{array} & \begin{array}{c} r_{(y-1)a+1,1} \\ \vdots \\ r_{ya,1} \end{array} & \begin{array}{c} r_{(y-1)a+1,1} \\ \vdots \\ r_{ya,1} \end{array} & \begin{array}{c} \cdots \\ \vdots \\ \cdots \end{array} & \begin{array}{c} r_{(y-1)a+1,6} \\ \vdots \\ r_{ya,6} \end{array} & \begin{array}{c} r_{(y-1)a+1,6} \\ \vdots \\ r_{ya,6} \end{array} & \begin{array}{c} r_{(y-1)a+1,6} \\ \vdots \\ r_{ya,6} \end{array} \\
\cline{3-9}
 & r\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. & \begin{array}{c} r_{ya+1,1} \\ r_{ya+2,1} \\ \vdots \\ r_{s,1} \end{array} & \begin{array}{c} r_{ya+1,1} \\ r_{ya+2,1} \\ \vdots \\ r_{s,1} \end{array} & \begin{array}{c} r_{ya+1,1} \\ r_{ya+2,1} \\ \vdots \\ r_{s,1} \end{array} & \begin{array}{c} \cdots \\ \cdots \\ \vdots \\ \cdots \end{array} & \begin{array}{c} r_{ya+1,6} \\ r_{ya+2,6} \\ \vdots \\ r_{s,6} \end{array} & \begin{array}{c} r_{ya+1,6} \\ r_{ya+2,6} \\ \vdots \\ r_{s,6} \end{array} & \begin{array}{c} r_{ya+1,6} \\ r_{ya+2,6} \\ \vdots \\ r_{s,6} \end{array} \\
\cline{3-9}
\end{array}
$$

Figure 6.13: Example of $\mathcal{R}_1$

Let $\mathcal{R}'_{i,j}$ be the $a \times y$ array from column j of $\mathcal{R}_i$. Let $z_{i,j}$ be the $r \times 1$ array from column j of $\mathcal{R}_i$. Let v be the $9s$-tuple obtained by taking each entry from $\mathcal{R}'_i$ going down the columns. Thus, from $\mathcal{R}'_1$ we get

$$v = (r_{1,1}, r_{2,1} \ldots, r_{s,1}, \ldots, r_{1,6}, \ldots, r_{s,6}) .$$

Now write v as $v = \left(v_0, v_1, \ldots, v_{\frac{36s}{a}-1}\right)$ where each v_i has a entries.

Let $\mathcal{R}''_i = \left(v_0^T, v_1^T, \ldots, v_{\frac{36s}{a}-1}^T\right)$. We now show that $\mathcal{R}''_1, \mathcal{R}''_2, \mathcal{R}''_3$ form a 3–MOELR$(a, b; n)$. First note that $\mathcal{R}''_1$ is an $a \times \frac{36s}{a}$ array. However, each entry represents a row of an LS(s). Thus, $\mathcal{R}''_i$ contains $\left(\frac{36s}{a}\right) s = \frac{36s^2}{a} = b$ columns. Because we apply the same structuring to each $\mathcal{R}_i$, it follows that orthogonality holds.

Case 1: $r = \frac{a}{36}$

We have that $\mathcal{R}''_i$ has the form similar to that of Figure 6.11. Thus the number of columns in $\mathcal{R}''_i$ is

$$s(36y + 1) = s\left(36\left(\frac{s-r}{a}\right) + 1\right) = s\left(\frac{36s}{a} - \frac{36r}{a} + 1\right) = \frac{36s^2}{a} = b.$$

It is easy to see that equitability holds in each column because each $z_{i,j}$ lies in a different column. Because $a < s$, it follows that each symbol appears exactly 0 or 1 time in each column. Consider any row of $\mathcal{R}''_i$. In this row, each symbol appears $6y + 1$ or $6y$ times. We have:

$$\begin{aligned} 6y &= 6\left(\frac{s-r}{a}\right) \\ &= \frac{6s}{a} - \frac{6r}{a} \\ &= \frac{6s}{a} - \frac{6}{a} \cdot \frac{a}{36} \\ &= \frac{6s}{a} - \frac{1}{36} \\ &= \left\lfloor \frac{6s}{a} \right\rfloor = \left\lfloor \frac{b}{n} \right\rfloor \end{aligned}$$

and

$$\begin{aligned} 6y+1 &= 6\left(\frac{s-r}{a}\right)+1 \\ &= \frac{6s}{a}-\frac{6r}{a}+1 \\ &= \frac{6s}{a}-\frac{1}{36}+1 \\ &= \frac{6s}{a}+\frac{35}{36} \\ &= \left\lceil\frac{6s}{a}\right\rceil = \left\lceil\frac{b}{n}\right\rceil, \end{aligned}$$

thus equitability holds.

Case 2: $r = \frac{35a}{36}$

In this case, $\mathcal{R}_i''$ has a form similar to that of Figure 6.12.
Thus the number of columns in $\mathcal{R}_i''$ is

$$s(36y+35) = s\left(36\left(\frac{s-r}{a}\right)+35\right) = s\left(\frac{36s}{a}-\frac{36r}{a}+35\right) = \frac{36s^2}{a} = b.$$

Again, it is easy to see that equitability holds in each column. As in Case 1, each symbol appears $6y$ or $6y+1$ times in each row. ✠

Example 6.19. 2–MOELR$(4, 81; 18)$ *using Lemma 6.16*

Let $A = \{a_0, \ldots, a_8\}$ and $B = \{b_0, \ldots, b_8\}$ where A and B are two unique symbol sets. We also will only use 2–MOLS(9) to demonstrate the idea. Let $r_{i,j}$ be the i^{th} row of the 9×9 Latin square based on the j^{th} symbol set. Figure 6.15 is the 2–MOLS(9) based on the first symbol set.

$$M_{1,1} = \boxed{\begin{array}{c} a_0a_1a_2a_3a_4a_5a_6a_7a_8 \\ a_1a_2a_0a_4a_5a_3a_7a_8a_6 \\ a_2a_0a_1a_5a_3a_4a_8a_6a_7 \\ a_3a_4a_5a_6a_7a_8a_0a_1a_2 \\ a_4a_5a_3a_7a_8a_6a_1a_2a_0 \\ a_5a_3a_4a_8a_6a_7a_2a_0a_1 \\ a_6a_7a_8a_0a_1a_2a_3a_4a_5 \\ a_7a_8a_6a_1a_2a_0a_4a_5a_3 \\ a_8a_6a_7a_2a_0a_1a_5a_3a_4 \end{array}} = \boxed{\begin{array}{c} r_{1,1} \\ r_{2,1} \\ r_{3,1} \\ r_{4,1} \\ r_{5,1} \\ r_{6,1} \\ r_{7,1} \\ r_{8,1} \\ r_{9,1} \end{array}} \quad M_{2,1} = \boxed{\begin{array}{c} a_0a_1a_2a_3a_4a_5a_6a_7a_8 \\ a_2a_0a_1a_5a_3a_4a_8a_6a_7 \\ a_1a_2a_0a_4a_5a_3a_7a_8a_6 \\ a_6a_7a_8a_0a_1a_2a_3a_4a_5 \\ a_8a_6a_7a_2a_0a_1a_5a_3a_4 \\ a_7a_8a_6a_1a_2a_0a_4a_5a_3 \\ a_3a_4a_5a_6a_7a_8a_0a_1a_2 \\ a_5a_3a_4a_8a_6a_7a_2a_0a_1 \\ a_4a_5a_3a_7a_8a_6a_1a_2a_0 \end{array}} = \boxed{\begin{array}{c} r_{1,1} \\ r_{2,1} \\ r_{3,1} \\ r_{4,1} \\ r_{5,1} \\ r_{6,1} \\ r_{7,1} \\ r_{8,1} \\ r_{9,1} \end{array}}$$

Figure 6.15: Converted 9×9 Latin square

Now we create a 2–MOELR$(9, 36; 18)$ using Lemma 6.14 with 2–MOLS(9) and the first two rows of an OA$(3, 2)$. Here $k = 2, n_1 = 2, n_2 = 9$.

$$\mathcal{R}_1 = \boxed{M_{1,1} \mid M_{1,1} \mid M_{1,2} \mid M_{1,2}} = \begin{array}{|cccc|}\hline r_{1,1} & r_{1,1} & r_{1,2} & r_{1,2} \\ r_{2,1} & r_{2,1} & r_{2,2} & r_{2,2} \\ r_{3,1} & r_{3,1} & r_{3,2} & r_{3,2} \\ r_{4,1} & r_{4,1} & r_{4,2} & r_{4,2} \\ r_{5,1} & r_{5,1} & r_{5,2} & r_{5,2} \\ r_{6,1} & r_{6,1} & r_{6,2} & r_{6,2} \\ r_{7,1} & r_{7,1} & r_{7,2} & r_{7,2} \\ r_{8,1} & r_{8,1} & r_{8,2} & r_{8,2} \\ r_{9,1} & r_{9,1} & r_{9,2} & r_{9,2} \\ \hline\end{array}$$

$$\mathcal{R}_2 = \boxed{M_{2,1} \mid M_{2,2} \mid M_{2,1} \mid M_{2,2}} = \begin{array}{|cccc|}\hline r_{1,1} & r_{1,2} & r_{1,1} & r_{1,2} \\ r_{2,1} & r_{2,2} & r_{2,1} & r_{2,2} \\ r_{3,1} & r_{3,2} & r_{3,1} & r_{3,2} \\ r_{4,1} & r_{4,2} & r_{4,1} & r_{4,2} \\ r_{5,1} & r_{5,2} & r_{5,1} & r_{5,2} \\ r_{6,1} & r_{6,2} & r_{6,1} & r_{6,2} \\ r_{7,1} & r_{7,2} & r_{7,1} & r_{7,2} \\ r_{8,1} & r_{8,2} & r_{8,1} & r_{8,2} \\ r_{9,1} & r_{9,2} & r_{9,1} & r_{9,2} \\ \hline\end{array}$$

We now apply Lemma 6.16 with $m = 2, s = 9, a = 4$, and $b = 81$. It follows that from $\mathcal{R}_1$, we get

$$\vec{v} = (r_{1,1}, r_{2,1}, r_{3,1}, \ldots, r_{9,1}, r_{1,1}, \ldots, r_{9,1}, r_{1,2}, \ldots, r_{9,2}, r_{1,2}, \ldots, r_{9,2}),$$

and

$$\begin{array}{rcl} \vec{v}_0^T & = & \vec{v}[1,2,3,4]^T = (r_{1,1}, r_{2,1}, r_{3,1}, r_{4,1})^T \\ \vec{v}_1^T & = & \vec{v}[5,6,7,8]^T = (r_{5,1}, r_{6,1}, r_{7,1}, r_{8,1})^T \\ \vec{v}_2^T & = & \vec{v}[9,10,11,12]^T = (r_{9,1}, r_{1,1}, r_{2,1}, r_{3,1})^T \\ \vec{v}_3^T & = & \vec{v}[13,14,15,16]^T = (r_{4,1}, r_{5,1}, r_{6,1}, r_{7,1})^T \\ \vec{v}_4^T & = & \vec{v}[17,18,19,20]^T = (r_{8,1}, r_{9,1}, r_{1,2}, r_{2,2})^T \\ \vec{v}_5^T & = & \vec{v}[21,22,23,24]^T = (r_{3,2}, r_{4,2}, r_{5,2}, r_{6,2})^T \\ \vec{v}_6^T & = & \vec{v}[25,26,27,28]^T = (r_{7,2}, r_{8,2}, r_{9,2}, r_{1,2})^T \\ \vec{v}_7^T & = & \vec{v}[29,30,31,32]^T = (r_{2,2}, r_{3,2}, r_{4,2}, r_{5,2})^T \\ \vec{v}_8^T & = & \vec{v}[33,34,35,36]^T = (r_{6,2}, r_{7,2}, r_{8,2}, r_{9,2})^T \end{array}$$

So we get $\mathcal{R}_1'' = (\vec{v}_0^T, \vec{v}_1^T, \vec{v}_2^T, \vec{v}_3^T, \vec{v}_4^T)$. Thus the 2–MOELR$(4, 81; 18)$ is given below.

$$\mathcal{R}_1'' = \boxed{\vec{v_0}^T \quad \vec{v_1}^T \quad \cdots \quad \vec{v_8}^T} = \begin{array}{|ccccccccc|}\hline r_{1,1} & r_{5,1} & r_{9,1} & r_{4,1} & r_{8,1} & r_{3,2} & r_{7,2} & r_{2,2} & r_{6,2} \\ r_{2,1} & r_{6,1} & r_{1,1} & r_{5,1} & r_{9,1} & r_{4,2} & r_{8,2} & r_{3,2} & r_{7,2} \\ r_{3,1} & r_{7,1} & r_{2,1} & r_{6,1} & r_{1,2} & r_{5,2} & r_{9,2} & r_{4,2} & r_{8,2} \\ r_{4,1} & r_{8,1} & r_{3,1} & r_{7,1} & r_{2,2} & r_{6,2} & r_{1,2} & r_{5,2} & r_{9,2} \\ \hline \end{array}$$

and

$$\mathcal{R}_2'' = \boxed{\vec{v_0}^T \quad \vec{v_1}^T \quad \cdots \quad \vec{v_8}^T} = \begin{array}{|ccccccccc|}\hline r_{1,1} & r_{5,2} & r_{9,1} & r_{4,2} & r_{8,2} & r_{3,1} & r_{7,1} & r_{2,2} & r_{6,2} \\ r_{2,1} & r_{6,2} & r_{1,2} & r_{5,2} & r_{9,2} & r_{4,1} & r_{8,1} & r_{3,2} & r_{7,2} \\ r_{3,1} & r_{7,2} & r_{2,2} & r_{6,2} & r_{1,1} & r_{5,1} & r_{9,1} & r_{4,2} & r_{8,2} \\ r_{4,1} & r_{8,2} & r_{3,2} & r_{7,2} & r_{2,1} & r_{6,1} & r_{1,2} & r_{5,2} & r_{9,2} \\ \hline \end{array}$$

Lemma 6.20. *If $t > 2$, and there exists y_1–MOLS(t) and a y_2–MOELR$(\frac{a}{t}, \frac{b}{t}; \frac{n}{t})$, then there exists a*

$$(\min\{y_1, y_2\})\text{–MOELR}(a, b; n).$$

Proof: Let $m = \min\{y_1, y_2\}$ and let $A_1, A_2, \ldots, A_m$ be m–MOLS(t). For $j = 1, 2, \ldots, t$, define S_j to be a set of $\frac{n}{t}$ symbols, so that $\bigcup_j S_j$ is a partition of a set of n distinct symbols. Define $B_{i,j}$ to be the i^{th} rectangle of an m–MOELR$(\frac{a}{t}, \frac{b}{t}; \frac{n}{t})$ on symbol set S_j. Now replace each entry $j \in A_i$ with $B_{i,j}$. This forms a set of m $(a \times b)$–rectangles, $M_1, M_2, \ldots, M_m$, which we now prove is an m–MOELR$(a, b; n)$.

For any pair of rectangles M_r and M_s, we had that A_r and A_s were orthogonal, so every pair of symbol sets (S_{j_1}, S_{j_2}) has occurred exactly once among the t^2 pairs, $1 \leq j_1, j_2 \leq t$. Furthermore, B_{r,j_1} and B_{s,j_2} are also orthogonal, so every ordered pair of symbols among the $\left(\frac{n}{t}\right)^2$ pairs has occurred exactly once. Thus, M_r and M_s are orthogonal.

Now consider any rectangle M_s. In each row, every symbol set S_j for $j = 1, 2, \ldots, t$ occurs exactly once because A_s is a Latin square. Also, every entry in symbol set S_j occurs an equitable number of times in $B_{s,j}$, so each row of M_s is equitable. The same argument holds for each column of M_s. So $M_1, M_2, \ldots, M_m$ forms an m–MOELR$(a, b; n)$. ✠

Example 6.21. 2–MOELR$(12, 27; 18)$ *using Lemma 6.20*

Note that $3|12$ and $3|27$ and there exists 3–MOELR$(4, 9; 6)$. Let $B_{i,j}$ be the i^{th} rectangle based on the j^{th} symbol set where $i = 1, 2, 3$ and $j = 1, 2, 3$. For example,

$$B_{1,1} = \begin{array}{|ccccccccc|}\hline 1 & 2 & 3 & 4 & 5 & 6 & 3 & 5 & 2 \\ 3 & 5 & 6 & 2 & 1 & 2 & 4 & 6 & 4 \\ 4 & 6 & 1 & 5 & 3 & 1 & 2 & 4 & 3 \\ 2 & 1 & 4 & 1 & 6 & 5 & 6 & 3 & 5 \\ \hline \end{array}$$

$$B_{2,1} = \begin{array}{|ccccccccc|}\hline 1 & 2 & 3 & 4 & 5 & 6 & 4 & 2 & 1 \\ 2 & 6 & 1 & 5 & 3 & 4 & 1 & 5 & 2 \\ 5 & 3 & 4 & 1 & 6 & 2 & 6 & 3 & 5 \\ 3 & 5 & 6 & 6 & 4 & 3 & 2 & 1 & 4 \\ \hline \end{array}$$

$$B_{3,1} = \begin{array}{|ccccccccc|} \hline 1 & 2 & 3 & 4 & 5 & 6 & 2 & 6 & 3 \\ 4 & 3 & 4 & 1 & 2 & 5 & 6 & 2 & 5 \\ 3 & 5 & 6 & 2 & 1 & 3 & 4 & 1 & 6 \\ 6 & 4 & 2 & 5 & 3 & 4 & 1 & 5 & 1 \\ \hline \end{array}.$$

Now using the structure of 2–MOLS(3) we get the following two matrices.

$$M_1 = \begin{array}{|ccc|} \hline B_{1,1} & B_{1,2} & B_{1,3} \\ B_{1,2} & B_{1,3} & B_{1,1} \\ B_{1,3} & B_{1,1} & B_{1,2} \\ \hline \end{array} \; M_2 = \begin{array}{|ccc|} \hline B_{2,1} & B_{2,2} & B_{2,3} \\ B_{2,3} & B_{2,1} & B_{2,2} \\ B_{2,2} & B_{2,3} & B_{2,1} \\ \hline \end{array}$$

Thus, since orthogonality and equitability hold for $B_{i,j}$, and there exists 2–MOLS(3), we have a 2–MOELR$(12, 27; 18)$.

Lemma 6.22. *Suppose $a \leq b$, $n = hk$, and $ab = n^2$. Let*

$$x = \min\{N_{\text{MOLS}}(h), N_{\text{MOLS}}(k)\}.$$

*If $h|a$, then there exists an x–*MOELR$(a, b; n)$.

Proof: Let $A_1, A_2, \ldots, A_x$ be x–MOLS(k). For $j = 1, 2, \ldots, k$, define S_j to be a set of h symbols, so that $\bigcup_{j=1}^{k} S_j$ is a partition of a set of n distinct symbols. Define $B_{i,j}$ to be the i^{th} Latin square of a set of x–MOLS(h) on symbol set S_j. Now replace each entry $j \in A_i$ with $B_{i,j}$. This forms a set of x–MOLS(n).

$$A_i = \begin{array}{|c|c|c|c|} \hline 1 & 2 & \cdots & k \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline B_{i,1} & B_{i,2} & \cdots & B_{i,k} \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline \end{array}$$

Permute the columns of each of these Latin squares according to the following permutation. For $m = 0, 1, \ldots, k-1$, let $mh + u \mapsto (u-1)k + (m+1)$ for $u = 1, \ldots, h$. This permutation creates $h \times k$ sub-squares which has each of the first columns of the $B_{i,j}$ sub-squares, followed by each second column and so on. Because there were k sub-squares in each row, we can now consider each Latin square being composed of $h \times k$ sub-squares H_j, $j = 1, \ldots, h$, in which each column contains entries from a different symbol set. Because there are k symbol sets, every row of the H_j contains exactly one entry from each symbol set as in Figure 6.16.

1^{st} columns	2^{nd} columns	$\cdots$	c^{th} columns
H_1	H_2	$\cdots$	H_h
H_{h+1}	H_{h+2}	$\cdots$	H_{2h}
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$H_{(k-1)h+1}$	$H_{(k-1)h+2}$	$\cdots$	H_{kh}

Figure 6.16: Representation of the first mutually orthogonal Latin square of the $n \times n$ matrix.

This matrix has $n = hk$ sub-squares. Because $h|a$, we can write $a = hr$ for some positive integer r. Arrange the sub-squares into an $r \times c$ matrix L_i as follows.

$$L_i = \begin{array}{|c|c|c|c|} \hline H_1 & H_2 & \cdots & H_c \\ \hline H_{c+1} & H_{c+2} & \cdots & H_{2c} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline H_{(r-1)c+1} & H_{(r-1)c+2} & \cdots & H_{rc} \\ \hline \end{array} \text{ for } i = 1, 2, \ldots, x.$$

Notice that L_i has $c = \frac{hk}{r}$ sub-squares in each row, and it has r sub-squares in each column. Therefore, L_i has $hr = a$ rows and ck columns. Because $n^2 = h^2k^2 = ab$, it follows that

$$b = \frac{h^2k^2}{a} = \frac{khk}{r} = k\left(\frac{hk}{r}\right) = kc.$$

Thus L_i has b columns.

We now show that $\{L_i : i = 1, 2, \ldots, x\}$ forms a x–MOELR$(a, b; n)$. Because we construct L_i in the same way for each L_i orthogonality holds. Because every row of H_j contains exactly 1 entry from each of the k symbol sets, it follows that every symbol occurs exactly $\left\lfloor \frac{c}{h} \right\rfloor$ or $\left\lceil \frac{c}{h} \right\rceil$ times in every row of L_i. Because $ck = b$, it follows that $\frac{c}{h} = \frac{b}{hk} = \frac{b}{n}$. Therefore each symbol occurs $\left\lfloor \frac{b}{n} \right\rfloor$ times or $\left\lceil \frac{b}{n} \right\rceil$ times in each row of L_i.

Now suppose there is a column in L_i that is not equitable. Then there would have to be a repeated entry in this column. This would mean we had some column in A_i with a repeated entry. But A_i was an LS(k), so that could not have happened. Therefore, each column is equitable. This shows that we have x–MOELR$(a, b; n)$.

✠

Example 6.23. *A* 2–MOELR$(16, 25; 20)$ *using Lemma 6.22*

Let $h = 4$ and $k = 5$. In Figure 6.17, we give 2–MOLS(20) by using 2–MOLS(4) and 2–MOLS(5).

Permute the columns according to the given permutation. Note that the first 5 columns were the 1^{st} columns of each 4×4 subsquare; the next 5 columns were the 2^{nd} column of each 4×4 subsquare, and so on. The permuted matrices are given in Figure 6.18

The 2–MOELR$(16, 25; 20)$ is given in Figure 6.19.

6.6 Main Theorem

Combining the results given in (Chowla, Erdös, and Straus [15]) and (Colbourn, Dinitz, [18]), we have that $N_{\text{MOLS}}(n) \geq 3$ for $n \geq 11$.

Lemma 6.24. *If* $n^2 = ab$ *and* $\gcd(n, 6) = 1$*, then* $N_{\text{MOELR}}(a, b; n) \geq 3$.

Proof: Without loss, assume $a \leq b$. If n is a prime power, then apply Corollary 6.13. Otherwise n is a product of prime powers not involving 2 or 3. Let $h = p$ be a prime such

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15	18	17	20	19
3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14	19	20	17	18
4	3	2	1	8	7	6	5	12	11	10	9	16	15	14	13	20	19	18	17
17	18	19	20	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
18	17	20	19	2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
19	20	17	18	3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
20	19	18	17	4	3	2	1	8	7	6	5	12	11	10	9	16	15	14	13
13	14	15	16	17	18	19	20	1	2	3	4	5	6	7	8	9	10	11	12
14	13	16	15	18	17	20	19	2	1	4	3	6	5	8	7	10	9	12	11
15	16	13	14	19	20	17	18	3	4	1	2	7	8	5	6	11	12	9	10
16	15	14	13	20	19	18	17	4	3	2	1	8	7	6	5	12	11	10	9
9	10	11	12	13	14	15	16	17	18	19	20	1	2	3	4	5	6	7	8
10	9	12	11	14	13	16	15	18	17	20	19	2	1	4	3	6	5	8	7
11	12	9	10	15	16	13	14	19	20	17	18	3	4	1	2	7	8	5	6
12	11	10	9	16	15	14	13	20	19	18	17	4	3	2	1	8	7	6	5
5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	1	2	3	4
6	5	8	7	10	9	12	11	14	13	16	15	18	17	20	19	2	1	4	3
7	8	5	6	11	12	9	10	15	16	13	14	19	20	17	18	3	4	1	2
8	7	6	5	12	11	10	9	16	15	14	13	20	19	18	17	4	3	2	1

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14	19	20	17	18
4	3	2	1	8	7	6	5	12	11	10	9	16	15	14	13	20	19	18	17
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15	18	17	20	19
9	10	11	12	13	14	15	16	17	18	19	20	1	2	3	4	5	6	7	8
11	12	9	10	15	16	13	14	19	20	17	18	3	4	1	2	7	8	5	6
12	11	10	9	16	15	14	13	20	19	18	17	4	3	2	1	8	7	6	5
10	9	12	11	14	13	16	15	18	17	20	19	2	1	4	3	6	5	8	7
17	18	19	20	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
19	20	17	18	3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
20	19	18	17	4	3	2	1	8	7	6	5	12	11	10	9	16	15	14	13
18	17	20	19	2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	1	2	3	4
7	8	5	6	11	12	9	10	15	16	13	14	19	20	17	18	3	4	1	2
8	7	6	5	12	11	10	9	16	15	14	13	20	19	18	17	4	3	2	1
6	5	8	7	10	9	12	11	14	13	16	15	18	17	20	19	2	1	4	3
13	14	15	16	17	18	19	20	1	2	3	4	5	6	7	8	9	10	11	12
15	16	13	14	19	20	17	18	3	4	1	2	7	8	5	6	11	12	9	10
16	15	14	13	20	19	18	17	4	3	2	1	8	7	6	5	12	11	10	9
14	13	16	15	18	17	20	19	2	1	4	3	6	5	8	7	10	9	12	11

Figure 6.17: A 2–MOLS(20)

that $p|a$ and $k = \frac{n}{p}$. Because $h > 3$ is prime, it follows that $N_{\text{MOLS}}(h) > 4$. We also have that $k \geq 35$ or k is prime, so $N_{\text{MOLS}}(k) \geq 3$. Now apply Lemma 6.22.

✠

Lemma 6.25. *If* $n^2 = ab$ *and* $\gcd(n, 3) = 1$*, then* $N_{\text{MOELR}}(a, b; n) \geq 3$*, except possibly when* $(a, b; n) \in \{(2, 50; 10), (49, 100; 70)\}$.

Proof: If n is odd then we apply Lemma 6.24. Otherwise let n be even and without loss assume that $a \leq b$.

If a has an odd prime divisor h, then $h > 3$ and $N_{\text{MOLS}}(h) \geq 3$. Also, h is a divisor of n and $k = \frac{n}{h}$ is such that $N_{\text{MOLS}}(k) \geq 3$ unless $k = 2$ or 10. Thus if $k \neq 2, 10$ we may apply Lemma 6.22 to obtain a 3–MOELR(n). If $k = 2$, then $n = 2p$ for an odd prime $p > 3$. Then there are 2 possibilities for a; $a = p$ or $2p$. In each case $a|n$ and we can apply Lemma 6.11. If $k = 10$, then $n = 10p$ for an odd prime $p > 3$. Then there are 5 possibilities for a. If $a = p, 2p, 5p$, or $2p$, then $a|n$ and we can apply Lemma 6.11. If $a = p^2$, then $b = 100$ and thus because $a < b, p = 5$ or $p = 7$. When $p = 5$ we may use

1	5	9	13	17	2	6	10	14	18	3	7	11	15	19	4	8	12	16	20
2	6	10	14	18	1	5	9	13	17	4	8	12	16	20	3	7	11	15	19
3	7	11	15	19	4	8	12	16	20	1	5	9	13	17	2	6	10	14	18
4	8	12	16	20	3	7	11	15	19	2	6	10	14	18	1	5	9	13	17
17	1	5	9	13	18	2	6	10	14	19	3	7	11	15	20	4	8	12	16
18	2	6	10	14	17	1	5	9	13	20	4	8	12	16	19	3	7	11	15
19	3	7	11	15	20	4	8	12	16	17	1	5	9	13	18	2	6	10	14
20	4	8	12	16	19	3	7	11	15	18	2	6	10	14	17	1	5	9	13
13	17	1	5	9	14	18	2	6	10	15	19	3	7	11	16	20	4	8	12
14	18	2	6	10	13	17	1	5	9	16	20	4	8	12	15	19	3	7	11
15	19	3	7	11	16	20	4	8	12	13	17	1	5	9	14	18	2	6	10
16	20	4	8	12	15	19	3	7	11	14	18	2	6	10	13	17	1	5	9
9	13	17	1	5	10	14	18	2	6	11	15	19	3	7	12	16	20	4	8
10	14	18	2	6	9	13	17	1	5	12	16	20	4	8	11	15	19	3	7
11	15	19	3	7	12	16	20	4	8	9	13	17	1	5	10	14	18	2	6
12	16	20	4	8	11	15	19	3	7	10	14	18	2	6	9	13	17	1	5
5	9	13	17	1	6	10	14	18	2	7	11	15	19	3	8	12	16	20	4
6	10	14	18	2	5	9	13	17	1	8	12	16	20	4	7	11	15	19	3
7	11	15	19	3	8	12	16	20	4	5	9	13	17	1	6	10	14	18	2
8	12	16	20	4	7	11	15	19	3	6	10	14	18	2	5	9	13	17	1

1	5	9	13	17	2	6	10	14	18	3	7	11	15	19	4	8	12	16	20
3	7	11	15	19	4	8	12	16	20	1	5	9	13	17	2	6	10	14	18
4	8	12	16	20	3	7	11	15	19	2	6	10	14	18	1	5	9	13	17
2	6	10	14	18	1	5	9	13	17	4	8	12	16	20	3	7	11	15	19
9	13	17	1	5	10	14	18	2	6	11	15	19	3	7	12	16	20	4	8
11	15	19	3	7	12	16	20	4	8	9	13	17	1	5	10	14	18	2	6
12	16	20	4	8	11	15	19	3	7	10	14	18	2	6	9	13	17	1	5
10	14	18	2	6	9	13	17	1	5	12	16	20	4	8	11	15	19	3	7
17	1	5	9	13	18	2	6	10	14	19	3	7	11	15	20	4	8	12	16
19	3	7	11	15	20	4	8	12	16	17	1	5	9	13	18	2	6	10	14
20	4	8	12	16	19	3	7	11	15	18	2	6	10	14	17	1	5	9	13
18	2	6	10	14	17	1	5	9	13	20	4	8	12	16	19	3	7	11	15
5	9	13	17	1	6	10	14	18	2	7	11	15	19	3	8	12	16	20	4
7	11	15	19	3	8	12	16	20	4	5	9	13	17	1	6	10	14	18	2
8	12	16	20	4	7	11	15	19	3	6	10	14	18	2	5	9	13	17	1
6	10	14	18	2	5	9	13	17	1	8	12	16	20	4	7	11	15	19	3
13	17	1	5	9	14	18	2	6	10	15	19	3	7	11	16	20	4	8	12
15	19	3	7	11	16	20	4	8	12	13	17	1	5	9	14	18	2	6	10
16	20	4	8	12	15	19	3	7	11	14	18	2	6	10	13	17	1	5	9
14	18	2	6	10	13	17	1	5	9	16	20	4	8	12	15	19	3	7	11

Figure 6.18: Permuted matrices of a 2–MOLS(20).

Lemma 6.11, however when $p = 7$ we have the exception $(a, b; n) = (49, 100; 70)$ to the theorem.

Now suppose $a = 2^\ell$ for some $\ell \geq 1$.

If $\ell = 1$, then $a = 2$ and divides n so we may apply Lemma 6.11, unless $n = 10$ where it is not known if 3–MOLS of order 10 exist. This is the exception $(a, b; n) = (2, 50; 10)$ to the theorem.

If $\ell > 2$, then we are guaranteed that $4|n$. However, if $\ell = 2$, then it is possible that $4 \nmid n$. In this case, $n = 2s$ where s is a product of prime powers not involving 2 or 3. There exists an OA$(3, 2)$ and 3–MOLS(s). So let $m = 2$ and apply Lemma 6.16 to obtain a 3–MOELR$(a, b; n)$ for all $s > 8$. If $s = 5$, then $n = 10$ and a 3–MOELR$(4, 25; 10)$ was constructed in Example 6.7. If $s = 7$, then $n = 14$ and there exists a 3–MOELR$(4, 49; 14)$ by Example 6.8. If $s = 8$, then $n = 16$ and is a prime power, so apply Corollary 6.11.

✠

Lemma 6.26. *If $n^2 = ab$ and n is odd, then $N_{\text{MOELR}}(a, b; n) \geq 3$, except possibly when $(a, b; n) \in \{(9, 25; 15), (9, 49; 21)\}$ or $a = 9$ and $n = 3s$ where $s \equiv 5, 7, 11,$ or 13*

$M_1=$

1	5	9	13	17	2	6	10	14	18	3	7	11	15	19	4	8	12	16	20	17	1	5	9	13
2	6	10	14	18	1	5	9	13	17	4	8	12	16	20	3	7	11	15	19	18	2	6	10	14
3	7	11	15	19	4	8	12	16	20	1	5	9	13	17	2	6	10	14	18	19	3	7	11	15
4	8	12	16	20	3	7	11	15	19	2	6	10	14	18	1	5	9	13	17	20	4	8	12	16
18	2	6	10	14	19	3	7	11	15	20	4	8	12	16	13	17	1	5	9	14	18	2	6	10
17	1	5	9	13	20	4	8	12	16	19	3	7	11	15	14	18	2	6	10	13	17	1	5	9
20	4	8	12	16	17	1	5	9	13	18	2	6	10	14	15	19	3	7	11	16	20	4	8	12
19	3	7	11	15	18	2	6	10	14	17	1	5	9	13	16	20	4	8	12	15	19	3	7	11
15	19	3	7	11	16	20	4	8	12	9	13	17	1	5	10	14	18	2	6	11	15	19	3	7
16	20	4	8	12	15	19	3	7	11	10	14	18	2	6	9	13	17	1	5	12	16	20	4	8
13	17	1	5	9	14	18	2	6	10	11	15	19	3	7	12	16	20	4	8	9	13	17	1	5
14	18	2	6	10	13	17	1	5	9	12	16	20	4	8	11	15	19	3	7	10	14	18	2	6
12	16	20	4	8	5	9	13	17	1	6	10	14	18	2	7	11	15	19	3	8	12	16	20	4
11	15	19	3	7	6	10	14	18	2	5	9	13	17	1	8	12	16	20	4	7	11	15	19	3
10	14	18	2	6	7	11	15	19	3	8	12	16	20	4	5	9	13	17	1	6	10	14	18	2
9	13	17	1	5	8	12	16	20	4	7	11	15	19	3	6	10	14	18	2	5	9	13	17	1

$M_2 =$

1	5	9	13	17	3	7	11	15	19	4	8	12	16	20	2	6	10	14	18	9	13	17	1	5
2	6	10	14	18	4	8	12	16	20	3	7	11	15	19	1	5	9	13	17	10	14	18	2	6
3	7	11	15	19	1	5	9	13	17	2	6	10	14	18	4	8	12	16	20	11	15	19	3	7
4	8	12	16	20	2	6	10	14	18	1	5	9	13	17	3	7	11	15	19	12	16	20	4	8
11	15	19	3	7	12	16	20	4	8	10	14	18	2	6	17	1	5	9	13	19	3	7	11	15
12	16	20	4	8	11	15	19	3	7	9	13	17	1	5	18	2	6	10	14	20	4	8	12	16
9	13	17	1	5	10	14	18	2	6	12	16	20	4	8	19	3	7	11	15	17	1	5	9	13
10	14	18	2	6	9	13	17	1	5	11	15	19	3	7	20	4	8	12	16	18	2	6	10	14
20	4	8	12	16	18	2	6	10	14	5	9	13	17	1	7	11	15	19	3	8	12	16	20	4
19	3	7	11	15	17	1	5	9	13	6	10	14	18	2	8	12	16	20	4	7	11	15	19	3
18	2	6	10	14	20	4	8	12	16	7	11	15	19	3	5	9	13	17	1	6	10	14	18	2
17	1	5	9	13	19	3	7	11	15	8	12	16	20	4	6	10	14	18	2	5	9	13	17	1
6	10	14	18	2	13	17	1	5	9	15	19	3	7	11	16	20	4	8	12	14	18	2	6	10
5	9	13	17	1	14	18	2	6	10	16	20	4	8	12	15	19	3	7	11	13	17	1	5	9
8	12	16	20	4	15	19	3	7	11	13	17	1	5	9	14	18	2	6	10	16	20	4	8	12
7	11	15	19	3	16	20	4	8	12	14	18	2	6	10	13	17	1	5	9	15	19	3	7	11

Figure 6.19: A 2–MOELR$(16, 25; 20)$

$\pmod{18}, s > 9$.

Proof: If $\gcd(n, 3) = 1$, then apply Lemma 6.25. Otherwise $3|n$ and without loss assume that $a \leq b$. If a has a prime divisor $h > 3$, then h is also a divisor of n so write $n = hk$. Then $N_{\text{MOLS}}(h) \geq 3$ and $N_{\text{MOLS}}(k) \geq 3$ unless $k = 3$. Thus, if $k \neq 3$, we may apply Lemma 6.22 to obtain a 3–MOELR$(a, b; n)$. If $k = 3$, then $n = 3p$ for a prime $p > 3$. There are 2 possibilities for a; $a = p$ or $3p$. In each case $a|n$, so we can apply Lemma 6.11.

Now suppose $a = 3^\ell$ for some $\ell \geq 1$.

If $\ell = 1$, then $a = 3$ and divides n, so we may apply Lemma 6.11.

If $\ell \geq 2$ and $9|n$, we set $h = 9$ and $k = \frac{n}{9}$. Then $\min\{N_{\text{MOLS}}(h), N_{\text{MOLS}}(k)\} \geq 3$, unless $k = 3$. So when $k \neq 3$ we may apply Lemma 6.22 to obtain $N_{\text{MOELR}}(a, b; n) \geq 3$. If $k = 3$, then $n = 27$ and $a \in \{9, 27\}$. In each case $a|n$, so apply Lemma 6.11.

If $\ell > 2$, then we are guaranteed that $9|n$. However, if $\ell = 2$, then it is possible that $9 \nmid n$. In this case, $n = 3s$ where s is a product of prime powers not involving 2 or 3. There exists an OA$(3, 3)$ and 3–MOLS(s). Let $m = 3$ and apply Lemma 6.17 to obtain a 3–MOELR$(a, b; n)$ for all $s > 9$ and $s \equiv 1, 8 \pmod 9$. If $s = 5$ or 7, then $n = 15$ or 21 and we have open cases $(a, b; n) \in \{(9, 25; 15), (9, 49; 21)\}$. If $s > 9$ and $s \equiv 2, 4, 5$, or

$7 \pmod 9$, then because $s \equiv 1$ or $5 \pmod 6$, we have open cases when $s \equiv 5, 7, 11, 13 \pmod{18}$.

✠

Lemma 6.27. *If $n^2 = ab$, then $N_{\text{MOELR}}(a, b; n) \geq 3$ except possibly when $(a, b; n) \in \{(2, 18; 6), (2, 50; 10), (8, 18; 12), (9, 64; 24), (9, 100; 30), (9, 256; 48), (12, 27; 18), (18, 32; 24), (18, 50; 30), (18, 98; 42), (18, 128; 48), (27, 300; 90), (36, 49; 42), (36, 121; 66), (36, 169; 78), (36, 289; 102), (36, 361; 114), (36, 529; 138), (36, 625; 150), (36, 841; 174), (36, 961; 186), (36, 1225; 210), (81, 100; 90)\}$ or*

> $a = 9$ *and* $n = 3s$, *where* $s \equiv 2, 4, 14, 16 \pmod{18}, s > 9$;
> $a = 18$ *and* $n = 3s$, *where* $s \equiv 4, 8, 10, 14 \pmod{18}, s > 18$,
> $a = 36$ *and* $n = 6s$, *where* $s \equiv 5, 7, 11, 13, 17, 19, 23, 25, 29, 31 \pmod{36}, s > 36$.

Proof: If $\gcd(n, 6) = 1$, then apply Lemma 6.24. If n is odd, then apply Lemma 6.26. If n is even and $\gcd(n, 3) = 1$, then apply Lemma 6.25. Otherwise $6|n$, and without loss assume $a \leq b$.

If a has a prime divisor $h > 3$, then h is also a divisor of n, so write $n = hk$. Then $N_{\text{MOLS}}(h) \geq 3$ and $N_{\text{MOLS}}(k) \geq 3$ unless $k = 2, 3, 6, 10$. Because $6|n$, we cannot have $k = 2, 3$, or 10. Thus, if $k \neq 6$, then we may apply Lemma 6.22 to obtain a 3–MOELR$(a, b; n) \geq 3$.

If $k = 6$, then $n = 6p$ for a prime $p > 3$. There are 5 possibilities for a. If $a = p, 2p, 3p$, or $6p$, then $a|n$ so apply Lemma 6.11. If $a = p^2$, then $b = 36$, and thus because $a < b$, $p = 5$, and we have the open case $(a, b; n) = (25, 36; 30)$.

Suppose $a = 2^\ell$ for some $\ell \geq 1$.

If $\ell = 1$, then $a = 2$ and divides n, so we may apply Lemma 6.11 unless $n = 6$ where 3–MOLS(6) does not exist. This is the possible exception $(a, b; n) = (2, 18; 6)$ to the theorem.

If $\ell \geq 2$ and $4|n$, then set $h = 4$ and $k = \frac{n}{4}$. Hence $\min\{N_{\text{MOLS}}(h), N_{\text{MOLS}}(k)\} \geq 3$ unless $k = 2, 3, 6$, or 10. Thus when $k \neq 2, 3, 6$, or 10 we can apply Lemma 6.22 to obtain $N_{\text{MOELR}}(a, b; n) \geq 3$. If $k = 2$, then $n = 8$ and $a = 4$, so apply Lemma 6.11. If $k = 3$, then $n = 12$ and $a = 4$ or 8. If $a = 4$, then apply Lemma 6.11. Otherwise, $a = 8$ and we have the open case $(a, b; n) = (8, 18; 12)$. If $k = 6$, then $n = 24$ and $a \in \{4, 8, 16\}$. If $a = 4$ or 8 then it divides n so apply Lemma 6.11. If $a = 16$, then $b = 36$. Here we can apply Lemma 6.20 with $t = 4$ because there exists a 3–MOELR$(4, 9; 6)$. If $k = 10$, then $n = 40$ and $a \in \{4, 8, 16, 32\}$. If $a = 4$ or 8, then it divides n so apply Lemma 6.11. If $a = 16$ or 32, then $8|a$ so we may apply Lemma 6.22 with $h = 8$ and $k = 5$ to obtain $N_{\text{MOELR}}(a, b; n) \geq 3$.

If $\ell > 2$, then we are guaranteed that $4|n$. However if $\ell = 2$, then it is possible that $4 \nmid n$. In this case, $n = 2s$ where s is an odd product of prime powers involving 3. There exists an OA$(3, 2)$ and 3–MOLS(s) if $s \neq 3$. So if $s \neq 3$, we can apply Lemma 6.16 with $m = 2$ to obtain a 3–MOELR$(a, b; n)$ for all $s > 4$. If $s = 3$, then $n = 6$ and 3–MOELR$(4, 9; 6)$ is given in Example 6.5.

Suppose $a = 3^\ell$ for some $\ell \geq 1$.

If $\ell = 1$, then $a = 3$ and divides n, so apply Lemma 6.11 unless $n = 6$ where 3–MOLS(6) does not exist. In this case, a 3–MOELR$(3, 12; 6)$ is given in Example 6.6.

If $\ell \geq 2$ and $9|n$, then set $h = 9$ and $k = \frac{n}{9}$. Then $\min\{N_{\text{MOLS}}(h), N_{\text{MOLS}}(k)\} \geq 3$ unless $k = 2, 3, 6$, or 10. Thus when $k \neq 2, 3, 6, 10$, we may apply Lemma 6.22 to obtain $N_{\text{MOELR}}(a, b; n) \geq 3$. If $k = 2$, then $n = 18$ and $a = 9$ in which case $a|n$, so apply Lemma 6.11. If $k = 3$, then $n = 27$ and $a = 9$ or 27, so $a|n$ and we may apply Lemma 6.11. If $k = 6$, then $n = 54$ and $a = 9$ or 27, so $a|n$ and we may again apply Lemma 6.11. If $k = 10$, then $n = 90$ and $a = 9, 27$, or 81. If $a = 9$, then $a|n$ so apply Lemma 6.11. Otherwise, we have two open cases: $(a, b; n) \in \{(27, 300; 90), (81, 100; 90)\}$.

If $\ell > 2$, then we are guaranteed that $9|n$. However, if $\ell = 2$, then it is possible that $9 \nmid n$. In this case, $n = 3s$ where s is a product of even prime powers not involving 3. There exists an OA$(3, 3)$ and 3–MOLS(s) if $s \neq 2$ or 10 and $s \equiv 1, 8 \pmod 9$. So if $s \neq 2$ or 10 and $s \equiv 1, 8 \pmod 9$, we may apply Lemma 6.17 with $m = 3$ to obtain a 3–MOELR$(a, b; n)$ for all $s > 9$. If $(s, n) = (4, 12)$, then we get the open case $(a, b; n) = (9, 16; 12)$. If $(s, n) = (2, 6)$ then $a = 9$ and $b = 4$ so we need not consider this case. If $(s, n) = (6, 18)$, then $(a, b; n) = (9, 36; 18)$. We may apply Lemma 6.20 with $t = 3$ because a 3–MOELR$(3, 12; 6)$ was given in Example 6.6. If $(s, n) = (8, 24)$ we get the open case $(a, b; n) = (9, 64; 24)$, and if $(s, n) = (10, 30)$, then we have the open case $(a, b; n) = (9, 100; 30)$. If $s > 9$ and $s \equiv 2, 4, 5, 7 \pmod 9$, then because $s \equiv 2$ or 4 $\pmod 6$, we also have open cases when $s \equiv 2, 4, 14, 16 \pmod{18}$.

Suppose $a = 2^{\ell_1}3^{\ell_2}$ for some $\ell_1, \ell_2 \geq 1$.

If $\ell_1 = \ell_2 = 1$, then $a = 6$ and divides n so we may apply Lemma 6.11.

If $\ell_1 = 1$ and $\ell_2 \geq 2$ and $18|n$, we set $h = 18$ and $k = \frac{n}{18}$. Then

$$\min\{N_{\text{MOLS}}(h), N_{\text{MOLS}}(k)\} \geq 3$$

unless $k = 2, 3, 6$, or 10. Thus if $k \neq 2, 3, 6$, or 10 we may apply Lemma 6.22 to obtain a 3–MOELR$(a, b; n)$. If $k = 2$, then $n = 36$ and $a = 18$. If $k = 3$, then $n = 54$ and $a \in \{18, 54\}$. If $k = 6$, then $n = 108$ and $a \in \{18, 54\}$. In each of these cases $a|n$, so apply Lemma 6.11. If $k = 10$, then $n = 180$ and $a \in \{18, 54, 162\}$. If $a = 18$, then $a|n$, so apply Lemma 6.11. If $a = 54$ or 162 let $h = 9$ and $k = 20$. Then $\min(N_{\text{MOLS}}(h), N_{\text{MOLS}}(k)\} \geq 3$, so apply Lemma 6.22.

If $\ell_1 = 1$ and $\ell_2 > 2$, then we are guaranteed that $18|n$. However, if $\ell_1 = 1$ and $\ell_2 = 2$, then it is possible that $18 \nmid n$. In this case $n = 3s$ where s is an even product of prime powers not involving the prime 3. There exists an OA$(3, 3)$ and 3–MOLS(s) if $s \neq 2$ or 10 and $s \equiv 2, 16 \pmod{18}$. Let $m = 3$ and apply Lemma 6.17 to obtain a 3–MOELR$(a, b; n)$ for all $s > 18$. If $(s, n) \in \{(2, 6), (4, 12)\}$, then $a > b$, so we need not consider these cases. If $(s, n) \in \{(8, 24), (10, 30), (14, 42), (16, 48)\}$, then we have the open cases $(a, b; n) \in \{(18, 32; 24), (18, 50; 30), (18, 98; 42), (18, 128; 48)\}$. If $s > 18$ and $s \equiv 4, 8, 10, 14 \pmod{18}$, then we have open cases.

If $\ell_2 = 1$ and $\ell_1 \geq 2$ and $12|n$, we set $h = 12$ and $k = \frac{n}{12}$. Then

$$\min\{N_{\text{MOLS}}(h), N_{\text{MOLS}}(k)\} \geq 3$$

unless $k = 2, 3, 6$, or 10. Thus if $k \neq 2, 3, 6$, or 10, we may apply Lemma 6.22 to obtain a 3–MOELR$(a, b; n)$. If $k = 2$, then $n = 24$ and $a = 12$. Here $a|n$, so apply Lemma 6.11. If $k = 3$, then $n = 36$ and $a \in \{12, 24\}$. If $a = 12$, then it divides n, so apply Lemma 6.11. If $a = 24$, then let $h = 4$ and $k = 9$ and apply Lemma 6.22. If $k = 6$, then $n = 72$ and $a \in \{12, 24, 48\}$. If $a = 12$ or 24, then it divides n, so apply Lemma 6.11. If $a = 48$, then let $h = 4$ and $k = 18$ and apply Lemma 6.22. If $k = 10$, then $n = 120$ and $a \in \{12, 24, 48, 96\}$. If $a = 12$ or 24, then it divides n so apply Lemma 6.11. If $a = 48$ or 96, then let $h = 4$ and $k = 30$ and apply Lemma 6.22.

If $\ell_2 = 1$ and $\ell_1 > 2$, then we are guaranteed that $12|n$. However, if $\ell_2 = 1$ and $\ell_1 = 2$, then it is possible that $12 \nmid n$. In this case, $n = 2s$ where s is an odd product of prime powers involving the prime 3. There exists an $\mathrm{OA}(3, 2)$ and a set of 3–MOLS(s) if $s \neq 3$. Let $m = 2$ and apply Lemma 6.16 to obtain a 3–MOELR$(a, b; n)$ for all $s > 12$. If $(s, n) = (9, 18)$, then we get the open case $(a, b; n) = (12, 27; 18)$. If $(s, n) = 36$, then $a > b$ and we need not consider it.

If $\ell_1 \geq 2$ and $\ell_2 \geq 2$ and $36|n$, then set $h = 36$ and $k = \frac{n}{36}$. Then

$$\min\{N_{\mathbf{MOLS}}(h), N_{\mathbf{MOLS}}(k)\} \geq 3$$

unless $k = 2, 3, 6$, or 10. Thus, when $k \neq 2, 3, 6$, or 10 we can apply Lemma 6.22 to obtain $N_{\mathbf{MOELR}}(a, b; n) \geq 3$. If $k = 2$, then $n = 72$ and $a = 36$ or 72. Here $a|n$ so apply Lemma 6.11. If $k = 3$, then $n = 108$ and $a = 36, 72$, or 108. If $a = 36$ or 108, then $a|n$ so apply Lemma 6.11. If $a = 72$, then $9|a$ so we may apply Lemma 6.22 with $h = 9$ and $k = 12$ to obtain $N_{\mathbf{MOELR}}(a, b; n) \geq 3$. If $k = 6$, then $n = 216$ and $a \in \{36, 72, 108, 144, 216\}$. If $a \in \{36, 72, 108, 216\}$, then $a|n$ and so apply Lemma 6.11. If $a = 144$, then $12|a$ so we may apply Lemma 6.22 with $h = 12$ and $k = 18$ to obtain $N_{\mathbf{MOELR}}(a, b; n) \geq 3$. If $k = 10$, then $n = 360$ and $a = \{36, 72, 108, 144, 216, 288, 324\}$. If $a = 36$ or 72, then $a|n$ so apply Lemma 6.11. Otherwise, we set $h = 4$ and $k = 90$ and apply Lemma 6.22 and apply Lemma 6.22 to obtain a 3–MOELR$(a, b; n)$.

If $\ell_1 > 2$ and $\ell_2 > 2$, then we are guaranteed that $36|n$. However, if $\ell_1 = 2$ or $\ell_2 = 2$, then it is possible that $36 \nmid n$.

Suppose $\ell_1 = \ell_2 = 2$ and $n = 6s$ where s is a product of primes powers not involving 2 or 3, then $36 \nmid n$. There exists an $\mathrm{OA}(3, 6)$ and 3–MOLS(s). Apply Lemma 6.18 with $m = 6$ for all $s > 36$ and $s \equiv 1$ or $35 \pmod{36}$. If $(s, n) \in \{(7, 42), (11, 66), (13, 78), (17, 102), (19, 114), (23, 138), (25, 150), (29, 174), (31, 186), (35, 210)\}$, then we get the following open cases: $(a, b; n) \in \{(36, 49; 42), (36, 121; 66), (36, 169; 78), (36, 289; 102), (36, 361; 114), (36, 529; 138), (36, 625; 150), (36, 841; 174), (36, 961; 186), (36, 1225; 210)\}$. If $(s, n) = (5, 30)$, then $a > b$ and we need not consider this case. If $s > 36$ and $s \equiv 5, 7, 11, 13, 17, 19, 23, 25, 29, 31 \pmod{36}$, then we have open cases.

Suppose $\ell_1 > 2$ but $\ell_2 = 2$. Then $a = 2^{\ell_1} \cdot 9$. If $n = 3s$ where s is a product of prime powers not involving 3 but involving 4, then $36 \nmid n$. However, in this case we may write $n = 12s_1$ where s_1 is a product of prime powers not involving 3. Hence, if $s_1 \neq 2$, then we may let $h = 4$ and $k = 3s_1$ and apply Lemma 6.22 to obtain $N_{\mathbf{MOELR}}(a, b; n) \geq 3$. If $s_1 = 2$, then $n = a = 72$, and because $a|n$, we may apply Lemma 6.11.

Suppose $\ell_2 > 2$ but $\ell_1 = 2$. Then $a = 4 \cdot 3^{\ell_2}$. If $n = 2s$ where s is an odd product of prime powers involving 9, then $36 \nmid n$. However, in this case we may write $n = 18s_1$ where s_1 is an odd product of prime powers. Then, if $s_1 \neq 3, 5$, we set $h = 9$ and $k = 2s_1$ and apply Lemma 6.22 to obtain $N_{\text{MOELR}}(a, b; n) \geq 3$. If $s_1 = 3$ or 5, then $n = 54$ or 90 respectively and $a > b$, so we need not consider these cases.

✠

Suppose $n = p_0^{a_1} p_1^{a_1} \cdots p_t^{a_t}$ where $p_0 < p_1 < \cdots < p_t$ are all prime. Without loss of generality, we can assume $a \leq b$, for if $a > b$, then we could apply Lemma 6.9. So because we assume $a \leq b$ and $n^2 = ab$, we need only consider a such that $a \leq n$. If $n = p_i^{a_i}$, then apply Corollary 6.13. If $p_i \neq 2$ or 3 for any i, then apply Lemma 6.24. If $p_0 = 2$ and $p_i \neq 3$ for any i, then apply Lemma 6.25. If $p_0 = 3$ and $p_i \neq 2$, then apply Lemma 6.26. Finally if $p_0 = 2$ and $p_1 = 3$, then apply Lemma 6.27.

By combining the results from the above lemmas, if $n \geq 216$ we can construct a 3–MOELR$(a, b; n)$ for all a and b such that $n^2 = ab$ with some exceptions. Notice that we can often get more than 3, but we are always guaranteed to get 3. Thus we have the main theorem.

Theorem 6.28. *If $n^2 = ab$ and $n \geq 216$, then $N_{\text{MOELR}}(a, b; n) \geq 3$ except possibly when*

> $a = 9$ *and* $n = 3s$*, where* $s \equiv 2, 4, 5, 7 \pmod 9$, $s > 9$*;*
> $a = 18$ *and* $n = 3s$*, where* $s \equiv 4, 8, 10, 14 \pmod{18}$, $s > 18$*;*
> $a = 36$ *and* $n = 6s$*, where* $s \equiv 5, 7, 11, 13, 17, 19, 23, 25, 29, 31 \pmod{36}$, $s > 36$*;*

Furthermore, if $n < 216$, then $N_{\text{MOELR}}(a, b; n) \geq 3$ except possibly when $(a, b; n)$ is one of the following.

$(2, 18; 6)$	$(18, 32; 24)$	$(36, 121; 66)$	$(36, 961; 186)$
$(2, 50; 10)$	$(18, 50; 30)$	$(36, 169; 78)$	$(36, 1225; 210)$
$(8, 18; 12)$	$(18, 98; 42)$	$(36, 289; 102)$	$(49, 100; 70)$
$(9, 16; 12)$	$(18, 128; 48)$	$(36, 361; 114)$	$(81, 100; 90)$
$(9, 25; 15)$	$(25, 36; 30)$	$(36, 529; 138)$	
$(9, 49; 21)$	$(27, 300; 90)$	$(36, 625; 150)$	
$(12, 27; 18)$	$(36, 49; 42)$	$(36, 841; 174)$	

Also, $N_{\text{MOELR}}(2, 2; 2) = 1$, $N_{\text{MOELR}}(3, 3; 3) = 2$, $N_{\text{MOELR}}(6, 6; 6) = 1$,
and $N_{\text{MOELR}}(10, 10; 10) = 2$.

Appendix A

Table of $\mathrm{MOELR}(a, b; n)$ where $n \leq 216$

The following is a list of the MOELRs that have been solved. Make note that ✓ implies that every dimension of A with the given n symbols has a solution. The notation 3–$(a, b; n)$ represents a 3–$\mathrm{MOELR}(a, b; n)$ and was used to conserve space. The list of $n \leq 216$ moves incrementally from 2 to 144 and then skips to n values where $a = 36$. The column headings named N indicate N_{MOELR}.

n	$a \times b$	N	Construction
2	2×2	✓	Corollary 6.13
3	3×3	✓	Corollary 6.13
4	All possible	✓	Corollary 6.13
5	5×5	✓	Corollary 6.13
6	2×18		
	3×12	3	C
	4×9	3	C
	6×6	1	[21]
7	7×7	✓	Corollary 6.13
8	All possible	✓	Corollary 6.13
9	All possible	✓	Corollary 6.13
10	2×50		
	4×25	3	Example 6.7
	5×20	3	Lemma 6.14
	10×10	2	[10]
11	11×11	✓	Corollary 6.13
12	2×72	5	Lemma 6.11
	3×48	5	Lemma 6.11
	4×36	5	Lemma 6.11
	6×24	5	Lemma 6.11
	8×18		
	9×16	3	Lemma 6.14
	12×12	5	[30]
13	13×13	✓	Corollary 6.13
14	2×98	3	Lemma 6.11
	4×49		
	7×28	3	Lemma 6.11
	14×14	3	[46]
15	3×75	4	Lemma 6.11
	5×45	4	Lemma 6.11
	9×25		
	15×15	4	[1, 44]
16	All possible	✓	Corollary 6.13
17	17×17	✓	Corollary 6.13
18	2×162	3	Theorem 6.11
	3×108	3	Lemma 6.11
	4×81	3	Lemma 6.16
	6×54	3	Lemma 6.11
	9×36	3	Lemma 6.11
	12×27		
	18×18	3	[49]
19	19×19	✓	Corollary 6.13

n	$a \times b$	N	Construction
20	2×200	4	Lemma 6.11
	4×100	4	Lemma 6.11
	5×80	4	Lemma 6.11
	8×50	3	Lemma 6.22
	10×40	4	Lemma 6.11
	16×25	3	Lemma 6.22
	20×20	4	[8, 47]
21	3×147	5	Lemma 6.11
	7×63	5	Lemma 6.11
	9×49		
	21×21	5	[40]
22	2×242	3	Lemma 6.11
	4×121	3	Lemma 6.16; $m = 2, s = 11$
	11×44	3	Lemma 6.11
	22×22	3	[2, 9]
23	23×23	✓	Corollary 6.13
24	2×288	6	Lemma 6.11
	3×192	6	Lemma 6.11
	4×144	6	Lemma 6.11
	6×96	6	Lemma 6.11
	8×72	6	Lemma 6.11
	9×64	3	Lemma 6.14; $k = 3, a = 9$ $b = 16, n = 12$
	12×48	6	Lemma 6.11
	16×36	3	Lemma 6.20; $t = 4, 3$–$(4, 9; 6)$
	18×32		
	24×24	6	[6]
25	All possible	✓	Corollary 6.13
26	2×338	4	Lemma 6.11
	4×169	3	Lemma 6.16; $m = 2, s = 13$
	13×52	4	Lemma 6.11
	26×26	4	[17]
27	All possible	✓	Corollary 6.13

n	$a \times b$	N	Construction
28	2×392	5	Lemma 6.11
	4×196	5	Lemma 6.11
	7×112	5	Lemma 6.11
	8×98	3	Lemma 6.22; $h = 4, k = 7$
	14×56	5	Lemma 6.11
	16×49	3	Lemma 6.22; $h = 4, k = 7$
	28×28	5	[1]
29	29×29	✓	Corollary 6.13
30	2×450	4	Lemma 6.11
	3×300	4	Lemma 6.11
	4×225	3	Lemma 6.16; $m = 2, s = 15$
	5×180	4	Lemma 6.11
	6×150	4	Lemma 6.11
	9×100		If 3–$(9, 25; 15)$ exists, then Lemma 6.14
	10×90	4	Lemma 6.11
	12×75	3	Lemma 6.16; $m = 2, s = 15$
	15×60	4	Lemma 6.11
	18×50		
	20×45	3	Lemma 6.20; $t = 5$
	25×36		If $\exists$ 3–$(25, 9; 15)$ then use Lemma 6.14.
	30×30	4	[8]
31	31×31	✓	Corollary 6.13
32	All possible	✓	Corollary 6.13
33	3×363	5	Lemma 6.11
	9×121		
	11×99	5	Lemma 6.11
	33×33	5	[7]
34	2×578	4	Lemma 6.11
	4×289	3	Lemma 6.16; $m = 2, s = 17$
	17×68	4	Lemma 6.11
	34×34	4	[1]

n	$a \times b$	N	Construction
35	5×245	5	Lemma 6.11
	7×175	5	Lemma 6.11
	25×49	4	Lemma 6.22; $h = 5, k = 7$
	34×34	5	[53]
36	2×648	8	Lemma 6.11
	3×432	8	Lemma 6.11
	4×324	8	Lemma 6.11
	8×162	3	Lemma 6.22; $h = 4, k = 9$
	9×144	8	Lemma 6.11
	12×108	8	Lemma 6.11
	16×81	3	Lemma 6.22; $h = 4, k = 9$
	18×72	8	Lemma 6.11
	24×54	3	Lemma 6.22; $h = 4, k = 9$
	27×48	3	Lemma 6.22; $h = 9, k = 4$
	36×36	8	[6]
37	37×37	✓	Corollary 6.13
38	2×722	4	Lemma 6.11
	4×361	3	Lemma 6.16; $m = 2, s = 19$
	19×76	4	Lemma 6.11
	38×38	4	[8]
39	3×507	5	Lemma 6.11
	9×169		
	13×117	5	Lemma 6.11
	39×39	5	[7]
40	2×800	7	Lemma 6.11
	4×400	7	Lemma 6.11
	5×320	7	Lemma 6.11
	8×200	7	Lemma 6.11
	10×160	7	Lemma 6.11
	16×100	3	Lemma 6.22; $h = 8, k = 5$
	20×80	7	Lemma 6.11
	25×64	4	Lemma 6.22; $h = 5, k = 8$
	32×50	4	Lemma 6.22; $h = 8, k = 5$
	28×28	7	[5]
41	41×41	✓	Corollary 6.13

n	$a \times b$	N	Construction
42	2×882	5	Lemma 6.11
	3×588	5	Lemma 6.11
	4×441	3	Lemma 6.16; $m = 2, s = 21$
	7×252	5	Lemma 6.11
	9×196		If $\exists$ 3–$(9, 49; 21)$ then use 6.14
	12×147	3	Lemma 6.16; $m = 2, s = 21$
	14×126	5	Lemma 6.11
	18×98		
	21×84	5	Lemma 6.11
	28×63	3	Lemma 6.20; $t = 7, 3$–$(4, 9; 6)$
	36×49		
	42×42	5	[1]
43	43×43	✓	Corollary 6.13
44	2×968	5	Lemma 6.11
	4×484	5	Lemma 6.11
	8×242	3	Lemma 6.16; $m = 2, s = 22$
	11×176	5	Lemma 6.11
	16×121	3	Lemma 6.22; $h = 4, k = 11$
	22×88	5	Lemma 6.11
	44×44	5	[1]
45	3×675	6	Lemma 6.11
	5×289	6	Lemma 6.11
	9×68	6	Lemma 6.11
	15×135	6	Lemma 6.11
	25×81	4	Lemma 6.22; $h = 5, k = 9$
	27×75	4	Lemma 6.22; $h = 9, k = 5$
	45×45	6	[3]
46	2×1058	4	Lemma 6.11
	4×529	3	Lemma 6.16; $m = 2, s = 23$
	23×92	4	Lemma 6.11
	46×46	4	[18]
47	47×47	✓	Corollary 6.13
48	2×1152	8	Lemma 6.11
	3×768	8	Lemma 6.11
	4×576	8	Lemma 6.11
	8×288	8	Lemma 6.11
	9×256		
	16×144	3	Lemma 6.22; $h = 4, k = 12$
	18×128		
	24×96	8	Lemma 6.11
	32×72	3	Lemma 6.22; $h = 4, k = 12$
	36×64	3	Lemma 6.22; $h = 4, k = 12$
	48×48	8	[4]
49	All possible	✓	Corollary 6.13
50	2×800	6	Lemma 6.11
	4×625	3	Lemma 6.16; $m = 2, s = 25$
	5×500	6	Lemma 6.11
	10×250	6	Lemma 6.11
	20×125	3	Lemma 6.20; $t = 5, 3$–$(4, 25; 10)$
	25×100	6	Lemma 6.11
	50×50	6	[18]
51	3×867	5	Lemma 6.11
	9×289	3	Lemma 6.17; $m = 3, s = 17$
	17×153	5	Lemma 6.11
	51×51	5	[7]
52	2×1352	5	Lemma 6.11
	4×676	5	Lemma 6.11
	8×338	3	Lemma 6.22; $h = 4, k = 13$
	13×208	5	Lemma 6.11
	16×169	3	Lemma 6.22; $h = 4, k = 13$
	26×104	5	Lemma 6.11
	52×52	5	[1]
53	53×53	✓	Corollary 6.13

n	$a \times b$	N	Construction
54	2×1458	5	Lemma 6.11
	3×972	5	Lemma 6.11
	4×729	3	Lemma 6.16; $m = 2, s = 27$
	6×486	5	Lemma 6.11
	9×324	3	Lemma 6.11
	12×243	3	Lemma 6.16; $m = 2, s = 27$
	18×162	3	Lemma 6.11;
	27×108	5	Lemma 6.11
	36×81	3	Lemma 6.20; $t = 9, 3$–$(4, 9; 6)$
	54×54	5	[1]
55	5×605	6	Lemma 6.11
	11×275	6	Lemma 6.11
	25×121	4	Lemma 6.22; $h = 5, k = 11$
	55×55	6	[54]
56	2×1568	7	Lemma 6.11
	4×784	7	Lemma 6.11
	7×448	7	Lemma 6.11
	8×392	7	Lemma 6.11
	14×224	7	Lemma 6.11
	16×196	6	Lemma 6.22; $h = 8, k = 7$
	28×112	7	Lemma 6.11
	32×98	6	Lemma 6.22; $h = 8, k = 7$
	49×64	6	Lemma 6.22; $h = 7, k = 8$
	56×56	7	[2, 37]
57	3×1083	7	Lemma 6.11
	9×361	4	Lemma 6.16; $m = 3, s = 19$
	19×171	7	Lemma 6.11
	57×57	7	[18]
58	2×1682	5	Lemma 6.11
	4×841	3	Lemma 6.16; $m = 2, s = 29$
	29×116	5	Lemma 6.11
	58×58	5	[18]
59	59×59	✓	Corollary 6.13

n	$a \times b$	N	Construction
60	2×1800	4	Lemma 6.11
	3×1200	4	Lemma 6.11
	4×900	4	Lemma 6.11
	5×720	4	Lemma 6.11
	6×600	4	Lemma 6.11
	8×450	3	Lemma 6.22; $h = 4, k = 15$
	9×400	4	Lemma 6.16; $m = 3, s = 20$
	10×360	4	Lemma 6.11
	12×300	4	Lemma 6.11
	15×240	4	Lemma 6.11
	16×225	3	Lemma 6.22; $h = 4, k = 15$
	18×200	3	Lemma 6.17; $m = 3, s = 20$
	20×180	4	Lemma 6.11
	24×150	4	Lemma 6.22; $h = 12, k = 5$
	25×144	4	Lemma 6.22; $h = 5, k = 12$
	30×120	4	Lemma 6.11
	36×100	4	Lemma 6.22; $h = 12, k = 5$
	45×80	4	Lemma 6.22; $h = 5, k = 12$
	48×75	4	Lemma 6.22; $h = 12, k = 5$
	50×72	4	Lemma 6.22; $h = 5, k = 12$
	60×60	4	[18]
61	61×61	✓	Corollary 6.13
62	2×1922	5	Lemma 6.11
	4×961	3	Lemma 6.16; $m = 2, s = 31$
	31×124	5	Lemma 6.11
	62×62	5	[1]
63	3×1323	6	Lemma 6.11
	7×567	6	Lemma 6.11
	9×441	6	Lemma 6.11;
	21×189	6	Lemma 6.11
	27×147	6	Lemma 6.22; $h = 9, k = 7$
	63×63	6	[18]
64	All possible	✓	Corollary 6.13

n	$a \times b$	N	Construction
65	5×845	6	Lemma 6.11
	13×325	6	Lemma 6.11
	25×169	4	Lemma 6.22; $h = 5, k = 13$
	65×65	6	[18]
66	2×2178	5	Lemma 6.11
	3×1452	5	Lemma 6.11
	4×1089	3	Lemma 6.16; $m = 2, s = 33$
	6×726	5	Lemma 6.11
	9×484		
	11×396	5	Lemma 6.11
	12×363	3	Lemma 6.16; $m = 2, s = 33$
	18×242		
	22×198	5	Lemma 6.11
	33×132	5	Lemma 6.11
	36×121		
	44×99	3	Lemma 6.20; $t = 11, 3\text{–}(4, 9; 6)$
	66×66	5	[18]
67	67×67	✓	Corollary 6.13
68	2×2312	5	Lemma 6.11
	4×1156	5	Lemma 6.11
	8×578	3	Lemma 6.22; $h = 4, k = 17$
	16×289	3	Lemma 6.22; $h = 4, k = 17$
	17×272	5	Lemma 6.11
	34×136	5	Lemma 6.11
	68×68	5	[18]
69	3×1587	6	Lemma 6.11
	9×529		
	23×207	6	Lemma 6.11
	69×69	6	[18]

n	$a \times b$	N	Construction
70	2×2450	6	Lemma 6.11
	4×1225	3	Lemma 6.16; $m = 2, s = 35$
	5×980	6	Lemma 6.11
	7×700	6	Lemma 6.11
	10×490	6	Lemma 6.11
	14×350	6	Lemma 6.11
	20×245	3	Lemma 6.22; $h = 5, k = 14$
	25×196	3	Lemma 6.22; $h = 5, k = 14$
	28×175	3	Lemma 6.22; $h = 14, h = 5$
	35×140	6	Lemma 6.11
	49×100		
	50×98	3	Lemma 6.22; $h = 5, k = 14$
	70×70	6	[18]
71	71×71	✓	Corollary 6.13
72	2×2592	7	Lemma 6.11
	3×1728	7	Lemma 6.11
	4×1296	7	Lemma 6.11;
	6×864	7	Lemma 6.11
	8×648	7	Lemma 6.11
	9×576	7	Lemma 6.11
	12×432	7	Lemma 6.11
	16×324	7	Lemma 6.22; $h = 8, k = 9$
	18×288	7	Lemma 6.11
	24×216	7	Lemma 6.11
	27×192	7	Lemma 6.22; $h = 9, k = 8$
	32×162	7	Lemma 6.22; $h = 8, k = 9$
	36×144	7	Lemma 6.11
	48×108	7	Lemma 6.22; $h = 8, k = 9$
	54×96	7	Lemma 6.22; $h = 9, k = 8$
	64×81	7	Lemma 6.22; $h = 8, k = 9$
	72×72	7	[18]
73	73×73	✓	Corollary 6.13

n	$a \times b$	N	Construction
74	2×2738	5	Lemma 6.11
	4×1369	3	Lemma 6.16; $m = 2, s = 37$
	37×148	5	Lemma 6.11
	74×74	5	[18]
75	3×1875	7	Lemma 6.11
	5×1125	7	Lemma 6.11
	9×625		
	15×375	7	Lemma 6.11
	25×225	4	Lemma 6.11
	45×125	4	Lemma 6.22; $h = 15, k = 5$
	75×75	7	[6]
76	2×2888	6	Lemma 6.11
	4×1444	6	Lemma 6.11
	8×722	3	Lemma 6.22; $h = 4, k = 19$
	16×361	3	Lemma 6.22; $h = 4, k = 19$
	19×304	6	Lemma 6.11
	38×152	6	Lemma 6.11
	76×76	6	[18]
77	7×847	6	Lemma 6.11
	11×539	6	Lemma 6.11
	49×121	6	Lemma 6.22; $h = 7, k = 11$
	77×77	6	[18]
78	2×3042	6	Lemma 6.11
	3×2028	6	Lemma 6.11
	4×1521	3	Lemma 6.16; $m = 2, s = 39$
	6×1014	6	Lemma 6.11
	9×676	4	Lemma 6.16; $m = 3, s = 26$
	12×507	3	Lemma 6.16; $m = 2, s = 39$
	13×468	6	Lemma 6.11
	18×338		
	26×234	6	Lemma 6.11
	36×169		
	39×156	6	Lemma 6.11
	52×117	3	Lemma 6.20; $t = 13, 3\text{–}(4, 9; 6)$
	78×78	6	[18]

n	$a \times b$	N	Construction
79	79×79	✓	Corollary 6.13
80	2×3200	9	Lemma 6.11
	4×1600	9	Lemma 6.11
	5×1280	9	Lemma 6.11
	8×800	9	Lemma 6.11
	10×640	9	Lemma 6.11
	16×400	9	Lemma 6.11
	20×320	9	Lemma 6.11
	25×256	4	Lemma 6.22; $h = 5, k = 16$
	32×200	3	Lemma 6.22; $h = 4, k = 20$
	40×160	9	Lemma 6.11
	50×128	4	Lemma 6.22; $h = 5, k = 16$
	64×100	3	Lemma 6.22; $h = 4, k = 20$
	80×80	9	[2, 5]
81	All possible	✓	Corollary 6.13
82	2×3362	8	Lemma 6.11
	4×1681	3	Lemma 6.16; $m = 2, s = 41$
	41×164	8	Lemma 6.11
	82×82	8	[18]
83	83×83	✓	Corollary 6.13

n	$a \times b$	N	Construction
84	2×3528	6	Lemma 6.11
	3×2352	6	Lemma 6.11
	4×1764	6	Lemma 6.11
	6×1176	6	Lemma 6.11
	7×1008	6	Lemma 6.11
	8×882	3	Lemma 6.22; $h = 4, k = 21$
	9×784	4	Lemma 6.16; $m = 3, s = 28$
	12×588	6	Lemma 6.11
	14×504	6	Lemma 6.11
	16×441	3	Lemma 6.22; $h = 4, k = 21$
	18×392		
	21×336	6	Lemma 6.11
	24×294	3	Lemma 6.22; $h = 4, k = 21$
	28×252	6	Lemma 6.11
	36×196	3	Lemma 6.22; $h = 4, k = 21$
	42×168	6	Lemma 6.11
	49×144	5	Lemma 6.22; $h = 7, k = 12$
	56×126	5	Lemma 6.22; $h = 7, k = 12$
	63×112	5	Lemma 6.22; $h = 7, k = 12$
	72×98	3	Lemma 6.22; $h = 4, k = 21$
	84×84	6	[18]
85	5×1445	6	Lemma 6.11
	17×425	6	Lemma 6.11
	25×289	4	Lemma 6.22; $h = 5, k = 17$
	85×85	6	[18]
86	2×3698	6	Lemma 6.11
	4×1849	3	Lemma 6.16; $m = 2, s = 43$
	43×172	6	Lemma 6.11
	86×86	6	[18]
87	3×2523	6	Lemma 6.11
	9×841		
	29×261	6	Lemma 6.11
	87×87	6	[18]
88	2×3872	7	Lemma 6.11
	4×1936	7	Lemma 6.11
	8×968	7	Lemma 6.11
	11×704	7	Lemma 6.11
	16×484	7	Lemma 6.22; $h = 8, k = 11$
	22×352	7	Lemma 6.11
	32×242	7	Lemma 6.22; $h = 8, k = 11$
	44×176	7	Lemma 6.11
	64×121	7	Lemma 6.22; $h = 8, k = 11$
	88×88	7	[18]
89	89×89	✓	Corollary 6.13

n	$a \times b$	N	Construction
90	2×4050	6	Lemma 6.11
	3×2700	6	Lemma 6.11
	4×2025	3	Lemma 6.16; $m = 2, s = 45$
	5×1620	6	Lemma 6.11
	6×1350	6	Lemma 6.11
	9×900	6	Lemma 6.11
	10×810	6	Lemma 6.11
	12×675	3	Lemma 6.16; $m = 2, s = 45$
	15×540	6	Lemma 6.11
	18×450	6	Lemma 6.11
	20×405	3	Lemma 6.22; $h = 5, k = 18$
	25×324	3	Lemma 6.22; $h = 5, k = 18$
	27×300		
	30×270	6	Lemma 6.11
	36×225	3	Lemma 6.22; $h = 18, k = 5$
	45×180	6	Lemma 6.11
	50×162	3	Lemma 6.22; $h = 5, k = 18$
	54×150	3	Lemma 6.22; $h = 18, k = 5$
	60×135	3	Lemma 6.22; $h = 5, k = 18$
	81×100		
	90×90	6	[18]
91	7×1183	7	Lemma 6.11
	13×637	7	Lemma 6.11
	49×169	6	Lemma 6.22; $h = 7, k = 13$
	91×91	7	[18]
92	2×4232	6	Lemma 6.11
	4×2116	6	Lemma 6.11
	8×1058	3	Lemma 6.22; $h = 4, k = 23$
	16×529	3	Lemma 6.22; $h = 4, k = 23$
	23×368	6	Lemma 6.11
	46×184	6	Lemma 6.11
	92×92	6	[18]

n	$a \times b$	N	Construction
93	3×2883	6	Lemma 6.11
	9×961		
	31×279	6	Lemma 6.11
	93×93	6	[18]
94	2×4418	6	Lemma 6.11
	4×2209	3	Lemma 6.16; $m = 2, s = 47$
	47×188	6	Lemma 6.11
	94×94	6	[18]
95	5×1805	6	Lemma 6.11
	19×475	6	Lemma 6.11
	25×361	4	Lemma 6.22; $h = 5, k = 19$
	95×95	6	[18]
96	2×4608	7	Lemma 6.11
	3×3072	7	Lemma 6.11
	4×2304	7	Lemma 6.11
	6×1536	7	Lemma 6.11
	8×1152	7	Lemma 6.11
	9×1024		
	12×768	7	Lemma 6.11
	16×576	7	Lemma 6.11
	18×512		
	24×384	7	Lemma 6.11
	32×288	7	Lemma 6.11
	36×256	5	Lemma 6.22; $h = 4, k = 24$
	48×192	7	Lemma 6.11
	64×144	5	Lemma 6.22; $h = 4, k = 24$
	72×128	5	Lemma 6.22; $h = 4, k = 24$
	96×96	7	[18]
97	97×97	✓	Corollary 6.13

n	$a \times b$	N	Construction
98	2×4802	6	Lemma 6.11
	4×2401	3	Lemma 6.16; $m = 2, s = 49$
	7×1372	6	Lemma 6.11
	14×686	6	Lemma 6.11
	28×343	3	Lemma 6.22; $h = 7, k = 14$
	49×196	6	Lemma 6.11
	98×98	6	[18]
99	3×3267	8	Lemma 6.11
	9×1089	8	Lemma 6.11
	11×891	8	Lemma 6.11
	27×363	8	Lemma 6.22; $h = 9, k = 11$
	33×297	8	Lemma 6.11
	81×121	8	Lemma 6.22; $h = 9, k = 11$
	99×99	8	[18]
100	2×5000	8	Lemma 6.11
	4×2500	8	Lemma 6.11
	5×2000	8	Lemma 6.11
	8×1250	3	Lemma 6.22; $h = 4, k = 25$
	10×1000	8	Lemma 6.11
	16×625	3	Lemma 6.22; $h = 4, k = 25$
	20×500	8	Lemma 6.11
	25×400	8	Lemma 6.11
	50×200	8	Lemma 6.11
	80×125	4	Lemma 6.22; $h = 5, k = 20$
	100×100	8	[18]
101	101×101	✓	Corollary 6.13

n	$a \times b$	N	Construction
102	2×5202	6	Lemma 6.11
	3×3468	6	Lemma 6.11
	4×2601	3	Lemma 6.16; $m = 2, s = 51$
	6×1734	6	Lemma 6.11
	9×1156		
	12×867	3	Lemma 6.16; $m = 2, s = 51$
	17×612	6	Lemma 6.11
	18×578	3	Lemma 6.17; $m = 3, s = 34$
	34×306	6	Lemma 6.11
	36×289		
	51×204	6	Lemma 6.11
	68×153	3	Lemma 6.20; $t = 17, 3\text{–}(4, 9; 6)$
	102×102	6	[18]
103	103×103	✓	Corollary 6.13
104	2×5408	7	Lemma 6.11
	4×2704	7	Lemma 6.11
	8×1352	7	Lemma 6.11
	13×832	7	Lemma 6.11
	16×676	7	Lemma 6.22; $h = 8, k = 13$
	26×416	7	Lemma 6.11
	32×338	7	Lemma 6.22; $h = 8, k = 13$
	52×208	7	Lemma 6.11
	104×104	7	[18]

n	$a \times b$	N	Construction
105	3×3675	7	Lemma 6.11
	5×2205	7	Lemma 6.11
	7×1575	7	Lemma 6.11
	9×1225	4	Lemma 6.16; $m = 3, s = 35$
	15×735	7	Lemma 6.11
	21×525	7	Lemma 6.11
	25×441	4	Lemma 6.22; $h = 5, k = 21$
	35×315	7	Lemma 6.11
	45×245	4	Lemma 6.22; $h = 5, k = 21$
	49×225	4	Lemma 6.22; $h = 7, k = 15$
	63×175	4	Lemma 6.22; $h = 7, k = 15$
	75×147	4	Lemma 6.22; $h = 5, k = 21$
	105×105	7	[18]
106	2×5618	6	Lemma 6.11
	4×2809	3	Lemma 6.16; $m = 2, s = 53$
	53×212	6	Lemma 6.11
	106×106	6	[18]
107	107×107	✓	Corollary 6.13
108	2×5832	6	Lemma 6.11
	3×3888	6	Lemma 6.11
	4×2916	6	Lemma 6.11
	6×1944	6	Lemma 6.11
	9×1296	6	Lemma 6.11
	12×972	6	Lemma 6.11
	16×729	3	Lemma 6.22; $h = 4, k = 27$
	18×648	6	Lemma 6.11
	24×486	3	Lemma 6.22; $h = 4, k = 27$
	27×432	6	Lemma 6.11
	36×324	6	Lemma 6.11
	48×243	3	Lemma 6.22; $h = 4, k = 27$
	54×216	6	Lemma 6.11
	72×162	5	Lemma 6.22; $h = 9, k = 12$
	81×144	5	Lemma 6.22; $h = 9, k = 12$
	108×108	6	[18]
109	109×109	✓	Corollary 6.13
110	2×6050	6	Lemma 6.11
	4×3025	3	Lemma 6.16; $m = 2, s = 55$
	5×2420	6	Lemma 6.11
	10×1210	6	Lemma 6.11
	11×1100	6	Lemma 6.11
	20×605	3	Lemma 6.22; $h = 5, k = 22$
	22×550	6	Lemma 6.11
	25×484	3	Lemma 6.22; $h = 5, k = 22$
	44×275		
	50×242	3	Lemma 6.22; $h = 5, k = 22$
	55×220	6	Lemma 6.11
	100×121	3	Lemma 6.22; $h = 5, k = 22$
	110×110	6	[18]

n	$a \times b$	N	Construction
111	3×4107	6	Lemma 6.11
	9×1369	4	Lemma 6.16; $m = 3, s = 37$
	37×333	6	Lemma 6.11
	111×111	6	[18]
112	2×6272	13	Lemma 6.11
	4×3136	13	Lemma 6.11
	7×1792	13	Lemma 6.11
	8×1568	13	Lemma 6.11
	14×896	13	Lemma 6.11
	16×784	13	Lemma 6.11
	28×448	13	Lemma 6.11
	32×392	6	Lemma 6.22; $h = 16, k = 7$
	49×256	6	Lemma 6.22; $h = 7, k = 16$
	56×224	13	Lemma 6.11
	64×196	6	Lemma 6.22; $h = 16, k = 7$
	98×128	6	Lemma 6.22; $h = 7, k = 16$
	112×112	13	[18]
113	113×113	✓	Corollary 6.13
114	2×6498	6	Lemma 6.11
	3×4332	6	Lemma 6.11
	4×3249	3	Lemma 6.16; $m = 2, s = 57$
	6×2166	6	Lemma 6.11
	9×1444		
	12×1083	3	Lemma 6.16; $m = 2, s = 57$
	18×722	4	Lemma 6.16; $m = 3, s = 38$
	19×684	6	Lemma 6.11
	36×361		
	38×342	6	Lemma 6.11
	57×228	6	Lemma 6.11
	76×171	3	Lemma 6.20; $t = 19, 3\text{–}(4, 9; 6)$
	114×114	6	[18]

n	$a \times b$	N	Construction
115	5×2645	7	Lemma 6.11
	23×575	7	Lemma 6.11
	25×529	4	Lemma 6.22; $h = 5, k = 23$
	115×115	7	[18]
116	2×6728	6	Lemma 6.11
	4×3364	6	Lemma 6.11
	8×1682	3	Lemma 6.22; $h = 4, k = 29$
	16×841	3	Lemma 6.22; $h = 4, k = 29$
	29×464	6	Lemma 6.11
	58×232	6	Lemma 6.11
	116×116	6	[18]
117	3×4563	8	Lemma 6.11
	9×1521	8	Lemma 6.11
	13×1053	8	Lemma 6.11
	27×507	8	Lemma 6.22; $h = 9, k = 13$
	39×351	8	Lemma 6.11
	81×169	8	Lemma 6.22; $h = 9, k = 13$
	117×117	8	[18]
118	2×6962	6	Lemma 6.11
	4×3481	3	Lemma 6.16; $m = 2, s = 59$
	59×236	6	Lemma 6.11
	118×118	6	[18]
119	7×2023	6	Lemma 6.11
	17×833	6	Lemma 6.11
	49×289	6	Lemma 6.22; $h = 7, k = 17$
	119×119	6	[18]

n	$a \times b$	N	Construction
120	2×7200	7	Lemma 6.11
	3×4800	7	Lemma 6.11
	4×3600	7	Lemma 6.11
	5×2880	7	Lemma 6.11
	6×2400	7	Lemma 6.11
	8×1800	7	Lemma 6.11
	9×1600		
	10×1440	7	Lemma 6.11
	12×1200	7	Lemma 6.11
	15×960	7	Lemma 6.11
	16×900	4	Lemma 6.22; $h = 8, k = 15$
	18×800		
	20×720	7	Lemma 6.11
	24×600	7	Lemma 6.11
	25×576	4	Lemma 6.22; $h = 5, k = 24$
	30×480	7	Lemma 6.11
	32×450	4	Lemma 6.22; $h = 8, k = 15$
	36×400	4	Lemma 6.22; $h = 4, k = 30$
	40×360	7	Lemma 6.11
	45×320	4	Lemma 6.22; $h = 5, k = 24$
	48×300	4	Lemma 6.22; $h = 8, k = 15$
	50×288	4	Lemma 6.22; $h = 5, k = 24$
	60×240	7	Lemma 6.11
	64×225	4	Lemma 6.22; $h = 8, k = 15$
	72×200	4	Lemma 6.22; $h = 8, k = 15$
	75×192	4	Lemma 6.22; $h = 5, k = 24$
	80×180	4	Lemma 6.22; $h = 8, k = 15$
	90×160	4	Lemma 6.22; $h = 5, k = 24$
	96×150	4	Lemma 6.22; $h = 8, k = 15$
	100×144	4	Lemma 6.22; $h = 5, k = 24$
	120×120	7	[18]

n	$a \times b$	N	Construction
121	121×121	✓	Corollary 6.13
122	2×7442	6	Lemma 6.11
	4×3721	3	Lemma 6.16; $m = 2, s = 61$
	61×244	6	Lemma 6.11
	122×122	6	[18]
123	3×5043	6	Lemma 6.11
	9×1681		
	41×369	6	Lemma 6.11
	123×123	6	[18]
124	2×4688	6	Lemma 6.11
	4×3844	6	Lemma 6.11
	8×1922	3	Lemma 6.22; $h = 4, k = 31$
	16×961	3	Lemma 6.22; $h = 4, k = 31$
	31×496	6	Lemma 6.11
	62×248	6	Lemma 6.11
	124×124	6	[18]
125	All possible	✓	Corollary 6.13

n	$a \times b$	N	Construction
126	2×7638	6	Lemma 6.11
	3×5292	6	Lemma 6.11
	4×3969	3	Lemma 6.16; $m = 2, s = 63$
	6×2646	6	Lemma 6.11
	7×2268	6	Lemma 6.11
	9×1764	6	Lemma 6.11
	12×1323	3	Lemma 6.16; $m = 2, s = 63$
	14×1134	6	Lemma 6.11
	18×882	6	Lemma 6.11
	21×756	6	Lemma 6.11
	27×588	3	Lemma 6.22; $h = 9, k = 14$
	28×567	3	Lemma 6.22; $h = 7, k = 18$
	36×441	3	Lemma 6.22; $h = 9, k = 14$
	42×378	6	Lemma 6.11
	49×324	3	Lemma 6.22; $h = 7, k = 18$
	54×294	3	Lemma 6.22; $h = 9, k = 14$
	63×252	6	Lemma 6.11
	81×196	3	Lemma 6.22; $h = 9, k = 14$
	84×189	3	Lemma 6.22; $h = 7, k = 18$
	98×162	3	Lemma 6.22; $h = 7, k = 18$
	126×126	6	[18]
127	127×127	✓	Corollary 6.13
128	All possible	✓	Corollary 6.13
129	3×5547	7	Lemma 6.11
	9×1849		
	43×387	7	Lemma 6.11
	129×129	7	[18]
130	2×8450	6	Lemma 6.11
	4×4225	3	Lemma 6.16; $m = 2, s = 65$
	5×3380	6	Lemma 6.11
	10×1690	6	Lemma 6.11
	13×1300	6	Lemma 6.11
	20×845	4	Lemma 6.22; $h = 5, k = 26$
	25×676	4	Lemma 6.22; $h = 5, k = 26$
	26×650	6	Lemma 6.11
	50×338	4	Lemma 6.22; $h = 5, k = 26$
	52×325	3	Lemma 6.20; $t = 13$, 3–$(4, 25; 10)$
	65×260	6	Lemma 6.11
	100×169	4	Lemma 6.22; $h = 5, k = 26$
	130×130	6	[18]
131	131×131	✓	Corollary 6.13

n	$a \times b$	N	Construction
132	2×8712	6	Lemma 6.11
	3×5808	6	Lemma 6.11
	4×4356	6	Lemma 6.11
	6×2904	6	Lemma 6.11
	8×2178	3	Lemma 6.22; $h = 4, k = 33$
	9×1936	4	Lemma 6.16; $m = 3, s = 44$
	11×1584	6	Lemma 6.11
	12×1452	6	Lemma 6.11
	16×1089	3	Lemma 6.22; $h = 4, k = 33$
	18×968		
	22×792	6	Lemma 6.11
	24×726	5	Lemma 6.22; $h = 12, k = 11$
	33×528	6	Lemma 6.11
	36×484	5	Lemma 6.22; $h = 12, k = 11$
	44×396	6	Lemma 6.11
	48×363	5	Lemma 6.22; $h = 12, k = 11$
	66×264	6	Lemma 6.11
	88×198	5	Lemma 6.22; $h = 11, k = 12$
	99×176	5	Lemma 6.22; $h = 11, k = 12$
	121×144	5	Lemma 6.22; $h = 11, k = 12$
	132×132	6	[18]
133	7×2527	7	Lemma 6.11
	19×937	7	Lemma 6.11
	49×361	6	Lemma 6.22; $h = 7, k = 19$
	133×133	7	[18]
134	2×8978	6	Lemma 6.11
	4×4489	3	Lemma 6.16; $m = 2, s = 67$
	67×268	6	Lemma 6.11
	134×134	6	[18]

n	$a \times b$	N	Construction
135	3×6075	7	Lemma 6.11
	5×3645	7	Lemma 6.11
	9×2025	7	Lemma 6.11
	15×1215	7	Lemma 6.11
	25×729	4	Lemma 6.22; $h = 5, k = 27$
	27×675	7	Lemma 6.11
	45×405	7	Lemma 6.11
	75×243	4	Lemma 6.22; $h = 5, k = 27$
	81×225	4	Lemma 6.22; $h = 27, k = 5$
	135×135	7	[18]
136	2×9248	7	Lemma 6.11
	4×4624	7	Lemma 6.11
	8×2312	7	Lemma 6.11
	16×1156	7	Lemma 6.22; $h = 8, k = 17$
	17×1088	7	Lemma 6.11
	32×578	7	Lemma 6.22; $h = 8, k = 17$
	34×544	7	Lemma 6.11
	64×289	7	Lemma 6.22; $h = 8, k = 17$
	68×272	7	Lemma 6.11
	136×136	7	[18]
137	137×137	✓	Corollary 6.13

n	$a \times b$	N	Construction
138	2×9522	6	Lemma 6.11
	3×6348	6	Lemma 6.11
	4×4761	3	Lemma 6.16; $m = 2, s = 69$
	6×3174	6	Lemma 6.11
	9×2116	4	Lemma 6.16; $m = 3, s = 46$
	12×1587	3	Lemma 6.16; $m = 2, s = 69$
	18×1058		
	23×828	6	Lemma 6.11
	36×529		
	46×414	6	Lemma 6.11
	69×276	6	Lemma 6.11
	92×207	3	Lemma 6.20; $t = 23, 3\text{–}(4, 9; 6)$
	138×138	6	[18]
139	139×139	✓	Corollary 6.13

n	$a \times b$	N	Construction
140	2×9800	6	Lemma 6.11
	4×4900	6	Lemma 6.11
	5×3920	6	Lemma 6.11
	7×2800	6	Lemma 6.11
	8×2450	3	Lemma 6.22; $h = 4, k = 35$
	10×1960	6	Lemma 6.11
	14×1400	6	Lemma 6.11
	16×1225	3	Lemma 6.22; $h = 4, k = 35$
	20×980	6	Lemma 6.11
	25×784	4	Lemma 6.22; $h = 5, k = 28$
	28×700	6	Lemma 6.11
	35×560	6	Lemma 6.11
	40×490	4	Lemma 6.22; $h = 5, k = 28$
	49×400	4	Lemma 6.22; $h = 7, k = 20$
	50×392	4	Lemma 6.22; $h = 5, k = 28$
	56×350	4	Lemma 6.22; $h = 7, k = 20$
	70×280	6	Lemma 6.11
	80×245	4	Lemma 6.22; $h = 5, k = 28$
	98×200	4	Lemma 6.22; $h = 7, k = 20$
	100×196	4	Lemma 6.22; $h = 5, k = 28$
	112×175	4	Lemma 6.22; $h = 7, k = 20$
	140×140	6	[18]
141	3×6627	7	Lemma 6.11
	9×2209		
	47×423	7	Lemma 6.11
	141×141	7	[18]
142	2×10082	6	Lemma 6.11
	4×5041	3	Lemma 6.16; $m = 2, s = 71$
	71×284	6	Lemma 6.11
	142×142	6	[18]

n	$a \times b$	N	Construction
143	11×1859	10	Lemma 6.11
	13×1573	10	Lemma 6.11
	121×169	10	Lemma 6.22; $h = 11, k = 13$
	143×143	10	[18]
144	2×10368	10	Lemma 6.11
	3×6912	10	Lemma 6.11
	4×5184	10	Lemma 6.11
	6×3456	10	Lemma 6.11
	8×2592	10	Lemma 6.11
	9×2304	10	Lemma 6.11
	12×1728	10	Lemma 6.11
	16×1296	10	Lemma 6.11
	18×1152	10	Lemma 6.11
	24×864	10	Lemma 6.11
	27×768	8	Lemma 6.22; $h = 9, k = 16$
	32×648	8	Lemma 6.22; $h = 16, k = 9$
	36×576	10	Lemma 6.11
	48×432	10	Lemma 6.11
	54×384	8	Lemma 6.22; $h = 9, k = 16$
	64×324	8	Lemma 6.22; $h = 16, k = 9$
	72×288	10	Lemma 6.11
	81×256	8	Lemma 6.22; $h = 9, k = 16$
	96×216	8	Lemma 6.22; $h = 16, k = 9$
	108×192	8	Lemma 6.22; $h = 9, k = 16$
	128×162	8	Lemma 6.22; $h = 16, k = 9$
	144×144	10	[18]
150	36×625		
174	36×841		
186	36×961		
210	36×1225		

$C =$ Computational construction

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